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A Comparison of the Generalized minC Combination and the Hybrid DSm Combination Rules

Published in:
Florentin Smarandache & Jean Dezert (Editors)
Advances and Applications of DSmT for Information Fusion
(Collected works), Vol. II
American Research Press, Rehoboth, 2006
ISBN: 1-59973-000-6
Chapter IV, pp. 113 - 129

Abstract: *A generalization of the minC combination to DSm hyper-power sets is presented. Both the special formulas for static fusion or dynamic fusion without non-existential constraints and the quite general formulas for dynamic fusion with non-existential constraints are included. Examples of the minC combination on several different hybrid DSm models are presented. A comparison of the generalized minC combination with the hybrid DSm rule is discussed and explained on examples.*

4.1 Introduction

Belief functions are one of the widely used formalisms for uncertainty representation and processing. Belief functions enable representation of incomplete and uncertain knowledge, belief updating and combination of evidence. Originally belief functions were introduced as a principal notion of Dempster-Shafer Theory (DST) or the Mathematical Theory of Evidence [19].

For combination of beliefs Dempster's rule of combinations is used in DST. Under strict probabilistic assumptions, its results are correct and probabilistically interpretable for any couple of belief functions. Nevertheless these assumptions are rarely fulfilled in real applications. There are not rare examples where the assumptions are not fulfilled and where results of Dempster's rule are counter intuitive, e.g. see [2, 3, 20], thus a rule with more intuitive results is required in such situations.

The support of the grant 1ET100300419 GA AV ČR is kindly announced.
The work was partly supported by the Institutional Research Plan AV0Z10300504 "Computer Science for the Information Society: Models, Algorithms, Applications".

Hence series of modifications of Dempster's rule were suggested and alternative approaches were created. The classical ones are Dubois-Prade's rule [13] and Yager's belief combination rule [23]. Among the others a wide class of operators [17] and an analogous idea proposed in [15], Smets' Transferable Belief Model (TBM) using so-called non-normalized Dempster's rule [22], disjunctive (or dual Dempster's) rule of combination [12], combination 'per elements' with its special case — minC combination, see [4, 8], and other combination rules. It is also necessary to mention the method for application of Dempster's rule in the case of partially reliable input beliefs [14].

A brand new approach performs the Dezert-Smarandache (or Dempster-Shafer modified) theory (DSmT) with its DSm rule of combination. There are two main differences: 1) mutual exclusivity of elements of a frame of discernment is not assumed in general; mathematically it means that belief functions are not defined on the power set of the frame, but on a so-called hyper-power set, i.e. on the Dedekind lattice defined by the frame; 2) a new combination mechanism which overcomes problems with conflict among the combined beliefs and which also enables a dynamic fusion of beliefs.

As the classical Shafer's frame of discernment may be considered the special case of a so-called hybrid DSm model, the DSm rule of combination is compared with the classic rules of combination in the publications about DSmT [11, 20]. For better and objective comparison with the DSm rule the classic Dempster's, Yager's, and Dubois-Prade's rules were generalized to DSm hyper-power sets [7].

In despite of completely different motivations, ideas and assumptions of minC combination and DSm rule, there is an analogy in computation mechanisms of these approaches described in the author's Chapter 10 in [20]. Unfortunately the minC combination had been designed for classic belief functions defined only on the power set of a frame of discernment in that time. Recently, formulas for computation of minC on general n-element frame discernment has been published [8], and the ideas of minC combination have been generalized to DSm hyper-power sets in [10].

A goal of this contribution is to continue [5] using the recent results from [10], and complete a comparison of minC combination and hybrid DSm rules.

4.2 MinC combination on classic frames of discernment

4.2.1 Basic Definitions

All the classic definitions suppose an exhaustive finite *frame of discernment* $\Theta = \{\theta_1, ..., \theta_n\}$, whose elements are mutually exclusive.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Theta) \longrightarrow [0,1]$, such that $\sum_{A \subseteq \Theta} m(A) = 1$, the values of bba are called *basic belief masses (bbm)*.[1] A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Theta) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$, belief function Bel uniquely corresponds to bba m and vice-versa. $\mathcal{P}(\Theta)$ is often denoted also by 2^{Θ}. A *focal element* is a subset X of the frame of discernment Θ, such that $m(X) > 0$.

Dempster's (conjunctive) rule of combination $\oplus$ is given as

$$(m_1 \oplus m_2)(A) = K \sum_{X \cap Y = A} m_1(X) m_2(Y)$$

[1] $m(\emptyset) = 0$ is often assumed in accordance with Shafer's definition [19]. A classical counter example is Smets' Transferable Belief Model (TBM) which admits positive $m(\emptyset)$ as it assumes $m(\emptyset) \geq 0$.

for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X \cap Y = \emptyset} m_1(X) m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [19]; putting $K = 1$ and $(m_1 \oplus m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\circledcirc$, see e. g. [22].

An algebra $\mathcal{L} = (L, \wedge, \vee)$ is called a *lattice* if $L \neq \emptyset$ and $\wedge$, $\vee$ are two binary operations *meet* and *join* on L with the following properties: $x \wedge x = x$, $x \vee x = x$ (idempotency), $x \wedge y = y \wedge x$, $x \vee y = y \vee x$ (commutativity), $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$ (associativity), and $x \wedge (y \vee x) = x$, $x \vee (y \wedge x) = x$ (absorption). If the operations $\wedge$, $\vee$ satisfy also distributivity, i.e. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ we speak about a *distributive lattice*.

We can equivalently write any element of $X \in L$ in *conjunctive normal form* (CNF):
$X = \bigwedge_{i=1,...,m}(\bigvee_{j=1,...,k_i} X_{ij})$ for some m, $k_1, ..., k_m$, $X_{ij} \in L$, i.e. meet of joins.

4.2.2 Ideas of the minC combination

The minC combination (the minimal conflict/contradiction combination) is a generalization of the non-normalized Dempster's rule $\circledcirc$. $m(\emptyset)$ from $\circledcirc$ is considered as a *conflict (or contradiction)* arising by the conjunctive combination. To handle it, a system of different types of conflicts is considered according to the basic belief masses producing it.

We distinguish contradictions (conflicts) according to the sets to which the original bbms were assigned by m_i. There is only one type of contradiction (conflict) $\times$ on the belief functions defined on a binary frame of discernment, $\times$ corresponds to $m(\emptyset)$; hence the generalized level of minC combination fully coincides with the (non-normalized) conjunctive rule there. In the case of an n-element frame of discernment we distinguish different types of conflicts, e.g. $A \times B$, $A \times BC$, $A \times B \times C$, if $m_i(\{A\}), m_j(\{B\}) > 0$, $m_i(\{A\}), m_j(\{B, C\}) > 0$, $m_i(\{A\})$, $m_j(\{B\})$, $m_k(\{C\}) > 0$ etc. A very important role is played by so-called *potential conflicts (contradictions)*, e.g. $AB \times BC$, which is not a conflict in the case of combination of two beliefs ($\{A, B\} \cap \{B, C\} = \{B\} \neq \emptyset$), but it can cause a conflict in a later combination with another belief, e.g. real conflict $AB \times BC \times AC$ because there is $\{A, B\} \cap \{B, C\} \cap \{A, C\} = \emptyset$ which is different from $B \times AC$. Not to have (theoretically) an infinite number of different conflicts, the conflicts are divided into classes of equivalence which are called *types of conflicts*, e.g. $A \times B \sim B \times A \sim A \times B \times B \times B \times A \times A \times A$, etc. For more detail see [4].

In full version of [8], it is shown that the structure of pure and potential conflicts forms a distributive lattice $\mathcal{L}(\Omega) = (L(\Omega), \wedge, \vee)$, where $X \in L(\Omega)$ iff either $X = \{\omega_i\}$, where $\omega_i \in \Omega$, or $X = \{\omega_{i1} \times \omega_{i2} \times .... \times \omega_{ik_i}\}$, where $\omega_{ij} \in \Omega$ for $1 \leq i \leq n$, $1 \leq j \leq k_i$, or $X = U \wedge V$ or $X = U \vee V$ for some couple $U, V \in \mathcal{L}(\Omega)$; $\wedge$ $\vee$ are defined as it follows:
$X \vee Y = \{w \mid w \in X \text{ or } w \in Y \text{ and } (\neg \exists w')(w' \in X \cup Y,\ w' \leq w)\}$,
$X \wedge Y = \{w \mid w \in X \cap Y \text{ or } [w = \omega_{w1} \times \omega_{w2} \times ... \times \omega_{wk_w}, \text{ where } (\exists x \in X)(x \leq w), (\exists y \in Y)(y \leq w) \text{ and } (\neg \exists w' \leq w)((\exists x \in X)(x \leq w'), (\exists y \in Y)(y \leq w'))]\}$.
Where it is further defined: $x \times x = x$, $y \times x = x \times y$, and $x_{11} \times x_{12} \times ... \times x_{1k_1} \leq x_{21} \times x_{22} \times ... \times x_{2k_2}$ iff $(\forall x_{1k})(\exists x_{2m})(x_{1k} = x_{2m})$. Note that $X \wedge Y = X \cap Y$ if $X \subseteq Y$ or $Y \subseteq X$.

