Structural Properties of Neutrosophic Abel-Grassmann’s Groupoids

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Abstract
In this paper, we have introduced the notion of neutrosophic \((2, 2)\)-regular, neutrosophic strongly regular neutrosophic AG-groupoids and investigated these structures. We have shown that neutrosophic regular, neutrosophic intra-regular and neutrosophic strongly regular AG-groupoid are the only generalized classes of neutrosophic AG-groupoid. Further we have shown that non-associative regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, \((2, 2)\)-regular and strongly regular AG* - neutrosophic groupoids do not exist.

Keyword
A neutrosophic AG-groupoid, left invertive law, medial law and paramedial law. [2000]20M10 and 20N99

Introduction
We know that in every branch of science there are lots of complications and problems appear which affluence the uncertainties and impaction. Most of these problems and complications are concerning with human life. These problems also play pivotal role for being subjective and classical. Common used methods are not sufficient to apply on these problems. To solve these complications, concept of fuzzy sets was published by Lotfi A. Zadeh in 1965, which has a wide range of applications in various fields such as engineering, artificial intelligence, control engineering, operation research, management science, robotics and many more. Zadeh introduced fuzzy sets to address uncertainties. By use of fuzzy sets the manipulated data and information of uncertainties can be processed. The idea of fuzzy sets was particularly designed to characterize uncertainty and vagueness and to present dignified tools in order to deal with the ambiguity intrinsic to the various problems. Fuzzy logic gives a conjecture morphology that enables approximate human reasoning capabilities to be applied to knowledge-based systems. The concept of fuzzy logic gives a mathematical potency to deal with the uncertainties associated with the human intellectual processes, such as reasoning and judgment.
In literature, a lot of theories have been developed to contend with uncertainty, imprecision and vagueness. In which, theory of probability, rough set theory, fuzzy set theory, intuitionistic fuzzy sets etc, have played imperative role to cope with diverse types of uncertainties and imprecision entrenched in a system. But all these above theories were not sufficient tool to deal with indeterminate and inconsistent information in belief system. F.Samrandache noticed that the law of excluded middle are presently inactive in the modern logics and also by getting inspired with sport games (winning/tie/defeating), voting system (yes/ NA/no), decision making (making a decision/hesitating/not making) etc, he developed a new concept called neutrosophic set (NS) which is basically generalization of fuzzy sets and intuitionistic fuzzy sets. NS can be described by membership degree, and indeterminate degree and non-membership degree. The neutrosophic logic is an extended idea of neutrosophy. Fuzzy theory is used when uncertainty is modeled and when there is indeterminancy involved we use neutrosophic theory. The neutrosophic algebraic structures have defined very recently. Basically, Vasantha K andasmy and Florentin Smarandache present the concept of neutrosophic algebraic structures by using neutrosophic theory. A number of the neutrosophic algebraic structures introduced and considered include neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic biseigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and neutrosophic AG-groupoids.

Preliminaries
Abel Grassmann's groupoid abbreviated as an AG-groupoid is a groupoid whose element satisfies the left invertive law i.e \((ab)c = (cb)a\) for all \(a, b, c \in S\). An AG-groupoid is a non associative and non-commutative algebraic structure mid way between a groupoid and commutative semigroup. AG-groupoids generalize the concept of commutative semigroup and have an important application within the theory of flocks.

An AG-groupoid, is a groupoid \(S\) holding the left invertive law \((ab)c = (cb)a\), for all \(a, b, c \in S\).

This left invertive law has been obtained by introducing braces on the left of ternary commutative law \(abc = cba\).

Basic Laws of AG-groupoid
In an AG-groupoid, the medial law holds
\[(ab)(cd) = (ac)(bd), \text{ for all } a, b, c, d \in S.\]

In an AG-groupoid \(S\) with left identity, the paramedial law holds
\[(ab)(cd) = (da)(bc), \text{ for all } a, b, c, d \in S.\]

Further if an AG-groupoid contains a left identity, the following law holds
\[a(bc) = b(ac), \text{ for all } a, b, c \in S.\]
Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. In 1995, Florentin Smarandache introduced the idea of neutrosophy. Neutrosophic logic is an extension of fuzzy logic. Madad Khan et al., for the first time introduced the idea of a neutrosophic LA-semigroup in [4]. Moreover $SUI = \{a + bI : a, b \in S \text{ and } I \text{ is literal indeterminacy such that } I^2 = I\}$ becomes neutrosophic LA-semigroup under the operation $* \text{ defined as: }$

\[(a + bI) * (c + dI) = ac + bdI \text{ For all } (a + bI), (c + dI) \in SUI \text{. That is } (SUI,*)\]

becomes neutrosophic LA-semigroup. They represented it by $N(S)$.

\[
[(a_1 + a_2I)(b_1 + b_2I)(c_1 + c_2I) = [(a_1 + a_2I)(b_1 + b_2I)](a_1 + a_2I),
\]

holds for all $(a_1 + a_2I), (b_1 + b_2I), (c_1 + c_2I) \in N(S)$.

It is since than called the neutrosophic left invertive law. A neutrosophic groupoid satisfying the left invertive law is called a neutrosophic left almost semigroup and is abbreviated as neutrosophic LA-semigroup.

In a neutrosophic LA-semigroup $N(S)$ medial law holds i.e

\[
[(a_1 + a_2I)(b_1 + b_2I)][(c_1 + c_2I)(d_1 + d_2I)]
\]

\[
= [(a_1 + a_2I)(c_1 + c_2I)][(b_1 + b_2I)(d_1 + d_2I)],
\]

holds for all $(a_1 + a_2I), (b_1 + b_2I), (c_1 + c_2I), (d_1 + d_2I) \in N(S)$.

There can be a unique left identity in a neutrosophic LA-semigroup. In a neutrosophic LA-semigroup $N(S)$ with left identity $(e + el)$ the following laws hold for all $(a_1 + a_2I), (b_1 + b_2I), (c_1 + c_2I), (d_1 + d_2I) \in N(S)$.

\[
[(a_1 + a_2I)(b_1 + b_2I)][(c_1 + c_2I)(d_1 + d_2I)]
\]

\[
= [(d_1 + d_2I)(b_1 + b_2I)][(c_1 + c_2I)(a_1 + a_2I)],
\]

\[
[(a_1 + a_2I)(b_1 + b_2I)][(c_1 + c_2I)(d_1 + d_2I)]
\]

\[
= [(d_1 + d_2I)(c_1 + c_2I)][(b_1 + b_2I)(a_1 + a_2I)],
\]

and

\[
(a_1 + a_2I)(b_1 + b_2I)(c_1 + c_2I) = (b_1 + b_2I)(a_1 + a_2I)(c_1 + c_2I)].
\]

for all $(a_1 + a_2I), (b_1 + b_2I), (c_1 + c_2I) \in N(S)$.

(3) is called neutrosophic paramedial law and a neutrosophic LA semigroup satisfies (5) is called Neutrosophic AG**-groupoid.

Now, $(a + bI)^2 = a + bI$ implies $a + bI$ is idempotent and if holds for all $a + bI \in N(S)$ then $N(S)$ is called idempotent neutrosophic LA-semigroup. This structure is closely related with a neutrosophic commutative semigroup, because if a Neutrosophic AG-groupoid contains a right identity, then it becomes a commutative semigroup.
A neutrosophic \( AG \)-groupoid \( N(S) \) with neutrosophic left identity becomes a Neutrosophic semigroup \( N(\mathcal{S}) \) under new binary operation \( "\circ" \) defined as
\[
(x_1 + x_2 I) \circ (y_1 + y_2 I) = ((x_1 + x_2 I)(a_1 + a_2 I))(y_1 + y_2 I)
\]
quasi regular if all elements of \( N(S) \) are left quasi regular. It is easy to show that for all \( x_1 + x_2 I, y_1 + y_2 I \in N(S) \).

for all \( (x_1 + x_2 I), (y_1 + y_2 I) \in N(S) \).

It is easy to show that \( \circ \) is associative
\[
[(x_1 + x_2 I) \circ (y_1 + y_2 I)] \circ (z_1 + z_2 I)
= [[[((x_1 + x_2 I)(a_1 + a_2 I))(y_1 + y_2 I))(a_1 + a_2 I)](z_1 + z_2 I)]
= [[[((z_1 + z_2 I)(a_1 + a_2 I)][[(x_1 + x_2 I)(a_1 + a_2 I))((y_1 + y_2 I))]
= [[(x_1 + x_2 I)(a_1 + a_2 I)][((z_1 + z_2 I)(a_1 + a_2 I)]((y_1 + y_2 I))]
= [(x_1 + x_2 I)(a_1 + a_2 I)]((y_1 + y_2 I)(a_1 + a_2 I))(z_1 + z_2 I)]
= (x_1 + x_2 I) \circ [(y_1 + y_2 I) \circ (z_1 + z_2 I)].
\]

Hence \( N(S) \) is a neutrosophic semigroup

**Regularities in Neutrosophic \( AG \) -groupoids**

An element \( a + bI \) of a neutrosophic \( AG \) -groupoid \( N(\mathcal{S}) \) is called a regular element of \( N(\mathcal{S}) \) if there exists \( x_1 + x_2 I \in N(\mathcal{S}) \) such that \( a + bI = [(a + bI) \star (x_1 + x_2 I)](a + bI) \) and \( N(\mathcal{S}) \) is called regular if all elements of \( N(\mathcal{S}) \) are regular.

An element \( a + bI \) of neutrosophic \( AG \)-groupoid \( N(\mathcal{S}) \) is called a weakly regular element of \( N(\mathcal{S}) \) if there exist \( (x_1 + x_2 I), (y_1 + y_2 I) \in N(\mathcal{S}) \) such that \( a + bI = [(a + bI)(x_1 + x_2 I)](a + bI)]((y_1 + y_2 I)) \) and \( N(\mathcal{S}) \) is called weakly regular if all elements of \( N(\mathcal{S}) \) are weakly regular.

