Structural Properties of Neutrosophic Abel-Grassmann's Groupoids

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Abstract .

In this paper, we have introduced the notion of neutrosophic $(2,2)$ -regular, neutrosophic strongly regular neutrosophic **AG** -groupoids and investigated these structures. We have shown that neutrosophic regular, neutrosophic intra-regular and neutrosophic strongly regular **AG** -groupoid are the only generalized classes of neutrosophic **AG** -groupoid. Further we have shown that non-associative regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$ -regular and strongly regular **AG*** - neutrosophic groupoids do not exist.

Keyword

A neutrosophic **AG** -groupoid, left invertive law, medial law and paramedial law.
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Introduction

We know that in every branch of science there are lots of complications and problems appear which affluence the uncertainties and impaction. Most of these problems and complications are concerning with human life. These problems also play pivotal role for being subjective and classical. Common used methods are not sufficient to apply on these problems. To solve these complications, concept of fuzzy sets was published by Lotfi A.Zadeh in 1965, which has a wide range of applications in various fields such as engineering, artificial intelligence, control engineering, operation research, management science, robotics and many more. Zadeh introduced fuzzy sets to address uncertainies. By use of fuzzy sets the manipulated data and information of uncertainies can be prossessed. The idea of fuzzy sets was particularly designed to characterize uncertainty and vagueness and to present dignified tools in order to deal with the ambiguity intrinsic to the various problems. Fuzzy logic gives a conjecture morphology that enables approximate human reasoning capabilities to be applied to knowledge-based systems. The concept of fuzzy logic gives a mathematical potency to deal with the uncertainties associated with the human intellectual processes, such as reasoning and judgment.

In literature, a lot of theories have been developed to contend with uncertainty, imprecision and vagueness. In which, theory of probability, rough set theory fuzzy set theory, intiutionistic fuzzy sets etc, have played imperative role to cope with diverse types of uncertainties and imprecision entrenched in a system. But all these above theories were not sufficient tool to deal with indeterminate and inconsistent information in believe system. F.Samrandache noticed that the law of excluded middle are presently inactive in the modern logics and also by getting inspired with sport games (winning/tie/defeating), voting system (yes/ NA/no), decision making (making a decision/hesitating/not making) etc, he developed a new concept called neutrosophic set (NS) which is basically generalization of fuzzy sets and intiutionistic fuzzy sets. NS can be described by membership degree, and indeterminate degree and non-membership degree.

The neutrosophic logic is an extended idea of neutrosophy. Fuzzy theory is used when uncertainty is modeled and when there is indeterminancy involved we use neutrosophic theory. The neutrosophic algebraic structures have defined very recently. Basically, Vasantha K andasmy and Florentin Smarandache present the concept of neutrosophic algebraic structures by using neutrosophic theory. A number of the neutrosophic algrebraic structures introduced and considered include neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and neutrosophic AG-groupoids.

Preliminaries

Abel Grassmann's groupoid abbreviated as an AG-groupoid is a groupoid whose element satisfies the left invertive law i.e $(ab)c = (cb)a$ for all $a,b,c \in S$ .An AG-groupoid is a non associative and non-commutative algebraic structure mid way between a groupoid and commutative semigroup. AG-groupoids generalizes the concept of commutative semigroup

and have an important application within the theory of flocks.

An **AG**-groupoid , is a groupoid **S** holding the left invertive law
$$(ab)c = (cb)a, \text{ for all } a,b,c \in \textbf{S}.$$

This left invertive law has been obtained by introducing braces on the left of ternary commutative law $abc = cba$.

Basic Laws of AG-groupoid

In an **AG** -groupoid, the medial law holds
$$(ab)(cd) = (ac)(bd), \text{ for all } a,b,c,d \in \textbf{S}.$$

In an **AG**-groupoid **S** with left identity, the paramedial law holds
$$(ab)(cd) = (dc)(ba), \text{ for all } a,b,c,d \in \textbf{S}.$$

Further if an **AG**-groupoid contains a left identity, the following law holds
$$a(bc) = b(ac), \text{ for all } a,b,c \in \textbf{S}.$$

Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. In 1995, Florentin Smarandache introduced the idea of neutrosophy. Neutrosophic logic is an extension of fuzzy logic. Madad Khan et al., for the first time introduced the idea of a neutrosophic LA-semigroup in [4]. Moreover $SUI = \{a + bI : \text{ where } a, b \in S \text{ and } I \text{ is literal}$ indeterminacy such that $I^2 = I$ becomes neutrosophic LA-semigroup under the operation $*$ defined as:

$(a + bI) * (c + dI) = ac + bdI$ For all $(a + bI), (c + dI) \in SUI$. That is $(SUI, *)$ becomes neutrosophic LA-semigroup. They represented it by $N(S)$.

$$[(a_1 + a_2 I)(b_1 + b_2 I)](c_1 + c_2 I) = [(c_1 + c_2 I)(b_1 + b_2 I)](a_1 + a_2 I), \quad (1)$$

holds for all $(a_1 + a_2 I), \ (b_1 + b_2 I), \ (c_1 + c_2 I) \in N(S)$.

It is since than called the neutrosophic left invertive law. A neutrosophic groupoid satisfying the left invertive law is called a neutrosophic left almost semigroup and is abbreviated as neutrosophic LA-semigroup.

In a neutrosophic LA-semigroup $N(S)$ medial law holds i.e

$$[(a_1 + a_2 I)(b_1 + b_2 I)][(c_1 + c_2 I)(d_1 + d_2 I)]$$
$$= [(a_1 + a_2 I)(c_1 + c_2 I)][(b_1 + b_2 I)(d_1 + d_2 I)], \quad (2)$$

holds for all $(a_1 + a_2 I), (b_1 + b_2 I), (c_1 + c_2 I), (d_1 + d_2 I) \in N(S)$.

There can be a unique left identity in a neutrosophic LA-semigroup. In a neutrosophic LA-semigroup $N(S)$ with left identity $(e + eI)$ the following

laws hold for all $(a_1 + a_2 I), (b_1 + b_2 I), (c_1 + c_2 I), (d_1 + d_2 I) \in N(S)$.

$$[(a_1 + a_2 I)(b_1 + b_2 I)][(c_1 + c_2 I)(d_1 + d_2 I)]$$
$$= [(d_1 + d_2 I)(b_1 + b_2 I)][(c_1 + c_2 I)(a_1 + a_2 I)], \quad (3)$$

$$[(a_1 + a_2 I)(b_1 + b_2 I)][(c_1 + c_2 I)(d_1 + d_2 I)]$$
$$= [(d_1 + d_2 I)(c_1 + c_2 I)][(b_1 + b_2 I)(a_1 + a_2 I)], \quad (4)$$

and

$$(a_1 + a_2 I)[(b_1 + b_2 I)(c_1 + c_2 I)] = (b_1 + b_2 I)[(a_1 + a_2 I)(c_1 + c_2 I)]. \quad (5)$$

for all $(a_1 + a_2 I), (b_1 + b_2 I), (c_1 + c_2 I) \in N(S)$.

(3) is called neutrosophic paramedial law and a neutrosophic LA semigroup satisfies (5) is called Neutrosophic AG**-groupoid.

Now, $(a + bI)^2 = a + bI$ implies $a + bI$ is idempotent and if holds for all $a + bI \in N(S)$ then $N(S)$ is called idempotent neutrosophic LA-semigroup. This structure is closely related with a neutrosophic commutative semigroup, because if a Neutrosophic **AG**-groupoid contains a right identity, then it becomes a commutative semigroup.

A neutrosophic **AG**-groupoid N(S) with neutrosophic left identity becomes a Neutrosophic semigroup N(**S**) under new binary operation "∘" defined as

$$(x_1 + x_2 I) \circ (y_1 + y_2 I) = [(x_1 + x_2 I)(a_1 + a_2 I)](y_1 + y_2 I)$$

quasi regular if all elements of N(S)) are left quasi regular.

for all $(x_1 + x_2 I), (y_1 + y_2 I) \in N(\textbf{S})$.

It is easy to show that ∘ is associative

$$[(x_1 + x_2 I) \circ (y_1 + y_2 I)] \circ (z_1 + z_2 I)$$

$$= [[[(x_1 + x_2 I)(a_1 + a_2 I)](y_1 + y_2 I)](a_1 + a_2 I)](z_1 + z_2 I)$$

$$= [[(z_1 + z_2 I)(a_1 + a_2 I)][[(x_1 + x_2 I)(a_1 + a_2 I)](y_1 + y_2 I)]]$$

$$= [(x_1 + x_2 I)(a_1 + a_2 II)[[(z_1 + z_2 I)(a_1 + a_2 I)](y_1 + y_2 I)]$$

$$= [(x_1 + x_2 I)(a_1 + a_2 I)[(y_1 + y_2 I)(a_1 + a_2 I)](z_1 + z_2 I)]$$

$$= (x_1 + x_2 I) \circ [(y_1 + y_2 I) \circ (z_1 + z_2 I)].$$

Hence N(S) is a neutrosophic semigroup

Regularities in Neutrosophic **AG** -groupoids

An element $a + bI$ of a neutrosophic **AG** -groupoid $N(\textbf{S})$ is called a **regular** element of $N(\textbf{S})$ if there exists $x_1 + x_2 I \in N(\textbf{S})$ such that $a + bI = [(a + bI) * (x_1 + x_2 I)](a + bI)$ and N(**S**) is called regular if all elements of N(**S**) are regular.

An element $a + bI$ of neutrosophic **AG**-groupoid N(**S**) is called a **weakly regular** element of N(**S**) if there exist $(x_1 + x_2 I), (y_1 + y_2 I) \in N(\textbf{S})$ such that $a + bI = [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$ and $N(\textbf{S})$ is called weakly regular if all elements of N(**S**) are weakly regular.

An element $a + bI$ of a neutrosophic **AG** -groupoid $N(\textbf{S})$ is called an **intra-regular** element of $N(\textbf{S})$ if there exist $x_1 + x_2 I, y_1 + y_2 I \in N(\textbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$ and $N(\textbf{S})$ is called intra-regular if all elements of $N(\textbf{S})$ are intra-regular.

An element $a + bI$ of a neutrosophic **AG** -groupoid $N(\textbf{S})$ is called a **right regular** element of $N(\mathcal{S})$ if there exists $x_1 + x_2 I \in N(\textbf{S})$ such that $a + bI = (a + bI)^2(x_1 + x_2 I) = [(a + bI)(a + bI)](x_1 + x_2 I)$ and $N(\textbf{S})$ is called right regular if all elements of $N(\textbf{S})$ are right regular.

An element $a + bI$ of a Neutrosophic **AG** -groupoid $N(\textbf{S})$ is called **left regular** element of **N(S)** if there exists $x_1 + x_2 I \in N(\mathcal{S})$ such that $a + bI = (x_1 + x_2 I)(a + bI)^2 = (x_1 + x_2 I)[(a + bI)(a + bI)]$ and $N(\textbf{S})$ is called left regular if all elements of $N(\textbf{S})$ are left regular.

An element $a + bI$ of a Neutrosophic **AG** -groupoid $N(\textbf{S})$ is called a **left quasi regular** element of $N(\textbf{S})$ if there exist $x_1 + x_2 I, y_1 + y_2 I \in N(\textbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)]$ and $N(\textbf{S})$ is called left quasi regular if all elements of N(S)) are left quasi regular.