We can extend $\mathcal{L}(\Omega)$ with $\emptyset$ to $\mathcal{L}_{\emptyset}(\Omega) = (L(\Omega) \cup \{\emptyset\}, \wedge, \vee)$, where $x \wedge \emptyset = \emptyset$ and $x \vee \emptyset = x$ for all $x \in L(\Omega)$. But we do no need it in classical case as no positive gbbm's are assigned to $\emptyset$ in input BF's and $x \wedge y \neq \emptyset$ and $x \vee y \neq \emptyset$ for any $x, y \in \mathcal{L}(\Omega)$.

The generalized level of minC combination gives non-negative weights to all elements of $\mathcal{L}(\Theta)$, i.e. also to the conflicts/contradictions and potential conflicts, i.e. it produces and combines so-called *generalized bba's* and *generalized* belief functions defined on the so-called *generalized frame of discernment* $\mathcal{L}(\Theta)$, which includes also all corresponding types of conflicts.

The generalized level of minC combination is associative and commutative operation and it commutes also with coarsening of frame of discernment. After performance of the generalized level of the minC, all bbms of both pure and potential conflicts should be reallocated / proportionalized among all corresponding non-conflicting elements of $\mathcal{P}(\Theta)$.

Unfortunately such proportionalizations break associativity of the minC combination. Hence all the input bba's must be combined on the generalized level at first, and the proportionalization may not be performed before finishing of the generalized level combination. So it is useful to keep also generalized level results because of to be prepared for possible additional source of belief, which we possibly want to combine together with the present input beliefs.

4.2.3 Formulas for the minC combination

Let $\bigcap X = X_1 \cap X_2 \cap ... \cap X_k$ and $c(X) = \{X_1, ..., X_k\}$, where $CNF(X) = X_1 \wedge X_2 \wedge ... \wedge X_k$, similarly let $\bigcup X = X_1 \cup X_2 \cup ... \cup X_k$, where $CNF(X) = X_1 \wedge X_2 \wedge ... \wedge X_k$, it holds that $X_i = X_{i1} \vee X_{i2} \vee ... \vee X_{ik_i}$ for any of these X_is thus it corresponds to $\{X_{i1}, X_{i2}, ..., X_{ik_i}\}$, and $\bigcup X \in \mathcal{P}(\Theta)$, let further $p(X) = \{Y_1 \cup ... \cup Y_m \mid 1 \leq m \leq k, Y_i \in c(X) \text{ for } i = 1, ..., m\}$. Let all X from $\mathcal{L}(\Theta)$ be in CNF in the following formulas, unless another form of X is explicitly specified.

The generalized level of the minC combination is computed for all $A \in \mathcal{L}(\Theta)$ as

$$m^0(A) = \sum_{X \wedge Y = A} m_1(X)\, m_2(Y).$$

Reallocation of gbbm's of potential conflicts: for all $\emptyset \neq A \in \mathcal{P}(\Theta)$,

$$m^1(A) = m^0(A) + \sum_{\substack{X \in \mathcal{L}(\Theta) \\ X \neq A,\ \bigcap X = A}} m^0(X) = \sum_{\substack{X \in \mathcal{L}(\Theta) \\ \bigcap X = A}} m^0(X).$$

Final classic bba m we obtain after proportionalization of gbbm's of pure conflicts.

$$m(A) = \sum_{\substack{X \in \mathcal{L}(\Theta) \\ \cap X = A}} m^0(X) \ + \sum_{\substack{X \in \mathcal{L}(\Theta) \\ \cap X = \emptyset,\ A \subseteq \cup X}} prop(A, X)\, m^0(X),$$

where
$prop_{11}(A, X) = prop_{12}(A, X) = \frac{m^1(A)}{\sum_{Y \in p(X)} m^1(Y)}$ for $A \in p(X)$, $\sum_{Y \in p(X)} m^1(Y) > 0$,
$prop_{11}(A, X) = prop_{12}(A, X) = 0$ for $A \notin p(X)$,
$prop_{11}(A, X) = \frac{1}{|p(X)|-1}$ for $A \in p(X)$, $\sum_{Y \in p(X)} m^1(Y) = 0$,
$prop_{12}(A, X) = 1$ for $A = \bigcup X$, $\sum_{Y \in p(X)} m^1(Y) = 0$,
$prop_{12}(A, X) = 0$ for $A \subset \bigcup X$, $\sum_{Y \in p(X)} m^1(Y) = 0$,
$prop_{21}(A, X) = prop_{22}(A, X) = \frac{m^1(A)}{cbel^1(X)}$ for $cbel^1(X) > 0$,
$prop_{21}(A, X) = \frac{1}{2^{|\bigcup X|}-1}$ for $cbel^1 = 0$,
$prop_{22}(A, X) = \frac{m^1(A)}{cbel^1(X)}$ for $cbel^1(X) > 0$,
$prop_{22}(A, X) = 1$ for $cbel^1(X) = 0$ and $A = \bigcup X$,
$prop_{22}(A, X) = 0$ for $cbel^1(X) = 0$ and $A \subset \bigcup X$,
where $cbel^1(X) = \sum_{\emptyset \neq Y \in \mathcal{P}(\Theta),\ Y \subseteq \bigcup X} m^1(Y)$, $m(\emptyset) = 0$ $(= m^0(\emptyset) = m^1(\emptyset))$.

Proportionalization coefficient function $prop_{ij}(\_,\_)$ determines the proportionalization ratio for distribution of conflicting gbbm's. The first index i indicates whether 1) $m^0(X)$ is proportionalized only among elements of $p(X)$, i.e, among all conjuncts from $CNF(X)$ and among all disjunctions of these conjuncts for $i = 1$, or 2) $m^0(X)$ is proportionalized among all subsets of $\bigcup X$ for $i = 2$. The second index indicates the way of proportionalization when the proportionalization ratio is "$\frac{0}{0}$": 1) division of $m^0(X)$ to the same parts and distribution of these parts among all conjuncts in question (for $i = 1, j = 1$) or among all subsets of $\bigcup X$ (for $i = 2, j = 1$) is used, or 2) whole conflicting gbbm $m^0(X)$ is relocated to $\bigcup X$ for $j = 2$. $prop_{1j}$ corresponds to proportionalization a) from [4, 5] and $prop_{2j}$ corresponds to proportionalization b) from [5] (resp. to c) from [4]). For another proportionalizations see the full version of [8].

Let us present the proportionalization on a small example $m^0(X)$, where $X = \theta_1 \wedge (\theta_2 \vee \theta_3)$: X is already in CNF, i.e. $CNF(X) = X$, it has two conjuncts singleton θ_1 and disjunction $\theta_2 \vee \theta_3$, we can construct the only nontrivial disjunction $\theta_1 \vee \theta_2 \vee \theta_3$ from these conjuncts, $\bigcup X = \theta_1 \vee \theta_2 \vee \theta_3$.