An element \( a + bI \) of a neutrosophic \( AG \) -groupoid \( N(\mathcal{S}) \) is called an intra-regular element of \( N(\mathcal{S}) \) if there exist \( x_1 + x_2 I, y_1 + y_2 I \in N(\mathcal{S}) \) such that \( a + bI = [(x_1 + x_2 I)(a + bI)](y_1 + y_2 I) \) and \( N(\mathcal{S}) \) is called intra-regular if all elements of \( N(\mathcal{S}) \) are intra-regular.

An element \( a + bI \) of a neutrosophic \( AG \) -groupoid \( N(\mathcal{S}) \) is called a right regular element of \( N(\mathcal{S}) \) if there exists \( x_1 + x_2 I \in N(\mathcal{S}) \) such that \( a + bI = (a + bI)^2 (x_1 + x_2 I) = [(a + bI)(a + bI)](x_1 + x_2 I) \) and \( N(\mathcal{S}) \) is called right regular if all elements of \( N(\mathcal{S}) \) are right regular.

An element \( a + bI \) of a Neutrosophic \( AG \) -groupoid \( N(\mathcal{S}) \) is called left regular element of \( N(\mathcal{S}) \) if there exists \( x_1 + x_2 I \in N(\mathcal{S}) \) such that \( a + bI = (x_1 + x_2 I)(a + bI)^2 = (x_1 + x_2 I)((a + bI)(a + bI)) \) and \( N(\mathcal{S}) \) is called left regular if all elements of \( N(\mathcal{S}) \) are left regular.

An element \( a + bI \) of a Neutrosophic \( AG \) -groupoid \( N(\mathcal{S}) \) is called a left quasi regular element of \( N(\mathcal{S}) \) if there exist \( x_1 + x_2 I, y_1 + y_2 I \in N(\mathcal{S}) \) such that \( a + bI = [(x_1 + x_2 I)(a + bI)]((y_1 + y_2 I)(a + bI)) \) and \( N(\mathcal{S}) \) is called left quasi regular if all elements of \( N(\mathcal{S}) \) are left quasi regular.
An element \(a + bI\) of a Neutrosophic \(\mathbb{AG}\)-groupoid \(N(S)\) is called a **completely regular** element of \(N(S)\) if \(a + bI\) is regular, left regular and right regular. \(N(S)\) is called completely regular if it is regular, left and right regular.

An element \(a + bI\) of a Neutrosopic \(\mathbb{AG}\)-groupoid \(N(S)\) is called a **\((2,2)\)-regular** element of \(N(S)\) if there exists \(x_1 + x_2I \in N(S)\) such that
\[
(a + bI)^3((x_1 + x_2I))(a + bI)^2
\]
and \(N(S)\) is called **\((2,2)\)**-regular if all elements of \(N(S)\) are \((2,2)\)-regular.

An element \(a + bI\) of a Neutrosophic \(\mathbb{AG}\)-groupoid \(N(S)\) is called a **strongly regular** element of \(N(S)\) if there exists
\[
(a_1 + a_2I)(b_1 + b_2I, c_1 + c_2I) = (b_1 + b_2I)(c_1 + c_2I)
\]
and \(N(S)\) is called strongly regular Neutrosophic \(\mathbb{AG}\)-groupoid if all elements of \(N(S)\) are strongly regular.

A Neutrosophic \(\mathbb{AG}\)-groupoid \(N(S)\) is called Neutrosophic \(\mathbb{AG}^*\)-groupoid if the following holds
\[
[(a_1 + a_2I)(b_1 + b_2I)](c_1 + c_2I) = (b_1 + b_2I)(a_1 + a_2I)(c_1 + c_2I)
\]
for all \(a_1 + a_2I, b_1 + b_2I, c_1 + c_2I \in N(S)\).

In Neutrosophic \(\mathbb{AG}^*\)-groupoid \(N(S)\) the following law holds

A Neutrosophic \(\mathbb{AG}\)-groupoid may or may not contains a left identity. The left identity of a Neutrosophic \(\mathbb{AG}\)-groupoid allow us to introduce the inverses of elements in a Neutrosophic \(\mathbb{AG}\)-groupoid. If an \(\mathbb{AG}\)-groupoid contains a left identity, then it is unique.

**Example 1** Let us consider a Neutrosophic \(\mathbb{AG}\)-groupoid
\[
N(S) = \{1 + I, 1 + 2I, 1 + 3I, 2 + I, 2 + 2I, 2 + 3I, 3 + I, 3 + 2I, 3 + 3I\}
\]
in the following multiplication table.

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Lemma 1 If $N(S)$ is a regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, (2,2)-regular or strongly regular neutrosophic AG -groupoid, then $N(S) = N(S)^2$.

Proof Let $N(S)$ be a Neutrosophic regular AG-groupoid, then $N(S)^2 \subseteq N(S)$ is obvious. Let $a + bI \in N(S)$, then since $N(S)$ is regular so there exists $x + yI \in N(S)$ such that $a + bI = [(a + bI)(x + yI)](a + bI)$.

Now $a + bI = [(a + b)(x + yI)](a + bI) \in N(S) N(S)$

$N(S) \subseteq N(S)^2$

Similarly if $N(S)$ is weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, (2,2)-regular or strongly regular, then we can show that $N(S) = N(S)^2$.

The converse is not true in general, because in Example lil, $N(S) = N(S)^2$ holds but $N(S)$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, (2,2)-regular and strongly regular, because $d_1 + d_2 I \in N(S)$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, (2,2)-regular and strongly regular.

Theorem 1 If $N(S)$ is a Neutrosophic AG-groupoid with left identity (AG"-groupoid), then $N(S)$ is intra-regular if and only if for all $a + bI \in N(S)$,

$$a + bI = [(x_1 + x_2 I)(a + bI)][(a + bI)(z_1 + z_2 I)]$$

holds for some $x_1 + x_2 I, z_1 + z_2 I \in N(S)$.

Proof Let $N(S)$ be an intra-regular Neutrosophic AG-groupoid with left identity (AG"-groupoid), then for any $a + bI \in N(S)$ there exist $x_i + x_2 I, y_1 + y_2 I \in N(S)$ such that

$$a + bI = [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$$

Now by using Lemma 1, $y_1 + y_2 I = (u_1 + u_2 I)(v_1 + v_2 I)$ for some $u_1 + u_2 I, v_1 + v_2 I \in N(S)$.

$$a + bI$$

$$= [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$$

$$= [(x_1 + x_2 I)((a + bI)(a + bI)](y_1 + y_2 I)$$

$$= [(a + bI)((x_1 + x_2 I)(a + bI)](y_1 + y_2 I)$$

$$= [(y_1 + y_2 I)((x_1 + x_2 I)(a + bI)](a + bI)$$
$$= [(y_1 + y_2 I)[(x_1 + x_2 I)(a + bI)]][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$$
$$= [[[u_1 + u_2 I](v_1 + v_2 I)][(x + y I)(a + bI)][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I),$$
$$= [[(a + bI)(x_1 + x_2 I)]((v_1 + v_2)(u_1 + u_2 I)][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$$
$$= [[[a + bI](x_1 + x_2 I)]((t_1 + t_2 I)][(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$$
$$= [[[x_1 + x_2 I](a + bI)](y_1 + y_2 I)(t_1 + t_2 I)][(a + bI)(x_1 + x_2 I)]$$
$$= [[[a + bI]^{2}(x_1 + x_2 I)][(y_1 + y_2 I)(t_1 + t_2 I)][(a + bI)(x_1 + x_2 I)]$$
$$= [[[a + bI]^{2}(x_1 + x_2 I)][(s_1 + s_2 I)][(a + bI)(x_1 + x_2 I),$$
$$= [[[s_1 + s_2 I](x_1 + x_2 I)(a + bI)^2][(a + bI)(x_1 + x_2 I)]$$
$$= [[[x_1 + x_2 I](a + bI)](a + bI)(a + bI)][(a + bI)(x_1 + x_2 I)]$$
$$= [[[a + bI][x_1 + x_2 I](s_1 + s_2 I)][(a + bI)(x_1 + x_2 I)]$$
$$= [[[a + bI][x_1 + x_2 I](w_1 + w_2 I)[(a + bI)(x_1 + x_2 I),$$
$$= [[[w_1 + w_2 I][a + bI][a + bI]][(a + bI)(x_1 + x_2 I)]$$
$$= [[[z_1 + z_2 I](a + bI)][(a + bI)(x_1 + x_2 I),$$
$$= [[[z_1 + z_2 I](a + bI)][(a + bI)(z_1 + z_2 I)]$$

where \((w_1 + w_2 I)(a + bI) = (z_1 + z_2 I) \in N(S)\)  
where \((x_1 + x_2 I)(s_1 + s_2 I) = (w_1 + w_2 I) \in N(S)\)  
where \((y_1 + y_2 I)(t_1 + t_2 I) = (s_1 + s_2 I) \in N(S)\)  
where \(((u_1 + u_2 I)(v_1 + v_2 I) = (y_1 + y_2 I) \in N(S)\)

Conversely, let for all \(a + bI \in N(S)\), \(a + bI = [(x_1 + x_2 I)(a + bI)][(a + bI)(z_1 + z_2 I)]\) holds for some \(x_1 + x_2 I, z_1 + z_2 I \in N(S)\). Now by using (4), (1), (2) and (3), we have
where \( ([x_1 + x_2 I]^2 (z_1 + z_2 I)^2]) = (t_1 + t_2 I) \in N(S) \) and 

\[(a + b)t = (u_1 + u_2 I) \in N(S) \text{ where } (a + b)t = (u_1 + u_2 I) \in N(S) \]

Thus \( N(S) \) is intra-regular.

