# Multi-Criteria Decision Making Method for *n*-wise Criteria Comparisons and Inconsistent Problems

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## Abstract

The purpose of this paper is to present an alternative of a hybrid method based on Saaty's Analytical Hierarchy Process and on the Technique for Order Preference by using the Similarity to Ideal Solution method (AHP-TOPSIS) and, based on the AHP and its use of pairwise comparisons, to extend it to a new method called α-D MCDM-TOPSIS (α-Discounting Method for multi-criteria decision making-TOPSIS). The new method overcomes the limits of AHP, which works only for pairwise comparisons of criteria, to any-wise (*n*-wise) comparisons, with crisp coefficients or with interval-valued coefficients. An extended MCMD method (called Extended α-D MCDM) of α-D MCDM, introduced by Smarandache to solve decision making problems, is developed. α-D MCDM-TOPSIS and Extended α-D MCDM are verified on several examples, to demonstrate how they work with consistent, weak inconsistent or strong inconsistent problems. Finally, we discuss and compare all methods.

## Keyword

Decision making, Extended α-D MCDM, Consistency, Inconsistency, n-wise criteria comparisons, AHP TOPSIS.

## 1 Introduction

Many economic, social or technological problems have been widely discussed and resolved in recent years by multi-criteria decision making methods [8]. However, the quantity of data, the complexity of the modern world and the recent technological advances have made obviously that MCDM methods are more challenging than ever, hence the necessity of developing other methods, able to give quality solutions.

Among MCDM methods, the most often used to improve the reliability of the decision making process is the combined method AHP-TOPSIS [12], [3], [2], [8], [10], [11] and [4].

AHP-TOPSIS is indeed a useful MCDM method to resolve difficult decision making problems and to select the best of the alternatives. Its applications are significant [8]: a support for management and planning of flight mission at NASA [12]; a method to study how the traffic congestion of urban roads is evaluated [3]; choosing logistics service provider in the mobile phone industry domains [10]; summarizing an e-SCM performance for management of supply chain [11]; evaluating faculty performance in engineering education, or sharing capacity assessment knowledge of supply chain [4].

Our paper is organized as follows. In the next section (*Section 2*), a literature survey for consistency problems is given. *Section 3* and *Section 4* focus on AHP-TOPSIS, and on the proposed α-D MCDM-TOPSIS model, respectively. The proposed method is tested on consistent, weak inconsistent and strong inconsistent examples (in *Section 5*). AHP method employed to rank the preferences is considered in *Section 6*. An extended α-D MCDM is introduced in *Section 7*, and it is shown how it can be applied for ranking preferences. We discuss developments via the use of an example to compare all methods. Finally, we draw conclusions and envisage some perspectives.

## 2 Comparison of characteristics between AHP and α-D MCDM: Consistency

### 2.1 A brief overview of Analytic Hierarchy Process (AHP)

AHP, introduced by the Saaty [6], is one of the most complete methods of multi-criteria decision making technique, determining the weights of criteria and ranking alternatives. The use of AHP only, or its hybrid use with other methods, proved its capacity to solve MCDM problems and to be a popular technique for determining weights — see more than a thousand references in [9]. Besides the performance of AHP and its added value at both levels, theoretical and practical, this method functions only if the problem is perfectly consistent, which is rarely checked in real MCDM problems.

### 2.2 Description of α-D MCDM

α-D MCDM (α-Discounting Method for Multi-Criteria Decision Making) was introduced by Smarandache — see [7]. The new method overcomes the limits of AHP, which work only for pairwise comparisons of criteria, expanding to any-wise (n-wise) comparisons.

Smarandache used the homogeneous linear mathematical equations to express the relationship between criteria with crisp coefficients or with interval-valued coefficients also for non-linear equations, with crisp coefficients or with interval-valued coefficients.

The two aims of α-D MCDM method were: firstly, to transform the equations of each criterion with respect to other criteria that has only a null solution into a linear homogeneous system having a non-null solution by multiplying each criteria of the right hand by non-null positive parameters $\alpha_1, \alpha_2, \cdots, \alpha_k$; secondly, to apply the "Fairness Principle" on the general solution of the above system by discounting each parameter by the same value $(\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_k)$.

### 2.2.1 α-D MCDM method

The general idea of the α-D MCDM is to transform any MCDM inconsistent problem (in which AHP does not work) to a MCDM consistent problem, by discounting each coefficient by the same percentage.

Let us assume that $C = \{C_1, C_2, \cdots, C_n\}$, with $n \geq 2$, is a set of criteria, and let's construct a linear homogeneous system of equations.

Each criterion $C_i$ can be expressed as linear homogeneous equation, or as non-linear equation, with crisp coefficients or with interval-valued coefficients of other criteria $C_1, \cdots C_j \cdots C_n$ —

$$C_i = f(C_1, \cdots C_j \cdots C_n).$$

Consequently, a comparisons matrix associated to this linear homogeneous system is constructed.

To determine the weights $w_i$ of the criteria, we solve the previous system.

The α-D MCDM method is not designed to rank preferences $P_i$ based on $C_i$ criteria, as AHP method does, but to determine only the weights of criteria in any type of problems (consistent, inconsistent).

AHP as cited above is a complete method designed to calculate the weights of criteria $C_i$ and to rank the preferences $P_i$. In addition, when the AHP is used with TOPSIS, or other MCDM method, we just benefit from the part of weight calculation criteria and we use TOPSIS to rank preferences — or other MCDM methods.

The same for α-D MCDM: firstly, it is just used to calculate the weight of criteria, that will be used later by TOPSIS to rank preferences, and, secondly, the α-D MCDM is extended to a complete method, in order to rank the preferences.

Therefore, we use α-D MCDM for calculating the weight of criteria $C_i$ and not to rank $P_i$ preferences.

We have —

$$C_i = f(\{C\} \setminus C_i).$$

Then, criteria $C_i$ is a linear equation of $C_j$ such as —

$$C_i = \sum_{j=1 \, j \neq i}^{n} x_{ij} C_j .$$

So, the comparisons criteria matrix has the number of criteria by rows and columns (rows number $n$ = number of criteria, and columns number $m$ = number of equations). In the result, we have a square matrix ($n = m$), consequently we can calculate the determinant of this matrix. At this point, we have an $n \times n$ linear homogeneous system and its associated matrix —

$$\begin{cases} x_{1,1} w_1 + x_{1,2} w_2 + \cdots + x_{1,n} w_n = 0 \\ \vdots \\ x_{n,1} w_1 + x_{m,2} w_2 + \cdots + x_{n,n} w_n = 0 \end{cases}$$

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}.$$

The difference between AHP and α-D MCDM is the ability of the latter to work with consistent and inconsistent problems, and if the problem is inconsistent, α-D MCDM method transforms it in a consistent problem, while AHP is unable to do that, managing strictly consistent problems.

In the following, we deal with the relationship between determinant of matrix and consistency, and the parameterization of system by $\alpha_i$, in order to get a consistent problem.

### Property 1

— If $det(X) = 0$, the system has a solution (i.e. MCDM problem is consistent).
— If $det(X) \neq 0$, the system has only the null solution (i.e. MCDM problem is inconsistent).

If the problem is inconsistent, then one constructs the parameterized matrix, denoted $X(\alpha)$, by parameterizing the right-hand in order to get $\det(X(\alpha)) = 0$ and use Fairness principe (set equal parameters to all criteria $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_k > 0$). To get priority vector, one resolves the new system obtained and set 1 to secondary variable, then normalize the vector by dividing the sum of all components.

## 2.3 Consistency of decision making problems

In this section, we discuss the consistency of the MCDM problems for both methods ($\alpha$-D MCDM and AHP).

For resolving a linear system of equations, in mathematics we use raw operations, such as substitution, interchange, … .