An element $a + bI$ of a Neutrosophic **AG**-groupoid $N(\mathbf{S})$ is called a **completely regular** element of $N(\mathbf{S})$ if $a + bI$ is regular, left regular and right regular. $N(\mathbf{S})$ is called completely regular if it is regular, left and right regular.

An element $a + bI$ of a Neutrosopic **AG**-groupoid $N(\mathbf{S})$ is called a **(2,2)-regular** element of $N(\mathbf{S})$ if there exists $x_1 + x_2 I \in N(\mathbf{S})$ such that $a + bI = [(a + bI)^2 (x_1 + x_2 I)](a + bI)^2$ and $N(\mathbf{S})$ is called $(2,2)$-regular **AG**-groupoid if all elements of $N(\mathbf{S})$ are $(2,2)$-regular.

An element $a + bI$ of a Neutrosophic **AG**-groupoid $N(\mathbf{S})$ is called a **strongly regular** element of $N(\mathbf{S})$ if there exists $x_1 + x_2 I \in N(\mathbf{S})$ such that $a + bI = [(a + bI)(x_1 + x_2 I)](a + bI)$ and $(a + bI)(x_1 + x_2 I) = (x_1 + x_2 I)(a + bI)$. $N(\mathbf{S})$ is called strongly regular Neutrosophic **AG**-groupoid if all elements of $N(\mathbf{S})$ are strongly regular.

A Neutrosophic **AG**-groupoid $N(\mathbf{S})$ is called Neutrosophic **AG***-groupoid if the following holds

$$[(a_1 + a_2 I)(b_1 + b_2 I)](c_1 + c_2 I) = (b_1 + b_2 I)[(a_1 + a_2 I)(c_1 + c_2 I)]$$

for all $a_1 + a_2 I, b_1 + b_2 I, c_1 + c_2 I \in N(\mathbf{S})$.

In Neutrosophic **AG***-groupoid N(S) the following law holds

A Neutrosophic **AG**-groupoid may or may not contains a left identity. The left identity of a Neutrosophic **AG**-groupoid allow us to introduce the inverses of elements in a Neutrosophic **AG**-groupoid. If an **AG**-groupoid contains a left identity, then it is unique.

Example 1 *Let us consider a Neutrosophic* **AG***-groupoid*
N(S) $= \{1 + 1I, \ 1 + 2I, \ 1 + 3I, \ 2 + 1I, \ 2 + 2I, \ 2 + 3I, \ 3 + 1I, \ 3 + 2I, \ 3 + 3I\}$ *in the following multiplication table.*

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+2I$	$1+3I$	$1+1I$	$1+2I$	$2+2I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$
$1+3I$	$1+2I$	$1+3I$	$1+1I$	$2+3I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$
$2+1I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$
$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$
$2+3I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$
$3+1I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$
$3+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$
$3+3I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$

Lemma 1 *If $N(\mathbf{S})$ is a regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular or strongly regular neutrosophic **AG** -groupoid，then $N(S) = N(\mathbf{S})^2$.*

Proof *Let $N(\mathbf{S})$ be a Neutrosophic regular **AG** -groupoid, then $N(\mathbf{S})^2 \subseteq N(\mathbf{S})$ is obvious. Let $a + bI \in N(\mathbf{S})$, then since $N(\mathbf{S})$ is regular so there exists $x + yI \in N(\mathbf{S})$ such that $a + bI = [(a + bI)(x + yI)](a + bI)$.*
Now
$$a + bI = [(a + b)(x + yI)](a + bI) \in N(\mathbf{S})\,\mathbf{N(S)}$$

$$N(\mathbf{S}) \subseteq N(\mathbf{S})^2$$

Similarly if $N(\mathbf{S})$ is weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular or strongly regular, then we can show that
$$N(\mathbf{S}) = N(\mathbf{S})^2 .$$

The converse is not true in general, because in Example lil, $N(\mathbf{S}) = N(\mathbf{S})^2$ holds but $N(\mathbf{S})$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$ -regular and strongly regular, because $d_1 + d_2 I \in N(\mathbf{S})$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular.

Theorem1 *If $N(\mathbf{S})$ is a Neutrosophic **AG** -groupoid with left identity ($\mathbf{AG}^{**}$ -groupoid ），then $N(\mathbf{S})$ is intra-regular if and only if for all $a + bI \in N(\mathbf{S})$, $a + bI = [(x_1 + x_2 I)(a + bI)][(a + bI)(z_1 + z_2 I)]$ holds   for   some $x_1 + x_2 I, z_1 + z_2 I \in N(\mathbf{S})$.*

Proof *Let $N(\mathbf{S})$ be an intra-regular Neutrosophic **AG** -groupoid with left identity ($\mathbf{AG}^{**}$-groupoid），then for any $a + bI \in N(\mathbf{S})$ there exist $x_1 + x_2 I, y_1 + y_2 I \in N(\mathbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$. Now by using Lemma1, $y_1 + y_2 I = (u_1 + u_2 I)(v_1 + v_2 I)$ for some $u_1 + u_2 I, v_1 + v_2 I \in N(\mathbf{S})$.*

$$
\begin{aligned}
a &+ bI \\
&= [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I) \\
&= [(x_1 + x_2 I)[(a + bI)(a + bI)]](y_1 + y_2 I) \\
&= [(a + bI)[(x_1 + x_2 I)(a + bI)]](y_1 + y_2 I) \\
&= [(y_1 + y_2 I)[(x_1 + x_2 I)(a + bI)]](a + bI)
\end{aligned}
$$

$$= [(y_1 + y_2 I)[(x_1 + x_2 I)(a + bI)]][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$$

$$= [[(u_1 + u_2 I)(v_1 + v_2)][(x + yI)(a + bI)]][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I),$$

$$= [[(a + bI)(x_1 + x_2 I)][(v_1 + v_2)(u_1 + u_2 I)]][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$$

$$= [[(a + bI)(x_1 + x_2 I)](t_1 + t_2 I)][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$$

$$= [[(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)](t_1 + t_2 I)][(a + bI)(x_1 + x_2 I)]$$

$$= [[(t_1 + t_2 I)(y_1 + y_2 I)][(x_1 + x_2 I)(a + bI)^2]][(a + bI)(x_1 + x_2 I)]$$

$$= [[(a + bI)^2(x_1 + x_2 I)][(y_1 + y_2 I)(t_1 + t_2 I)]][(a + bI)(x_1 + x_2 I)]$$

$$= [[(a + bI)^2(x_1 + x_2 I)][(s_1 + s_2 I)][(a + bI)(x_1 + x_2 I)],$$

$$= [[(s_1 + s_2 I)(x_1 + x_2 I)](a + bI)^2][(a + bI)(x_1 + x_2 I)]$$

$$= [[(s_1 + s_2 I)(x_1 + x_2 I)][(a + bI)(a + bI)]][(a + bI)(x_1 + x_2 I)]$$

$$= [[(a + bI)(a + bI)][(x_1 + x_2 I)(s_1 + s_2 I)]][(a + bI)(x_1 + x_2 I)]$$

$$= [[(a + bI)(a + bI)](w_1 + w_2 I)[(a + bI)(x_1 + x_2 I)],$$

$$= [[(w_1 + w_2 I)(a + bI)](a + bI)][(a + bI)(x_1 + x_2 I)]$$

$$= [(z_1 + z_2 I)(a + bI)][(a + bI)(x_1 + x_2 I)],$$

$$= [(x_1 + x_2 I)(a + bI)][(a + bI)(z_1 + z_2 I)]$$

where $(w_1 + w_2 I)(a + bI) = (z_1 + z_2 I) \in N(S)$ where
$(x_1 + x_2 I)(s_1 + s_2 I) = (w_1 + w_2 I) \in N(S)$ where
$(y_1 + y_2 I)(t_1 + t_2 I) = (s_1 + s_2 I) \in N(S)$ where
$(v_1 + v_2)(u_1 + u_2 I) = (t_1 + t_2 I) \in N(S)$ where $[(u_1 + u_2 I)(v_1 + v_2) = (y_1 + y_2 I)$
$\in N(S)$

Conversely, let for all $a + bI \in N(\mathbf{S})$,
$a + bI = [(x_1 + x_2 I)(a + bI)][(a + bI)(z_1 + z_2 I)]$ holds for some
$x_1 + x_2 I, z_1 + z_2 I \in N(\mathbf{S})$. *Now by using* (4), (1), (2) *and* (3), *we have*

$bI + a$

$])z_2 I + z_1 ()bI + a ([])bI + a ()x_2 I + x_1 ([=$

$])z_2 I + z_1 ()bI + a ()x_2 I + x_1 ([)bI + a ([=$

$])z_2 I + z_1 ()bI + a ()x_2 I + x_1 ([])z_2 I + z_1 ()bI + a ([])bI + a ()x_2 I + x_1 ([=])$

$z_2 I + z_1 ()bI + a ()x_2 I + x_1 ([])z_2 I + z_1 ()bI + a ()x_2 I + x_1 ([)bI + a ([=)bI$

$+a ([])z_2 I + z_1 ()bI + a ()x_2 I + x_1 ([])z_2 I + z_1 ()bI + a ()x_2 I + x_1 ([[=)bI +$

$a (^2])z_2 I + z_1 ()bI + a ()x_2 I + x_1 ([[=$

$)bI + a (]^2)z_2 I + z_1 (^2])bI + a ()x_2 I + x_1 ([[=$

$)bI + a ([])z_2 I + z_1 ()z_2 I + z_1 ([]^2)bI + a (^2)x_2 I + x_1 ([[=$

$)bI + a ([])z_2 I + z_1 (^2)bI + a ([])z_2 I + z_1 (^2)x_2 I + x_1 ([[=$

$)bI + a ([)z_2 I + z_1 ([)z_2 I + z_1 (^2)x_2 I + x_1 ([[^2)bI + a ([=$

$)bI + a ([)z_2 I + z_1 ([)z_2 I + z_1 (^2)x_2 I + x_1 ([])bI + a ()bI + a ([[=$

$)bI + a (]]^2)x_2 I + x_1 ([)z_2 I + z_1 ()z_2 I + z_1 ([[])bI + a ()bI + a ([[=$

$)bI + a (]]^2)x_2 I + x_1 (^2)z_2 I + z_1 ([])bI + a ()bI + a ([[=$

$)bI + a ([])bI + a ()bI + a (]]^2)z_2 I + z_1 (^2)x_2 I + x_1 ([[=$

$)bI + a ([])bI + a ()bI + a ([)t_2 I + t_1 ([=,$

$)u_2 I + u_1 (]^2)bI + a ()t_2 I + t_1 ([=$

where $\qquad [(x_1 + x_2 I)^2 (z_1 + z_2 I)^2] = (t_1 + t_2 I) \in N(S) \qquad$ *and*
$(a + bI) = (u_1 + u_2 I) \in N(S)$ *where* $(a + bI) = (u_1 + u_2 I) \in N(S)$

Thus $N(\mathbf{S})$ is intra-regular.

Theorem 2 *If* $N(\mathbf{S})$ *is a Neutrosophic* **AG** *-groupoid with left identity (* **AG**** *-groupoid), then the following are equivalent.*

(*i*) $N(\mathbf{S})$ is weakly regular.

(*ii*) $N(\mathbf{S})$ is intra-regular.