**Theorem 2** If \( N(S) \) is a Neutrosophic \( AG^-\)-groupoid with left identity \( (AG^-)^p \)-groupoid, then the following are equivalent.

(i) \( N(S) \) is weakly regular.

(ii) \( N(S) \) is intra-regular.

**Proof** (i) \( \Rightarrow \) (ii) Let \( N(S) \) be a weakly regular Neutrosophic \( AG^-\)-groupoid with left identity (Neutrosophic \( AG^-\)-groupoid), then for any \( a + bI \in N(S) \) there exist \( x_1 + x_2 I, y_1 + y_2 I \in N(S) \) such that 

\[ a + bI = [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)] \text{ and by Lemma 1, } \]

\[ x_1 + x_2 I = (u_1 + u_2 I)(v_1 + v_2 I) \text{ for some } (u_1 + u_2 I), (v_1 + v_2 I) \in N(S) \]

Let
Thus (1 + vI)(uI + u2I) = (tI + t2I) ∈ N(S). Now by using (3), (1), (4) and (2), we have

\[ a + bI = [(a + bI)(xI + x2I)][(a + bI)(yI + y2I)] \]
\[ = [(a + bI)(a + bI)][(xI + x2I)(yI + y2I)] \]
\[ = (xI + x2I)[(a + bI)]^2(yI + y2I) \]
\[ = [(uI + u2I)(vI + v2I)][(a + bI)^2(yI + y2I)] \]
\[ = [(yI + y2I)(a + bI)^2][(vI + v2I)(uI + u2I)] \]
\[ = [(yI + y2I)(a + bI)^2][tI + t2I] \]

Thus N(S) is intra-regular.

(ii) ⇒ (i) Let N(S) be a intra regular Neutrosophic AG-groupoid with left identity (Neutrosophic AG*-groupoid), then for any \( a + bI \in N(S) \)

\[ a + bI = [(vI + y2I)(a + bI)^2][tI + t2I] \]
\[ = [(vI + y2I)(a + bI)^2][(vI + v2I)(uI + u2I)] \]
\[ = [(uI + u2I)(vI + v2I)][(a + bI)^2(yI + y2I)] \]
\[ = (xI + x2I)[(a + bI)^2(yI + y2I)] \]
\[ = [(a + bI)(xI + x2I)][(a + bI)(yI + y2I)] \]

⇒ Thus N(S) is weakly regular.

Theorem 3 If N(S) is a Neutrosophic AG-groupoid (Neutrosophic AG*-groupoid), then the following are equivalent.
(i) N(S) is weakly regular.
(ii) N(S) is right regular.

Proof (i) ⇒ (ii) Let N(S) be a weakly regular Neutrosophic AG-groupoid (AG*-groupoid), then for any \( a + bI \in N(S) \) there exist \( xI + x2I, yI + y2I \in N(S) \) such that

\[ a + bI = (a + bI)[xI + x2I](yI + y2I) \]

\[ \text{and let } \]
\[ (xI + x2I)(yI + y2I) = (tI + t2I) \text{ for some } (tI + t2I) \in N(S). \]

Now by using (2), we have

\[ a + bI = [(a + bI)(xI + x2I)][(a + bI)(yI + y2I)] \]
\[ = [(a + bI)(a + bI)][(xI + x2I)(yI + y2I)] \]
\[ = (a + bI)^2[tI + t2I] \]

Thus N(S) is right regular.

(ii) ⇒ (i) It follows from Lemma1 and (2).

\[ a + bI = (a + bI)^2[tI + t2I] \]
\[ = [(a + bI)(a + bI)][tI + t2I] \]
\[ = [(a + bI)(a + bI)][(xI + x2I)(yI + y2I)] \]
\[ = [(a + bI)(xI + x2I)][(a + bI)(yI + y2I)] \]
where \((t_1 + t_2I) = (x_1 + x_2I)(y_1 + y_2I) \in N(S)\). Thus \(N(S)\) is weakly regular.

**Theorem 4** If \(N(S)\) is a Neutrosophic \(\mathcal{AG}\) -groupoid with left identity (Neutrosophic \(\mathcal{AG}\)**-groupoid), then the following are equivalent.

(i) \(N(S)\) is weakly regular.

(ii) \(N(S)\) is left regular.

**Proof** (i) \(\Rightarrow\) (ii) Let \(N(S)\) be a weakly regular Neutrosophic \(\mathcal{AG}\) -groupoid with left identity (Neutrosophic \(\mathcal{AG}\)**-groupoid), then for any \(a + bI \in N(S)\) there exist \((x_1 + x_2I), (y_1 + y_2I) \in N(S)\) such that

\[
a + bI = [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)].
\]

Now by using (2) and (3), we have

\[
a + bI = [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)]
= [(a + bI)(a + bI)][(x_1 + x_2I)(y_1 + y_2I)]
= [(y_1 + y_2I)(x_1 + x_2I)][(a + bI)(a + bI)]
= [(y_1 + y_2I)(x_1 + x_2I)][(a + bI)]^2
= (t_1 + t_2I)[(a + bI)]^2,
\]

where \([(y_1 + y_2I)(x_1 + x_2I)] = (t_1 + t_2I)\) for some \((t_1 + t_2I) \in N(S)\). Thus \(N(S)\) is left regular.

(ii) \(\Rightarrow\) (i) It follows from Lemma1, (3) and (2).

\[
a + bI = (t_1 + t_2I)[(a + bI)]^2
= [(y_1 + y_2I)(x_1 + x_2I)][(a + bI)]^2
= [(y_1 + y_2I)(x_1 + x_2I)][(a + bI)(a + bI)]
= [(a + bI)(a + bI)][(x_1 + x_2I)(y_1 + y_2I)]
= [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)],
\]

Where \([(y_1 + y_2I)(x_1 + x_2I)] = (t_1 + t_2I)\) for some \((t_1 + t_2I) \in N(S)\). Thus \(N(S)\) is weakly regular.

**Theorem 5** If \(N(S)\) is a Neutrosophic \(\mathcal{AG}\) -groupoid with left identity (Neutrosophic \(\mathcal{AG}\)**-groupoid), then the following are equivalent.

(i) \(N(S)\) is weakly regular.

(ii) \(N(S)\) is left quasi regular

**Proof** (i) \(\Rightarrow\) (ii) Let \(N(S)\) be a weakly regular Neutrosophic \(\mathcal{AG}\) -groupoid with left identity, then for \(a + bI \in N(S)\) there exists \((x_1 + x_2I), (y_1 + y_2I) \in N(S)\) such that

\[
a + bI = [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)]
= [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)]
= [(y_1 + y_2I)(a + bI)][(x_1 + x_2I)(a + bI)],
\]
Thus $N(S)$ is left quasi regular.

(i) $N(S)$ is a Neutrosophic $\text{AG}$-groupoid with left identity, then for $a + bl \in N(S)$ there exists $(x_1 + x_2 I), (y_1 + y_2 I) \in N(S)$ such that

$$a + bl = [(y_1 + y_2 I)(a + b)][(x_1 + x_2 I)(a + b)]$$

$$a + bl = [(y_1 + y_2 I)(a + b)][(x_1 + x_2 I)(a + b)]$$

$$= [(a + b)(x_1 + x_2 I)][(a + b)(y_1 + y_2 I)].$$

Thus $N(S)$ is weakly regular.

**Theorem 6** If $N(S)$ is a Neutrosophic $\text{AG}$-groupoid with left identity, then the following are equivalent.

(i) $N(S)$ is $(2,2)$-regular.

(ii) $N(S)$ is completely regular.

**Proof** (i) $\Rightarrow$ (ii) Let $N(S)$ be a $(2,2)$-regular Neutrosophic $\text{AG}$-groupoid with left identity, then for $a + bl \in N(S)$ there exists $(x_1 + x_2 I) \in N(S)$ such that

$$a + bl = [(a + bI)^2(x + yI)][a + bl]^2.$$ Now

$$a + bl = [(a + bI)^2(x + x_1 I)][a + bl]^2$$

$$= [(y_1 + y_2 I)(a + bI)^2],$$

where $(a + bI)^2(x + x_2 I) = (y_1 + y_2 I) \in N(S),$ and by using (3), we have

$$a + bl = [(a + bI)^2(x_1 + x_2 I)][(a + bI)(a + bl)]$$

$$= [(a + bl)(a + bI)][(x_1 + x_2 I)][(a + bI)^2]$$

$$= [(a + bI)(a + bI)][(a + bl)(a + bI)][(x_1 + x_2 I)(e_1 + e_2 I)]$$

$$= [((a + bl)^2)(a + bl)][(a + bl)^2][(t_1 + t_2 I)]$$

$$= [((a + bl)^2)(t_1 + t_2 I)][(a + bl)][a + bl]$$

$$= [((a + bl)(a + bl)][(t_1 + t_2 I)][(a + bl)][a + bl]$$

$$= [((a + bl)(t_1 + t_2 I)][(a + bl)(a + bl)][(a + bl)]$$

$$= [(a + bl)][(a + bl)(t_1 + t_2 I)][(a + bl)]$$

$$= [(a + bl)][(a + bl)(u_1 + u_2 I)][(a + bl)]$$
where \[(t_1 + t_2 I) = (x_1 + x_2 I)\in N(S)\] and \[(u_1 + u_2 I) = (a + bI)^2 (t_1 + t_2 I).\] Thus \(N(S)\) is neutrosophic left regular, right regular and regular, so \(N(S)\) is completely regular.