$prop_{1j}$ proportionalizes conflicting $m^0(X)$ among conjuncts θ_1, $\theta_2 \vee \theta_3$, and their disjunction $\theta_1 \vee \theta_2 \vee \theta_3$:
if $m^1(\theta_1) + m^1(\theta_2 \vee \theta_3) + m^1(\theta_1 \vee \theta_2 \vee \theta_3) > 0$ we have:

$$prop_{1j}(\theta_1, X) = \frac{m^1(\theta_1)}{m^1(\theta_1) + m^1(\theta_2 \vee \theta_3) + m^1(\theta_1 \vee \theta_2 \vee \theta_3)}$$

$$prop_{1j}(\theta_2 \vee \theta_3, X) = \frac{m^1(\theta_2 \vee \theta_3)}{m^1(\theta_1) + m^1(\theta_2 \vee \theta_3) + m^1(\theta_1 \vee \theta_2 \vee \theta_3)}$$

$$prop_{1j}(\theta_1 \vee \theta_2 \vee \theta_3, X) = \frac{m^1(\theta_1 \vee \theta_2 \vee \theta_3)}{m^1(\theta_1) + m^1(\theta_2 \vee \theta_3) + m^1(\theta_1 \vee \theta_2 \vee \theta_3)}$$

if $m^1(\theta_1) + m^1(\theta_2 \vee \theta_3) + m^1(\theta_1 \vee \theta_2 \vee \theta_3) = 0$ we have:

$$prop_{11}(\theta_1, X) = prop_{11}(\theta_2 \vee \theta_3, X) = prop_{11}(\theta_1 \vee \theta_2 \vee \theta_3) = 1/3$$

$$prop_{12}(\theta_1, X) = prop_{11}(\theta_2 \vee \theta_3, X) = 0, prop_{12}(\theta_1 \vee \theta_2 \vee \theta_3) = 1.$$

$prop_{2j}$ proportionalizes conflicting $m^0(X)$ among all subsets of $\bigcup X = \theta_1 \vee \theta_2 \vee \theta_3$, i.e. among $\theta_1, \theta_2, \theta_3, \theta_1 \vee \theta_2, \theta_1 \vee \theta_3, \theta_2 \vee \theta_3, \theta_1 \vee \theta_2 \vee \theta_3$:
if $S = m^1(\theta_1) + m^1(\theta_2) + m^1(\theta_3) + m^1(\theta_1 \vee \theta_2) + m^1(\theta_1 \vee \theta_3) + m^1(\theta_2 \vee \theta_3) + m^1(\theta_1 \vee \theta_2 \vee \theta_3) > 0$ we have, $prop_{2j}(A, X) = \frac{m^1(A)}{S}$ for all $A \subseteq \bigcup X$;
if $S = 0$ we have, $prop_{21}(A, X) = 1/7$ for all $A \subseteq \bigcup X$. $prop_{22}(A, X) = 0$ for all $A \subset \bigcup X$, $prop_{22}(\bigcup X) = 1$.

4.3 Introduction to DSm theory

Because DSmT is a new theory which is in permanent dynamic evolution, we have to note that this text is related to its state described by formulas and text presented in the basic publication on DSmT — in the DSmT book Vol. 1 [20]. Rapid development of the theory is demonstrated by appearing of the current second volume of the book. For new advances of DSmT see other chapters of this volume.

4.3.1 Dedekind lattice and other basic DSm notions

Dempster-Shafer modified Theory or Dezert-Smarandache Theory (DSmT) by J. Dezert and F. Smarandache [11, 20] allows mutually overlapping elements of a frame of discernment. Thus a frame of discernment is a finite exhaustive set of elements $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}$, but not necessarily exclusive in DSmT. As an example we can introduce a three-element set of colours $\{Red, Green, Blue\}$ from the DSmT homepage[2]. DSmT allows that an object can have 2 or 3 colours in the same time: e.g. it can be both red and blue, or red and green and blue in the same time, it corresponds to a composition of general colours from the 3 basic ones.

DSmT uses basic belief assignments and belief functions defined analogically to the classic Dempster-Shafer theory (DST), but they are defined on so-called hyper-power set or Dedekind lattice instead of the classic power set of the frame of discernment. To be distinguished from the classic definitions they are called generalized basic belief assignments and generalized belief functions[3].

The *Dedekind lattice*, more frequently called *hyper-power set* D^Θ in DSmT, is defined as the set of all composite propositions built from elements of Θ with union and intersection operators $\cup$ and $\cap$ such that $\emptyset, \theta_1, \theta_2, ..., \theta_n \in D^\Theta$, and if $A, B \in D^\Theta$ then also $A \cup B \in D^\Theta$ and $A \cap B \in D^\Theta$, no other elements belong to D^Θ ($\theta_i \cap \theta_j \neq \emptyset$ in general, $\theta_i \cap \theta_j = \emptyset$ iff $\theta_i = \emptyset$ or $\theta_j = \emptyset$).

Thus the hyper-power set D^Θ of Θ is closed to $\cup$ and $\cap$ and $\theta_i \cap \theta_j \neq \emptyset$ in general. Whereas the classic power set 2^Θ of Θ with exclusive elements is closed to $\cup$, $\cap$ and complement, and $\theta_i \cap \theta_j = \emptyset$ for every $i \neq j$.

Examples of hyper-power sets. Let $\Theta = \{\theta_1, \theta_2\}$, we have $D^\Theta = \{\emptyset,\ \theta_1 \cap \theta_2,\ \theta_1,\ \theta_2,\ \theta_1 \cup \theta_2\}$, i.e. $|D^\Theta| = 5$. For $\Theta = \{\theta_1, \theta_2, \theta_3\}$ we have $|\Theta| = 3$, $|D^\Theta| = 19$.

A *DSm generalized basic belief assignment (DSm gbba)* m is a mapping $m : D^\Theta \longrightarrow [0, 1]$, such that $\sum_{A \in D^\Theta} m(A) = 1$ and $m(\emptyset) = 0$. The quantity $m(A)$ is called the *DSm generalized basic belief mass (DSm gbbm) of A*. A *DSm generalized belief function (DSm gBF) Bel* is a mapping $Bel : D^\Theta \longrightarrow [0, 1]$, such that $Bel(A) = \sum_{X \subseteq A, X \in D^\Theta} m(X)$.

4.3.2 DSm models

If we assume a Dedekind lattice (hyper-power set) according to the above definition without any other assumptions, i. e. all elements of an exhaustive frame of discernment can mutually overlap themselves, we speak about the *free DSm model* $\mathcal{M}^f(\Theta)$, i. e. about DSm model free of constraints.

In general it is possible to add exclusivity or non-existential constraints into DSm models, we speak about *hybrid DSm models* in such cases.

An exclusivity constraint $\theta_1 \cap \theta_2 \overset{\mathcal{M}_1}{\equiv} \emptyset$ says that elements θ_1 and θ_2 are mutually exclusive in model $\mathcal{M}_1$, whereas both of them can overlap with θ_3. If we assume exclusivity constraints $\theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_1 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, another exclusivity constraint directly follows them: $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$. In this case all the elements of the 3-element frame of discernment

[2] www.gallup.unm.edu/~smarandache/DSmT.htm

[3] If we want to distinguish these generalized notions from the generalized level of minC combination we use DSm generalized basic belief assignment, DSm generalized belief mass and function, and analogically minC generalized basic belief assignment and minC gbbm further in this text, on the other hand no minC generalized BF has been defined.

$\Theta = \{\theta_1, \theta_2, \theta_3\}$ are mutually exclusive as in the classic Dempster-Shafer theory, and we call such hybrid DSm model as *Shafer's model* $\mathcal{M}^0(\Theta)$.

A non-existential constraint $\theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$ brings an additional information about a frame of discernment saying that θ_3 is impossible, it forces all the gbbm of $X \subseteq \theta_3$ to be equal to zero for any gbba in model $\mathcal{M}_3$. It represents a sure meta-information with respect to generalized belief combination, which is used in a dynamic fusion.

In a degenerated case of the *degenerated DSm model* $\mathcal{M}_\emptyset$ we always have $m(\emptyset) = 1$, $m(X) = 0$ for $X \neq \emptyset$. It is the only gbbm on $\mathcal{M}_\emptyset$, and it is the only case, where $m(\emptyset) > 0$ is allowed in DSmT.

The total ignorance on Θ is the union $I_t = \theta_1 \cup \theta_2 \cup ... \cup \theta_n$. $\boldsymbol{\emptyset} = \{\boldsymbol{\emptyset}_\mathcal{M}, \emptyset\}$, where $\boldsymbol{\emptyset}_\mathcal{M}$ is the set of all elements of D^Θ which are forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical empty set[4]. Because we will not work with $\mathcal{M}_\emptyset$ in the present contribution, we will work only $\emptyset \neq X \in D^\Theta$, thus $X \in \boldsymbol{\emptyset}$ is the same as $X \in \boldsymbol{\emptyset}_\mathcal{M}$ in this text.

For a given DSm model we can define (in addition to [20]) $\Theta_\mathcal{M} = \{\theta_i | \theta_i \in \Theta, \theta_i \notin \boldsymbol{\emptyset}_\mathcal{M}\}$, $\Theta_\mathcal{M} \overset{\mathcal{M}}{\equiv} \Theta$, and $I_\mathcal{M} = \bigcup_{\theta_i \in \Theta_\mathcal{M}} \theta_i$, i.e. $I_\mathcal{M} \overset{\mathcal{M}}{\equiv} I_t$, $I_\mathcal{M} = I_t \cap \Theta_\mathcal{M}$, $I_{\mathcal{M}_\emptyset} = \emptyset$. $D^{\Theta_\mathcal{M}}$ is a hyper-power set on the DSm frame of discernment $\Theta_\mathcal{M}$, i.e. on Θ without elements which are excluded by the constraints of model $\mathcal{M}$. It holds $\Theta_\mathcal{M} = \Theta$, $D^{\Theta_\mathcal{M}} = D^\Theta$and $I_\mathcal{M} = I_t$ for any DSm model without non-existential constraint. Whereas *reduced hyper-power set* $D^\Theta_\mathcal{M}$ from Chapter 4 in [20] arises from D^Θ by identifying of all $\mathcal{M}$-equivalent elements. $D^\Theta_{\mathcal{M}^0}$ corresponds to classic power set 2^Θ.