Proof (*i*) $\Rightarrow$ (*ii*) Let $N(\mathbf{S})$ be a weakly regular *Neutrosophic* **AG** -groupoid with left identity (Neutrosophic AG**-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $x_1 + x_2 I, y_1 + y_2 I \in N(\mathbf{S})$ such   that $a + bI = [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$ and   by   Lemma1, $x_1 + x_2 I = (u_1 + u_2 I)(v_1 + v_2 I)$ for   some $(u_1 + u_2 I), (v_1 + v_2 I) \in N(\mathbf{S})$ Let

$(v_1 + v_2 I)(u_1 + u_2 I) = (t_1 + t_2 I) \in N(\mathbf{S})$. Now by using (3), (1), (4) and (2), we have

$$a + bI = [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$$
$$= [(a + bI)(a + bI)][(x_1 + x_2 I)(y_1 + y_2 I)]$$
$$= (x_1 + x_2 I)[(a + bI)^2 (y_1 + y_2 I)]$$
$$= [(u_1 + u_2 I)(v_1 + v_2 I)][(a + bI)^2 (y_1 + y_2 I)]$$
$$= [(y_1 + y_2 I)(a + bI)^2][(v_1 + v_2 I)(u_1 + u_2 I)]$$
$$= [(y_1 + y_2 I)(a + bI)^2](t_1 + t_2 I)$$

Thus $N(\mathbf{S})$ is intra-regular.

$(ii) \Rightarrow (i)$ Let $N(S)$ be a intra regular Neutrosophic AG-groupoid with left identity (Neutrosophic $A\mathbf{G}^{**}$-groupoid), then for any $a + bI \in N(S)$

$$a + bI = [(y_1 + y_2 I)(a + bI)^2](t_1 + t_2 I)$$
$$= [(y_1 + y_2 I)(a + bI)^2][(v_1 + v_2 I)(u_1 + u_2 I)]$$
$$= [(u_1 + u_2 I)(v_1 + v_2 I)][(a + bI)^2 (y_1 + y_2 I)]$$
$$= (x_1 + x_2 I)[(a + bI)^2 (y_1 + y_2 I)]$$
$$= [(a + bI)(a + bI)][(x_1 + x_2 I)(y_1 + y_2 I)]$$
$$= [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$$

$\Rightarrow$ Thus $N(S)$ is weakly regular ·

Theorem 3 *If $N(\mathbf{S})$ is a Neutrosophic $\mathbf{AG}$-groupoid ($Neutrosophic\mathbf{AG}^{**}$-groupoid), then the following are equivalent.*

(i) $N(\mathbf{S})$ is weakly regular.

(ii) $N(\mathbf{S})$ is right regular.

Proof $(i) \Rightarrow (ii)$ Let $N(\mathbf{S})$ be a weakly regular Neutrosophic $\mathbf{AG}$-groupoid ($\mathbf{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $x_1 + x_2 I, y_1 + y_2 I \in N(\mathbf{S})$ such that $a + bI = (a + bI)(x_1 + x_2 I)(a + bI)(y_1 + y_2 I)$ and let $(x_1 + x_2 I)(y_1 + y_2 I) = (t_1 + t_2 I)$ for some $(t_1 + t_2 I) \in N(\mathbf{S})$. Now by using (2), we have

$$a + bI = [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$$
$$= [(a + bI)(a + bI)][(x_1 + x_2 I)(y_1 + y_2 I)]$$
$$= (a + bI)^2 (t_1 + t_2 I)$$

Thus $N(\mathbf{S})$ is right regular.

$(ii) \Rightarrow (i)$ It follows from Lemma1 and (2) .

$$a + bI = (a + bI)^2 (t_1 + t_2 I)$$
$$= [(a + bI)(a + bI)](t_1 + t_2 I)$$
$$= [(a + bI)(a + bI)][(x_1 + x_2 I)(y_1 + y_2 I)]$$
$$= [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$$

where $(t_1 + t_2 I) = (x_1 + x_2 I)(y_1 + y_2 I) \in N(\mathbf{S})$. Thus $N(S)$ is weakly regular.

Theorem 4 *If $N(\mathbf{S})$ is a Neutrosophic **AG** -groupoid with left identity (Neutrosophic AG**-groupoid), then the following are equivalent.*

(*i*) $N(\mathbf{S})$ *is weakly regular.*

(*ii*) $N(\mathbf{S})$ *is left regular.*

Proof $(i) \Rightarrow (ii)$ Let $N(\mathbf{S})$ be a weakly regular Neutrosophic **AG** -groupoid with left identity (Neutrosophic AG**-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I), (y_1 + y_2 I) \in N(\mathbf{S})$ such that $a + bI = [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$. Now by using (2) and (3), we have

$$a + bI = [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$$
$$= [(a + bI)(a + bI)][(x_1 + x_2 I)(y_1 + y_2 I)]$$
$$= [(y_1 + y_2 I)(x_1 + x_2 I)][(a + bI)(a + bI)]$$
$$= [(y_1 + y_2 I)(x_1 + x_2 I)](a + bI)^2$$
$$= (t_1 + t_2 I)(a + bI)^2,$$

where $[(y_1 + y_2 I)(x_1 + x_2 I)] = (t_1 + t_2 I)$ for some $(t_1 + t_2 I) \in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is left regular.

$(ii) \Rightarrow (i)$ It follows from Lemma1, (3) and (2).

$$a + bI = (t_1 + t_2 I)(a + bI)^2$$
$$= [(y_1 + y_2 I)(x_1 + x_2 I)](a + bI)^2$$
$$= [(y_1 + y_2 I)(x_1 + x_2 I)][(a + bI)(a + bI)]$$
$$= [(a + bI)(a + bI)][(x_1 + x_2 I)(y_1 + y_2 I)]$$
$$= [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)],$$

Where $(y_1 + y_2 I)(x_1 + x_2 I) = (t_1 + t_2 I)$ for some $(t_1 + t_2 I) \in N(\mathbf{S})$. Thus $N(S)$ is weakly regular.

Theorem 5 *If $N(\mathbf{S})$ is a Neutrosophic **AG** -groupoid with left identity (Neutrosophic AG**-groupoid)， then the following are equivalent.*

(*i*) $N(\mathbf{S})$ *is weakly regular.*

(*ii*) $N(\mathbf{S})$ *is left quasi regular*

Proof $(i) \Rightarrow (ii)$ Let $N(\mathbf{S})$ be a weakly regular $Neutrosophic \mathbf{AG}$ -groupoid with left identity, then for $a + bI \in N(\mathbf{S})$ there exists $(x_1 + x_2 I), (y_1 + y_2 I) \in N(\mathbf{S})$ such that $a + bI = [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$

$$a + bI = [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$$
$$= [(y_1 + y_2 I)(a + bI)][(x_1 + x_2 I)(a + bI)].$$

Thus $N(S)$ is left quasi regular.

$(ii) \Rightarrow (i)$ Let $N(\mathbf{S})$ be a left quasi regular Neutrosophic $\mathbf{AG}$-groupoid with left identity then.for $a + bI \in N(\mathbf{S})$ there exists $(x_1 + x_2 I), (y_1 + y_2 I) \in N(\mathbf{S})$ such that $a + bI = [(y_1 + y_2 I)(a + b)][(x_1 + x_2 I)(a + b)]$

$$a + bI = [(y_1 + y_2 I)(a + b)][(x_1 + x_2 I)(a + b)]$$
$$= [(a + b)(x_1 + x_2 I)][(a + b)(y_1 + y_2 I)].$$

Thus $N(S)$ is weakly regular.

Theorem 6 *If $N(\mathbf{S})$ is a Neutrosophic $\mathbf{AG}$-groupoid with left identity, then the following are equivalent.*

(i) $N(\mathbf{S})$ is $(2, 2)$-regular.

(ii) $N(\mathbf{S})$ is completely regular.

Proof $(i) \Rightarrow (ii)$ Let $N(\mathbf{S})$ be a $(2, 2)$-regular *Neutrosophic$\mathbf{AG}$*-groupoid with left identity, then for $a + bI \in N(\mathbf{S})$ there exists $(x_1 + x_2 I) \in N(\mathbf{S})$ such that $a + bI = [(a + bI)^2 (x + yI)](a + bI)^2$. Now

$$a + bI = [(a + bI)^2 (x_1 + x_2 I)](a + bI)^2$$
$$= [(y_1 + y_2 I)(a + bI)^2],$$

where $(a + bI)^2 (x_1 + x_2 I) = (y_1 + y_2 I) \in N(\mathbf{S})$, and by using (3), we have

$$a + bI = [(a + bI)^2 (x_1 + x_2 I)][(a + bI)(a + bI)]$$
$$= [(a + bI)(a + bI)][(x_1 + x_2 I)[(a + bI)^2]$$
$$= (a + bI)^2 (z_1 + z_2 I),$$

where $(x_1 + x_2 I)(a + bI)^2 = (z_1 + z_2 I) \in N(\mathbf{S})$. and by using (3), (1) and (4), we have

$$a + bI$$
$$= [(a + bI)^2 [(x_1 + x_2 I)[(a + bI)(a + bI)]]$$
$$= [(a + bI)(a + bI)][(x_1 + x_2 I)(a + bI)^2]$$
$$= [(a + bI)(a + bI)][[(e_1 + e_2 I)(x_1 + x_2 I)][a + bI)(a + bI)]]$$
$$= [(a + bI)(a + bI)][[(a + bI)(a + bI)][(x_1 + x_2 I)(e_1 + e_2 I)]]$$
$$= [(a + bI)(a + bI)][(a + bI)^2 (t_1 + t_2 I)]$$
$$= [[(a + bI)^2 (t_1 + t_2 I)](a + bI)](a + bI)$$
$$= [[[(a + bI)(a + bI)](t_1 + t_2 I)](a + bI)](a + bI)$$
$$= [[(t_1 + t_2 I)(a + bI)][(a + bI)(a + bI)]](a + bI)$$
$$= [[(a + bI)(a + bI)][(t_1 + t_2 I)(a + bI)]](a + bI)$$
$$= [[(a + bI)(t_1 + t_2 I)][(a + bI)(a + bI)]](a + bI)$$
$$= [(a + bI)[[(a + bI)(t_1 + t_2 I)](a + bI)]](a + bI)$$
$$= [(a + bI)(u_1 + u_2 I)](a + bI)$$

where $\qquad (t_1 + t_2 I) = (x_1 + x_2 I)(e_1 + e_2 I) \in N(\mathbf{S})$ &   where $(u_1 + u_2 I) = (a + bI)^2 (t_1 + t_2 I). \in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is neutrosophic left regular, right regular and regular, so $N(\mathbf{S})$ is completely regular.