(ii) \(\Rightarrow\) (i) Assume that \(N(S)\) is a completely regular neutrosophic \(A_G\)-groupoid with left identity, then for any \(a + bI \in N(S)\) there exist \((x_1 + x_2 I),(y_1 + y_2 I),(z_1 + z_2 I)\in N(S)\) such that
\[
(a + bI) = [(a + bI)(x_1 + x_2 I)](a + bI), \quad (a + bI) = (a + bI)^2 (y_1 + y_2 I) \quad \text{and} \quad (a + bI) = (z_1 + z_2 I)(a + bI)^2.
\]
Now by using (1), (4) and (3), we have
\[
a + bI = [(a + bI)(x_1 + x_2 I)](a + bI) = [[(a + bI)^2 (y_1 + y_2 I)(x_1 + x_2 I)](a + bI)^2]
\]
where \([(z_1 + z_2 I)(y_1 + y_2 I)](x_1 + x_2 I) = (v_1 + v_2 I)\in N(S).\] This shows that \(N(S)\) is \((2,2)\)-regular.

**Lemma 2** Every weakly regular neutrosophic \(A_G\)-groupoid with left identity (Neutrosophic \(A_G^*\)-groupoid) is regular.

**Proof** Assume that \(N(S)\) is a weakly regular Neutrosophic \(A_G\)-groupoid with left identity (Neutrosophic \(A_G^*\)-groupoid), then for any \((a + bI) \in N(S)\) there exist \((x_1 + x_2 I),(y_1 + y_2 I)\in N(S)\) such that
\[
a + bI = [(a + bI)(x_1 + x_2 I)](a + bI)(y_1 + y_2 I).
\]
Let \((x_1 + x_2 I)(y_1 + y_2 I) = t_1 + t_2 I \in N(S)\) and
\[
[(t_1 + t_2 I)(y_1 + y_2 I)(x_1 + x_2 I)](a + bI) = (u_1 + u_2 I) \in N(S).\] Now by using (1), (2), (3) and (4), we have
\[ a + bI = [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)] \]
\[ = [[(a + bI)(y_1 + y_2 I)](x_1 + x_2 I)](a + bI) \]
\[ = [(x_1 + x_2 I)(y_1 + y_2 I)](a + bI) \]
\[ = [(t_1 + t_2 I)(a + bI)](a + bI) \]
\[ = [(t_1 + t_2 I)][((a + bI)(x_1 + x_2 I))][(a + bI)] \]
\[ = [(t_1 + t_2 I)][((a + bI)(a + bI)](x_1 + x_2 I)(y_1 + y_2 I))]\]
\[ = [(t_1 + t_2 I)][((a + bI)(a + bI)](x_1 + x_2 I)(y_1 + y_2 I))]\]
\[ = [(a + bI)][(a + bI)](x_1 + x_2 I)(y_1 + y_2 I)]\]
\[ = [(a + bI)(a + bI)](a + bI) \]
\[ = [(a + bI)(u_1 + u_2 I)](a + bI) \]

where \([(t_1 + t_2 I)][((y_1 + y_2 I)(x_1 + x_2 I)](a + bI)]) = u_1 + u_2 I \in N(S) \). Thus \( N(S) \) is regular.
The converse of Lemma 2 is not true in general, as can be seen from the following example.

**Example 2** [ref10] Let us consider a Neutrosophic \( \textbf{AG} \) -groupoid \( N(S) = \{1 + I_1, 1 + 2I_1, 3I_1, 3I_2 + 4I_1, 2 + 2I_1, 2 + 2I_2, 3I_1, 3I_2 + 4I_1 \}\] with left identity \( 3 \) in the following Cayley’s table.

<table>
<thead>
<tr>
<th>( \cdot )</th>
<th>1 + I_1</th>
<th>1 + 2I_1</th>
<th>3I_1</th>
<th>3I_2 + 4I_1</th>
<th>2 + 2I_1</th>
<th>2 + 2I_2</th>
<th>3I_1</th>
<th>3I_2 + 4I_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 + I_1</td>
<td>1 + I_1</td>
<td>1 + 2I_1</td>
<td>3I_1</td>
<td>3I_2 + 4I_1</td>
<td>2 + 2I_1</td>
<td>2 + 2I_2</td>
<td>3I_1</td>
<td>3I_2 + 4I_1</td>
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<tr>
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<td>1 + 2I_1</td>
<td>1 + 2I_1</td>
<td>3I_1</td>
<td>3I_2 + 4I_1</td>
<td>2 + 2I_1</td>
<td>2 + 2I_2</td>
<td>3I_1</td>
<td>3I_2 + 4I_1</td>
</tr>
<tr>
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<td>3I_1</td>
<td>3I_1</td>
<td>3I_1</td>
<td>3I_1</td>
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<td>3I_1</td>
<td>3I_1</td>
</tr>
<tr>
<td>3I_2 + 4I_1</td>
<td>3I_2 + 4I_1</td>
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<td>3I_2 + 4I_1</td>
<td>3I_2 + 4I_1</td>
<td>3I_2 + 4I_1</td>
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<td>3I_2 + 4I_1</td>
<td>3I_2 + 4I_1</td>
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<tr>
<td>2 + 2I_1</td>
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</tr>
<tr>
<td>2 + 2I_2</td>
<td>2 + 2I_2</td>
<td>2 + 2I_2</td>
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<td>2 + 2I_2</td>
<td>2 + 2I_2</td>
<td>2 + 2I_2</td>
<td>2 + 2I_2</td>
</tr>
</tbody>
</table>

**Theorem 7** If \( N(S) \) is a Neutrosophic \( \textbf{AG} \) -groupoid with left identity (Neutrosophic \( \textbf{AG}^{**} \)-groupoid ) then the following are equivalent.

(i) \( N(S) \) is weakly regular.

(ii) \( N(S) \) is completely regular.

**Proof** (i) \( \Rightarrow \) (ii)
Let \( N(S) \) be a weakly regular Neutrosophic \( \mathbf{AG} \)-groupoid (\( \mathbf{AG}^{**} \)-groupoid), then for any \( a + bl \in N(S) \) there exist \((x_i + x_I), (y_i + y_I) \in N(S)\) such that

\[
a + bl = (a + bl)(x_i + x_I) = (a + bl)(y_i + y_I)
\]

and let \((x_i + x_I)(y_i + y_I) = (t_i + t_I)\) for some \((t_i + t_I) \in N(S)\). Now by using (2), we have

\[
a + bl = [(a + bl)(x_i + x_I)][(a + bl)(y_i + y_I)]
\]

and

\[
= [(a + bl)(a + bl)][(x_i + x_I)(y_i + y_I)]
\]

\[
= [(y_i + y_I)(x_i + x_I)][(a + bl)(a + bl)]
\]

\[
= [(y_i + y_I)(x_i + x_I)][(a + bl)^2]
\]

\[
= (t_i + t_I)(a + bl)^2,
\]

where \((x_i + x_I)(y_i + y_I) = (t_i + t_I)\) for some \((t_i + t_I) \in N(S)\). Thus \( N(S) \) is right regular.

Let \( N(S) \) be a weakly regular Neutrosophic \( \mathbf{AG} \)-groupoid with left identity (\( \text{Neutrosophic } \mathbf{AG}^{**-\text{groupoid}} \)), then for any \( a + bl \in N(S) \) there exist \((x_i + x_I), (y_i + y_I) \in N(S)\) such that

\[
a + bl = [(a + bl)(x_i + x_I)][(a + bl)(y_i + y_I)].
\]

Now by using (2) and (3), we have

\[
a + bl = [(a + bl)(x_i + x_I)][(a + bl)(y_i + y_I)]
\]

\[
= [(a + bl)(a + bl)][(x_i + x_I)(y_i + y_I)]
\]

\[
= [(y_i + y_I)(x_i + x_I)][(a + bl)(a + bl)]
\]

\[
= [(y_i + y_I)(x_i + x_I)][(a + bl)^2]
\]

\[
= (t_i + t_I)(a + bl)^2,
\]

where \((y_i + y_I)(x_i + x_I) = (t_i + t_I)\) for some \((t_i + t_I) \in N(S)\). Thus \( N(S) \) is left regular.

Assume that \( N(S) \) is a weakly regular Neutrosophic \( \mathbf{AG} \)-groupoid with left identity (\( \text{Neutrosophic } \mathbf{AG}^{**-\text{groupoid}} \)), then for any \( a + bl \in N(S) \) there exist \((x_i + x_I), (y_i + y_I) \in N(S)\) such that

\[
a + bl = [(a + bl)(x_i + x_I)][(a + bl)(y_i + y_I)].
\]

Let \((x_i + x_I)(y_i + y_I) = (t_i + t_I) \in N(S)\) and

\[
[(t_i + t_I)(y_i + y_I)(x_i + x_I)][(a + bl) = (u_i + u_I) \in N(S). \]

Now by using (1), (2), (3) and (4), we have

\[
a + bl = [(a + bl)(x_i + x_I)][(a + bl)(y_i + y_I)]
\]

\[
= [(a + bl)(y_i + y_I)(x_i + x_I)][(a + bl)]
\]

\[
= [(x_i + x_I)(y_i + y_I)][(a + bl)][(a + bl)]
\]

\[
= [(t_i + t_I)(a + bl)][(a + bl)]
\]

\[
= [(t_i + t_I)(a + bl)][(a + bl)(x_i + x_I)][(a + bl)(y_i + y_I)]
\]

\[
= [(t_i + t_I)(a + bl)][(x_i + x_I)(y_i + y_I)(a + bl)]
\]

\[
= [(t_i + t_I)(a + bl)][(y_i + y_I)(x_i + x_I)(a + bl)]
\]

\[
= [(a + bl)(t_i + t_I)][(y_i + y_I)(x_i + x_I)(a + bl)]
\]

\[
= [(a + bl)(u_i + u_I)(a + bl)]
\]

\[
= [(a + bl)(u_i + u_I)(a + bl)],
\]
where \([t_1 + t_2I][(y_1 + y_2I)(x_1 + x_2I)(a + bI)] = (u_1 + u_2I) \in N(S)\). Thus \(N(S)\) is regular. Thus \(N(S)\) is completely regular.