4.3.3 The DSm rule of combination

The *classic DSm rule* (DSmC) is defined for belief combination on the free DSm model as it follows[5]:

$$m_{\mathcal{M}^f(\Theta)}(A) = (m_1 \circledcirc m_2)(A) = \sum_{X \cap Y = A,\ X,Y \in D^\Theta} m_1(X)\, m_2(Y).$$

Since D^Θ is closed under operators $\cap$ and $\cup$ and all the $\cap$s are non-empty, the classic DSm rule guarantees that $(m_1 \circledcirc m_2)$ is a proper generalized basic belief assignment. The rule is commutative and associative. For n-ary version of the rule see [20].

When the free DSm model $\mathcal{M}^f(\Theta)$ does not hold due to the nature of the problem under consideration, which requires to take into account some known integrity constraints, one has to work with a proper hybrid DSm model $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$. In such a case, the *hybrid DSm rule of combination DSmH* based on the hybrid model $\mathcal{M}(\Theta)$, $\mathcal{M}^f(\Theta) \neq \mathcal{M}(\Theta) \neq \mathcal{M}_\emptyset(\Theta)$, for $k \geq 2$ independent sources of information is defined as: $m_{\mathcal{M}(\Theta)}(A) = (m_1 \circledcirc m_2 \circledcirc ... \circledcirc m_k)(A) = \phi(A)[S_1(A)+S_2(A)+S_3(A)]$, in full generality, see [20]. For a comparison with minC combination we use binary version of the rule, thus we have:

$$m_{\mathcal{M}(\Theta)}(A) = (m_1 \circledcirc m_2)(A) = \phi(A)[S_1(A) + S_2(A) + S_3(A)],$$

where $\phi(A)$ is a *characteristic non-emptiness function* of a set A, i. e. $\phi(A) = 1$ if $A \notin \boldsymbol{\emptyset}$ and $\phi(A) = 0$ otherwise. $S_1 \equiv m_{\mathcal{M}^f(\Theta)}$, $S_2(A)$, and $S_3(A)$ are defined by

$$S_1(A) = \sum_{X,Y \in D^\Theta,\, X \cap Y = A} m_1(X)\ , m_2(Y)$$

[4] $\boldsymbol{\emptyset}$ should be $\boldsymbol{\emptyset}_\mathcal{M}$ extended with the classical empty set $\emptyset$, thus more correct should be the expression $\boldsymbol{\emptyset} = \boldsymbol{\emptyset}_\mathcal{M} \cup \{\emptyset\}$.

[5] To distinguish the DSm rule from Dempster's rule, we use $\circledcirc$ instead of $\oplus$ for the DSm rule in this text.

$$S_2(A) = \sum_{X,Y \in \emptyset,\ [\mathcal{U}=A] \vee [(\mathcal{U} \in \emptyset) \wedge (A=I_t)]} m_1(X)\, m_2(Y)$$

$$S_3(A) = \sum_{X,Y \in D^\Theta,\ X \cup Y = A,\ X \cap Y \in \emptyset} m_1(X)\, m_2(Y)$$

with $\mathcal{U} = u(X) \cup u(Y)$, where $u(X)$ is the union of all singletons θ_i that compose X and Y; all the sets A, X, Y are supposed to be in some canonical form, e.g. CNF. Unfortunately no mention about the canonical form is included in [20].

As size of hyper-power set D^Θ rapidly increase with cardinality of the frame of discernment Θ some readers may be interested in Chapter 2 of [20] on the generation of hyper-power sets, including subsection about memory size and complexity. For applications of DSmT see contributions in second parts of both the volumes of DSmT book.

In [20], is was shown that DSm hyper-power set corresponds to minC generalized frame of discernment extended with $\emptyset$, where overlappings of elements in DSm hyper-power set correspond to elementary conflicts in minC generalized frame of discernment and that the classic DSm rule numerically coincides with the generalized level of minC combination.

4.4 MinC combination on hyper-power sets

4.4.1 Generalized level of minC combination on hyper-power set

From the correspondence of hyper-power set (Dedekind Lattice) D^Θ with distributive lattice $\mathcal{L}_\emptyset(\Theta)$ representing extended minC generalized frame of discernment and from numerical coincidence of the classic DSm rule with generalized level of minC combination, we obtain coincidence of generalized level of minC on the hyper-power set with the generalized level of the classic minC combination and with the classic DSm rule (DSmC). Hence the generalized level of the minC combination on the hyper-power set is given by the following formula:

$$(m_1 \circledcirc m_2)^0(A) = m^0(A) = \sum_{X \wedge Y = A} m_1(X)\, m_2(Y) = \sum_{X \cap Y = A} m_1(X)\, m_2(Y).$$

4.4.2 MinC combination on the free DSm model $\mathcal{M}^f$

There are no constraints on the free DSm model, all elements of hyper-power set are allowed to have a positive (DSm generalized) bbm. It means that there are no conflicting bbms in minC combination generalized to the free DSm model. Thus no reallocation of bbms is necessary in minC combination generalized to the free DSm model. Thus minC combination generalized to the free DSm model coincides with its generalized level from the previous subsection:

$$m(A) = m^0(A) = \sum_{X \cap Y = A} m_1(X)\, m_2(Y).$$

Hence the generalized level of the minC combination and the minC combination on the free DSm model is associative and commutative operation on DSm generalized belief functions. The combination also commutes with coarsening of the frame of discernment.

Let us note that $m(\emptyset) = 0 = m^0(\emptyset)$ always holds as $X \cap Y \neq \emptyset$ for any $X, Y \in D^\Theta$, and $m_i(\emptyset) = 0$ for any DSm gbba on D^Θ.

4.4.3 Static minC combination on hybrid DSm models

Let us continue our generalization with a static combination, where DSm model is not changed within the combination process, i.e. all input belief functions are defined on a hybrid model in question. Let us suppose a fixed DSm model $\mathcal{M}$, thus we can use $\equiv$ instead of $\overset{\mathcal{M}}{\equiv}$ for simplification of generalized minC formulas.

As some of the elements of D^Θ are equal to other ones in hybrid DSm model $\mathcal{M}$, we have to reallocate their m^0 gbbm's to a corresponding elements $D^\Theta_\mathcal{M}$ as it follows:

$$m^1(A) = m^0(A) + \sum_{X \neq A,\ X \in D^\Theta,\ X \equiv A} m^0(X) \ = \sum_{X \in D^\Theta,\ X \equiv A} m^0(X),$$

for all $\emptyset \not\equiv A \in D^\Theta$, (i.e. for all $A \notin \emptyset_\mathcal{M}$). This step corresponds to relocation of potential conflicts in classic minC combination.

The rest is reallocation of m^0 bbms of sets which are equivalent to $\emptyset$; such sets correspond to pure conflicts in the classic case. Analogically to the proportionalization of gbbm of pure conflict X to its power set $\mathcal{P}(\bigcup X)$ in the classic minC combination, we proportionalize[6] conflicting gbbm $m^0(X)$ to substructure of the DSm model $\mathcal{M}$ defined by $\bigcup X$, i.e. to $D^{\bigcup X}_\mathcal{M}$, we do not care about $Y \equiv \emptyset_\mathcal{M}$ because they are not allowed by model $\mathcal{M}$.