### Definition 1 [7]

Applying any substitution raw operations on two equations, if it does not influence the system consistency and there is an agreement of all equations, we say that the linear system of equations (of the linear MCDM problem) is consistent.

### Definition 2 [7]

Applying any substitution raw operations on two equations, if equation result is in disagreement with another, we say that the linear system of equations (of the linear MCDM problem) is weakly consistent.

### Definition 3 [7]

Applying any substitution raw operations on two equations, if equation result is in opposition with another, we say that the linear system of equations (of the linear MCDM problem) is strongly inconsistent.

## 2.4 Consistency

AHP provides the decision maker with a way of examining the consistency of entries in a pairwise comparison matrix; the problem of accepting/rejecting matrices has been largely discussed [5], [1], [13], especially regarding the relation between the consistency and the scale used to represent the decision maker's judgments. AHP is too restrictive when the size of the matrix increases, and when order $n$ of judgment matrix is large; the satisfying consistency is more difficult to be met [5], [1].

This problem may become a very difficult one when the decision maker is not perfectly consistent, moreover, it seems impossible (AHP does not work) when there are not pairwise comparisons, but all kind of comparisons between criteria, such as $n-$wise, because there is set a strict consistency condition in the AHP, in order to keep the rationality of preference intensities between compared elements.

In addition, the inconsistency exists in all judgments [5]; comparing three alternatives — or more, it is possible that inconsistency exists when there are more than 25 percent of the $3-by-3$ reciprocal matrices with a consistency ratio less than or equal to ten percent. Consequently, as the matrix size increases, the percentage of inconsistency decreases dramatically [1], [5].

Furthermore, the AHP method sets a consistency ratio (CR) threshold ( $CR(X) > 0.1$ ), which should not be exceeded, by examining the inconsistency of the pairwise comparison matrix, but this requirement for the Saaty's matrix is not achievable in the real situations.

In order to overcome this deficiency, instead of the AHP we suggest employing an α-D MCDM, which is very natural and more suitable for the linguistic descriptions of the Saaty's scale and, as a result of it, it is easier to reach this requirement in the real situations.

Moreover, the attractiveness of α-D MCDM is due to its potential to overcome limits of AHP, which works only for pairwise comparisons of criteria, expanding to $n$-wise (with $n \geq 2$) comparisons, with crisp coefficients or with interval-valued coefficients. Therefore, α-D MCDM method works for inconsistent, weak inconsistent and strong inconsistent problems.

As previously shown, in α-D MCDM method, in order to transform a inconsistent MCDM problem to a consistent problem — we multiply each criteria of the right hand by non-null positive parameters $\alpha_1, \alpha_2, \cdots, \alpha_k$ and we use "Fairness Principle" assigning to each parameter the same value $(\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_k)$.

## Property 2

In α-D MCDM (and Fairness-Principle for coefficients $\alpha_i$), the parameter $\alpha$ (or $\frac{1}{\alpha}$) signifies the degree of consistency and $\beta$ ($\beta = f(\alpha)$) represents the degree of inconsistency.

— If $0 < \alpha < 1$, then $\alpha$ and $\beta = 1 - \alpha$ represent the degree of consistency and the degree of inconsistency, respectively, of the decision-making problem.

— If $\alpha > 1$, then $\frac{1}{\alpha}$ and $\beta = 1 - \frac{1}{\alpha}$ represent the degree of consistency and the degree of inconsistency, respectively, of the decision-making problem.

Property 3

In AHP method, RI — consistency index, CR — consistency ratio and $\lambda$ ($\lambda_{max}$ largest) — the eigenvalue of the $(X)_{n\times n}$ pairwise comparison matrix.

— We say that MCDM problem is consistent (pairwise comparison matrix $X$ is consistent), if $Rank(X) = 1$ and $\lambda = n$ (ideal case).
— We say that MCDM problem is consistent too (pairwise comparison matrix $X$ is consistent), if consistency ratio $CR(X) \leq 0.1$, $CR(X) = \frac{CI(X)}{RI(X)}$, where $CI(X) = \frac{\lambda_{max} - n}{n-1}$, and RI values are given (simulation parameter).
— If $CR(X) \leq 0.1$, the MCDM problem is inconsistent and the pairwise comparison matrix should be improved.

| **Characteristics** | **AHP** | **$\alpha$ -D MCDM** |
|---|---|---|
| Weight elicitation | Pairwise comparison | $n$ -wise comparison ( $n \geq 2$ ) |
| Number of attributes accommodated | $7 \pm 2$ | Large inputs |
| Consistent problems | Provided | Not provided and α-D MCDM gives same result as AHP |
| Weakly inconsistent problems | Does not work | Justifiable results |
| Strongly inconsistent problems | Does not work | Justifiable results |

*Table 1:* Comparison of characteristics of both methods (AHP, α-D MCDM)

## 3 Description of data structure decision problems under consideration

Taking into account that pertinent data is frequently very high-priced to collect, we can’t change real life problems to obtain a specific form of data. In addition, information from real world certainly includes imperfection — such as uncertainty, conflict, etc.

Hence, the choice of the MCDM method is based, firstly, on the structure of decision problem considered, secondly, on the types of data that can be obtained, and, finally, on the capability to get accurate results. For this reason, we detail the different types of all data structure decision problem, for example:

— If decision matrix illustrates the importance of alternatives with respect of criteria, the pairwise (or $n$-wise) comparison can't be used directly in the hybrid AHP-TOPSIS approach. Firstly, priority weights for criteria are calculated using AHP technique, and then the alternatives are prioritized using TOPSIS approach.

The derivation of weights is a central step in eliciting the decision-maker's preferences, but the hybrid AHP-TOPSIS method is more difficult to be met: on one hand, AHP does not work in inconsistent problems, on the other hand it cannot be employed for the $n$-wise comparisons criteria cases.

The problem can be abstracted as how to derive weights for a set of activities according to their impact on the situation and the objective of decisions to be made.

Hence, this study will extend AHP-TOPSIS to a MCDM to fit real world. A complete and efficient procedure for decision making will then be provided. The developed model has been analyzed to select the best alternative using α-D MCDM and the technique for order preference by similarity to ideal solution (TOPSIS) as a hybrid approach.

Let us assume that $C=\{C_1,C_2,\cdots,C_n\}$ is a set of Criteria, with $n\geq 2$, and $A=\{A_1,A_2,\cdots,A_m\}$ is the set of Preferences (Alternatives), with $m\geq 1$.

$$\begin{array}{ccc}
\begin{array}{ccc} C_1 & & C_n \\ \downarrow & \cdots & \downarrow \end{array} & & \\
\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} & \begin{array}{c} \leftarrow C_1 \\ \vdots \\ \leftarrow C_n \end{array}
\end{array}$$

| | $C_1$ | $C_2$ | $\cdots$ | $C_n$ |
|---|---|---|---|---|
| | $w_1$ | $w_2$ | $\cdots$ | $w_n$ |
| $A_1$ | $a_{11}$ | $a_{12}$ | $\cdots$ | $a_{1n}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | $\cdots$ | $a_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_m$ | $a_{m1}$ | $a_{m2}$ | $\cdots$ | $a_{mn}$ |

*Table 2:* Decision matrix

If the data cannot be obtained directly to construct the decision matrix $A=(a_{ij})$ above, we should have, for each criteria $C_i$, a pairwise (or $n$-wise) comparison matrix of the preferences (not just for the criterion).

The comparison matrix of the preferences gives the relative importance ($b_{ij}$) of each alternative $A_i$ compared with another $A_j$ with respect to criterion $C_k$.

As mentioned, the comparison matrices of the preferences should be given, but for comparing the results, we will demonstrate how we can obtain it from decision matrix.