\((ii) \Rightarrow (i)\)

Assume that \(N(S)\) is completely regular Neutrosophic \(AG\)-groupoid with left identity (Neutrosophic \(AG^{*}\)-groupoid), then for any \((a + bI) \in N(S)\) there exist \((t_1 + t_2I) \in N(S)\) such that 
\[
(a + bI) = (a + bI)^2(x_1 + x_2I),
\]
\[
(a + bI) = (y_1 + y_2I)(a + bI)^2, \quad a + bI = [(a + bI)(z_1 + z_2I)](a + bI)
\]
\[
a + bI = (a + bI)^2(x_1 + x_2I)
\]
\[
(a + bI) = [(a + bI)(a + bI)][(y_1 + y_2I)(u_1 + u_2I)]
\]
\[
(a + bI) = [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)],
\]

where \((x_1 + x_2I) = (v_1 + v_2I)(u_1 + u_2I) \in N(S)\). Thus \(N(S)\) is Neutrosophic weakly regular.

\[
a + bI = (x_1 + x_2I)(a + bI)^2
\]
\[
= [(u_1 + u_2I)(v_1 + v_2I)](a + bI)^2
\]
\[
= [(u_1 + u_2I)(v_1 + v_2I)][(a + bI)(a + bI)]
\]
\[
= [(a + bI)(a + bI)][(u_1 + u_2I)(v_1 + v_2I)]
\]
\[
= [(a + bI)(u_1 + u_2I)][(a + bI)(v_1 + v_2I)],
\]

where \((x_1 + x_2I) = (u_1 + u_2I)(v_1 + v_2I)\) for some \((x_1 + x_2I) \in N(S)\). Thus \(N(S)\) is weakly regular.

\[
a + bI
\]
\[
= [(a + bI)(z_1 + z_2I)](a + bI)
\]
\[
= [[(a + bI)^2(x_1 + x_2I)][(z_1 + z_2I)]][(y_1 + y_2I)(a + bI)^2]
\]
\[
= [[[z_1 + z_2I](x_1 + x_2I)][(a + bI)(a + bI)]][(y_1 + y_2I)(a + bI)(a + bI)]
\]
\[
= [(a + bI)][(z_1 + z_2I)(x_1 + x_2I)][(a + bI)][(a + bI)(y_1 + y_2I)(a + bI)]
\]
\[
= [(a + bI)(t_1 + t_2I)][(a + bI)(w_1 + w_2I)],
\]

where \((t_1 + t_2I) = [[(z_1 + z_2I)(x_1 + x_2I)](a + bI)] \in N(S)\) & \((w_1 + w_2I) = [(y_1 + y_2I)(a + bI)] \in N(S)\). Thus \(N(S)\) is weakly regular.

**Lemma 3** Every strongly regular Neutrosophic \(AG\)-groupoid with left identity (Neutrosophic \(AG^{**}\)-groupoid) is completely regular.
Proof Assume that \( N(S) \) is a strongly regular Neutrosophic AG-groupoid with left identity \((\text{Neutrosophic AG}^{**}\text{-groupoid})\), then for any \( (a + bI) \in N(S) \) there exists \( (x_1 + x_2I) \in N(S) \) such that \( a + bI = [(a + bI)(x_1 + x_2I)](a + bI) \) and \( (a + bI)(x_1 + x_2I) = (x_1 + x_2I)(a + bI) \). Now by using (1), we have
\[
\begin{align*}
  a + bI &= [(a + bI)(x_1 + x_2I)](a + bI) \\
  &= [(x_1 + x_2I)(a + bI)](a + bI) \\
  &= [(a + bI)(a + bI)](x_1 + x_2I) \\
  &= (a + bI)^2(x_1 + x_2I).
\end{align*}
\]
This shows that \( N(S) \) is right regular and by Theorems 4 and 7, it is clear to see that \( N(S) \) is completely regular.

Theorem 8 In a Neutrosophic AG-groupoid \( N(S) \) with left identity \((\text{Neutrosophic AG}^{**}\text{-groupoid})\) the following are equivalent.

(i) \( N(S) \) is weakly regular.

(ii) \( N(S) \) is intra-regular.

(iii) \( N(S) \) is right regular.

(iv) \( N(S) \) is left regular.

(v) \( N(S) \) is left quasi regular.

(vi) \( N(S) \) is completely regular.

(vii) For all \( a + bI \in N(S) \), there exist \( (x_1 + x_2I), (y_1 + y_2I) \in N(S) \) such that \( a + bI = [(x_1 + x_2I)(a + bI)][(a + bI)(y_1 + y_2I)] \).

Proof (i) \( \Rightarrow \) (ii) Let \( N(S) \) be weakly regular Neutrosophic AG-groupoid with left identity \((\text{Neutrosophic AG}^{**}\text{-groupoid})\), then for any \( a + bI \in N(S) \) there exist \( (x_1 + x_2I), (y_1 + y_2I) \in N(S) \) such that
\[
\begin{align*}
  a + bI &= [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)] \\
  &= [(a + bI)(a + bI)][(x_1 + x_2I)(y_1 + y_2I)] \\
  &= [(x_1 + x_2I)(a + bI)^2(y_1 + y_2I)] \\
  &= [(u_1 + u_2I)(v_1 + v_2I)][(a + bI)^2(y_1 + y_2I)] \\
  &= [(v_1 + v_2I)(a + bI)^2][(v_1 + v_2I)(u_1 + u_2I)] \\
  &= [(y_1 + y_2I)(a + bI)^2(t_1 + t_2I)]
\end{align*}
\]
where \( (v_1 + v_2I)(u_1 + u_2I) = (t_1 + t_2I) \in N(S) \). Thus \( N(S) \) is intra-regular.
(ii) $\Rightarrow$ (iii) Let $N(S)$ be a weakly regular Neutrosophic AG\-groupoid with left identity (Neutrosophic AG**\-groupoid), then for any $a + bI \in N(S)$ there exist $(x_1 + x_2 I), (y_1 + y_2 I) \in N(S)$ such that $a + bI = [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$

$$a + bI = [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$$

$$= [[(u_1 + u_2 I)(v_1 + v_2 I)][(a + bI)(a + bI)](y_1 + y_2 I)]$$

$$= [(a + bI)^2((v_1 + v_2 I)(u_1 + u_2 I))](y_1 + y_2 I)$$

$$= [(y_1 + y_2 I)((v_1 + v_2 I)(u_1 + u_2 I))](a + bI)^2$$

$$= [(a + bI)(a + bI)][[(v_1 + v_2 I)(u_1 + u_2 I)](y_1 + y_2 I)]$$

$$= (a + bI)^2(s_1 + s_2 I),$$

where $x_1 + x_2 I = (u_1 + u_2 I)(v_1 + v_2 I) \in N(S)$ and $s_1 + s_2 I = [(y_1 + y_2 I)(v_1 + v_2 I)(u_1 + u_2 I)] \in N(S).$ Thus $N(S)$ is right regular.

(iii) $\Rightarrow$ (iv) Let $N(S)$ be a right regular Neutrosophic AG\-groupoid with left identity (Neutrosophic AG**\-groupoid), then for any $a + bI \in N(S)$ there exist $x_1 + x_2 I \in N(S)$ such that $a + bI = (a + b)^2(x_1 + x_2 I)$

$$a + bI = (a + b)^2(x_1 + x_2 I)$$

$$= [(a + bI)(a + bI)](x_1 + x_2 I)$$

$$= [(a + bI)(a + bI)][(u_1 + u_2 I)(v_1 + v_2 I)]$$

$$= [(v_1 + v_2 I)(u_1 + u_2 I)][(a + bI)(a + bI)]$$

$$= (y_1 + y_2 I)(a + bI)^2$$

Where $(y_1 + y_2 I) = [(v_1 + v_2 I)(u_1 + u_2 I)] \in N(S).$ Thus $N(S)$ is left regular.