$(ii) \Rightarrow (i)$ Assume that $N(\mathbf{S})$ is a completely regular neutrosophic **AG** -groupoid with left identity, then for any $a + bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I), (y_1 + y_2 I), (z_1 + z_2 I) \in N(\mathbf{S})$ such   that $(a + bI) = [(a + bI)(x_1 + x_2 I)](a + bI)$, $(a + bI) = (a + bI)^2 (y_1 + y_2 I)$ and $(a + bI) = (z_1 + z_2 I)(a + bI)^2$. Now by using (1), (4) and (3), we have

$$a + bI$$
$$= [(a + bI)(x_1 + x_2 I)](a + bI)$$
$$= [[(a + bI)^2 (y_1 + y_2 I)](x_1 + x_2 I)][(z_1 + z_2 I)(a + bI)^2]$$
$$= [[(x_1 + x_2 I)(y_1 + y_2 I)](a + bI)^2][(z_1 + z_2 I)(a + bI)^2]$$
$$= [[(x_1 + x_2 I)(y_1 + y_2 I)][(a + bI)(a + bI)]][(z_1 + z_2 I)(a + bI)^2]$$
$$= [[(a + bI)(a + bI)][(y_1 + y_2 I)(x_1 + x_2 I)]][(z_1 + z_2 I)(a + bI)^2]$$
$$= [(a + bI)^2[(y_1 + y_2 I)(x_1 + x_2 I)]][(z_1 + z_2 I)(a + bI)^2]$$
$$= [[(z_1 + z_2 I)(a + bI)^2][(y_1 + y_2 I)(x_1 + x_2 I)]](a + bI)^2$$
$$= [[(z_1 + z_2 I)(y_1 + y_2 I)][(a + bI)^2(x_1 + x_2 I)]](a + bI)^2$$
$$= [(a + bI)^2[[(z_1 + z_2 I)(y_1 + y_2 I)](x_1 + x_2 I)]](a + bI)^2$$
$$= [(a + bI)^2 (v_1 + v_2 I)](a + bI)^2,$$

where $[(z_1 + z_2 I)(y_1 + y_2 I)][(x_1 + x_2 I)] = (v_1 + v_2 I) \in N(\mathbf{S})$. This shows that $N(\mathbf{S})$ is $(2,2)$-regular.

Lemma2 *Every weakly regular neutrosophic* **AG** *-groupoid with left identity (Neutrosophic AG**-groupoid) is regular.*

Proof Assume that $N(\mathbf{S})$ is a weakly regular Neutrosophic **AG** -groupoid with left identity *(Neutrosophic AG**-groupoid)*, then for any $(a + bI) \in N(\mathbf{S})$ there exist $(x_1 + x_2 I), (y_1 + y_2 I) \in N(\mathbf{S})$ such   that $a + bI = [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$.   Let $(x_1 + x_2 I)(y_1 + y_2 I) = t_1 + t_2 I \in N(\mathbf{S})$ and $[(t_1 + t_2 I)][(y_1 + y_2 I)(x_1 + x_2 I)]](a + bI) = (u_1 + u_2 I) \in N(\mathbf{S})$. Now by using (1), (2), (3) and (4), we have

$$a + bI$$

$$= [(a+bI)(x_1 + x_2 I)][(a+bI)(y_1 + y_2 I)]$$

$$= [[(a+bI)(y_1 + y_2 I)](x_1 + x_2 I)](a+bI)$$

$$= [[(x_1 + x_2 I)(y_1 + y_2 I)](a+bI)](a+bI)$$

$$= [(t_1 + t_2 I)(a+bI)](a+bI)$$

$$= [(t_1 + t_2 I)[[(a+bI)(x_1 + x_2 I)][(a+bI)(y_1 + y_2 I)]]](a+bI)$$

$$= [(t_1 + t_2 I)[[(a+bI)(a+bI)][(x_1 + x_2 I)(y_1 + y_2 I)]]](a+bI)$$

$$= [(t_1 + t_2 I)[[(y_1 + y_2 I)(x_1 + x_2 I)][(a+bI)(a+bI)]]](a+bI)$$

$$= [(t_1 + t_2 I)[(a+bI)[[(y_1 + y_2 I)(x_1 + x_2 I)](a+bI)]]](a+bI)$$

$$= [(a+bI)[(t_1 + t_2 I)[[(y_1 + y_2 I)(x_1 + x_2 I)](a+bI)]]](a+bI)$$

$$= [(a+bI)(u_1 + u_2 I)](a+bI),$$

where $[(t_1 + t_2 I)[[(y_1 + y_2 I)(x_1 + x_2 I)](a+bI)]] = u_1 + u_2 I \in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is regular.

The converse of Lemma 2 is not true in general, as can be seen from the following example.

Example2 *[ref10] Let us consider a Neutrosophic* **AG** *-groupoid*

$$\mathbf{N(S)} = \begin{Bmatrix} 1+1I, 1+2I, 1+3I, 1+4I, 2+1I, 2+2I, 2+3I, 2+4I, \\ 3+1I, 3+2I, 3+3I, 3+4I, 4+1I, 4+2I, 4+3I, 4+4I \end{Bmatrix} \text{ with left identity } 3$$

in the following Cayley's table.

•	1+1I	1+2I	1+3I	1+4I	2+1I	2+2I	2+3I	2+4I	3+1I	3+2I	3+3I	3+4I	4+1I	4+2I	4+3I	4+4I
1+1I	2+2I	2+2I	2+4I	2+4I	2+2I	2+2I	2+4I	2+4I	4+2I	4+2I	4+4I	4+4I	4+2I	4+2I	4+4I	4+4I
1+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	4+2I	4+2I	4+2I	4+2I	4+2I	4+2I	4+2I	4+2I
1+3I	2+1I	2+2I	2+3I	2+4I	2+1I	2+2I	2+3I	2+4I	4+1I	4+2I	4+3I	4+4I	4+1I	4+2I	4+3I	4+4I
1+4I	2+1I	2+2I	2+1I	2+2I	2+1I	2+2I	2+1I	2+2I	4+1I	4+2I	4+1I	4+2I	4+1I	4+2I	4+1I	4+2I
2+1I	2+2I	2+2I	2+4I	2+4I	2+2I	2+2I	2+4I	2+4I	2+2I	2+2I	2+4I	2+4I	2+2I	2+2I	2+4I	2+4I
2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I	2+2I
2+3I	2+1I	2+2I	2+3I	2+4I	2+1I	2+2I	2+3I	2+4I	2+1I	2+2I	2+3I	2+4I	2+1I	2+2I	2+3I	2+4I
2+4I	2+1I	2+2I	2+1I	2+2I	2+1I	2+2I	2+1I	2+2I	2+1I	2+2I	2+1I	2+2I	2+1I	2+2I	2+1I	2+2I
3+1I	1+2I	1+2I	1+4I	1+4I	2+2I	2+2I	2+4I	2+4I	3+2I	3+2I	3+4I	3+4I	4+2I	4+2I	4+4I	4+4I
3+2I	1+2I	1+2I	1+2I	1+2I	2+2I	2+2I	2+2I	2+2I	3+2I	3+2I	3+2I	3+2I	4+2I	4+2I	4+2I	4+2I
3+3I	1+1I	1+2I	1+3I	1+4I	2+1I	2+2I	2+3I	2+4I	3+1I	3+2I	3+3I	3+4I	4+1I	4+2I	4+3I	4+4I
3+4I	1+1I	1+2I	1+1I	1+2I	2+1I	2+2I	2+1I	2+2I	3+1I	3+2I	3+1I	3+2I	4+1I	4+2I	4+1I	4+2I
4+1I	1+2I	1+2I	1+4I	1+4I	2+2I	2+2I	2+4I	2+4I	1+1I	1+2I	1+4I	1+4I	2+2I	2+2I	2+4I	2+4I
4+2I	1+2I	1+2I	1+2I	1+2I	2+2I	2+2I	2+2I	2+2I	1+1I	1+2I	1+2I	1+2I	2+2I	2+2I	2+2I	2+2I
4+3I	1+1I	1+2I	1+3I	1+4I	2+1I	2+2I	2+3I	2+4I	1+2I	1+2I	1+3I	1+4I	2+1I	2+2I	2+3I	2+4I
4+4I	1+1I	1+2I	1+1I	1+2I	2+1I	2+2I	2+1I	2+2I	1+2I	1+2I	1+1I	1+2I	2+1I	2+2I	2+1I	2+2I

Theorem 7 *If $N(\mathbf{S})$ is a Neutrosophic* **AG** *-groupoid with left identity (Neutrosophic AG**-groupoid) then the following are equivalent.*

(i) $N(\mathbf{S})$ *is weakly regular.*

(ii) $N(\mathbf{S})$ *is completely regular.*

Proof $(i) \Rightarrow (ii)$

Let $N(\mathbf{S})$ be a weakly regular Neutrosophic $\mathbf{AG}$-groupoid ($\mathbf{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I),(y_1 + y_2 I) \in N(\mathbf{S})$ such that $a+bI = (a+bI)(x_1 + x_2 I)(a+bI)(y_1 + y_2 I)$ and let $(x_1 + x_2 I)(y_1 + y_2 I) = (t_1 + t_2 I)$ for some $(t_1 + t_2 I) \in N(\mathbf{S})$. Now by using (2), we have

$$
\begin{aligned}
a+bI &= [(a+bI)(x_1 + x_2 I)][(a+bI)(y_1 + y_2 I)] \\
&= [(a+bI)(a+bI)][(x_1 + x_2 I)(y_1 + y_2 I)] \\
&= (a+bI)^2 (t_1 + t_2 I),
\end{aligned}
$$

where $(x_1 + x_2 I)(y_1 + y_2 I) = t_1 + t_2 I \in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is right regular.

Let $N(\mathbf{S})$ be a weakly regular Neutrosophic $\mathbf{AG}$-groupoid with left identity *(Neutrosophic AG**-groupoid)* then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I),(y_1 + y_2 I) \in N(\mathbf{S})$ such that $a+bI = [(a+bI)(x_1 + x_2 I)][(a+bI)(y_1 + y_2 I)]$. Now by using (2) and (3), we have

$$
\begin{aligned}
a+bI &= [(a+bI)(x_1 + x_2 I)][(a+bI)(y_1 + y_2 I)] \\
&= [(a+bI)(a+bI)][(x_1 + x_2 I)(y_1 + y_2 I)] \\
&= [(y_1 + y_2 I)(x_1 + x_2 I)][(a+bI)(a+bI)] \\
&= [(y_1 + y_2 I)(x_1 + x_2 I)](a+bI)^2 \\
&= (t_1 + t_2 I)(a+bI)^2,
\end{aligned}
$$

Where $(y_1 + y_2 I)(x_1 + x_2 I) = (t_1 + t_2 I)$ for some $(t_1 + t_2 I) \in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is left regular.

Assume that $N(\mathbf{S})$ is a weakly regular Neutrosophic $\mathbf{AG}$-groupoid with left identity *(Neutrosophic AG**-groupoid)*, then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I),(y_1 + y_2 I) \in N(\mathbf{S})$ such that $a+bI = [(a+bI)(x_1 + x_2 I)][(a+bI)(y_1 + y_2 I)]$. Let $(x_1 + x_2 I)(y_1 + y_2 I) = (t_1 + t_2 I) \in N(\mathbf{S})$ and

$[(t_1 + t_2 I)[(y_1 + y_2 I)(x_1 + x_2 I)]](a+bI) = (u_1 + u_2 I) \in N(\mathbf{S})$. Now by using (1), (2), (3) and (4), we have

$$
\begin{aligned}
a+bI &= [(a+bI)(x_1 + x_2 I)][(a+bI)(y_1 + y_2 I)] \\
&= [[(a+bI)(y_1 + y_2 I)](x_1 + x_2 I)](a+bI) \\
&= [[(x_1 + x_2 I)(y_1 + y_2 I)](a+bI)](a+bI) \\
&= [(t_1 + t_2 I)(a+bI)](a+bI) \\
&= [(t_1 + t_2 I)[[(a+bI)(x_1 + x_2 I)][(a+bI)(y_1 + y_2 I)]]](a+bI) \\
&= [(t_1 + t_2 I)[[(a+bI)(a+bI)][(x_1 + x_2 I)(y_1 + y_2 I)]]](a+bI) \\
&= [(t_1 + t_2 I)[[(y_1 + y_2 I)(x_1 + x_2 I)][(a+bI)(a+bI)]]](a+bI) \\
&= [(t_1 + t_2 I)[(a+bI)[[(y_1 + y_2 I)(x_1 + x_2 I)](a+bI)]]](a+bI) \\
&= [(a+bI)[(t_1 + t_2 I)[[(y_1 + y_2 I)(x_1 + x_2 I)](a+bI)]]](a+bI) \\
&= [(a+bI)(u_1 + u_2 I)](a+bI),
\end{aligned}
$$

where $[(t_1 + t_2 I)[[(y_1 + y_2 I)(x_1 + x_2 I)](a + bI)]] = (u_1 + u_2 I) \in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is regular. Thus $N(S)$ is completely regular.