$$m(A) = m^1(A) + \ reallocated\ gbbm's\ of\ conflicts.$$

$$m(A) = \sum_{\substack{X \in D^\Theta \\ X \equiv A}} m^0(X) \ + \sum_{\substack{X \in D^\Theta \\ X \equiv \emptyset,\, A \subseteq \cup X}} prop(A,X)\, m^0(X),$$

where proportionalization coefficient function *prop* is analogous to the *prop* in the classic version; there are only the following differences in notation: we use $X \in D^\Theta$ instead of $X \in \mathcal{L}(\Theta)$, $X \in D^\Theta_\mathcal{M}$ instead of $X \in \mathcal{P}(\Theta)$, $bel^1_\mathcal{M}$ instead of $cbel^1$, $|D^\Theta_\mathcal{M}|$ .... $\mathcal{P}(\Theta) = 2^{|\Theta|}$, $|D^{\bigcup X}_\mathcal{M}|$ .... $2^{|\bigcup X|} = |p(X)|$, $A \in D^{c(X)}_\mathcal{M}$ .... $A \in p(X)$, $Z \in \boldsymbol{\emptyset}_\mathcal{M}$ .... $\bigcup Z = \emptyset$, and similarly. Where $D^{c(X)}_\mathcal{M} = \{Y \in D^\Theta_\mathcal{M} \mid c(Y) \subseteq c(X)\}$, i.e. elements of $D^{c(X)}_\mathcal{M}$ are all unions and intersections constructed from conjuncts from $CNF(X)$ (from X_i such that $CNF(X) = X_1 \cap ... \cap X_k$). Let X be such that $CNF(X) = (\theta_1 \cup \theta_2) \cap (\theta_1 \cup \theta_3) \cap \theta_4$ for example, thus $c(X) = \{\theta_1 \cup \theta_2,\ \theta_1 \cup \theta_3,\ \theta_4\}$, and $D^{c(X)}_{\mathcal{M}^f}$ contains e.g. $\theta_1 \cup \theta_2 \cup \theta_4$ and $(\theta_1 \cup \theta_2) \cap \theta_4$, but neither $\theta_1 \cup \theta_4$ or $\theta_1 \cap \theta_4$ nor $\theta_2 \cup \theta_3 \cup \theta_4$ as $\theta_1, \theta_2, \theta_3, \theta_2 \cup \theta_3, \theta_2 \cup \theta_4, \theta_3 \cup \theta_4$ are not elements of $c(X)$.

For $m^0(X) > 0$ we have that $(\bigcup X) \notin \emptyset_\mathcal{M}$ in static combination, because $X \subseteq \bigcup X$ and similarly for all input focal elements X_i from which $m^0(X)$ is computed $X_i \subseteq \bigcup X$. Thus we have no problem with cardinality $|D^{\cup X}_\mathcal{M}|$ which is always ≥ 2.
It is possible to show that $\sum_{X \in D^\Theta_\mathcal{M}} m(X) = 1$, i.e. $m(A)$ correctly defines static combination of gbba's on hybrid DSm model $\mathcal{M}$. We can also show that the above definition coincides with the classic minC combination on the Shafer's DSm model $\mathcal{M}_0$. Hence the above definition really generalizes the classic minC combination.

4.4.4 Dynamic minC combination

To make a full generalization of minC combination in the DSm nature. We have to allow also a change of a DSm model during combination, i.e. to allow input belief functions which

[6] If a proportionalization ratio is not defined, i.e. if it should be "$\frac{0}{0}$" then either 1) division to the same parts or 2) reallocation to $\bigcup X$ is used, analogically to the classic case.

are defined on more general model that is the resulting one, i.e. we have to be prepared to prohibition of some input focal elements. In such a case we have no immediate suggestion how reallocate $m^0(X)$ for $X \equiv \emptyset$ such that also $\bigcup X \equiv \emptyset$. In correspondence to non-defined proportionalization ratios we can distribute it among all non-empty elements of DSm model $\mathcal{M}$ or to relocate it to whole $I_{\mathcal{M}}$. We can represent both these proportionalizations with coefficient functions $prop(A, X)$ for computation of proportion of conflicting gbbm $m^0(X)$ which to be reallocated to $\emptyset \not\equiv A \in D^{\cup X}_{\mathcal{M}}$ and analogical $dyn(A, X)$ for dynamic fusion proportionalization of $m^0(X)$ where $\bigcup X \equiv \emptyset$. With respect to two types of proportionalization and two variants of non-defined proportionalizaton ratios managing we obtain four variants of coefficient function $prop$ and two variants coefficient function dyn: of $prop_{11}(A, X)$, $prop_{12}(A, X)$, $prop_{21}(A, X)$, $prop_{22}(A, X)$, $dyn_1(A, X)$, and $dyn_2(A, X)$. We can summarize the dynamic minC combination as it follows:

$$m^0(A) = \sum_{\substack{X,Y \in D^\Theta \\ X \cap Y = A}} m_1(X)\, m_2(Y)$$

$$m_{ij}(A) = \sum_{\substack{X \in D^\Theta \\ X \equiv A}} m^0(X) + \sum_{\substack{\emptyset \equiv X \in D^\Theta \\ A \subseteq \bigcup X}} prop_{ij}(A, X)\, m^0(X) + \sum_{\substack{X \in D^\Theta \\ \bigcup X \equiv \emptyset}} dyn_j(A, X)\, m^0(X)$$

for all $\emptyset \not\equiv A \in D^\Theta_{\mathcal{M}}$, where $|D^\Theta_{\mathcal{M}}| > 1$ and where $prop_{ij}(A, X), dyn_j(A, X)$ are defined as it follows:

$prop_{11}(A, X) = prop_{12}(A, X) = \frac{m^1(A)}{\sum_{A,Y \in D^{c(X)}_{\mathcal{M}}} m^1(Y)}$ for $A \in D^{c(X)}_{\mathcal{M}}$, $\sum_{Y \in D^{c(X)}_{\mathcal{M}}} m^1(Y) > 0$,
$prop_{11}(A, X) = prop_{12}(A, X) = 0$ for $A \notin D^{c(X)}_{\mathcal{M}}$,
$prop_{11}(A, X) = \frac{1}{|D^{c(X)}_{\mathcal{M}}| - 1}$ for $A \in D^{c(X)}_{\mathcal{M}}$, $\sum_{Y \in D^{c(X)}_{\mathcal{M}}} m^1(Y) = 0$,
$prop_{12}(A, X) = 1$ for $A = \bigcup X$, $\sum_{Y \in D^{c(X)}_{\mathcal{M}}} m^1(Y) = 0$,
$prop_{12}(A, X) = 0$ for $A \subset \bigcup X$, $\sum_{Y \in D^{c(X)}_{\mathcal{M}}} m^1(Y) = 0$,

$prop_{21}(A, X) = \frac{m^1(A)}{bel^1_{\mathcal{M}}(X)}$ for $bel^1_{\mathcal{M}}(X) > 0$,
$prop_{21}(A, X) = \frac{1}{|D^{\cup X}_{\mathcal{M}}| - 1}$ for $bel^1_{\mathcal{M}}(X) = 0$,
$prop_{22}(A, X) = \frac{m^1(A)}{bel^1_{\mathcal{M}}(X)}$ for $bel^1_{\mathcal{M}}(X) > 0$,
$prop_{22}(A, X) = 1$ for $bel^1_{\mathcal{M}}(X) = 0$ and $A = \bigcup X$,
$prop_{22}(A, X) = 0$ for $bel^1_{\mathcal{M}}(X) = 0$ and $A \subset \bigcup X$,

$dyn_1(A, \_) = \frac{m^1(A)}{\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z)}$, if $\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z) > 0$,
$dyn_1(A, \_) = \frac{1}{|D^\Theta_{\mathcal{M}}| - 1}$, if $\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z) = 0$,
$dyn_2(A, \_) = \frac{m^1(A)}{\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z)}$, if $\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z) > 0$,
$dyn_2(I_{\mathcal{M}}, \_) = 1$, if $\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z) = 0$,
$dyn_2(A, \_) = 0$, if $\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z) = 0$, $A \neq I_{\mathcal{M}}$,

$m_{ij}(A) = 0$ for $A \equiv \emptyset$.

Similarly to the classic case we can show that $\sum_{X \in D^\Theta_{\mathcal{M}}} m(X) = 1$ hence the above formulas produce a correct gbba also for dynamic combination.