(iv) $\Rightarrow$ (v) Let $N(S)$ be a left regular Neutrosophic AG\-groupoid with left identity (Neutrosophic AG**\-groupoid), then for any $a + bI \in N(S)$ there exist $(x_1 + x_2 I) \in N(S)$ such that $(a + bI) = (x_1 + x_2 I)(a + bI)^2$

$$a + bI = (x_1 + x_2 I)(a + bI)^2$$

$$= [(u_1 + u_2 I)(v_1 + v_2 I)][(a + bI)(a + bI)]$$

$$= [(u_1 + u_2 I)(a + bI)][(v_1 + v_2 I)(a + bI)]$$
Thus $N(S)$ is left quasi regular

$(v) \Rightarrow (vi)$ Let $N(S)$ be a left quasi regular Neutrosophic AG-groupoid with left identity (Neutrosophic AG-groupoid), then for any $a + bI \in N(S)$ there exist $x_{i} + x_{2}I \in N(S)$ such that $a + bI = [(x_{i} + x_{2}I)(a+bI)][(y_{i} + y_{2}I)(a+bI)]$

\[
\begin{align*}
    a + bI &= [(x_{i} + x_{2}I)(a+bI)][(y_{i} + y_{2}I)(a+bI)] \\
    &= [(a+bI)(a+bI)][(y_{i} + y_{2}I)(x_{i} + x_{2}I)] \\
    &= (a+bI)^2(v_{i} + v_{2}I)
\end{align*}
\]

where $v_{i} + v_{2}I = [(y_{i} + y_{2}I)(x_{i} + x_{2}I)] \in N(S)$

Thus $N(S)$ is right regular $\Rightarrow (1)$

Let $N(S)$ be a left quasi regular Neutrosophic AG-groupoid with left identity (Neutrosophic AG-groupoid), then for any $a + bI \in N(S)$ there exist $x_{i} + x_{2}I \in N(S)$ such that $a + bI = [(x_{i} + x_{2}I)(a+bI)][(y_{i} + y_{2}I)(a+bI)]$

\[
\begin{align*}
    a + bI &= [(x_{i} + x_{2}I)(a+bI)][(y_{i} + y_{2}I)(a+bI)] \\
    &= [(x_{i} + x_{2}I)(y_{i} + y_{2}I)][(a+bI)(a+bI)] \\
    &= (a+bI)^2(x_{i} + x_{2}I)
\end{align*}
\]

where $(u_{i} + u_{2}I) = [(x_{i} + x_{2}I)(y_{i} + y_{2}I)] \in N(S)$

Thus $N(S)$ is left regular $\Rightarrow (2)$

Let $N(S)$ be a left quasi regular Neutrosophic AG-groupoid with left identity (Neutrosophic AG-groupoid), then for any $a + bI \in N(S)$ there exist $(x_{i} + x_{2}I) \in N(S)$ such that $a + bI = [(x_{i} + x_{2}I)(a+bI)][(y_{i} + y_{2}I)(a+bI)]$

\[
\begin{align*}
    a + bI &= [(x_{i} + x_{2}I)(a+bI)][(y_{i} + y_{2}I)(a+bI)] \\
    &= [(a+bI)(a+bI)][(y_{i} + y_{2}I)(x_{i} + x_{2}I)] \\
    &= [(y_{i} + y_{2}I)(x_{i} + x_{2}I)(a+bI)](a+bI) \\
    &= [(v_{i} + v_{2}I)(a+bI)](a+bI), \\
    &= [(v_{i} + v_{2}I)(a+bI)][(x_{i} + x_{2}I)(a+bI)](a+bI) \\
    &= [(v_{i} + v_{2}I)(a+bI)][(y_{i} + y_{2}I)(a+bI)](a+bI) \\
    &= [(a+bI)(a+bI)][(v_{i} + v_{2}I)(a+bI)](a+bI) \\
    &= [(a+bI)(a+bI)(t_{1} + t_{2}I)(a+bI)](a+bI)
\end{align*}
\]

where $(v_{i} + v_{2}I) = (y_{i} + y_{2}I)(x_{i} + x_{2}I) \in N(S)$

By (1),(2) & (3) $N(S)$ is completely regular.
Let \( N(S) \) be a complete regular Neutrosophic \( \mathbb{AG} \)-groupoid with left identity (Neutrosophic \( \mathbb{AG} \)-groupoid), then for any \( a + bI \in N(S) \) there exist \( (x_1 + x_2I) \in N(S) \) such that \( a + bI = (a + bI)^2 (x_1 + x_2I) \), where \( (x_1 + x_2I) = ((u_1 + u_2I)(v_1 + v_2I)) \in N(S) \) is weakly regular.

\[
(a + bI) = [(a + bI)(a + bI)]((u_1 + u_2I)(v_1 + v_2I))
\]

where \( (x_1 + x_2I) = ((u_1 + u_2I)(v_1 + v_2I)) \in N(S) \). Thus \( N(S) \) is weakly regular.

\[
(a + bI) = [(a + bI)(a + bI)]\left(\begin{array}{c}
\left((u_1 + u_2I)(v_1 + v_2I)\right)
\end{array}\right)
\]

where \( (y_1 + y_2I) = [(v_1 + v_2I)(u_1 + u_2I)] \in N(S) \). Thus \( N(S) \) is neutrosophic weakly regular.

\[
(a + bI) = [(a + bI)(a + bI)]\left(\begin{array}{c}
\left((x_1 + x_2I)(y_1 + y_2I)\right)
\end{array}\right)
\]

where \( (x_1 + x_2I) = [(z_1 + z_2I)(v_1 + v_2I)] \in N(S) \) & \( (w_1 + w_2I) = [(y_1 + y_2I)(a + bI)] \in N(S) \). Thus \( N(S) \) is neutrosophic weakly regular.

(ii) \( \Rightarrow \) (vii) Let \( N(S) \) be an intra-regular Neutrosophic \( \mathbb{AG} \)-groupoid with left identity (\( \mathbb{AG} \)-groupoid), then for any \( a + bI \in N(S) \) there exist \( (x_1 + x_2I), (y_1 + y_2I) \in N(S) \) such that \( a + bI = [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \).

Now by using Lemma 1, \( (y_1 + y_2I) = (u_1 + u_2I)(v_1 + v_2I) \) for some \( u_1 + u_2I, v_1 + v_2I \in N(S) \). Thus by using (4), (1) and (3), we have

\[
(a + bI) = [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I)
\]

\[
(a + bI) = [(x_1 + x_2I)((a + bI)(a + bI))](y_1 + y_2I)
\]

\[
(a + bI) = [(a + bI)((x_1 + x_2I)(a + bI))](y_1 + y_2I)
\]

\[
(a + bI) = [(y_1 + y_2I)((x_1 + x_2I)(a + bI))](a + bI)
\]
\[ (y_1 + y_2 I)[(x_1 + x_2 I)(a + bI)]((x_1 + x_2 I)(a + bI)^2) = (y_1 + y_2 I)[(x_1 + x_2 I)(a + bI)]((x_1 + x_2 I)(a + bI)^2) \]
\[ = [(u_1 + u_2 I)(v_1 + v_2 I)][(x + yI)(a + bI)][((x_1 + x_2 I)(a + bI)^2)] \]
\[ = [(a + bI)(x_1 + x_2 I)][(v_1 + v_2 I)(u_1 + u_2 I)][((x_1 + x_2 I)(a + bI)^2)] \]
\[ = [(a + bI)(x_1 + x_2 I)(t_1 + t_2 I)][(x_1 + x_2 I)(a + bI)^2] \]
\[ = [(a + bI)^2(x_1 + x_2 I)(t_1 + t_2 I)][(a + bI)(x_1 + x_2 I)] \]
\[ = [(a + bI)^2(x_1 + x_2 I)][(a + bI)(x_1 + x_2 I)] \]
\[ = [(s_1 + s_2 I)(a + bI)]((a + bI)(x_1 + x_2 I)] \]
\[ = [(s_1 + s_2 I)(x_1 + x_2 I)][(a + bI)(a + bI)] \]
\[ = [(a + bI)(a + bI)][(x_1 + x_2 I)(s_1 + s_2 I)] \]
\[ = [(a + bI)(a + bI)][(x_1 + x_2 I)(s_1 + s_2 I)] \]
\[ = [(a + bI)(a + bI)][(w_1 + w_2 I)] \]
\[ = [(z_1 + z_2 I)(a + bI)] \]
\[ = [(x_1 + x_2 I)(a + bI)] \]

where \((u_1 + u_2 I)(v_1 + v_2 I) = (y_1 + y_2 I) \in N(S)\) \& where
\[(v_1 + v_2 I)(u_1 + u_2 I) = (t_1 + t_2 I) \in N(S)\] \& where
\[(y_1 + y_2 I)(t_1 + t_2 I) = (s_1 + s_2 I) \in N(S)\] \& where
\[(x_1 + x_2 I)(s_1 + s_2 I) = (w_1 + w_2 I) \in N(S)\]
\[(w_1 + w_2 I)(a + bI) = (z_1 + z_2 I) \in N(S)\]

\((vii) \Rightarrow (ii)\) let for all \(a + bI \in N(S)\), holds for some
\[(x_1 + x_2 I),(z_1 + z_2 I) \in N(S)\). Now by using (4), (1), (2) and (3) we have...
The converse of above is not true in general. Indeed, from Example 1, regular neutrosophic intra regular completely regular groupoid with left identity is not necessarily intra-regular.

**Remark**

\[
(a + bI) = (u_1 + u_2 I) \in N(S) \quad \text{where} \quad (a + bI) = (u_1 + u_2 I) \in N(S).
\]

Thus \( N(S) \) is neutrosophic intra regular.

**Remark**

Every intra-regular, right regular, left regular, left quasi regular and completely regular \( \mathbf{AG} \)-groupoids with left identity (\( \mathbf{AG}^* \)-groupoids) are regular.

The converse of above is not true in general. Indeed, from Example 1, regular \( \mathbf{AG} \)-groupoid with left identity is not necessarily intra-regular.

**Theorem 9**

In a Neutrosophic \( \mathbf{AG} \)-groupoid \( N(S) \) with left identity, the following are equivalent.

(i) \( N(S) \) is weakly regular.

(ii) \( N(S) \) is intra-regular.

(iii) \( N(S) \) is right regular.

(iv) \( N(S) \) is left regular.

(v) \( N(S) \) is left quasi regular.

(vi) \( N(S) \) is completely regular.
(vii) For all \( a + bI \in N(S) \), there exist \((x_1 + x_2I, y_1 + y_2I) \in N(S) \) such that \( a + bI = [(x_1 + x_2I)(a + bI)](y_1 + y_2I) \).

(viii) \( N(S) \) is \((2,2)\)-regular.