$(ii) \Rightarrow (i)$

Assume that $N(S)$ is completely regular Neutrosophic **AG** -groupoid with left identity ($Neutrosophic$**AG****-groupoid$)$, then for any $(a + bI) \in N(\mathbf{S})$ there exist $(t_1 + t_2 I) \in N(\mathbf{S})$ such that $(a + bI) = (a + bI)^2 (x_1 + x_2 I)$, $(a + bI) = (y_1 + y_2 I)(a + bI)^2$, $a + bI = [(a + bI)(z_1 + z_2 I)](a + bI)$

$$a + bI = (a + bI)^2 (x_1 + x_2 I)$$
$$= [(a + bI)(a + bI)][(v_1 + v_2 I)(u_1 + u_2 I)]$$
$$= [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)],$$

where $(x_1 + x_2 I) = (v_1 + v_2 I)(u_1 + u_2 I) \in N(\mathbf{S})$. Thus $N(S)$ is Neutrosophic weakly regular

$$a + bI = (x_1 + x_2 I)(a + bI)^2$$
$$= [(u_1 + u_2 I)(v_1 + v_2 I)](a + bI)^2$$
$$= [(u_1 + u_2 I)(v_1 + v_2 I)][(a + bI)(a + bI)]$$
$$= [(a + bI)(a + bI)][(u_1 + u_2 I)(v_1 + v_2 I)]$$
$$= [(a + bI)(u_1 + u_2 I)][(a + bI)(v_1 + v_2 I)],$$

where $(x_1 + x_2 I) = (u_1 + u_2 I)(v_1 + v_2 I)$ for some $(x_1 + x_2 I) \in N(\dot{\mathbf{S}})$. Thus $N(S)$ is weakly regular.

$a + bI$
$= [(a + bI)(z_1 + z_2 I)](a + bI)$
$= [[(a + bI)^2 (x_1 + x_2 I)](z_1 + z_2 I)][(y_1 + y_2 I)(a + bI)^2]$
$= [[(z_1 + z_2 I)(x_1 + x_2 I)][(a + bI)(a + bI)]][(y_1 + y_2 I)[(a + bI)(a + bI)]]$
$= [(a + bI)[[(z_1 + z_2 I)(x_1 + x_2 I)](a + bI)]][(a + bI)[(y_1 + y_2 I)(a + bI)]]$
$= [(a + bI)(t_1 + t_2 I)][(a + bI)(w_1 + w_2 I)],$

where $(t_1 + t_2 I) = [[(z_1 + z_2 I)(x_1 + x_2 I)](a + bI)]] \in N(\mathbf{S})$ & $(w_1 + w_2 I) = [(y_1 + y_2 I)(a + bI)] \in N(S)$. Thus $N(\mathbf{S})$ is weakly regular.

Lemma 3 *Every strongly regular Neutrosophic **AG** -groupoid with left identit (Neutrosophic AG**-groupoid) is completely regular.*

Proof Assume that $N(\mathbf{S})$ is a strongly regular Neutrosophic $\mathbf{AG}$-groupoid with left identity (*Neutrosophic*$\mathbf{AG}^{**}$-groupoid), then for any $(a+bI) \in N(\mathbf{S})$ there exists $(x_1 + x_2 I) \in N(\mathbf{S})$ such that $a+bI = [(a+bI)(x_1 + x_2 I)](a+bI)$ and $(a+bI)(x_1 + x_2 I) = (x_1 + x_2 I)(a+bI)$. Now by using (1), we have

$$a + bI = [(a+bI)(x_1 + x_2 I)](a+bI)$$
$$= [(x_1 + x_2 I)(a+bI)](a+bI)$$
$$= [(a+bI)(a+bI)](x_1 + x_2 I)$$
$$= (a+bI)^2 (x_1 + x_2 I).$$

This shows that $N(\mathbf{S})$ is right regular and by Theorems 4 and 7, it is clear to see that $N(\mathbf{S})$ is completely regular.

Theorem 8 *In a Neutrosophic* $\mathbf{AG}$-*groupoid* $\mathbf{N(S)}$ *with left identity (Neutrosophic AG**-groupoid) the following are equivalent.*

(*i*) $N(\mathbf{S})$ *is weakly regular.*

(*ii*) $N(\mathbf{S})$ *is intra-regular.*

(*iii*) $N(\mathbf{S})$ *is right regular.*

(*iv*) $N(\mathbf{S})$ *is left regular.*

(*v*) $N(\mathbf{S})$ *is left quasi regular.*

(*vi*) $N(\mathbf{S})$ *is completely regular.*

(*vii*) *For all* $a+bI \in N(\mathbf{S})$, *there exist* $(x_1 + x_2 I),(y_1 + y_2 I) \in N(\mathbf{S})$ *such that* $a + bI = [(x_1 + x_2 I)(a+bI)][(a+bI)(y_1 + y_2 I)].$

Proof $(i) \Rightarrow (ii)$ Let $N(\mathbf{S})$ be weakly regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity *(Neutrosophic AG**-groupoid)*, then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I),(y_1 + y_2 I) \in N(\mathbf{S})$ such that $a+bI = [(a+bI)(x_1 + x_2 I)][(a+bI)(y_1 + y_2 I)]$ and by Lemma 1, $(x_1 + x_2 I) = (u_1 + u_2 I)(v_1 + v_2 I)$ for some $(u_1 + u_2 I),(v_1 + v_2 I) \in N(\mathbf{S})$. Let $(v_1 + v_2 I)(u_1 + u_2 I) = (t_1 + t_2 I) \in N(\mathbf{S})$. Now by using (3), (1), (4) and (2), we have

$$a + bI = [(a+bI)(x_1 + x_2 I)][(a+bI)(y_1 + y_2 I)]$$
$$= [(a+bI)(a+bI)][(x_1 + x_2 I)(y_1 + y_2 I)]$$
$$= (x_1 + x_2 I)[(a+bI)^2 (y_1 + y_2 I)]$$
$$= [(u_1 + u_2 I)(v_1 + v_2 I)][(a+bI)^2 (y_1 + y_2 I)]$$
$$= [(y_1 + y_2 I)(a+bI)^2][(v_1 + v_2 I)(u_1 + u_2 I)]$$
$$= [(y_1 + y_2 I)(a+bI)^2](t_1 + t_2 I)$$

where $(v_1 + v_2 I)(u_1 + u_2 I) = (t_1 + t_2 I) \in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is intra-regular

$(ii) \Rightarrow (iii)$ Let $N(\mathbf{S})$ be a weakly regular *Neutrosophic* **AG**-groupoid with left identity (*Neutrosophic***AG****-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I),(y_1 + y_2 I) \in N(\mathbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$

$$\begin{aligned}
a + bI &= [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I) \\
&= [[(u_1 + u_2 I)(v_1 + v_2 I)][(a + bI)(a + bI)]](y_1 + y_2 I) \\
&= [(a + bI)^2((v_1 + v_2 I)(u_1 + u_2 I))](y_1 + y_2 I) \\
&= [(y_1 + y_2 I)((v_1 + v_2 I)(u_1 + u_2 I))](a + bI)^2 \\
&= [(y_1 + y_2 I)((v_1 + v_2 I)(u_1 + u_2 I))][(a + bI)(a + bI)] \\
&= [(a + bI)(a + bI)][((v_1 + v_2 I)(u_1 + u_2 I))(y_1 + y_2 I)] \\
&= (a + bI)^2(s_1 + s_2 I),
\end{aligned}$$

where $\quad\quad\quad\quad x_1 + x_2 I = (u_! + u_2 I)(v_1 + v_2 I) \in N(\mathcal{S}) \quad\quad\quad$ & $s_1 + s_2 I = [(y_1 + y_2 I)[v_1 + v_2 I)(u_1 + u_2 I)]] \in N(\mathbf{S})$. Thus $N(S)$ is right regular

$(iii) \Rightarrow (iv)$ Let $N(\mathbf{S})$ be a right regular *Neutrosophic* **AG**-groupoid with left identity (*Neutrosophic***AG****-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist

$x_1 + x_2 I \in N(\mathbf{S})$ such that $a + bI = (a + b)^2(x_1 + x_2 I)$

$$\begin{aligned}
a + bI &= (a + b)^2(x_1 + x_2 I) \\
&= [(a + bI)(a + bI)](x_1 + x_2 I) \\
&= [(a + bI)(a + bI)][(u_1 + u_2 I)(v_1 + v_2 I)] \\
&= [(v_1 + v_2 I)(u_1 + u_2 I)][a + bI)(a + bI)] \\
&= (y_1 + y_2 I)(a + bI)^2
\end{aligned}$$

Where $(y_1 + y_2 I) = [(v_1 + v_2 I)(u_1 + u_2 I)] \in N(\mathbf{S})$. Thus $N(S)$ is left regular

$(iv) \Rightarrow (v)$ Let $N(\mathbf{S})$ be a left regular *Neutrosophic* **AG**-groupoid with left identity *(Neutrosophic AG**-groupoid)*, then for any $a + bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I) \in N(\mathbf{S})$ such that $(a + bI) = (x_1 + x_2 I)(a + bI)^2$

$$\begin{aligned}
a + bI &= (x_1 + x_2 I)(a + bI)^2 \\
&= [(u_1 + u_2 I)(v_1 + v_2 I)][(a + bI)(a + bI)] \\
&= [(u_1 + u_2 I)(a + bI)][(v_1 + v_2 I)(a + bI)]
\end{aligned}$$