For each criterion $C_k$, the comparison matrix of the preferences is defined by $B_k=(b_{ij})$ such as $b_{ij}=\frac{a_{ik}}{a_{jk}}$, with $i=1,2\cdots,n$ (for each criterion a comparison matrix of preferences, consequently $n$ comparisons preferences matrices will be constructed).

| $C_i$ | $A_1$ | $A_2$ | $\cdots$ | $A_n$ |
|---|---|---|---|---|
| $A_1$ | $b_{11}$ | $b_{12}$ | $\cdots$ | $b_{1m}$ |
| $A_2$ | $b_{21}$ | $b_{22}$ | $\cdots$ | $b_{2m}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_m$ | $b_{m1}$ | $b_{m2}$ | $\cdots$ | $b_{mm}$ |

*Table 3:* Relative importance of alternative comparison matrix

Hence, we need to construct $n$ (number of criteria) matrices with pairwise or $n$-wise comparisons of size $m*m$ each, with $m$ the number of preferences, in these cases we can use AHP or $\alpha$-D MCDM.

In this case, AHP method is used both to calculate the weights of criteria and to ranking preferences by calculate the priority.

AHP being more difficult to be met, we will extend $\alpha$-D MCDM to work for the calculation of the weights criteria and ranking preferences.

## 4 AHP-TOPSIS method

In the real word decisions problems (*case 1, Section 3*) we have multiple preferences and diverse criteria. The MCDM problem can be summarized as it follows:

— Calculate weights $w_i$ of criteria $C_i$;

— Rank preferences (alternatives) $A_i$.

Let us assume there are $n$ criteria and their pairwise relative importance is $x_{ij}$.

TOPSIS assumes that we have $n$ alternatives (preferences) $A_i (i = 1, 2, \cdots, m)$ and $n$ attributes/criteria $C_j (j = 1, 2, \cdots, n)$ and comparison matrix $a_{ij}$ of preference $i$ with respect to criterion $j$.

The AHP-TOPSIS method is described in the following steps:

**Step 4.1.** *Construct decision matrix denoted by* $A = (a_{ij})_{m \times n}$

| | $C_1$ | $C_2$ | $\cdots$ | $C_n$ |
|---|---|---|---|---|
| | $w_1$ | $w_2$ | $\cdots$ | $w_n$ |
| $A_1$ | $a_{11}$ | $a_{12}$ | $\cdots$ | $a_{1n}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | $\cdots$ | $a_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_m$ | $a_{m1}$ | $a_{m2}$ | $\cdots$ | $a_{mn}$ |

*Table 4:* Decision matrix

**Step 4.2.** *Determine weights* ($w_j$) *of each criterion using AHP Method*, where

$$\sum_{j=1}^{n} w_j = 1,\ j = 1, 2, \cdots, n.$$

**Step 4.2.1.** *Build a pairwise comparison matrix of criteria*

The pairwise comparison of criterion $i$ with respect to criterion $j$ gives a square matrix $(X)_{n\times n} = (x_{ij})$ where $x_{ij}$ represents the relative importance of criterion $i$ over the criterion $j$. In the matrix, $x_{ij} = 1$ when $i = j$ and $x_{ij} = 1/x_{ji}$. So, we get a $n \times n$ pairwise comparison matrix $(X)_{n\times n}$.

**Step 4.2.2.** *Find the relative normalized weight* ($w_j$) *of each criterion* defined by following formula —

$$w_j = \frac{\prod_{j=1}^{n} (x_{ij})^{1/n}}{\sum \prod_{j=1}^{n} (x_{ij})^{1/n}}$$

Then, get $w_i$ weight of the $i^{th}$ criterion.

**Step 4.2.3.** *Calculate matrix $X_3$ and $X_4$* — such that $X_3 = X_1 \times X_2$ and where $X_4 = X_3 / X_2$ —

$$X2 = [w_1, w_2, \cdots, w_j]^T .$$

**Step 4.2.4.** *Find the largest eigenvalue of pairwise comparison matrix*

For simplified calculus, the largest eigenvalue of pairwise comparison matrix is the average of $X_4$. Furthermore, according to the Perron-Frobenius theorem, principal eigenvalue $\lambda_{max}$ always exists for the Saaty's matrix and it holds $\lambda_{max} \geq n$; for fully consistent matrix $\lambda_{max} = n$.

Consistency check is then performed to ensure that the evaluation of the pair-wise comparison matrix is reasonable and acceptable.

**Step 4.2.5.** *Determine the consistency ratio* $(CR)$

After calculation consistency ratio (RC) using equation (eq.), and in order to verify the consistency of the matrix that is considered to be consistent if CR is less than threshold and not otherwise, according to Saaty and search. At this point, we have the weights of criteria and if the consistency is checked, we will be using TOPSIS to rank preferences.

**Step 4.3.** *Normalize decision matrix*

The normalize decision matrix is obtained, which is given here with $r_{ij}$

$$r_{ij} = a_{ij} / \left( \sum_{i=1}^{m} a_{ij}^2 \right)^{0.5} ;\ j = 1, 2, \cdots, n;\ i = 1, 2 \cdots, m .$$

**Step 4.4.** *Calculate the weighted decision matrix*

Weighting each column of obtained matrix by its associated weight.

$$v_{ij} = w_j r_{ij};\ j = 1, 2, \cdots, n;\ i = 1, 2 \cdots, m.$$

**Step 4.5.** *Determine the positive ideal solution (PIS) and negative ideal solution (NIS)*

$$A^+ = (v_1^+, v_2^+, \cdots, v_n^+) = \left\{ \left( max_i \left\{ v_{ij} \mid j \in B \right\} \right), \left( min_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\};$$

$$A^- = (v_1^-, v_2^-, \cdots, v_n^-) = \left\{ \left( min_i \left\{ v_{ij} \mid j \in B \right\} \right), \left( max_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\}.$$

The benefit and cost solutions are represented by $B$ and $C$ respectively.

**Step 4.6.** *Calculate the distance measure for each alternative from the PIS and NIS*

The distance measure for each alternative from the PIS is —

$$S_i^+ = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^+)^2 \right\}^{0.5} ; i = 1, 2 \cdots, m$$

Also, the distance measure for each alternative from the NIS is —

$$S_i^- = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^-)^2 \right\}^{0.5} ; i = 1, 2 \cdots, m$$

**Step 4.7.** *Determine the values of relative closeness measure*

For each alternative we calculate the relative closeness measure as it follows:

$$T_i = \frac{S_i^-}{(S_i^+ + S_i^-)} ; i = 1, 2 \cdots, m .$$

Rank alternatives set according to the order of relative closeness measure values $T_i$.