Proof (i) \(\Rightarrow\) (ii) Let \( N(S) \) be a weakly regular Neutrosophic AG-groupoid with left identity \((\text{Neutrosophic AG}^\ast\text{-groupoid})\), then for any \( a + bI \in N(S) \) there exist \((x_1 + x_2I, y_1 + y_2I) \in N(S) \) such that
\[
 a + bI = [(a + bI)(x_1 + x_2I)](y_1 + y_2I)
 = [(a + bI)(a + bI)](x_1 + x_2I)(y_1 + y_2I)
 = (x_1 + x_2I)(a + bI)(y_1 + y_2I)
 = (u_1 + u_2I)(v_1 + v_2I)(a + bI)(y_1 + y_2I)
 = [(y_1 + y_2I)(a + bI)^2](v_1 + v_2I)(u_1 + u_2I)
 = [(y_1 + y_2I)(a + bI)^2](u_1 + u_2I)(v_1 + v_2I)
 = (a + bI)^2(t_1 + t_2I)
\]

where \((v_1 + v_2I)(u_1 + u_2I) = (t_1 + t_2I) \in N(S)\). Thus \( N(S) \) is intra-regular.

(ii) \(\Rightarrow\) (iii) Let \( N(S) \) be a intra regular Neutrosophic AG-groupoid with left identity \((\text{Neutrosophic AG}^\ast\text{-groupoid})\), then for any \( a + bI \in N(S) \) there exist \(x_1 + x_2I, y_1 + y_2I \in N(S) \) such that
\[
 a + bI = [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I)
 = [(u_1 + u_2I)(v_1 + v_2I)(a + bI)(a + bI)](y_1 + y_2I)
 = (a + bI)^2((v_1 + v_2I)(u_1 + u_2I))(y_1 + y_2I)
 = [(y_1 + y_2I)(v_1 + v_2I)(u_1 + u_2I)](a + bI)(a + bI)
 = [(a + bI)(a + bI)](v_1 + v_2I)(u_1 + u_2I)(y_1 + y_2I)
 = (a + bI)^2(s_1 + s_2I)
\]

where \((x_1 + x_2I) = (u_1 + u_2I)(v_1 + v_2I) \in N(S)\) \& where
\[
(s_1 + s_2I) = [(y_1 + y_2I)(v_1 + v_2I)(u_1 + u_2I)] \in N(S)\). Thus \( N(S) \) is right regular.

(iii) \(\Rightarrow\) (iv) Let \( N(S) \) be a right regular Neutrosophic AG-groupoid with left identity \((\text{Neutrosophic AG}^\ast\text{-groupoid})\), then for any \( a + bI \in N(S) \) there exist \((x_1 + x_2I) \in N(S) \) such that
\[
 a + bI = (a + bI)^2(x_1 + x_2I)
\]
\[
a + bl = (a + b)^2 (x_1 + x_2 I)
\]
\[
= [(a + bl)(a + bl)](x_1 + x_2 I)
\]
\[
= [(a + bl)(a + bl)][(u_1 + u_2 I)(v_1 + v_2 I)]
\]
\[
= [(v_1 + v_2 I)(u_1 + u_2 I)][a + bl](a + bl)]
\]
\[
= (v_1 + v_2 I)(a + bl)^2
\]

where \((v_1 + v_2 I) = [(v_1 + v_2 I)(u_1 + u_2 I)] \in N(S)\). Thus \(N(S)\) is left regular

(iv) \(\Rightarrow\) (v) Let \(N(S)\) be a left regular Neutrosophic AG-groupoid with left identity \((Neutrosophic \text{AG}^{\ast} \text{-groupoid})\), then for any \(a + bl \in N(S)\) there exist \((x_1 + x_2 I) \in N(S)\) such that \(a + bl = (x_1 + x_2 I)(a + bl)^2\)

\[
a + bl = (x_1 + x_2 I)(a + bl)^2
\]
\[
= [(u_1 + u_2 I)(v_1 + v_2 I)][(a + bl)(a + bl)]
\]
\[
= [(u_1 + u_2 I)(a + bl)][(v_1 + v_2 I)(a + bl)]
\]

Thus \(N(S)\) is left quasi regular

(v) \(\Rightarrow\) (vi) Let \(N(S)\) be a left quasi regular Neutrosophic AG-groupoid with left identity \((Neutrosophic \text{AG}^{\ast} \text{-groupoid})\), then for any \(a + bl \in N(S)\) there exist \((x_1 + x_2 I) \in N(S)\) such that \(a + bl = [(x_1 + x_2 I)(a + bl)][(y_1 + y_2 I)(a + bl)]\)

\[
a + bl = [(x_1 + x_2 I)(a + bl)][(y_1 + y_2 I)(a + bl)]
\]
\[
= [(a + bl)(a + bl)][(y_1 + y_2 I)(x_1 + x_2 I)]
\]
\[
= (a + bl)^2 (v_1 + v_2 I)
\]

where \((v_1 + v_2 I) = [(v_1 + v_2 I)(x_1 + x_2 I)] \in N(S)\). Thus \(N(S)\) is neutrosophic right regular. Let \(N(S)\) be a left quasi regular Neutrosophic AG-groupoid with left identity \((Neutrosophic \text{AG}^{\ast} \text{-groupoid})\), then for any \(a + bl \in N(S)\) there exist \((x_1 + x_2 I) \in N(S)\) such that \(a + bl = [(x_1 + x_2 I)(a + bl)][(y_1 + y_2 I)(a + bl)]\)

\[
a + bl = [(x_1 + x_2 I)(a + bl)][(y_1 + y_2 I)(a + bl)]
\]
\[
= [(x_1 + x_2 I)(y_1 + y_2 I)][(a + bl)(a + bl)]
\]
\[
= (u_1 + u_2 I)(a + bl)^2
\]

where \((u_1 + u_2 I) = [(x_1 + x_2 I)(y_1 + y_2 I)] \in N(S)\). Thus \(N(S)\) is neutrosophic left regular.
Let \( N(S) \) be a neutrosophic left quasi regular Neutrosophic \( AG \)-groupoid with left identity (Neutrosophic \( AG^\ast \)-groupoid), then for any \( a + bI \in N(S) \) there exist \( x_i + x_2I \in N(S) \) such that \( a + bI = [(x_i + x_2I)(a + bI)][(y_i + y_2I)(a + bI)] \)

\[
a + bI = [(x_i + x_2I)(a + bI)][(y_i + y_2I)(a + bI)]
\]

\[
= [(a + bI)(a + bI)][(y_i + y_2I)(x_i + x_2I)]
\]

\[
= [(y_i + y_2I)(x_i + x_2I)][(a + bI)](a + bI)
\]

where \( (v_1 + v_2I) = (y_i + y_2I)(x_i + x_2I) \in N(S) \) & where \( (t_1 + t_2I) = [(v_1 + v_2I)[((x_i + x_2I)(y_i + y_2I)](a + bI))] \in N(S) \). Thus \( N(S) \) is regular \( \Rightarrow (3) \)

By (1), (2) & (3) \( N(S) \) is Neutrosophic completely regular.

\((vi) \Rightarrow (i)\) Assume that \( N(S) \) is neutrosophic completely regular Neutrosophic \( AG \)-groupoid with left identity (Neutrosophic \( AG^\ast \)-groupoid), then for any \( a + bI \in N(S) \) there exist \( t_i + t_2I \in N(S) \) such that \( a + bI = (a + bI)^2(x_i + x_2I), \)

\( a + bI = (y_i + y_2I)(a + bI)^2, \)

\( a + bI = [(a + bI)(z_i + z_2I)](a + bI) \)

\[
a + bI = [(a + bI)(z_i + z_2I)](a + bI)
\]

\[
= [[(a + bI)^2(x_i + x_2I)](z_i + z_2I)][(y_i + y_2I)(a + bI)^2]
\]

\[
= [(z_i + z_2I)(x_i + x_2I)][(a + bI)(a + bI)]][(y_i + y_2I)](a + bI)]\]

\[
= [(a + bI)][(z_i + z_2I)(x_i + x_2I)][(a + bI)][(y_i + y_2I)](a + bI)]\]

\[
= [(a + bI)(t_i + t_2I)][(a + bI)(w_i + w_2I)]
\]

Thus \( N(S) \) is Neutrosophic weakly regular.

\((ii) \Rightarrow (vii)\) Let \( N(S) \) be a Neutrosophic intra-regular Neutrosophic \( AG \)-groupoid with left identity (Neutrosophic \( AG^\ast \)-groupoid), then for any \( a + bI \in N(S) \) there exist \( (x_i + x_2I), (y_i + y_2I) \in N(S) \) such that \( a + bI = [(x_i + x_2I)(a + bI)^2][(y_i + y_2I)]. \)