If we want to combine 3 or more (k) gBF's, we apply twice or more times (k times) the binary combination on the generalized level (in the classic minC terminology), i.e. on the free

			$\mathcal{M}^f$	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_7$
	m_1	m_2	m^0	m_{ij}	m_{1j}	m_{2j}	m_{1j}	m_{2j}	m_{1j}	m_{2j}	m_{1j}	m_{2j}	m_{ij}	m_{1j}	m_{2j}
$\theta_1\cap\theta_2\cap\theta_3$	0	0	0.16												
$\theta_1 \cap \theta_2$	0.10	0.20	0.22	0.26											
$\theta_1 \cap \theta_3$	0.10	0	0.12	0.14	0.15	0.20	0.18	0.41							
$\theta_2 \cap \theta_3$	0	0.20	0.19	0.23	0.30	0.41					0.42	0.70			
$\theta_1\cap(\theta_2\cup\theta_3)$	0	0	0	0.00											
$\theta_2\cap(\theta_1\cup\theta_3)$	0	0	0.05	0.06											
$\theta_3\cap(\theta_1\cup\theta_2)$	0	0	0.01	0.01	0.01	0.02									
$\Box$	0	0	0	0											
θ_1	0.10	0.20	0.08	0.10	0.26	0.14	0.31	0.24	0.39	0.41				0.62	0.67
θ_2	0.20	0.10	0.03	0.04	0.10	0.05	0.17	0.09	0.17	0.15	0.27	0.07		0.37	0.33
θ_3	0.30	0.10	0.10	0.12	0.13	0.13	0.29	0.23	0.37	0.39	0.31	0.23	1.00		
$\Box\theta_1$	0	0	0.02	0.02	0.025	0.03									
$\Box\theta_2$	0	0	0	0	0	0									
$\Box\theta_3$	0	0	0	0											
$\theta_1 \cup \theta_2$	0.10	0	0	0	0	0	0	0	0	0				0.01	0
$\theta_1 \cup \theta_3$	0.10	0.20	0.02	0.02	0.025	0.03	0.05	0.03	0.06	0.05					
$\theta_2 \cup \theta_3$	0	0	0	0	0	0	0	0	0	0	0	0			
$\theta_1\cup\theta_2\cup\theta_3$	0	0	0	0	0	0	0	0	0	0					

Table 4.1: MinC combination of gbba's m_1 and m_2 on hybrid DSm models $\mathcal{M}_1, ..., \mathcal{M}_7$.

DSm model, (or equivalently k-ary combination on the free DSm model), and after it we use some proportionalization in the same way as in the case of the minC combination of two gBF's. Hence we can see that the minC combination is defined on any DSm model for any k generalized belief functions.

4.5 Examples of minC combination

Three simple examples for both the static and dynamic fusion on Shafer's DSm model $\mathcal{M}^0$ have been presented in [10]. Nevertheless, for an illustration of all main properties of the generalized minC rule it is necessary to see, how the rule works on general hybrid DSm models. Therefore we present examples of fusion on seven different hybrid DSm models $\mathcal{M}_1, ..., \mathcal{M}_7$ in this text, see Table 4.1.

For easier comparison of the generalized minC combination with the hybrid DSm rule we use the models from Examples 1 — 7, see DSm book Vol. 1 [20], Chapter 4. All the combinations are applied to two generalized belief functions on a 3-element frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3\}$. The hybrid DSm models from the examples are given as it follows:

$\mathcal{M}_1$: $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_1}{\equiv} \emptyset$,

$\mathcal{M}_2$: $\theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$, thus also $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$,

$\mathcal{M}_3$: $\theta_2 \cap (\theta_1 \cup \theta_3) \overset{\mathcal{M}_3}{\equiv} \emptyset$ and hence also $\theta_1 \cap \theta_2 \overset{\mathcal{M}_3}{\equiv} \theta_2 \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$,

$\mathcal{M}_4 = \mathcal{M}^0$: $\theta_1 \cap \theta_2 \overset{\mathcal{M}_4}{\equiv} \theta_2 \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \theta_1 \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \emptyset$ and hence also $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \theta_1 \cap (\theta_2 \cup \theta_3) \overset{\mathcal{M}_4}{\equiv} \theta_2 \cap (\theta_1 \cup \theta_3) \overset{\mathcal{M}_4}{\equiv} \theta_3 \cap (\theta_2 \cup \theta_3) \overset{\mathcal{M}_4}{\equiv} \Box \overset{\mathcal{M}_4}{\equiv} \emptyset$, and further $\Box\theta_1 \overset{\mathcal{M}_4}{\equiv} \theta_1$, $\Box\theta_2 \overset{\mathcal{M}_4}{\equiv} \theta_2$, $\Box\theta_3 \overset{\mathcal{M}_4}{\equiv} \theta_3$,

$\mathcal{M}_5$: $\theta_1 \overset{\mathcal{M}_5}{\equiv} \emptyset$ (θ_1 is removed from $\Theta = \{\theta_1, \theta_2, \theta_2\}$ in fact) thus all $X \in D^\Theta$ which include intersection with θ_1 are forced to be empty (i.e. $X \overset{\mathcal{M}_5}{\equiv} \emptyset$), and all $Y \in D^\Theta$ which include union with θ_1 are forced to be equivavent to some element of $D^\Theta_{\mathcal{M}_5}$,

$\mathcal{M}_6$: $\theta_1 \overset{\mathcal{M}_6}{\equiv} \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset$, thus $\theta_1 \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \theta_2 \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \theta_1 \cup \theta_2 \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \theta_3 \cup (\theta_1 \cap \theta_2) \overset{\mathcal{M}_6}{\equiv} \theta_3$, and all

other $X \in D^{\Omega}$ are forced to be empty (i.e. $X \overset{\mathcal{M}_6}{\equiv} \emptyset$),
$\mathcal{M}_7$: $\theta_3 \cup (\theta_1 \cap \theta_2) \overset{\mathcal{M}_7}{\equiv} \emptyset$, i.e. also $\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$ and $\theta_1 \cap \theta_2 \overset{\mathcal{M}_7}{\equiv} \emptyset$, thus only $\theta_1 \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}_7}{\equiv} \theta_1 \overset{\mathcal{M}_7}{\not\equiv} \emptyset$, $\theta_2 \cup (\theta_1 \cap \theta_3) \overset{\mathcal{M}_7}{\equiv} \theta_2 \overset{\mathcal{M}_7}{\not\equiv} \emptyset$, $\theta_3 \cup (\theta_1 \cap \theta_2) \overset{\mathcal{M}_7}{\equiv} \theta_3 \overset{\mathcal{M}_7}{\not\equiv} \emptyset$, and all the other $X \in D^{\Theta}$ are constrained, for more details see [20].

We use the following abbreviations for 4 elements of D^{Θ}: $\Box$ for $(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) = (\theta_1 \cup \theta_2) \cap (\theta_1 \cup \theta_3) \cap (\theta_2 \cup \theta_3)$, $\Box\theta_1$ for $\theta_1 \cup (\theta_2 \cap \theta_3) = (\theta_1 \cup \theta_2) \cap (\theta_1 \cup \theta_3)$, $\Box\theta_2$ for $\theta_2 \cup (\theta_1 \cap \theta_3)$, and $\Box\theta_3$ for $\theta_3 \cup (\theta_1 \cap \theta_2)$. Thus $\Box$ is not any operator here, but just a symbol for abbreviation; it has its origin in the papers about minC combination [4, 10], see also Chapter 10 in DSm book Vol. 1 [20].

The generalized BF's Bel_1 and Bel_2 are represented by generalized bba's m_1 and m_2 from the referred Examples 1—7 again. For the values of gbbm's $m_i(A)$ see the 2nd and 3rd column of Table 4.1. All elements of the hyper-power set D^{Θ}, which correspond to the given frame of the discernment Θ, are placed in first column of the table.

For better comparison of different results of the generalized minC combination on different DSm models we put all the results into one table. Every row of the table body contain an element A of D^{Θ}, corresponding values of source gbba's $m_i(A)$, value $m^0(A)$, which corresponds to the free DSm model $\mathcal{M}^f$, and gbbm's $m_{ij}(A)$ corresponding to hybrid DSm models $\mathcal{M}_1$ — $\mathcal{M}_7$ referred in the first row of the table head. The fourth column of Table 4.1 present values $m^0(A)$ of the generalized level of the generalized minC combination. These values coincide with the resulting values $m(A)$ on the free DSm model $\mathcal{M}^f$, where values for all elements $A \in D^{\Theta}$ are defined and printed.