Thus $N(S)$ is left quasi regular

$(v) \Rightarrow (vi)$ Let $N(\mathbf{S})$ be a left quasi regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity ($Neutrosophic\mathbf{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $x_1 + x_2 I \in N(\mathbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)]$

$$
\begin{aligned}
a + bI &= [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)] \\
&= [(a + bI)(a + bI)][(y_1 + y_2 I)(x_1 + x_2 I)] \\
&= (a + bI)^2 (v_1 + v_2 I)
\end{aligned}
$$

where $v_1 + v_2 I = [(y_1 + y_2 I)(x_1 + x_2 I)] \in N(\mathbf{S})$

Thus $N(S)$ is right regular $\Rightarrow (1)$

Let $N(\mathbf{S})$ be a left quasi regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity ($Neutrosophic\mathbf{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $x_1 + x_2 I \in N(\mathbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)]$

$$
\begin{aligned}
a + bI &= [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)] \\
&= [(x_1 + x_2 I)(y_1 + y_2 I)][(a + bI)(a + bI)] \\
&= (u_1 + u_2 I)(a + bI)^2
\end{aligned}
$$

where $(u_1 + u_2 I) = [(x_1 + x_2 I)(y_1 + y_2 I)] \in N(\mathbf{S})$

Thus $N(S)$ is left regular $\Rightarrow (2)$

Let $N(\mathbf{S})$ be a left quasi regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity ($Neutrosophic\mathbf{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I) \in N(\mathbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)]$

$$
\begin{aligned}
a + bI &= [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)] \\
&= [(a + bI)(a + bI)][(y_1 + y_2 I)(x_1 + x_2 I)] \\
&= [((y_1 + y_2 I)(x_1 + x_2 I))(a + bI)](a + bI) \\
&= [(v_1 + v_2 I)(a + bI)](a + bI), \\
&= [(v_1 + v_2 I)[(x_1 + x_2 I)(a + bI))((y_1 + y_2 I)(a + bI)]](a + bI) \\
&= [(v_1 + v_2 I)[[(x_1 + x_2 I)(y_1 + y_2 I)][(a + bI)(a + bI)]](a + bI) \\
&= [(v_1 + v_2 I)[(a + bI)[[(x_1 + x_2 I)(y_1 + y_2 I)](a + bI)]]](a + bI) \\
&= [(a + bI)[(v_1 + v_2 I)[[(x_1 + x_2 I)(y_1 + y_2 I)](a + bI)]]](a + bI) \\
&= [(a + bI)(t_1 + t_2 I)](a + bI)
\end{aligned}
$$

where $\quad (v_1 + v_2 I) = (y_1 + y_2 I)(x_1 + x_2 I) \in N(\mathbf{S}) \quad$ & $\quad$ where $t_1 + t_2 I = [(v_1 + v_2 I)[[(x_1 + x_2 I)(y_1 + y_2 I)](a + bI)]] \in N(\mathbf{S})$

Thus $N(S)$ is regular $\Rightarrow (3)$.

By (1).(2) & (3) $N(S)$ is completely regular.

$(vi) \Rightarrow (i)$ Let $N(\mathbf{S})$ be a complete regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity (*Neutrosophic*$\mathbf{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I) \in N(\mathbf{S})$ such that $a+bI = (a+bI)^2(x_1 + x_2 I)$, $a+bI = (y_1 + y_2 I)(a+bI)^2$, $a+bI = [(a+bI)(z_1 + z_2 I)] \, (a+bI)$

$$a+bI = (a+bI)^2(x_1 + x_2 I)$$
$$= [(a+bI)(a+bI)][(u_1 + u_2 I)(v_1 + v_2 I)]$$
$$= [(a+bI)(u_1 + u_2 I)][(a+bI)(v_1 + v_2 I)]$$

where $(x_1 + x_2 I) = [(u_1 + u_2 I)(v_1 + v_2 I)] \in N(\mathbf{S})$. Thus $N(S)$ is weakly regular.

$$a+bI = (y_1 + y_2 I)(a+bI)^2$$
$$= [(v_1 + v_2 I)(u_1 + u_2 I)][(a+bI)(a+bI)]$$
$$= [(a+bI)(u_1 + u_2 I)][(a+bI)(v_1 + v_2 I)]$$

where $(y_1 + y_2 I) = [(v_1 + v_2 I)(u_1 + u_2 I)] \in N(\mathbf{S})$. Thus $N(S)$ is neutrosophic weakly regular.

$$a+bI$$
$$= [(a+bI)(z_1 + z_2 I)](a+bI)$$
$$= [[(a+bI)^2(x_1 + x_2 I)](z_1 + z_2 I)][(y_1 + y_2 I)(a+bI)^2]$$
$$= [[(z_1 + z_2 I)(x_1 + x_2 I)][(a+bI)(a+bI)]][(y_1 + y_2 I)[(a+bI)(a+bI)]]$$
$$= [(a+bI)[[(z_1 + z_2 I)(x_1 + x_2 I)](a+bI)]][(a+bI)[(y_1 + y_2 I)(a+bI)]]$$
$$= [(a+bI)(t_1 + t_2 I)][(a+bI)(w_1 + w_2 I)]$$

where $t_1 + t_2 I = [[(z_1 + z_2 I)(x_1 + x_2 I)](a+bI)]] \in N(\mathbf{S})$ & $(w_1 + w_2 I) = [(y_1 + y_2 I)(a+bI)] \in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is neutrosophic weakly regular.

$(ii) \Rightarrow (vii)$ Let $N(\mathbf{S})$ be an intra-regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity ($\mathbf{AG}^{**}$-groupoid) , then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I), (y_1 + y_2 I) \in N(\mathbf{S})$ such that $a+bI = [(x_1 + x_2 I)(a+bI)^2](y_1 + y_2 I)$. Now by using Lemma1, $(y_1 + y_2 I) = (u_1 + u_2 I)(v_1 + v_2 I)$ for some $u_1 + u_2 I, v_1 + v_2 I \in N(\mathbf{S})$. Thus by using (4), (1) and (3), we have

$$a+bI$$
$$= [(x_1 + x_2 I)(a+bI)^2](y_1 + y_2 I)$$
$$= [(x_1 + x_2 I)[(a+bI)(a+bI)]](y_1 + y_2 I)$$
$$= [(a+bI)[(x_1 + x_2 I)(a+bI)]](y_1 + y_2 I)$$
$$= [(y_1 + y_2 I)[(x_1 + x_2 I)(a+bI)]](a+bI)$$

$$= [(y_1 + y_2 I)[(x_1 + x_2 I)(a + bI)]][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$$

$$= [[(u_1 + u_2 I)(v_1 + v_2)][(x + yI)(a + bI)]][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$$

$$= [[(a + bI)(x_1 + x_2 I)][(v_1 + v_2)(u_1 + u_2 I)]][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$$

$$= [[(a + bI)(x_1 + x_2 I)](t_1 + t_2 I)][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$$

$$= [[(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)](t_1 + t_2 I)][(a + bI)(x_1 + x_2 I)]$$

$$= [[(t_1 + t_2 I)(y_1 + y_2 I)][(x_1 + x_2 I)(a + bI)^2]][(a + bI)(x_1 + x_2 I)]$$

$$= [[(a + bI)^2(x_1 + x_2 I)][(y_1 + y_2 I)(t_1 + t_2 I)]][(a + bI)(x_1 + x_2 I)]$$

$$= [[(a + bI)^2(x_1 + x_2 I)][(s_1 + s_2 I)]][(a + bI)(x_1 + x_2 I)]$$

$$= [[(s_1 + s_2 I)(x_1 + x_2 I)](a + bI)^2][(a + bI)(x_1 + x_2 I)]$$

$$= [[(s_1 + s_2 I)(x_1 + x_2 I)][(a + bI)(a + bI)]][(a + bI)(x_1 + x_2 I)]$$

$$= [[(a + bI)(a + bI)][(x_1 + x_2 I)(s_1 + s_2 I)]][(a + bI)(x_1 + x_2 I)]$$

$$= [[(a + bI)(a + bI)](w_1 + w_2 I)[(a + bI)(x_1 + x_2 I)]$$

$$= [[(w_1 + w_2 I)(a + bI)](a + bI)][(a + bI)(x_1 + x_2 I)]$$

$$= [(z_1 + z_2 I)(a + bI)][(a + bI)(x_1 + x_2 I)]$$

$$= [(x_1 + x_2 I)(a + bI)][(a + bI)(z_1 + z_2 I)]$$

where $[(u_1 + u_2 I)(v_1 + v_2) = (y_1 + y_2 I)$ $\in N(S)$ & where
$(v_1 + v_2)(u_1 + u_2 I) = (t_1 + t_2 I) \in N(S)$ & where
$(y_1 + y_2 I)(t_1 + t_2 I) = (s_1 + s_2 I) \in N(S)$ & where
$(x_1 + x_2 I)(s_1 + s_2 I) = (w_1 + w_2 I) \in N(S)$ & where
$(w_1 + w_2 I)(a + bI) = (z_1 + z_2 I) \in N(S)$

$(vii) \Rightarrow (ii)$ let for all $a + bI \in N(\mathbf{S})$,
$a + bI = [(x_1 + x_2 I)(a + bI)][(a + bI)(z_1 + z_2 I)]$ holds for some

$(x_1 + x_2 I), (z_1 + z_2 I) \in N(\mathbf{S})$. Now by using (4), (1), (2) and (3) we have

$$a + bI$$

$$= [(x_1 + x_2 I)(a + bI)][(a + bI)(z_1 + z_2 I)]$$

$$= [(a + bI)[(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)]$$

$$= [(x_1 + x_2 I)(a + bI)][(a + bI)(z_1 + z_2 I)][(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)]$$

$$= [(a + bI)[(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)][(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)]$$

$$= [[(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)][(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)]](a + bI)$$

$$= [[(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)]^2 (a + bI)$$

$$= [[(x_1 + x_2 I)(a + bI)]^2 (z_1 + z_2 I)^2](a + bI)$$

$$= [[(x_1 + x_2 I)^2 (a + bI)^2][(z_1 + z_2 I)(z_1 + z_2 I)]](a + bI)$$

$$= [[(x_1 + x_2 I)^2 (z_1 + z_2 I)][(a + bI)^2 (z_1 + z_2 I)]](a + bI)$$

$$= [(a + bI)^2 [[(x_1 + x_2 I)^2 (z_1 + z_2 I)](z_1 + z_2 I)](a + bI)$$

$$= [[(a + bI)(a + bI)][(x_1 + x_2 I)^2 (z_1 + z_2 I)](z_1 + z_2 I)](a + bI)$$

$$= [[(a + bI)(a + bI)][[(z_1 + z_2 I)(z_1 + z_2 I)](x_1 + x_2 I)^2]](a + bI)$$

$$= [[(a + bI)(a + bI)][(z_1 + z_2 I)^2 (x_1 + x_2 I)^2]](a + bI)$$

$$= [[(x_1 + x_2 I)^2 (z_1 + z_2 I)^2][(a + bI)(a + bI)]](a + bI)$$

$$= [(t_1 + t_2 I)[(a + bI)(a + bI)]](a + bI),$$

$$= [(t_1 + t_2 I)(a + bI)^2](u_1 + u_2 I)$$

where $[(x_1 + x_2 I)^2 (z_1 + z_2 I)^2] = (t_1 + t_2 I) \in N(S)$ and $(a + bI) = (u_1 + u_2 I) \in N(S)$ where $(a + bI) = (u_1 + u_2 I) \in N(S)$. Thus $N(S)$ is neutrosophic intra regular.

Remark *Every intra-regular, right regular, left regular, left quasi regular and completely regular* **AG** *-groupoids with left identity* ($\mathbf{AG}^{**}$ *-groupoids*) *are regular.*

The converse of above is not true in general. Indeed, from Example 1, regular **AG** -groupoid with left identity is not necessarily intra-regular.