Now by using Lemma1, \( (y_i + y_2I) = (u_i + u_2I)(v_i + v_2I) \) for some \( (u_i + u_2I), (v_i + v_2I) \in N(S). \)
\[a + bl\]
\[= [(x_1 + x_2 I)(a + bl)^2](y_1 + y_2 I)\]
\[= [(x_1 + x_2 I)(a + bl)(a + bl)](y_1 + y_2 I)\]
\[= [(a + bl)(x_1 + x_2 I)(a + bl)](y_1 + y_2 I)\]
\[= [(y_1 + y_2 I)(x_1 + x_2 I)(a + bl)](a + bl)\]
\[= [(y_1 + y_2 I)(x_1 + x_2 I)(a + bl)][(x_1 + x_2 I)(a + bl)^2](y_1 + y_2 I)\]
\[= [[[a + bl][x_1 + x_2 I]]((x_1 + x_2 I)(a + bl)^2)](y_1 + y_2 I)\]
\[= [[[a + bl][x_1 + x_2 I]](a + bl)(x_1 + x_2 I)](y_1 + y_2 I)\]
\[= [[[a + bl][x_1 + x_2 I]]((y_1 + y_2 I)(a + bl)^2)](x_1 + x_2 I)\]
\[= [[[a + bl][x_1 + x_2 I]]((y_1 + y_2 I)(t_1 + t_2 I)](a + bl)(x_1 + x_2 I)\]
\[= [[[a + bl][x_1 + x_2 I]]((s_1 + s_2 I)](a + bl)((x_1 + x_2 I)\]
\[where \text{ (w}_1 + w_2 I)(a + bl) = (z_1 + z_2 I) \in N(S) \quad \text{ where} \]
\[(w_1 + w_2 I)(a + bl) = (w_1 + w_2 I) \in N(S) \quad \text{ where} \]
\[(y_1 + y_2 I)(t_1 + t_2 I) = (s_1 + s_2 I) \in N(S) \quad \text{ where} \]
\[(v_1 + v_2)(u_1 + u_2 I) = (t_1 + t_2 I) \in N(S) \quad \text{ where} \]
\[(u_1 + u_2 I)(v_1 + v_2) = (y_1 + y_2 I) \in N(S) \quad \text{ (vi)} \Rightarrow (ii) \quad \text{let for all} \quad a + bl \in N(S), \]
\[a + bl = [(x_1 + x_2 I)(a + bl)](z_1 + z_2 I) \quad \text{holds for some} \]
\[(x_1 + x_2 I), (z_1 + z_2 I) \in N(S). \text{ Now by using (4), (1), (2) and (3), we have} \]
\[a + bI = ((x_1 + x_2, I)(a + bI)\{(a + bI)(z_1 + z_2, I)\}
\]
\[= [(a + bI)(x_1 + x_2, I)(a + bI)]((z_1 + z_2, I))\]
\[= [[(x_1 + x_2, I)(a + bI)]((a + bI)(z_1 + z_2, I))(x_1 + x_2, I)(a + bI)]((z_1 + z_2, I))\]
\[= [(a + bI)[(x_1 + x_2, I)(a + bI)]((z_1 + z_2, I))(x_1 + x_2, I)(a + bI)]((z_1 + z_2, I))\]
\[= [[(x_1 + x_2, I)(a + bI)]((z_1 + z_2, I))(x_1 + x_2, I)(a + bI)]((z_1 + z_2, I))\]
\[= [[(x_1 + x_2, I)(a + bI)]^2((z_1 + z_2, I))^2)((a + bI))\]
\[= [[(x_1 + x_2, I)^2((a + bI)^2)((z_1 + z_2, I))^2)((a + bI))\]
\[= ((a + bI)^2((z_1 + z_2, I))^2)((a + bI))\]
\[= [[(a + bI)(a + bI)]((x_1 + x_2, I)^2((z_1 + z_2, I))^2)((a + bI))\]
\[= [[(a + bI)(a + bI)]((z_1 + z_2, I)^2((x_1 + x_2, I)^2))((a + bI))\]
\[= [[(a + bI)(a + bI)]((z_1 + z_2, I)^2((a + bI)(a + bI))((a + bI))\]
\[= [(t_1 + t_2, I)((a + bI)(a + bI)]((a + bI))\]
\[= [(t_1 + t_2, I)(a + bI)^2(u_1 + u_2, I)]\]

where \[\{(x_1 + x_2, I)^2((z_1 + z_2, I)^2)\} = (t_1 + t_2, I) \in N(S)\] and \[(a + bI) = (u_1 + u_2, I) \in N(S)\] where \[(a + bI) = (u_1 + u_2, I) \in N(S)\]. Thus \(N(S)\) is intra regular. 

\[(vi) \Rightarrow (viii)\] Assume that \(N(S)\) is completely regular Neutrosophic AG-groupoid with left identity (Neutrosophic AG’-groupoid), then for any \(a + bI \in N(S)\) there exist \((t_1 + t_2, I) \in N(S)\) such that \(a + bI = (a + bI^2((x_1 + x_2, I))\), \(a + bI = ((y_1 + y_2, I)(a + bI)^2), \) and \(a + bI = (a + bI)(z_1 + z_2, I))\)(a + bI))

\[a + bI = ((x_1 + x_2, I)(a + bI)^2\]
\[= (u_1 + u_2, I)(v_1 + v_2, I)]\]
\[= ((a + bI)(a + bI)]((x_1 + x_2, I))((a + bI)(a + bI)]\]
\[= ((a + bI)(a + bI)]((v_1 + v_2, I)(u_1 + u_2, I))\]

where \((x_1 + x_2, I) = (v_1 + v_2, I)(u_1 + u_2, I) \in N(S)\) Thus \(N(S)\) is weakly regular
where \( (x_i + x_j)(v_i + v_j) = (u_i + u_j)(v_i + v_j) \) for some \( (x_i + x_j) \in N(S) \). Thus \( N(S) \) is weakly regular.

\[
\begin{align*}
a + bl &= [(a + bl)(z_1 + z_2 I)](a + bl) \\
&= [((a + bl)^2(x_i + x_j I)(z_1 + z_2 I)][(y_1 + y_2 I)(a + bl)^2] \\
&= [[(z_1 + z_2 I)(x_i + x_j I)][(a + bl)(a + bl)]][(y_1 + y_2 I)(a + bl)(a + bl)] \\
&= [(a + bl)([(z_1 + z_2 I)(x_i + x_j I)][(a + bl)]][(y_1 + y_2 I)(a + bl)(a + bl)] \\
&= [(a + bl)(t_1 + t_2 I)][(a + bl)(w_1 + w_2 I)]
\end{align*}
\]

where \( (t_1 + t_2 I) = [[(z_1 + z_2 I)(x_i + x_j I)][(a + bl)]]\in N(S) \)

\[& & \& \]

\( (w_1 + w_2 I) = [(y_1 + y_2 I)(a + bl)]\in N(S) \). Thus \( N(S) \) is weakly regular.

\((viii) \Rightarrow (vi)\) Let \( N(S) \) be a \((2,2)\)-regular Neutrosophic AG-groupoid with left identity, then for \( a + bl \in N(S) \) there exists \( x_i + x_j \in N(S) \) such that

\[
\begin{align*}
a + bl &= [(a + bl)^2(x + y I)][(a + bl)^2].\text{Now}
\end{align*}
\]

\[
\begin{align*}
a + bl &= [(a + bl)^2(x_i + x_j I)][(a + bl)^2] \\
&= [(y_1 + y_2 I)(a + bl)^2]
\end{align*}
\]

where \( (a + bl)^2(x_i + x_j I) = (y_1 + y_2 I) \in N(S) \) and by using (3), we have

\[
\begin{align*}
a + bl &= [(a + bl)^2[(x_i + x_j I)][(a + bl)(a + bl)]] \\
&= [(a + bl)(a + bl)][(x_i + x_j I)(a + bl)^2] \\
&= (a + bl)^2(z_1 + z_2 I)
\end{align*}
\]

where \( (x_i + x_j I)(a + bl)^2 = (z_1 + z_2 I) \in N(S) \). and by using (3), (1) and (4), we have

\[
\begin{align*}
a + bl &= [(a + bl)^2[(x_i + x_j I)][(a + bl)(a + bl)]] \\
&= [(a + bl)(a + bl)][(x_i + x_j I)(a + bl)^2] \\
&= [(a + bl)(a + bl)][(a + bl)(a + bl)][(x_i + x_j I)(e_1 + e_2 I)] \\
&= [(a + bl)(a + bl)][(a + bl)^2(t_1 + t_2 I)] \\
&= [[(a + bl)^2(t_1 + t_2 I)][(a + bl)](a + bl)] \\
&= [[(a + bl)(a + bl)][(t_1 + t_2 I)][(a + bl)](a + bl)] \\
&= [[(a + bl)(a + bl)][(t_1 + t_2 I)(a + bl)](a + bl)] \\
&= [[(a + bl)(t_1 + t_2 I)][(a + bl)(a + bl)](a + bl)] \\
&= [(a + bl)][(a + bl)(t_1 + t_2 I)(a + bl)](a + bl)] \\
&= [(a + bl)(u_i + u_2 I)(a + bl)]
\end{align*}
\]
Thus $N(S)$ is left regular, right regular and regular, so $N(S)$ is completely regular.

(vi) $\Rightarrow$ (viii) Assume that $N(S)$ is a completely regular Neutrosophic AG-groupoid with left identity, then for any $a + bl \in N(S)$ there exist $(x_1 + x_2 I), (y_1 + y_2 I), (z_1 + z_2 I) \in N(S)$ such that

$a + bl = [(a + bl)(x_1 + x_2 I)](a + bl)$, \quad $a + bl = (a + bl)^2 (y_1 + y_2 I)$ and

$a + bl = (z_1 + z_2 I)(a + bl)^2$. Now by using (1), (4) and (3), we have

$$a + bl = [(a + bl)(x_1 + x_2 I)][(a + bl)]$$

$$= [[[a + bl]^2 (y_1 + y_2 I))(x_1 + x_2 I)][(z_1 + z_2 I)(a + bl)^2]]$$

$$= [[[a + bl](y_1 + y_2 I)(x_1 + x_2 I)][(z_1 + z_2 I)(a + bl)^2]]$$

$$= [[[a + bl][z_1 + z_2 I][a + bl]^2]]$$

$$= [[[a + bl](y_1 + y_2 I)(x_1 + x_2 I)][(z_1 + z_2 I)(a + bl)^2]]$$

$$= (z_1 + z_2 I)(a + bl)^2$$

$$= (z_1 + z_2 I)(y_1 + y_2 I)(x_1 + x_2 I)(a + bl)^2$$

$$= (a + bl)^2 (n_1 + v_2 I)(a + bl)^2$$

where $[(z_1 + z_2 I)(y_1 + y_2 I)](x_1 + x_2 I) = (v_1 + v_2 I) \in N(S)$. Thus $N(S)$ is (2,2) - regular.

References