## 5 α-D MCDM-TOPSIS method

The MCDM problem description is the same as the one used in AHP-TOPSIS method (*Section 4*), but in this case we have $n$-wise comparisons matrix of criteria. Let us assume that $C = \{C_1, C_2, \cdots, C_n\}$, with $n \geq 2$, and $\{A_1, A_2, \cdots, A_m\}$, with $m \geq 1$, are a set of criteria and a set of preferences, respectively. Let us assume that each criterion $C_i$ is a linear homogeneous equation of the other criteria $C_1, C_2, \cdots, C_n$:

$$C_i = f(\{C\} \setminus C_i).$$

The α-D MCDM-TOPSIS method is described in the following steps:

**Step 5.1.** *Construct decision matrix denoted by* $A = (a_{ij})_{m \times n}$

| | $C_1$ | $C_2$ | $\cdots$ | $C_n$ |
|---|---|---|---|---|
| | $w_1$ | $w_2$ | $\cdots$ | $w_n$ |
| $A_1$ | $a_{11}$ | $a_{12}$ | $\cdots$ | $a_{1n}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | $\cdots$ | $a_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_m$ | $a_{m1}$ | $a_{m2}$ | $\cdots$ | $a_{mn}$ |

*Table 5:* Decision matrix

**Step 5.2.** *Determine weights ($w_i$) of each criterion using $\alpha$-D MCDM Method*

**Step 5.2.1.** *Using $\alpha$-D MCDM to determine the importance weight ($w_i$) of the criteria*, where —

$$\sum_{j=1}^{n} w_j = 1,\ j = 1, 2, \cdots, n.$$

**Step 5.2.2.** *Build a system of equations and its associated matrix*

To construct linear system of equations, each criterion $C_i$ is expressed as a linear equation of $C_j$ such as —

$$C_i = \sum_{j=1\, j\neq i}^{n} x_{ij} C_j$$

Consequently, we have a system of *n* linear equations (one equation of each criterion) with *n* variables (variable $w_i$ is weight of criterion).

$$\begin{cases} x_{1,1}w_1 + x_{1,2}w_2 + \cdots + x_{1,n}w_n = 0 \\ \vdots \\ x_{n,1}w_1 + x_{m,2}w_2 + \cdots + x_{n,n}w_n = 0 \end{cases}$$

In mathematics, each linear system can be associated to a matrix, in this case, denoted by $X = (x_{ij})$, $1 \le i \le n$ and $1 \le j \le n$ where —

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}.$$

**Step 5.2.3.** *Solve system of equation using whose associated matrix*

Solve the system of equation; the different cases are discussed in *Property 1* in that we compute the determinant of $X$ (find strictly positive solution $w_i > 0$).

Solving this homogeneous linear system, in different cases above the general solution that we set as a solution vector —

$$S = [s_1, s_2, \cdots, s_n]$$

and set 1 to secondary variable, we get —

$$W = [w_1, w_2, \cdots, w_n].$$

Dividing each vector element on sum of all components of vector to get normalized vector, where —

$$w_j = \frac{s_j}{\sum_{k=1}^{n} s_k}; i = 1, 2 \cdots, n.$$

**Step 5.2.4.** *Build a pairwise comparison matrix of criteria*

The pairwise comparison of criterion $i$ with respect to criterion $j$ gives a square matrix $(X)_{n\times n} = (x_{ij})$ where $x_{ij}$ represents the relative importance of criterion $i$ over the criterion $j$. In the matrix, $x_{ij} = 1$ when $i = j$ and $x_{ij} = 1/x_{ji}$. So we get a $n \times n$ pairwise comparison matrix $(X)_{n\times n}$.

**Step 5.2.5.** *Find the relative normalized weight ( $w_j$ ) of each criterion* defined by the following formula —

$$w_j = \frac{\prod_{j=1}^{n} (x_{ij})^{1/n}}{\sum \prod_{j=1}^{n} (x_{ij})^{1/n}}.$$

Then, get $w_i$ weight of the $i^{th}$ criterion.

**Step 5.2.6.** *Calculate matrix* $X_3$ *and* $X_4$ — such that $X_3 = X_1 \times X_2$ and where $X_4 = X_3 / X_2$ —

$$X2 = [w_1, w_2, \cdots, w_j]^T.$$

**Step 5.2.7.** *Find the largest eigenvalue of pairwise comparison matrix*

For simplifying the calculus, the largest eigenvalue of pairwise comparison matrix is the average of $X_4$. Furthermore, according to the Perron-Frobenius theorem, principal eigenvalue $\lambda_{max}$ always exists for the Saaty's matrix and it holds $\lambda_{max} \geq n$; for fully consistent matrix $\lambda_{max} = n$.

Consistency check is then performed to ensure that the evaluation of the pair-wise comparison matrix is reasonable and acceptable.

**Step 5.2.8.** *Determine the consistency ratio* $(CR)$

After calculating consistency ratio (RC) using equation (eq.), and in order to verify the consistency of the matrix that is considered to be consistent if CR is less than threshold and not otherwise, according to Saaty and search.

At this point, we have the weights of criteria and if the consistency is checked, we will be using TOPSIS to rank preferences.

**Step 5.3.** *Normalize decision matrix*

The normalized decision matrix is obtained, which is given here with $r_{ij}$

$$r_{ij} = a_{ij} / \left( \sum_{i=1}^{m} a_{ij}^2 \right)^{0.5} ; \; j = 1, 2, \cdots, n; i = 1, 2 \cdots, m.$$

**Step 5.4.** *Calculate the weighted decision matrix*

Weighting each column of obtained matrix by its associated weight —

$$v_{ij} = w_j r_{ij}; \; j = 1, 2, \cdots, n; i = 1, 2 \cdots, m.$$

**Step 5.5.** *Determine the positive ideal solution (PIS) and negative ideal solution (NIS)*

$$A^+ = (v_1^+, v_2^+, \cdots, v_n^+) = \left\{ \left( max_i \left\{ v_{ij} \mid j \in B \right\} \right), \left( min_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\};$$

$$A^- = (v_1^-, v_2^-, \cdots, v_n^-) = \left\{ \left( min_i \left\{ v_{ij} \mid j \in B \right\} \right), \left( max_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\}.$$

The benefit and cost solutions are represented by $B$ and $C$, respectively.

**Step 5.6.** *Calculate the distance measure for each alternative from the PIS and NIS*

The distance measure for each alternative from the PIS is —

$$S_i^+ = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^+)^2 \right\}^{0.5} ; i = 1, 2 \cdots, m.$$

Also, the distance measure for each alternative from the NIS is —

$$S_i^- = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^-)^2 \right\}^{0.5} ; i = 1, 2 \cdots, m.$$

**Step 5.7.** *Determine the values of relative closeness measure*

For each alternative, we calculate the relative closeness measure as it follows:

$$T_i = \frac{S_i^-}{(S_i^+ + S_i^-)}; i = 1, 2 \cdots, m.$$

Rank alternatives set according to the order of relative closeness measure values $T_i$.

# 6 Numerical examples

We examine a numerical example in which a synthetic evaluation desire to rank four alternatives $A_1$, $A_2$, $A_3$ and $A_4$ with respect to three benefit attribute $C_1$, $C_2$ and $C_3$.

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| | $w_1$ | $w_2$ | $w_3$ |
| $A_1$ | 7 | 9 | 9 |
| $A_2$ | 8 | 7 | 8 |
| $A_3$ | 9 | 6 | 8 |
| $A_4$ | 6 | 7 | 8 |

*Table 6:* Decision matrix

In the examples below, we use α-D MCDM and AHP (if it works) to calculate the weights of the criteria $w_1$, $w_2$ and $w_3$. After we used TOPSIS to rank the four alternatives, the decision matrix (*Table 6*) is used for the three following examples.

## 6.1 Consistent Example 1

We use the $\alpha$ -D MCDM. Let the Set of Criteria be $\{C_1, C_2, C_3\}$ with $w_1 = w(C_1) = x$ and $w_3 = w(C_3) = z$.