Theorem 9 *In a Neutrosophic* **AG** *-groupoid* **N(S)** *with left identity，the following are equivalent.*

(i) $N(\mathbf{S})$ *is weakly regular.*

(ii) $N(\mathbf{S})$ *is intra-regular.*

(iii) $N(\mathbf{S})$ *is right regular.*

(iv) $N(\mathbf{S})$ *is left regular.*

(v) $N(\mathbf{S})$ *is left quasi regular.*

(vi) $N(\mathbf{S})$ *is completely regular.*

(vii) For all $a + bI \in N(\mathbf{S})$*, there exist* $(x_1 + x_2 I), (y_1 + y_2 I) \in N(\mathbf{S})$ *such that*
$a + bI = [(x_1 + x_2 I)(a + bI)][(a + bI)(y_1 + y_2 I)]$.

(viii) $\mathbf{N(S)}$ *is* $(2,2)$*-regular.*

Proof $(i) \Rightarrow (ii)$ Let $N(\mathbf{S})$ be a weakly regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity (*Neutrosophic*$\mathbf{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I), (y_1 + y_2 I) \in N(\mathbf{S})$ such that $a + bI = [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$ and by Lemma1, $(x_1 + x_2 I) = (u_1 + u_2 I)(v_1 + v_2 I)$ for some $(u_1 + u_2 I), (v_1 + v_2 I) \in N(\mathbf{S})$. Let $(v_1 + v_2 I)(u_1 + u_2 I) = (t_1 + t_2 I) \in N(\mathbf{S})$. Now by using (3), (1), (4) and (2), we have

$$
\begin{aligned}
a + bI &= [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)] \\
&= [(a + bI)(a + bI)][(x_1 + x_2 I)(y_1 + y_2 I)] \\
&= (x_1 + x_2 I)[(a + bI)^2 (y_1 + y_2 I)] \\
&= [(u_1 + u_2 I)(v_1 + v_2 I)][(a + bI)^2 (y_1 + y_2 I)] \\
&= [(y_1 + y_2 I)(a + bI)^2][(v_1 + v_2 I)(u_1 + u_2 I)] \\
&= [(y_1 + y_2 I)(a + bI)^2](t_1 + t_2 I)
\end{aligned}
$$

where $(v_1 + v_2 I)(u_1 + u_2 I) = (t_1 + t_2 I) \in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is intra-regular.

$(ii) \Rightarrow (iii)$ Let $N(\mathbf{S})$ be a intra regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity (*Neutrosophic*$\mathbf{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $x_1 + x_2 I, y_1 + y_2 I \in N(\mathbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$

$$
\begin{aligned}
a + bI &= [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I) \\
&= [[(u_1 + u_2 I)(v_1 + v_2 I)][(a + bI)(a + bI)]](y_1 + y_2 I), \\
&= [(a + bI)^2 ((v_1 + v_2 I)(u_1 + u_2 I))](y_1 + y_2 I) \\
&= [(y_1 + y_2 I)((v_1 + v_2 I)(u_1 + u_2 I))](a + bI)^2 \\
&= [(y_1 + y_2 I)((v_1 + v_2 I)(u_1 + u_2 I))][(a + bI)(a + bI)] \\
&= [(a + bI)(a + bI)][((v_1 + v_2 I)(u_1 + u_2 I))(y_1 + y_2 I)] \\
&= (a + bI)^2 (s_1 + s_2 I)
\end{aligned}
$$

where $(x_1 + x_2 I) = (u_1 + u_2 I)(v_1 + v_2 I) \in N(\mathbf{S})$ & where $(s_1 + s_2 I) = [(y_1 + y_2 I)[v_1 + v_2 I)(u_1 + u_2 I)]] \in N(\mathbf{S})$. Thus $N(S)$ is right regular

$(iii) \Rightarrow (iv)$ Let $N(\mathbf{S})$ be a right regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity (*Neutrosophic*$\mathbf{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I) \in N(\mathbf{S})$ such that $a + bI = (a + b)^2 (x_1 + x_2 I)$

$$a + bI = (a + b)^2(x_1 + x_2 I)$$
$$= [(a + bI)(a + bI)](x_1 + x_2 I)$$
$$= [(a + bI)(a + bI)][(u_1 + u_2 I)(v_1 + v_2 I)]$$
$$= [(v_1 + v_2 I)(u_1 + u_2 I)][(a + bI)(a + bI)]$$
$$= (y_1 + y_2 I)(a + bI)^2$$

where $(y_1 + y_2 I) = [(v_1 + v_2 I)(u_1 + u_2 I)] \in N(\mathbf{S})$. Thus $N(S)$ is left regular
$(iv) \Rightarrow (v)$ Let $N(\mathbf{S})$ be a left regular *Neutrosophic* **AG** -groupoid with left identity (*Neutrosophic***AG**** -groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I) \in N(\mathbf{S})$ such that $a + bI = (x_1 + x_2 I)(a + bI)^2$

$$a + bI = (x_1 + x_2 I)(a + bI)^2$$
$$= [(u_1 + u_2 I)(v_1 + v_2 I)][(a + bI)(a + bI)]$$
$$= [(u_1 + u_2 I)(a + bI)][(v_1 + v_2 I)(a + bI)]$$

Thus $N(S)$ is left quasi regular

$(v) \Rightarrow (vi)$ Let $N(\mathbf{S})$ be a left quasi regular *Neutrosophic* **AG** -groupoid with left identity (*Neutrosophic***AG**** -groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $x_1 + x_2 I \in N(\mathbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)]$

$$a + bI = [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)]$$
$$= [(a + bI)(a + bI)][(y_1 + y_2 I)(x_1 + x_2 I)]$$
$$= (a + bI)^2(v_1 + v_2 I)$$

where $(v_1 + v_2 I) = [(y_1 + y_2 I)(x_1 + x_2 I)] \in N(\mathbf{S})$. Thus $N(S)$ is neutrosophic right regular. Let $N(\mathbf{S})$ be a left quasi regular *Neutrosophic* **AG** -groupoid with left identity (*Neutrosophic***AG**** -groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist

$(x_1 + x_2 I) \in N(\mathbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)]$

$$a + bI = [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)]$$
$$= [(x_1 + x_2 I)(y_1 + y_2 I)][(a + bI)(a + bI)]$$
$$= (u_1 + u_2 I)(a + bI)^2$$

where $(u_1 + u_2 I) = [(x_1 + x_2 I)(y_1 + y_2 I)] \in N(\mathbf{S})$. Thus $N(S)$ is neutrosophic left regular ·

Let $N(\mathbf{S})$ be a neutrosophic left quasi regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity (*Neutrosophic*$\mathbf{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $x_1 + x_2 I \in N(\mathbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)]$

$$a + bI$$
$$= [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)]$$
$$= [(a + bI)(a + bI)][(y_1 + y_2 I)(x_1 + x_2 I)]$$
$$= [((y_1 + y_2 I)(x_1 + x_2 I))(a + bI)](a + bI)$$
$$= [(v_1 + v_2 I)(a + bI)](a + bI)$$
$$= [(v_1 + v_2 I)[(x_1 + x_2 I)(a + bI))((y_1 + y_2 I)(a + bI)]](a + bI)$$
$$= [(v_1 + v_2 I)[[(x_1 + x_2 I)(y_1 + y_2 I)][(a + bI)(a + bI)]](a + bI)$$
$$= [(v_1 + v_2 I)[(a + bI)[[(x_1 + x_2 I)(y_1 + y_2 I)](a + bI)]]](a + bI)$$
$$= [(a + bI)[(v_1 + v_2 I)[[(x_1 + x_2 I)(y_1 + y_2 I)](a + bI)]]](a + bI)$$
$$= [(a + bI)(t_1 + t_2 I)](a + bI)$$

where $(v_1 + v_2 I) = (y_1 + y_2 I)(x_1 + x_2 I) \in N(\mathbf{S})$ & where $(t_1 + t_2 I) = [(v_1 + v_2 I)[[(x_1 + x_2 I)(y_1 + y_2 I)](a + bI)]] \in N(\mathbf{S})$. Thus $N(S)$ is regular $\Rightarrow$ (3)

By (1).(2) & (3) $N(S)$ is Neutrosophic completely regular.

$(vi) \Rightarrow (i)$ Assume that $N(S)$ is neutrosophic completely regular Neutrosophic $\mathbf{AG}$-groupoid with left identity (*Neutrosophic*$\mathbf{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $t_1 + t_2 I \in N(\mathbf{S})$ such that $a + bI = (a + bI)^2 (x_1 + x_2 I)$, $a + bI = (y_1 + y_2 I)(a + bI)^2$, $a + bI = [(a + bI)(z_1 + z_2 I)](a + bI)$

$$a + bI$$
$$= [(a + bI)(z_1 + z_2 I)](a + bI)$$
$$= [[(a + bI)^2 (x_1 + x_2 I)](z_1 + z_2 I)][(y_1 + y_2 I)(a + bI)^2]$$
$$= [[(z_1 + z_2 I)(x_1 + x_2 I)][(a + bI)(a + bI)]][(y_1 + y_2 I)[(a + bI)(a + bI)]]$$
$$= [(a + bI)[[(z_1 + z_2 I)(x_1 + x_2 I)](a + bI)]][(a + bI)[(y_1 + y_2 I)(a + bI)]]$$
$$= [(a + bI)(t_1 + t_2 I)][(a + bI)(w_1 + w_2 I)]$$

Thus $N(S)$ is Neutrosophic weakly regular.

$(ii) \Rightarrow (vii)$ Let $N(\mathbf{S})$ be a Neutrosophic intra-regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity ($\mathbf{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I), (y_1 + y_2 I) \in N(\mathbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$. Now by using Lemma1, $(y_1 + y_2 I) = (u_1 + u_2 I)(v_1 + v_2 I)$ for some $(u_1 + u_2 I), (v_1 + v_2 I) \in N(\mathbf{S})$.

$a + bI$

$= [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$

$= [(x_1 + x_2 I)[(a + bI)(a + bI)]](y_1 + y_2 I)$

$= [(a + bI)[(x_1 + x_2 I)(a + bI)]](y_1 + y_2 I)$

$= [(y_1 + y_2 I)[(x_1 + x_2 I)(a + bI)]](a + bI)$

$= [(y_1 + y_2 I)[(x_1 + x_2 I)(a + bI)]][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$

$= [[(u_1 + u_2 I)(v_1 + v_2)][(x + yI)(a + bI)]][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I),$

$= [[(a + bI)(x_1 + x_2 I)][(v_1 + v_2)(u_1 + u_2 I)]][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$

$= [[(a + bI)(x_1 + x_2 I)](t_1 + t_2 I)][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$

$= [[(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)](t_1 + t_2 I)][(a + bI)(x_1 + x_2 I)]$

$= [[(t_1 + t_2 I)(y_1 + y_2 I)][(x_1 + x_2 I)(a + bI)^2]][(a + bI)(x_1 + x_2 I)]$

$= [[(a + bI)^2(x_1 + x_2 I)][(y_1 + y_2 I)(t_1 + t_2 I)]][(a + bI)(x_1 + x_2 I)]$