To space economizing, we present the DSm models $\mathcal{M}_i$ together with the resulting gbbm values $m_{ij}(A)$ in the corresponding columns of Table 4.1: only values for $A \in D^{\Theta}_{\mathcal{M}_i}$ are printed. The 0 values for $A \in D^{\Theta}$ which are constrained (forced by constraints to be empty) are not printed, similarly the 0 values for $X \in D^{\Theta}$ which are $\mathcal{M}_i$-equivalent to some $A \in D^{\Theta}_{\mathcal{M}_i}$ ($A \overset{\mathcal{M}_i}{\equiv} X \neq A$) are also not printed. Thus for example $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_1}{\equiv} \emptyset$, $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$ consequently $m_{ij}(\theta_1 \cap \theta_2 \cap \theta_3) = 0$ in both models $\mathcal{M}_1$ and $\mathcal{M}_2$ and $m_{ij}(\theta_1 \cap \theta_2) = 0$ in model $\mathcal{M}_2$, hence the corresponding cells in the table are blank. Similarly $\theta_1 \cap (\theta_2 \cup \theta_3) \overset{\mathcal{M}_2}{\equiv} \theta_1 \cap \theta_3$, $\theta_2 \cap (\theta_1 \cup \theta_3) \overset{\mathcal{M}_2}{\equiv} \theta_2 \cap \theta_3$, $\Box\theta_3 = \theta_3 \cup (\theta_1 \cap \theta_2) \overset{\mathcal{M}_2}{\equiv} \theta_3$, and $\Box \overset{\mathcal{M}_2}{\equiv} \theta_3 \cap (\theta_1 \cup \theta_2)$, thus values $m^0(X)$ are added to values $m^0(A)$ and $m^1(X) = m_{ij}(X) = 0$ for all such Xs and corresponding As ($A \overset{\mathcal{M}_2}{\equiv} X \neq A$), i.e. $m_{ij}(\theta_1 \cap (\theta_2 \cup \theta_3)), m_{ij}(\theta_2 \cap (\theta_1 \cup \theta_3)), m_{ij}(\Box\theta_3) m_{ij}(\Box)$ are forced to be 0 in DSm model $\mathcal{M}_2$, hence the corresponding cells in the 6th and 7th columns of the table are also blank. On the other hand there are printed 0 values for $m_{ij}(\theta_1 \cup \theta_2) = m_{ij}(\theta_2 \cup \theta_3) = m_{ij}(\theta_1 \cup \theta_2 \cup \theta_3) = m_{ij}(\Box\theta_2) = 0$ because these 0 values are not forced by constraints of the model $\mathcal{M}_2$ but they follow values of input gbba's m_1 and m_2. $\mathcal{M}_4 \equiv \mathcal{M}^0$ is Shafer's DSm model thus the values are printed just for $A \in 2^{\Theta}$ in the 10-th and 11-th columns. For details on equivalence of $A \in D^{\Theta}$ on hybrid DSm models $\mathcal{M}_3, \mathcal{M}_5, \mathcal{M}_6, \mathcal{M}_7$, see Chapter 4 in DSm book Vol. 1 [20]; for the model $\mathcal{M}_3$ see also Example 6 in Chapter 3 of this volume, specially the 5-th column of Table 3.6 as the model $\mathcal{M}_3$ coincides with DSm model $\mathcal{M}_{4.3}$ there. There is no row for $\emptyset$ in Table 4.1 as all the cells should be blank there.

Because of the values $m_i(A)$ of the used gbba's m_1 and m_2, there is no difference between m_{i1} and m_{i2} on all the models $\mathcal{M}_1, ..., \mathcal{M}_7$, moreover, there is also no difference between m_{1j} and m_{2j} on model $\mathcal{M}_1$. Trivially, there is no difference on trivial DSm model $\mathcal{M}_6$ which have the only element θ_3 not equivalent to empty set ($D^{\Theta}_{\mathcal{M}_6} = \{\theta_3, \emptyset\}$) thus there is the only possible gbbm $m(\theta_3) = 1$ on the model $\mathcal{M}_6$. Trivially, there is also no difference among m_{ij} on the free DSm model $\mathcal{M}^f$ because there is no constraint and consequently no proportionalization there.

To economize space in the table again, only columns with different values are printed. Results m^1 of the combination step which groups together values $m^0(X)$ of $\mathcal{M}_i$-equivalent elements of D^Θ are not presented from the same reason.

4.6 Comparison of the generalized minC combination and hybrid DSm combination rules

There is presented minC combination of generalized BF's Bel_1 and Bel_2 on the free DSm model and on 7 hybrid DSm models in the previous section. For a comparison of the generalized minC combination rule with the hybrid DSm rule (DSmH rule), we compute or recall the DSm rule results on the same DSm models from the examples in DSm book 1 [20], Chapter 4. We present the results in the same way as there were presented the results of the generalized minC combination in the previous section, see Table 4.2. From the definitions of the both the rules

			$\mathcal{M}^f$	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$
	m_1	m_2	$m_{\mathcal{M}^f}$	m_{DSmH}	m_{DSmH}	m_{DSmH}	m_{DSmH}	m_{DSmH}	m_{DSmH}	m_{DSmH}
$\theta_1\cap\theta_2\cap\theta_3$	0	0	0.16							
$\theta_1\cap\theta_2$	0.10	0.20	0.22	0.22						
$\theta_1\cap\theta_3$	0.10	0	0.12	0.12	0.14	0.17				
$\theta_2\cap\theta_3$	0	0.20	0.19	0.19	0.26			0.33		
$\theta_1\cap(\theta_2\cup\theta_3)$	0	0	0	0.02						
$\theta_2\cap(\theta_1\cup\theta_3)$	0	0	0.05	0.07						
$\theta_3\cap(\theta_1\cup\theta_2)$	0	0	0.01	0.03	0.03					
$\square$	0	0	0	0						
θ_1	0.10	0.20	0.08	0.08	0.12	0.16	0.18			0.43
θ_2	0.20	0.10	0.03	0.03	0.08	0.12	0.13	0.24		0.24
θ_3	0.30	0.10	0.10	0.10	0.17	0.23	0.24	0.39	1.00	
$\square\theta_1$	0	0	0.02	0.04	0.04					
$\square\theta_2$	0	0	0	0.01	0.01	0.01				
$\square\theta_3$	0	0	0	0.07						
$\theta_1\cup\theta_2$	0.10	0	0	0	0.09	0.11	0.11			0.33
$\theta_1\cup\theta_3$	0.10	0.20	0.02	0.02	0.06	0.08	0.17			
$\theta_2\cup\theta_3$	0	0	0	0	0	0.05	0.05	0.04		
$\theta_1\cup\theta_2\cup\theta_3$	0	0	0	0	0	0.07	0.12			

Table 4.2: DSmH combination of gbba's m_1 and m_2 on hybrid DSm models $\mathcal{M}_1, ..., \mathcal{M}_7$.

it is obvious that the minC and DSmH rules coincide themselves on the free DSm model and that they coincide also with the classic DSm (DSmC) rule and with the conjunctive rule of combination of gBF's on DSm hyper-power sets. In the examples we can compare the fourth columns in both the tables.

Trivially, both the rules coincide also on trivial DSm models with the only non-empty element, see e.g. $\mathcal{M}_6$ and the corresponding columns in the tables.

The presented examples are not enough conflicting to present differences between proportionalizations $prop_{i1}$ and $prop_{i2}$. Therefore we add another example for presentation of their differences and for better presentation of their relation to DSmH rule. For this reason we use a modified Zadeh's example on Shafer's model on 4-element frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$: $\mathcal{M}_8 = \mathcal{M}^f(\Theta)$. The small non-conflicting element is split to two parts θ_3 and θ_4 and similarly its bbms. In the same time, it is a modification of the example from subsection 5.4.1 from Chapter 5 in DSm book Vol. 1, where small parts of $m(\theta_3 \cup \theta_4)$ are more specified to θ_3 and θ_4 in inputs bba's, see Table 4.3. When coarsening $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ to $\{\theta_1, \theta_2, \theta_3 \equiv \theta_4\}$ in our present example, we obtain an instance of the classic Zadeh's example. Hence our example in Table 4.3 is just one of many possible refinements of Zadeh's example.

The structure of the table is analogous to that of previous tables. As the whole table representing D^Θ has 167 rows, all the rows which include only 0s and blank cells are skipped. Different results of minC using 4 proportionalizations are presented in 5-8th columns of the table. DSmH results are presented in 9-th column. As it is already mentioned in the introduction, we cannot forget that Dempster's rule produces correct results for combination of any 2 belief functions which correctly represent mutually probabilistically independent evidences, which are not in full contradiction, on Shafer's model. Therefore we present also the result of application of Dempster's rule in the last column of Table 4.3.