Let us consider the system of equations associated to MCDM problem and its associated matrix.

$$\begin{cases} x = & 4y \\ y = & 3z \\ z = & \frac{x}{12} \end{cases} \quad X1 = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 3 \\ \frac{1}{12} & 0 & 1 \end{pmatrix}.$$

We calculate $\det(X)$ (in this case, equal = 0); then MCDM problem is consistent; we solve the system; we get the following solution —

$$S = [12z \quad 3z \quad z].$$

Setting 1 to secondary variable, the general solution becomes —

$$S = [12 \quad 3 \quad 1],$$

and normalizing the vector (dividing by sum=12+3+1), the weights vector is:

$$W = \begin{bmatrix} \frac{12}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}.$$

*Using AHP, we get the same result.*

The pairwise comparison matrix of criteria is:

$$X1 = \begin{pmatrix} 1 & 4 & 12 \\ \frac{1}{4} & 1 & 3 \\ \frac{1}{12} & \frac{1}{3} & 1 \end{pmatrix},$$

whose maximum eigenvalue is $\lambda_{max} = 3$ and its corresponding normalized eigenvector (Perron-Frobenius vector) is —

$$W = \begin{bmatrix} \frac{12}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}.$$

*We use TOPSIS to rank the four alternatives.*

| $a_{ij}^2$ | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $w_i$ | 12/16 | 3/16 | 1/16 |
| $A_1$ | 49 | 81 | 81 |
| $A_2$ | 64 | 49 | 64 |
| $A_3$ | 81 | 36 | 64 |
| $A_4$ | 36 | 49 | 64 |
| $\sum_{i=1}^{n} a_{ij}^2$ | 230 | 215 | 273 |

*Table 7:* Calculate $(a_{ij}^2)$ for each column

| $r_{ij}$ | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $w_i$ | 12/16 | 3/16 | 1/16 |
| $A_1$ | 0.4616 | 0.6138 | 0.5447 |
| $A_2$ | 0.5275 | 0.4774 | 0.4842 |
| $A_3$ | 0.5934 | 0.4092 | 0.4842 |
| $A_4$ | 0.3956 | 0.4774 | 0.4842 |
| $\sum_{i=1}^{n} a_{ij}^2$ | 230 | 215 | 273 |

*Table 8:* Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get $r_{ij}$

| $v_{ij}$ | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $w_i$ | 12/16 | 3/16 | 1/16 |
| $A_1$ | 0.3462 | 0.1151 | 0.0340 |
| $A_2$ | 0.3956 | 0.0895 | 0.0303 |
| $A_3$ | 0.4451 | 0.0767 | 0.0303 |
| $A_4$ | 0.2967 | 0.0895 | 0.0303 |
| $v_{max}$ | 0.4451 | 0.1151 | 0.0340 |
| $v_{min}$ | 0.2967 | 0.0767 | 0.0303 |

*Table 9:* Multiply each column by $w_j$ to get $v_{ij}$

| Alternative | $S_i^+$ | $S_i^-$ | $T_i$ | Rank |
|---|---|---|---|---|
| $A_1$ | 0.0989 | 0.0627 | 0.3880 | 3 |
| $A_2$ | 0.0558 | 0.0997 | 0.6412 | 2 |
| $A_3$ | 0.0385 | 0.1484 | 0.7938 | 1 |
| $A_4$ | 0.1506 | 0.0128 | 0.0783 | 4 |

*Table 10:* The separation measure values and the final rankings
for decision matrix (*Table 4*) using AHP-TOPSIS and α-D MCDM-TOPSIS

*Table 10* presents the rank of alternatives ($A_1$, $A_2$, $A_3$, $A_4$) and separation measure values of each alternative from the PIS and from NIS in which the weighted values are calculated by AHP or α-D MCDM. Both methods, AHP and α-D MCDM with Fairness Principle, give the same weights as proven above methods together give same result in consistent problem.

## 6.2 Weak inconsistent Example 2 where AHP does not work

Let us consider another example investigated by [7] for which AHP does not work (i.e. AHP-TOPSIS does not work too); we use the α-D MCDM to calculate the weights values and ranking the four alternatives by TOPSIS (see *Table 14*).

Let the Set of Criteria be $\{C_1, C_2, C_3\}$ with $w_1 = w(C_1) = x$ and $w_3 = w(C_3) = z$. Let us consider the system of equations associated to MCDM problem and its associated matrix.

$$\begin{cases} x = & 2y + 3z \\ y = & \dfrac{x}{2} \\ z = & \dfrac{x}{3} \end{cases}$$

$$X1 = \begin{pmatrix} 1 & -2 & -3 \\ \frac{-1}{2} & 1 & 0 \\ \frac{-1}{3} & 0 & 1 \end{pmatrix}.$$

The solution of this system is $x = y = z = 0$; be the sum of weights = 1, then this solution is not acceptable.

Parameterizing the right-hand side coefficient of each equation by $\alpha_i$ we get:

$$\begin{cases} x = 2\alpha_1 y + 3\alpha_2 z \\ y = \dfrac{\alpha_3 x}{2} \\ z = \dfrac{\alpha_4 x}{3} \end{cases}$$

We solve the system and we get the following solution —

$$S = \begin{cases} y = \frac{\alpha_3 x}{2} \\ z = \frac{\alpha_4 x}{3} \end{cases} \text{ or } S = \begin{bmatrix} x & \dfrac{\alpha_3 x}{2} & \dfrac{\alpha_4 x}{3} \end{bmatrix}.$$

Setting 1 to secondary variable, the general solution becomes —

$$S = \begin{bmatrix} 1 & \dfrac{\alpha_3}{2} & \dfrac{\alpha_4}{3} \end{bmatrix}.$$

Applying Fairness Principle, then replacing $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha$, whence $\alpha = \dfrac{\sqrt{2}}{2}$.

$$S = \begin{bmatrix} 1 & \dfrac{\sqrt{2}}{4} & \dfrac{\sqrt{2}}{6} \end{bmatrix}.$$

Normalizing vector (dividing by sum), the weights vector is:

$$W = [0.62923 \quad 0.22246 \quad 0.14831].$$

*TOPSIS is used to rank the four alternative:* application of TOPSIS method is in the same manner as in the previous example (the four alternatives ( $Ai$ ) are ranked in the following *Table 14*).

| $a_{ij}^2$ | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| | 0.62923 | 0.22246 | 0.14831 |
| $A_1$ | 49 | 81 | 81 |
| $A_2$ | 64 | 49 | 64 |
| $A_3$ | 81 | 36 | 64 |
| $A_4$ | 36 | 49 | 64 |
| $\sum_{i=1}^{n} a_{ij}^2$ | 230 | 215 | 273 |

*Table 11:* Calculate $(a_{ij}^2)$ for each column

| $r_{ij}$ | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| | 0.62923 | 0.22246 | 0.14831 |
| $A_1$ | 0.4616 | 0.6138 | 0.5447 |
| $A_2$ | 0.5275 | 0.4774 | 0.4842 |
| $A_3$ | 0.5934 | 0.4092 | 0.4842 |
| $A_4$ | 0.3956 | 0.4774 | 0.4842 |
| $\sum_{i=1}^{n} a_{ij}^2$ | 230 | 215 | 273 |

*Table 12:* Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get $r_{ij}$

| $v_{ij}$ | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| | 0.62923 | 0.22246 | 0.14831 |
| $A_1$ | 0.2904 | 0.1365 | 0.0808 |
| $A_2$ | 0.3319 | 0.1062 | 0.0718 |
| $A_3$ | 0.3734 | 0.0910 | 0.0718 |
| $A_4$ | 0.2489 | 0.1062 | 0.0718 |
| $v_{max}$ | 0.3734 | 0.1365 | 0.0808 |
| $v_{min}$ | 0.2489 | 0.0910 | 0.0718 |

*Table 13:* Multiply each column by $w_j$ to get $v_{ij}$

| Alternative | $S_i^+$ | $S_i^-$ | $T_i$ | Rank |
|---|---|---|---|---|
| $A_1$ | 0.0830 | 0.0622 | 0.4286 | 3 |
| $A_2$ | 0.0522 | 0.0844 | 0.6178 | 2 |
| $A_3$ | 0.0464 | 0.1245 | 0.7285 | 1 |
| $A_4$ | 0.1284 | 0.0152 | 0.1057 | 4 |

*Table 14:* The separation measure values and the final rankings for decision matrix (*Table 4*) using $\alpha$ -D MCDM-TOPSIS