$= [[(a + bI)^2(x_1 + x_2 I)][(s_1 + s_2 I)][(a + bI)(x_1 + x_2 I)],$

$= [[(s_1 + s_2 I)(x_1 + x_2 I)](a + bI)^2][(a + bI)(x_1 + x_2 I)]$

$= [[(s_1 + s_2 I)(x_1 + x_2 I)][(a + bI)(a + bI)]][(a + bI)(x_1 + x_2 I)]$

$= [[(a + bI)(a + bI)][(x_1 + x_2 I)(s_1 + s_2 I)]][(a + bI)(x_1 + x_2 I)]$

$= [[(a + bI)(a + bI)](w_1 + w_2 I)[(a + bI)(x_1 + x_2 I)],$

$= [[(w_1 + w_2 I)(a + bI)](a + bI)][(a + bI)(x_1 + x_2 I)]$

$= [(z_1 + z_2 I)(a + bI)][(a + bI)(x_1 + x_2 I)],$

$= [(x_1 + x_2 I)(a + bI)][(a + bI)(z_1 + z_2 I)]$

where $\qquad (w_1 + w_2 I)(a + bI) = (z_1 + z_2 I) \in N(S)$ $\qquad$ where

$(x_1 + x_2 I)(s_1 + s_2 I) = (w_1 + w_2 I) \in N(S)$ $\qquad$ where

$(y_1 + y_2 I)(t_1 + t_2 I) = (s_1 + s_2 I) \in N(S)$ $\qquad$ where

$(v_1 + v_2)(u_1 + u_2 I) = (t_1 + t_2 I) \in N(S)$ $\qquad$ where

$(u_1 + u_2 I)(v_1 + v_2) = (y_1 + y_2 I) \in N(S)$

$(vii) \Rightarrow (ii)$ $\qquad$ let $\qquad$ for $\qquad$ all $\qquad$ $a + bI \in N(\mathbf{S}),$

$a + bI = [(x_1 + x_2 I)(a + bI)][(a + bI)(z_1 + z_2 I)]$ $\qquad$ holds $\qquad$ for $\qquad$ some

$(x_1 + x_2 I), (z_1 + z_2 I) \in N(\mathbf{S})$. Now by using $(4), (1), (2)$ and (3), we have

$$a + bI$$
$$= [(x_1 + x_2 I)(a + bI)][(a + bI)(z_1 + z_2 I)]$$
$$= [(a + bI)[(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)]$$
$$= [(x_1 + x_2 I)(a + bI)][(a + bI)(z_1 + z_2 I)][(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)]$$
$$= [(a + bI)[(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)][(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)]$$
$$= [[(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)][(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)]](a + bI)$$
$$= [[(x_1 + x_2 I)(a + bI)](z_1 + z_2 I)]^2 (a + bI)$$
$$= [[(x_1 + x_2 I)(a + bI)]^2 (z_1 + z_2 I)^2](a + bI)$$
$$= [[(x_1 + x_2 I)^2 (a + bI)^2][(z_1 + z_2 I)(z_1 + z_2 I)]](a + bI)$$
$$= [[(x_1 + x_2 I)^2 (z_1 + z_2 I)][(a + bI)^2 (z_1 + z_2 I)]](a + bI)$$
$$= [(a + bI)^2 [[(x_1 + x_2 I)^2 (z_1 + z_2 I)](z_1 + z_2 I)](a + bI)$$
$$= [[(a + bI)(a + bI)][(x_1 + x_2 I)^2 (z_1 + z_2 I)](z_1 + z_2 I)](a + bI)$$
$$= [[(a + bI)(a + bI)][[(z_1 + z_2 I)(z_1 + z_2 I)](x_1 + x_2 I)^2]](a + bI)$$
$$= [[(a + bI)(a + bI)][(z_1 + z_2 I)^2 (x_1 + x_2 I)^2]](a + bI)$$
$$= [[(x_1 + x_2 I)^2 (z_1 + z_2 I)^2][(a + bI)(a + bI)]](a + bI)$$
$$= [(t_1 + t_2 I)[(a + bI)(a + bI)]](a + bI),$$
$$= [(t_1 + t_2 I)(a + bI)^2](u_1 + u_2 I)$$

where $[(x_1 + x_2 I)^2 (z_1 + z_2 I)^2] = (t_1 + t_2 I) \in N(S)$ and $(a + bI) = (u_1 + u_2 I) \in N(S)$ where $(a + bI) = (u_1 + u_2 I) \in N(S)$. Thus $N(S)$ is intra regular.

$(vi) \Rightarrow (viii)$ Assume that $N(S)$ is completely regular Neutrosophic **AG**-groupoid with left identity (*Neutrosophic***AG****-*groupoid*),then for any $a + bI \in N(\mathbf{S})$ there exist $(t_1 + t_2 I) \in N(\mathbf{S})$ such that $a + bI = (a + bI)^2 (x_1 + x_2 I)$, $a + bI = (y_1 + y_2 I)(a + bI)^2$, $a + bI = [(a + bI)(z_1 + z_2 I)](a + bI)$

$$a + bI = (a + bI)^2 (x_1 + x_2 I)$$
$$= [(a + bI)(a + bI)][(v_1 + v_2 I)(u_1 + u_2 I)]$$
$$= [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$$

where $(x_1 + x_2 I) = (v_1 + v_2 I)(u_1 + u_2 I) \in N(\mathbf{S})$ Thus $N(S)$ is weakly regular

$$a + bI = (x_1 + x_2 I)(a + bI)^2$$
$$= [(u_1 + u_2 I)(v_1 + v_2 I)](a + bI)^2$$
$$= [(u_1 + u_2 I)(v_1 + v_2 I)][(a + bI)(a + bI)]$$
$$= [(a + bI)(a + bI)][(u_1 + u_2 I)(v_1 + v_2 I))]$$
$$= [(a + bI)(u_1 + u_2 I)][(a + bI)(v_1 + v_2 I)]$$

where $(x_1 + x_2 I) = (u_1 + u_2 I)(v_1 + v_2 I)$ for some $(x_1 + x_2 I) \in N(S)$. Thus $N(S)$ is weakly regular.

$a + bI$

$= [(a + bI)(z_1 + z_2 I)](a + bI)$

$= [[(a + bI)^2(x_1 + x_2 I)](z_1 + z_2 I)][(y_1 + y_2 I)(a + bI)^2]$

$= [[(z_1 + z_2 I)(x_1 + x_2 I)][(a + bI)(a + bI)]][(y_1 + y_2 I)[(a + bI)(a + bI)]]$

$= [(a + bI)[[(z_1 + z_2 I)(x_1 + x_2 I)](a + bI)]][(a + bI)[(y_1 + y_2 I)(a + bI)]]$

$= [(a + bI)(t_1 + t_2 I)][(a + bI)(w_1 + w_2 I)]$

where $(t_1 + t_2 I) = [[(z_1 + z_2 I)(x_1 + x_2 I)](a + bI)]] \in N(\mathbf{S})$ & $(w_1 + w_2 I) = [(y_1 + y_2 I)(a + bI)] \in N(\mathbf{S})$. Thus $N(S)$ is weakly regular.

$(viii) \Rightarrow (vi)$ Let $N(\mathbf{S})$ be a $(2, 2)$-regular $Neutrosophic\mathbf{AG}$-groupoid with left identity, then for $a + bI \in N(\mathbf{S})$ there exists $x_1 + x_2 I \in N(\mathbf{S})$ such that $a + bI = [(a + bI)^2(x + yI)](a + bI)^2$. Now

$a + bI = [(a + bI)^2(x_1 + x_2 I)](a + bI)^2$

$\quad = [(y_1 + y_2 I)(a + bI)^2]$

where $(a + bI)^2(x_1 + x_2 I) = (y_1 + y_2 I) \in N(\mathbf{S})$ and by using (3), we have

$a + bI = (a + bI)^2[(x_1 + x_2 I)[(a + bI)(a + bI)]]$

$\quad\quad = [(a + bI)(a + bI)][(x_1 + x_2 I)[(a + bI)^2]]$

$\quad\quad = (a + bI)^2(z_1 + z_2 I)$

where $(x_1 + x_2 I)(a + bI)^2 = (z_1 + z_2 I) \in N(\mathbf{S})$. and by using (3), (1) and (4), we have

$a + bI$

$= [(a + bI)^2[(x_1 + x_2 I)[(a + bI)(a + bI)]]$

$= [(a + bI)(a + bI)][(x_1 + x_2 I)(a + bI)^2]$

$= [(a + bI)(a + bI)][[(e_1 + e_2 I)(x_1 + x_2 I)][a + bI)(a + bI)]]$

$= [(a + bI)(a + bI)][[(a + bI)(a + bI)][(x_1 + x_2 I)(e_1 + e_2 I)]]$

$= [(a + bI)(a + bI)][(a + bI)^2(t_1 + t_2 I)]$

$= [[(a + bI)^2(t_1 + t_2 I)](a + bI)](a + bI)$

$= [[[(a + bI)(a + bI)](t_1 + t_2 I)](a + bI)](a + bI)$

$= [[(t_1 + t_2 I)(a + bI)][(a + bI)(a + bI)]](a + bI)$

$= [[(a + bI)(a + bI)][(t_1 + t_2 I)(a + bI)]](a + bI)$

$= [[(a + bI)(t_1 + t_2 I)][(a + bI)(a + bI)]](a + bI)$

$= [(a + bI)[[(a + bI)(t_1 + t_2 I)](a + bI)]](a + bI)$

$= [(a + bI)(u_1 + u_2 I)](a + bI)$

Thus N(S) is left regular, right regular and regular, so N(S) is completely regular.

$(vi) \Rightarrow (viii)$ Assume that $N(\mathbf{S})$ is a completely regular Neutrosophic **AG** - groupoid with left identity, then for any $a + bI \in N(\mathbf{S})$ there exist $(x_1 + x_2 I), (y_1 + y_2 I), (z_1 + z_2 I) \in N(\mathbf{S})$ such that $a + bI = [(a + bI)(x_1 + x_2 I)](a + bI)$, $a + bI = (a + bI)^2 (y_1 + y_2 I)$ and $a + bI = (z_1 + z_2 I)(a + bI)^2$. Now by using (1), (4) and (3), we have

$$a + bI$$
$$= [(a + bI)(x_1 + x_2 I)](a + bI)$$
$$= [[(a + bI)^2 (y_1 + y_2 I)](x_1 + x_2 I)][(z_1 + z_2 I)(a + bI)^2]$$
$$= [[(x_1 + x_2 I)(y_1 + y_2 I)](a + bI)^2][(z_1 + z_2 I)(a + bI)^2]$$
$$= [[(x_1 + x_2 I)(y_1 + y_2 I)][(a + bI)(a + bI)]][(z_1 + z_2 I)(a + bI)^2]$$
$$= [[(a + bI)(a + bI)][(y_1 + y_2 I)(x_1 + x_2 I)]][(z_1 + z_2 I)(a + bI)^2]$$
$$= [(a + bI)^2 [(y_1 + y_2 I)(x_1 + x_2 I)]][(z_1 + z_2 I)(a + bI)^2]$$
$$= [[(z_1 + z_2 I)(a + bI)^2][(y_1 + y_2 I)(x_1 + x_2 I)]](a + bI)^2$$
$$= [[(z_1 + z_2 I)(y_1 + y_2 I)][(a + bI)^2 (x_1 + x_2 I)]](a + bI)^2$$
$$= [(a + bI)^2 [[(z_1 + z_2 I)(y_1 + y_2 I)](x_1 + x_2 I)]](a + bI)^2$$
$$= [(a + bI)^2 (v_1 + v_2 I)](a + bI)^2$$

where $[(z_1 + z_2 I)(y_1 + y_2 I)][(x_1 + x_2 I)] = (v_1 + v_2 I) \in N(\mathbf{S})$. Thus $N(S)$ is $(2,2)$ - regular.

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