			$\mathcal{M}^f$	$\mathcal{M}_8$	$\mathcal{M}_8$	$\mathcal{M}_8$	$\mathcal{M}_8$	$\mathcal{M}_8$	$\mathcal{M}_8$
	m_1	m_2	m_0	m_{11}	m_{12}	m_{21}	m_{22}	m_{DSmH}	$m_\oplus$
$\theta_1 \cap \theta_2$	0	0	0.9506						
$\theta_1 \cap \theta_3$	0	0	0.0098						
$\theta_2 \cap \theta_4$	0	0	0.0097						
$\theta_3 \cap \theta_4$	0	0	0.0001						
$\theta_1 \cap (\theta_3 \cup \theta_4)$	0	0	0.0196						
$\theta_2 \cap (\theta_3 \cup \theta_4)$	0	0	0.0097						
θ_1	0.98	0	0	$0.3168\overline{6}$	0	$0.3168\overline{6}$	0	0	0
θ_2	0	0.97	0	$0.3168\overline{6}$	0	$0.3168\overline{6}$	0	0	0
θ_3	0	0.01	0.0001	0.00992	0.00992	0.01578	0.01578	0.0001	0.20
θ_4	0.01	0	0.0002	0.00994	0.00994	0.02166	0.02166	0.0002	0.40
$\theta_1 \cup \theta_2$	0	0	0	$0.3168\overline{6}$	0.95060	$0.3168\overline{6}$	0.95060	0.9506	0
$\theta_1 \cup \theta_3$	0	0	0	0	0	0	0	0.0098	0
$\theta_2 \cup \theta_4$	0	0	0	0	0	0	0	0.0097	0
$\theta_3 \cup \theta_4$	0.01	0.02	0.0002	0.02954	0.02954	0.01196	0.01196	0.0003	0.40
$\theta_1 \cup \theta_3 \cup \theta_4$	0	0	0	0	0	0	0	0.0196	0
$\theta_2 \cup \theta_3 \cup \theta_4$	0	0	0	0	0	0	0	0.0097	0

Table 4.3: Comparison of minC combination, hybrid DSm and Dempster's rules on a modified Zadeh's example on Shafer's model $\mathcal{M}_8 \equiv \mathcal{M}^0(\Theta)$ for a 4-element frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$. (Only non-empty non-zero rows of the table are printed.)

Results of the minC combination are usually more specified (i.e. gbbm's are located to less focal elements) in general cases, compare the columns corresponding to the same DSm models in Tables 4.1 and 4.2, see also comparison in Table 4.3. It holds more when using proportionalizations $prop_{i1}$, which produce more specified results than proportionalizations $prop_{i2}$ do. There are also examples, where it is not possible to say which rule produces more of less specified results. It is in cases of totally conflicting focal elements, where all input gbbm's corresponding to these elements are assigned to $X \equiv \emptyset$ by $m^0 \equiv m_{\mathcal{M}^f}$.

Moreover the counter examples arise in a special cases of input gBF's with focal elements which are all totally conflicting and some of them assign(s) gbba to overlapping element(s) of frame of discernment. For example, let us assume hybrid DSm model $\mathcal{M}_2$ and gBF's Bel_3, Bel_4, Bel_5 given by gbba's $m_3(\theta_1) = 1$, $m_4(\theta_2 \cap \theta_3) = 1$ and $m_5(\theta_1 \cap \theta_2 \cap \theta_3) = 1$.

When combining Bel_3 and Bel_4 using $prop_{22}$ we obtain a counter example for static fusion: $m_{11}(\theta_1) = m_{11}(\theta_2 \cap \theta_3) = m_{11}(\theta_1 \cup (\theta_2 \cap \theta_3)) = 1/3$, $m_{12}(\theta_1 \cup (\theta_2 \cap \theta_3)) = 1$, $m_{21}(X) = 1/12$, $m_{22}(\theta_1 \cup \theta_2 \cup \theta_3) = 1$, whereas for DSmH we obtain $m_{DSmH}(\theta_1 \cup (\theta_2 \cap \theta_3)) = 1$, i.e. $m_{DSmH}(\Box\theta_1) = 1$. We can immediately see that $\theta_1 \cup \theta_2 \cup \theta_3 \supset \theta_1 \cup (\theta_2 \cap \theta_3)$. When using $prop_{21}$ it not possible to say which of the rules produces more specified results as m_{21} assigns $1/12$ to every element of model $\mathcal{M}$: one of them is equal to $\Box\theta_1 = \theta_1 \cup (\theta_2 \cap \theta_3)$ (to what DSmH assigns 1), 4 of them are subset of $\Box\theta_1$, 3 of them are supersets of $\Box\theta_1$ and 4 of them are incomparable.

When combining Bel_3 and Bel_5 using $prop_{21}$ we obtain a similar case for dynamic fusion: $m_{11}(\theta_1) = m_{12}(\theta_1) = m_{22}(\theta_1) = m_{DSmH}(\theta_1) = 1$ and $m_{21}(X) = 1/12$ for all $\emptyset \not\equiv X \in D^{\Theta}_{\mathcal{M}_2}$. m_{21} assigns 1/12 to every element of model $\mathcal{M}$ again: one of them is equal to θ_1 (to what DSmH assigns 1, 1 of them $\theta_1 \cap \theta_3$ is subset of θ_1, 4 of them $(\Box\theta_1, \theta_1 \cup \theta_2, \theta_1 \cup \theta_3, \theta_1 \cup \theta_2 \cup \theta_3)$ are supersets of θ_1 and other 6 of them are incomparable.

A detail study of situations where it is not possible say whether minC combination produces more specified results and situations where DSmH rule produces more specified results is an open problem for future.

The principal difference between the minC combination and the hybrid DSm rule is the following: DSmH rule handles separately individual multiples of gbbm's $m_1(X)m_2(Y)$ and assign them to intersection (if non-empty) or to union (if non-empty) of focal elements X and Y. Whereas the minC combination groups together all the multiples, where $X \cap Y$ are mutually $\mathcal{M}$-equivalent and assigns the result to $X \cap Y$ (if non-empty) or proportionalizes it to focal elements derived from $\bigcup(X \cap Y)$. Hence multiples $m_i(\theta_1)m_j(\theta_2 \cap \theta_3), m_i(\theta_1)m_j(\theta_1 \cap \theta_2 \cap \theta_3), m_i(\theta_1 \cap \theta_2)m_j(\theta_2 \cap \theta_3), m_i(\theta_1 \cap \theta_2)m_j(\theta_1 \cap \theta_2 \cap \theta_3)$ and other $\mathcal{M}$-equivalent are reallocated all together in the minC combination. Similarly multiples $m_i(\theta_1)m_j(\theta_2), m_i(\theta_1)m_j(\theta_1 \cap \theta_2), m_i(\Box\theta_1)m_j(\theta_1 \cap \theta_2), m_i(\theta_1 \cap \theta_2)m_j(\theta_1 \cup \theta_2), m_i(\theta_1 \cap \theta_2)m_j(\theta_1 \cup \theta_2 \cup \theta_3)$ and other $\mathcal{M}$-equivalent are reallocated also all together in the minC combination. This is also the reason of minC results in the special cases, where $X \cup Y \subset \bigcup(X \cap Y)$ and $m^1(Z) = 0$ for all $Z \in D^{\Theta}_{\mathcal{M}}$, as in the previous paragraph.

The other principal difference is necessity of n-ary version of the rule for DSmH. Whereas we can apply (n-1) times computation of binary m^0 and some proportionalization after, in the case of the binary minC combination.

4.7 Related works.

We have to remember again the comparison of classic minC with DSmH on Shafer's DSm model at first, see Chapter 10 in [20].

To have a solid theoretical background for comparison of DSm rules with the classic ones, a generalization of Dempster's rule, Yager's rule [23], and Dubois-Prade rule [13] has been presented in [6, 7], see also Chapter 3 in this volume, and the generalized minC combination in [8].

We cannot forget for new types of DSm rules, especially Proportional Conflict Redistribution Rules [21], which are "between" DSmC and DSmH rules on one side and minC approach on the other side. Comparison of these rules with the generalized minC approach is a very interesting task for forthcoming research.

We have to mention also works by Besnard [1] and his collaborators Jaouen [16] and Perin [18], who propose to replace the classical Boolean algebras with a distributive lattice, hoping it might solve Smets' bomb issue. Their distributed lattice generated on a frame of discernment is the free DSm model in fact, it also coincides with a lattice $\mathcal{L}(\Theta)$ in minC combination. Moreover these authors use a conflicting relation for a construction of their evidential structure. There is no concept of negation similarly to DSm approach. Comparison of the conflicting relation with DSm constraints and of the evidential structures with hybrid DSm models is still an open problem for future research to formulate a relation between the two independently developed approaches to belief combination on distributive lattices. Nevertheless neither this issue really new as it has been started and unfortunately unfinished by Philippe Smets in 2004/2005.

4.8 Conclusion

The minC combination rule generalized to DSm hyper-power sets and general hybrid DSm models has been presented both for static and dynamic fusion of generalized belief functions.

Examples of the generalized minC combination on several hybrid DSm models have been presented and discussed. After it, a comparison of the generalized minC combination and the hybrid DSm rule has been performed and several open problems for a future research has been defined.

A step for inclusion of minC combination into family of DSm combination rules has been done.

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