## 6.3 Jean Dezert's strong inconsistent Example 3

Smarandache [7] introduced a Jean Dezert's Strong Inconsistent example. Let us consider the system of equations associated to MCDM problem and its associated matrix.

$$X = \begin{pmatrix} 1 & 9 & \frac{1}{9} \\ \frac{1}{9} & 1 & 9 \\ 9 & \frac{1}{9} & 1 \end{pmatrix} \qquad \begin{cases} x = & 9y, x > y \\ x = & \frac{1}{9}z, x < z \\ y = & 9z, y > z \end{cases}$$

We follow the same process as in the example above to get the general solution:

$$W = \begin{bmatrix} \frac{1}{6643} & \frac{81}{6643} & \frac{6561}{6643} \end{bmatrix}.$$

*We use TOPSIS to rank the four alternatives.*

| $a_{ij}^2$ | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| | 0.0002 | 0.0122 | 0.9877 |
| $A_1$ | 49 | 81 | 81 |
| $A_2$ | 64 | 49 | 64 |
| $A_3$ | 81 | 36 | 64 |
| $A_4$ | 36 | 49 | 64 |
| $\sum_{i=1}^{n} a_{ij}^2$ | 230 | 215 | 273 |

*Table 15:* Calculate $(a_{ij}^2)$ for each column

| $r_{ij}$ | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| | 0.0002 | 0.0122 | 0.9877 |
| $A_1$ | 0.503 | 0.699 | 0.623 |
| $A_2$ | 0.574 | 0.543 | 0.553 |
| $A_3$ | 0.646 | 0.466 | 0.553 |
| $A_4$ | 0.431 | 0.543 | 0.553 |
| $\sum_{i=1}^{n} a_{ij}^2$ | 230 | 215 | 273 |

*Table 16:* Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get $r_{ij}$

| $v_{ij}$ | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| | 0.0002 | 0.0122 | 0.9877 |
| $A_1$ | 0.0001 | 0.0075 | 0.5380 |
| $A_2$ | 0.0001 | 0.0058 | 0.4782 |
| $A_3$ | 0.0001 | 0.0050 | 0.4782 |
| $A_4$ | 0.0001 | 0.0058 | 0.4782 |
| $v_{max}$ | 0.0001 | 0.0075 | 0.5380 |
| $v_{min}$ | 0.0001 | 0.0050 | 0.4782 |

*Table 17:* Multiply each column by $w_j$ to get $v_{ij}$

| Alternative | $S_i^+$ | $S_i^-$ | $T_i$ | Rank |
|---|---|---|---|---|
| $A_1$ | 0.0000 | 0.0598 | 0.999668 | 1 |
| $A_1$ | 0.0598 | 0.0008 | 0.013719 | 2 |
| $A_1$ | 0.0598 | 0.0000 | 0.000497 | 4 |
| $A_1$ | 0.0598 | 0.0008 | 0.013715 | 3 |

*Table 18:* The separation measure values and the final rankings for decision matrix (*Table 4*) using α-D MCDM-TOPSIS

## 7 AHP method

As we proved in *Section 3. Description of data structure decision problems under consideration*, AHP method can be used in the second case, in which data is structured as pairwise comparisons of matrices of preferences.

Let us assume that $X$ is the comparison matrix of criteria; for each criterion $C_k$ $(k = 1, 2, \cdots, n)$ we have a comparison matrix of preferences $B_k$ and the consistency condition is perfect.

We use AHP to determine the importance weight $(w_i)$ of the criteria.

We apply again AHP method for each comparison matrices of preferences $B_k$ to determine the maximum eigenvalue and its associate eigenvector (same that is used to determine the weights of criteria).

For each matrix $B_k$ (associated to $C_k$), we calculate the $\lambda_{max}$ and priority vector (eigenvector).

| $C_i$ | $A_1$ | $A_2$ | $\cdots$ | $A_n$ | Priority vector |
|---|---|---|---|---|---|
| $A_1$ | $a_{11}$ | $a_{12}$ | $\cdots$ | $a_{1m}$ | $p_{1i}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | $\cdots$ | $a_{2m}$ | $p_{2i}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_m$ | $a_{m1}$ | $a_{m2}$ | $\cdots$ | $a_{mm}$ | $p_{mi}$ |

*Table 19:* Comparison matrix of the preferences with priority vector in latest column

We get a decision matrix (different from the decision matrix above), formed using priority vectors, in which the entries of the decision matrix are $p_{ij}$, and not $a_{ij}$.

| | $C_1$ | $C_2$ | $\cdots$ | $C_n$ |
|---|---|---|---|---|
| $w_i$ | $w_1$ | $w_2$ | $\cdots$ | $w_n$ |
| $A_1$ | $p_{11}$ | $p_{12}$ | $\cdots$ | $p_{1n}$ |
| $A_2$ | $p_{21}$ | $p_{22}$ | $\cdots$ | $p_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_m$ | $p_{m1}$ | $p_{m2}$ | $\cdots$ | $p_{mn}$ |

*Table 20:* Decision matrix of priority of preferences

The last step of AHP method is to rank the preferences using the following formula —

$$\sum_{j=1}^{n} w_j p_{ij}.$$

| | $C_1$ | $C_2$ | $\cdots$ | $C_n$ | $R_i$ |
|---|---|---|---|---|---|
| $w_i$ | $w_1$ | $w_2$ | $\cdots$ | $w_n$ | |
| $A_1$ | $p_{11}$ | $p_{12}$ | $\cdots$ | $p_{1n}$ | $\sum_{j=1}^{n} w_j p_{1j}$ |
| $A_2$ | $p_{21}$ | $p_{22}$ | $\cdots$ | $p_{2n}$ | $\sum_{j=1}^{n} w_j p_{2j}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_m$ | $p_{m1}$ | $p_{m2}$ | $\cdots$ | $p_{mn}$ | $\sum_{j=1}^{n} w_j p_{mj}$ |

*Table 21:* Ranking decision matrix

## 7.1 Numerical examples

Let us consider the three numerical examples (*Section 6*) and the decision matrix mentioned above.

For weak inconsistent and strong inconsistent examples, AHP does not work, as proved above, consequently the AHP method will be applied on consistent example 1.

The weights of criteria are calculated by using AHP (consistent example 1, *Section 6*).

As mentioned above (case 2, *Section 3*), for applying AHP we need to construct three pairwise comparisons matrices of size $4*4$ each.

| $C_1$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| $A_1$ | 1 | 7/8 | 7/9 | 7/6 |
| $A_2$ | 8/7 | 1 | 8/9 | 8/6 |
| $A_3$ | 9/7 | 7/8 | 1 | 9/6 |
| $A_4$ | 6/7 | 7/8 | 6/9 | 1 |

| $C_2$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| $A_1$ | 1 | 9/7 | 9/6 | 9/7 |
| $A_2$ | 7/9 | 1 | 7/6 | 1 |
| $A_3$ | 6/9 | 6/7 | 1 | 6/7 |
| $A_4$ | 7/9 | 1 | 7/6 | 1 |

| $C_3$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| $A_1$ | 1 | 9/8 | 9/8 | 9/8 |
| $A_2$ | 8/9 | 1 | 1 | 1 |
| $A_3$ | 8/9 | 1 | 1 | 1 |
| $A_4$ | 8/9 | 1 | 1 | 1 |

*Table 22:* Relative importance comparison matrix of alternatives for each criteria

We apply AHP method on three pairwise comparison matrices to rank the preferences, and calculate priority (i.e., normalized eigenvector), consistency index (CI) and consistency ratio (CR) for each matrix.

The eigenvalue, consistency index (CI) and consistency ratio (CR) for each matrix are: C1 — (CR =0, CI=0, $\lambda$ = 4), C2 — (CR =0, CI=0, $\lambda$ = 4) and C3 — (CR =0, CI=0, $\lambda$ = 4).

The priority vectors for three matrices are listed respectively in *Table 23*.

| $C_1$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | Priority vector |
|---|---|---|---|---|---|
| $A_1$ | 1 | 7/8 | 7/9 | 7/6 | 0.2333 |
| $A_2$ | 8/7 | 1 | 8/9 | 8/6 | 0.2667 |
| $A_3$ | 9/7 | 7/8 | 1 | 9/6 | 0.3000 |
| $A_4$ | 6/7 | 7/8 | 6/9 | 1 | 0.2000 |
| $C_2$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | Priority vector |
| $A_1$ | 1 | 9/7 | 9/6 | 9/7 | 0.3103 |

| | | | | | |
|---|---|---|---|---|---|
| $A_2$ | 7/9 | 1 | 7/6 | 1 | 0.2414 |
| $A_3$ | 6/9 | 6/7 | 1 | 6/7 | 0.2069 |
| $A_4$ | 7/9 | 1 | 7/6 | 1 | 0.2414 |

| $C_3$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | Priority vector |
|---|---|---|---|---|---|
| $A_1$ | 1 | 9/8 | 9/8 | 9/8 | 0.2727 |
| $A_2$ | 8/9 | 1 | 1 | 1 | 0.2424 |
| $A_3$ | 8/9 | 1 | 1 | 1 | 0.2424 |
| $A_4$ | 8/9 | 1 | 1 | 1 | 0.2424 |

*Table 23:* Relative importance comparison matrix of alternatives for each criteria

The last step of AHP method is applying it to rank the preferences using the following formula $\sum_{j=1}^{n} w_j p_{ij}$ (resulted as listed in the last column of the matrix above).

| | $C_1$ | $C_2$ | $C_3$ | $\sum_{j=1}^{n} w_j p_{ij}$ | Rank |
|---|---|---|---|---|---|
| $w_i$ | 12/16 | 3/16 | 1/16 | | |
| $A_1$ | 0.2333 | 0.3103 | 0.2727 | 0.2502 | 3 |
| $A_2$ | 0.2667 | 0.2414 | 0.2424 | 0.2604 | 2 |
| $A_3$ | 0.3000 | 0.2069 | 0.2424 | 0.2789 | 1 |
| $A_4$ | 0.2000 | 0.2414 | 0.2424 | 0.2104 | 4 |

*Table 24:* Decision matrix of priority of preferences and its ranking

## 8 Extended α-D MCDM

α-D MCDM introduced by Smarandache is not designed to rank preferences, but only for generating the weights for preferences or criteria, based on their $n$-wise matrix comparison. Hence, we proposed above a new method, called α-D MCDM-TOPSIS, employed to calculate the criteria weights for pairwise comparison matrices and for $n$-wise comparison matrices of criteria, in which α-D MCDM is used for calculate criteria weights, and TOPSIS — to rank the preferences.

In this section, we do not focus on criteria weights problem of $\alpha$-D MCDM, as discussed above and calculated for the three examples, but we propose an extension of α-D MCDM benefiting the skills of α-D MCDM to calculate maximum eigenvalue and its associate eigenvector of $n$-wise comparison matrix, in order to apply it again on $\alpha$-wise matrices of preferences.

An extended $\alpha$-D MCDM can be described as it follows:

Let us consider the second case of data structure decision problem (*Section 3*) — for each criterion $C_i$ corresponds a pairwise (or $n$-wise) comparison matrix of preferences and criteria.

**Step 8.1** *We use $\alpha$-D MCDM to calculate the weight ($w_i$) of the criteria*

**Step 8.2** *We apply again α-D MCDM method* for each comparison matrices of preferences $B_k$ to determine the maximum eigenvalue and its associate eigenvector (the same that is used to determine the weights of criteria).

For each matrix $B_k$ (associated to $C_k$), we calculate the $\lambda_{max}$ and priority vector (eigenvector).

We get —

| $C_i$ | $A_1$ | $A_2$ | $\cdots$ | $A_n$ | Priority vector |
|---|---|---|---|---|---|
| $A_1$ | $a_{11}$ | $a_{12}$ | $\cdots$ | $a_{1m}$ | $p_{1i}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | $\cdots$ | $a_{2m}$ | $p_{2i}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_m$ | $a_{m1}$ | $a_{m2}$ | $\cdots$ | $a_{mm}$ | $p_{mi}$ |

*Table 25:* Comparison matrix of the preferences with priority vector in latest column

**Step 8.3** *We get a decision matrix* (different from the decision matrix above), formed using priority vectors, in which the entries of the decision matrix are $p_{ij}$, and not $a_{ij}$.

| | $C_1$ | $C_2$ | $\cdots$ | $C_n$ |
|---|---|---|---|---|
| $w_i$ | $w_1$ | $w_2$ | $\cdots$ | $w_n$ |
| $A_1$ | $p_{11}$ | $p_{12}$ | $\cdots$ | $p_{1n}$ |
| $A_2$ | $p_{21}$ | $p_{22}$ | $\cdots$ | $p_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_m$ | $p_{m1}$ | $p_{m2}$ | $\cdots$ | $p_{mn}$ |

*Table 26:* Decision matrix of priority of preferences

**Step 8.4** *We employ the following formula (simple additive weighting) to rank the preferences* —

$$\sum_{j=1}^{n} w_j p_{ij}.$$

| | $C_1$ | $C_2$ | $\cdots$ | $C_n$ | $R_i$ |
|---|---|---|---|---|---|
| $w_i$ | $w_1$ | $w_2$ | $\cdots$ | $w_n$ | |
| $A_1$ | $p_{11}$ | $p_{12}$ | $\cdots$ | $p_{1n}$ | $\sum_{j=1}^{n} w_j p_{1j}$ |
| $A_2$ | $p_{21}$ | $p_{22}$ | $\cdots$ | $p_{2n}$ | $\sum_{j=1}^{n} w_j p_{2j}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_m$ | $p_{m1}$ | $p_{m2}$ | $\cdots$ | $p_{mn}$ | $\sum_{j=1}^{n} w_j p_{mj}$ |

*Table 27:* Ranking decision matrix

## 8.1 Numerical examples

Let us consider the three examples mentioned above (numerical consistent, weak inconsistent and strong inconsistent examples (*Section 6*).

We do not repeat the calculation of weights criteria by using α-D MCDM, because it was already done in *Section 6*, and we got the following priority vectors of criteria:

Consistent example 1 —

$$W = \begin{bmatrix} \frac{12}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}.$$

Weak inconsistent example 2 —

$$W = \begin{bmatrix} 0.62923 & 0.22246 & 0.14831 \end{bmatrix}.$$

Strong inconsistent example 3 —

$$W = \begin{bmatrix} \frac{1}{6643} & \frac{81}{6643} & \frac{6561}{6643} \end{bmatrix}.$$

We construct three comparisons matrices of size $4*4$ each (or three linear homogeneous systems), based on decision matrix of *Section 6*, and apply extended α-D MCDM to the three examples.

Let

$$m(A_1) = x,\ m(A_2) = y,\ m(A_3) = z \text{ and } m(A_4) = t.$$

For criteria $C_1$ —

- $A_2$ is $8$ seventh as important as $A_1$,

- $A_3$ is 9 seventh as important as $A_1$ ,
- $A_4$ is 6 seventh as important as $A_1$ .

The linear homogeneous system associated is —

$$\begin{cases} y \quad \frac{8}{7}x \\ z \quad \frac{9}{7}x \\ t \quad \frac{6}{7}x \end{cases}$$

and its general solution is —

$$W = \begin{bmatrix} \frac{7}{30} & \frac{8}{30} & \frac{9}{30} & \frac{6}{30} \end{bmatrix}.$$

For criteria $C_2$ —

- $A_1$ is 9 sixth as important as $A_3$ ,
- $A_3$ is 7 sixth as important as $A_3$ ,
- $A_4$ is 7 sixth as important as $A_3$ .

The linear homogeneous system associated is —

$$\begin{cases} x \quad \frac{9}{6}z \\ y \quad \frac{7}{6}z \\ t \quad \frac{7}{6}z \end{cases}$$

and its general solution is —

$$W = \begin{bmatrix} \frac{9}{29} & \frac{7}{29} & \frac{6}{29} & \frac{7}{29} \end{bmatrix}.$$

For criteria $C_3$ —

- $A_1$ is 9 eighth as important as $A_2$ ,
- $A_2$ , $A_3$ and $A_4$ have the same importance.

The associated linear homogeneous system is —

$$\begin{cases} x & \frac{9}{8}y \\ y & z \\ t & z \end{cases}$$

and its general solution is —

$$W = \begin{bmatrix} \frac{9}{33} & \frac{8}{33} & \frac{8}{33} & \frac{8}{33} \end{bmatrix}.$$

The results of extended α-D MCDM are summarized in the *Table 24*:

Consistent example 1 —

| | $C_1$ | $C_2$ | $C_3$ | $R_i$ | *Rank* |
|---|---|---|---|---|---|
| $w_i$ | 12/16 | 3/16 | 1/16 | | |
| $A_1$ | 7/30 | 9/29 | 9/33 | 0.2502 | 3 |
| $A_2$ | 8/30 | 7/29 | 8/33 | 0.2604 | 2 |
| $A_3$ | 9/30 | 6/29 | 8/33 | 0.2789 | 1 |
| $A_4$ | 6/30 | 7/29 | 8/33 | 0.2104 | 4 |

Weak inconsistent example 2 —

| | $C_1$ | $C_2$ | $C_3$ | $R_i$ | *Rank* |
|---|---|---|---|---|---|
| $w_i$ | 0.62923 | 0.22246 | 0.14831 | | |
| $A_1$ | 7/30 | 9/29 | 9/33 | 0.2563 | 3 |
| $A_2$ | 8/30 | 7/29 | 8/33 | 0.2574 | 2 |
| $A_3$ | 9/30 | 6/29 | 8/33 | 0.2707 | 1 |
| $A_4$ | 6/30 | 7/29 | 8/33 | 0.2155 | 4 |

Strong inconsistent example 3 —

| | $C_1$ | $C_2$ | $C_3$ | $R_i$ | *Rank* |
|---|---|---|---|---|---|
| $w_i$ | 1/6643 | 81/6643 | 6561/6643 | | |
| $A_1$ | 7/30 | 9/29 | 9/33 | 0.27318 | 1 |
| $A_2$ | 8/30 | 7/29 | 8/33 | 0.24242 | 2 |
| $A_3$ | 9/30 | 6/29 | 8/33 | 0.24200 | 4 |
| $A_4$ | 6/30 | 7/29 | 8/33 | 0.24241 | 3 |

*Table 24:* Decision matrix of priority of preferences and its ranking using Extended -DMCDM

| Example | Alternative | AHP-TOPSIS | | $\alpha$-DMCDM-TOPSIS | | AHP | | Extended $\alpha$-DMCDM | |
|---|---|---|---|---|---|---|---|---|---|
| *Consistent example 1* | $A_1$ | *0.3880* | *3* | *0.3880* | *3* | *0.2502* | *3* | *0.2502* | *3* |
| | $A_2$ | *0.6412* | *2* | *0.6412* | *2* | *0.2604* | *2* | *0.2604* | *2* |
| | $A_3$ | *0.7938* | *1* | *0.7938* | *1* | *0.2789* | *1* | *0.2789* | *1* |
| | $A_4$ | *0.0783* | *4* | *0.0783* | *4* | *0.2104* | *4* | *0.2104* | *4* |
| *Weak Inconsistent Example 2* | $A_1$ | *Does not works* | | *0.3880* | *3* | *Does not works* | | *0.2563* | *3* |
| | $A_2$ | | | *0.6412* | *2* | | | *0.2574* | *2* |
| | $A_3$ | | | *0.7938* | *1* | | | *0.2707* | *1* |
| | $A_4$ | | | *0.0783* | *4* | | | *0.2155* | *4* |
| *Strong Inconsistent Example 3* | $A_1$ | *Does not works* | | *0.999668* | *1* | *Does not works* | | *0.27318* | *1* |
| | $A_2$ | | | *0.013719* | *2* | | | *0.24242* | *2* |
| | $A_3$ | | | *0.000497* | *4* | | | *0.24200* | *4* |
| | $A_4$ | | | *0.013715* | *3* | | | *0.24241* | *3* |

*Table 25:* Summary of the results of three examples of all methods

For the three examples presented in this paper, the *Table 25*, summarizing all results of all methods, illustrates that the AHP and AHP-TOPSIS methods work just for the first example, in which criteria and alternatives are consistent in their pairwise comparisons. Our proposed methods — Extended α-D MCDM and α-D MCDM-TOPSIS — work not only for consistent example 1, giving the same results as AHP and AHP-TOPSIS methods, but also for weak inconsistent and strong inconsistent examples.

In the example 1, it is recorded that the alternative A3 has the first rank with the value (0.2789), the alternative A2 gets second rank with the coefficient value (0.2604), the alternative A1 realizes following rank with value (0.2502) and the alternative A4 lowers rank with the coefficient (0.2104), by using the AHP and our proposed method Extended α-DMCDM. Results indicate that all considered alternatives have near score values, for example 0.0685 ((A3)0.2789 - (A4) 0.2104). As a difference between the first and the latest ranking alternative, it is not sufficient to make a founded decision making, hence that can have a strong impact in practice to choose the best alternatives.

The results claimed that AHP-TOPSIS and our α-D MCDM-TOPSIS methods preserves the ranking order of the alternatives and overcome the near score values problem. By using AHP-TOPSIS and our α-D MCDM-TOPSIS methods, the score value of A3 was changed from 0.2789 to 0.7938, the score value of A2 was changed from 0.2604 to 0.6412, and the score value of A4 was changed from 0.2104 to 0.0783.

The bigger differences between the score values of alternatives 0.7155 ((A3) 0.7938 - (A4) 0.0783) is also subject to gain additional insights.

In the two last examples (weak inconsistent and strong inconsistent), one sees that the importance of discounting in our approaches suggest that they can be used to solve real-life problems in which criteria are not only pairwise, but $n$-wise comparisons, and the problems are not perfectly consistent. It is however worth to note that the ranking order of the four alternatives obtained by both methods is similar, but score values are slightly different. Both Extended α-D MCDM and α-D MCDM-TOPSIS methods allow taking into consideration any numbers of alternatives and any weights of criteria.

## 9 Conclusions

We have proposed two multi-criteria decision making methods, Extended α-D MCDM and α-D MCDM-TOPSIS models that allow to work for consistent and inconsistent MCDM problems. In addition, three examples have demonstrated that the α-D MCDM-TOPSIS model is efficient and robust.

Our approaches, Extended α-D MCDM and α-D MCDM-TOPSIS, give the same result as AHP-TOPSIS and AHP in consistent MCDM problems and elements of decision matrix are pairwise comparisons, but for weak inconsistent and strong inconsistent MCDM problems in which AHP and AHP-TOPSIS are limited and unable, our proposed methods — Extended α-D MCDM and α-D MCDM-TOPSIS — give justifiable results.

Furthermore, our proposed approaches — α-D MCDM-TOPSIS and Extended α-D MCDM — can be used to solve real-life problems in which criteria are not only pairwise, but $n$-wise comparisons, and the problems are not perfectly consistent.

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