RESEARCH ON NUMBER THEORY AND SMARANDACHE NOTIONS

(PROCEEDINGS OF THE FIFTH INTERNATIONAL CONFERENCE ON NUMBER THEORY AND SMARANDACHE NOTIONS)

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Preface

This Book is devoted to the proceedings of the fifth International Conference on Number Theory and Smarandache Notions held in Shangluo during March 27-30, 2009. The organizers were myself and Professor Chao Li from Shangluo University. The conference was supported by Shangluo University and there were more than 100 participants. We had three foreign guests, Professor K. Chakraborty from India, Professor Imre Katai from Hungary, Professor S. Kanemitsu from Japan. The conference was a great success and will give a strong impact on the development of number theory in general and Smarandache Notions in particular. We hope this will become a tradition in our country and will continue to grow. And indeed we are planning to organize the sixth conference in coming March which will be held in Tianshui, a beautiful city of Gansu.

In the volume we assemble not only those papers which were presented at the conference but also those papers which were submitted later and are concerned with the Smarandache type problems or other mathematical problems.

There are a few papers which are not directly related to but should fall within the scope of Smarandache type problems. They are 1. Y. Wang, Smarandache sequence of ulam numbers; 2. H. Gunarto and A. A. K. Majumdar, On numerical values of Z(n); 3. K. Nagarajan, A. Nagarajan and S. Somasundaram, M-graphoidal path covers of a graph; etc.

Other papers are concerned with the number-theoretic Smarandache problems and will enrich the already rich stock of results on them.

Readers can learn various techniques used in number theory and will get familiar with the beautiful identities and sharp asymptotic formulas obtained in the volume.

Researchers can download books on the Smarandache notions from the following open source Digital Library of Science:

www.gallup.unm.edu/~smarandache/eBooks-otherformats.htm.

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The opening ceremony of the conference is occurred in Shangluo University (http://www.slxy.cn), which is attracting more and more people from other countries to study Chinese Calligraphy, Chinese Drawing and Chinese Culture.

Professor Chao Li
An equation related to the Smarandache power function

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Abstract For any positive integer \( x \), let \( \varphi(x) \) and \( f(x) \) denote the Euler totient function and the Smarandache power function of \( x \) respectively. Let \( n \) be a fixed positive integer. Using elementary method, we prove that the equation \( f(x^n) = \varphi(x) \) has at least two positive integer solutions \( x \), but it has only finitely many positive integer solutions.

Keywords Euler totient function, Smarandache power function, equation, solution.

§1. Introduction and Results

Let \( N^+ \) be the set of all positive integers. For any positive integer \( x \), let \( P(x) \) denotes the set of distinct prime divisors of \( x \), and let

\[
f(x) = \min \{ m \mid m \in N^+, \, x \mid m^m, \, P(m) = P(x) \}.
\]

Then \( f(x) \) is called the Smarandache power function of \( x \). Many properties of this arithmetical function are discussed, see references [1], [2], [3] and [4].

Let \( n \) be a positive integer. Let \( \varphi(x) \) denotes the Euler totient function of \( x \). In this paper, we consider the equation

\[
f(x^n) = \varphi(x), \, x \in N^+.
\]

Recently, Chengliang Tian and Xiaoyan Li [4] gave all positive integer solutions of equation (2) for \( n = 1, \, 2 \) and 3. Simultaneously, they proposed the following:

Conjecture. For any fixed positive integer \( n \), the equation (2) has only finitely many positive integer solutions \( x \).

In this paper, we use the elementary method to prove the following general results.

Theorem 1. For any fixed positive integer \( n \), let

\[
r = \min \left\{ m \mid m \in N^+, \, n \leq 2^{m-1} \left( 1 - \frac{1}{m} \right) \right\}.
\]

Then \( x = 2^r \) is a solution of equation (2).

Since \( x = 1 \) is a solution of (2), by Theorem 1, we get the following Corollary immediately.

Corollary 1. For any fixed positive integer \( n \), the equation (2) has at least two positive integer solutions.
On the other hand, we can also obtain an explicit upper bound estimate for the solutions of equation (2) as follows:

**Theorem 2.** All positive integer solutions \( x \) of equation (2) satisfy \( x < 256n^2 \).

From Theorem 2, we may immediately deduce that the above Conjecture is correct. That is, we have the following:

**Corollary 2.** For any fixed positive integer \( n \), the equation (2) has only finite many positive integer solutions.

### §2. Proof of the theorem 1

For any positive integer \( m \), let

\[
g(m) = 2^{m-1} \left( 1 - \frac{1}{m} \right), \quad h(m) = 2^{m-2} \left( 1 - \frac{2}{m} \right).
\]  

(4)

**Lemma 1.** If \( m > 1 \), then \( h(m + 1) < g(m) < h(m + 2) \).

**Proof.** By (4), if \( h(m + 1) \geq g(m) \), then \( 1 - \frac{2}{m + 1} \geq 1 - \frac{1}{m} \). But, since \( m \geq 2 \), it is impossible. So we have \( h(m + 1) < g(m) \). Similarly, we can prove that \( g(m) < h(m + 2) \).

**Lemma 1 is proved.**

**Lemma 2.** Let \( s \) be a positive integer. Then \( x = 2^s \) is a solution of (2) if and only if \( s \) satisfies

\[
h(s) < n \leq g(s).
\]

(5)

**Proof.** Let \( x = 2^s \). Then we have \( \varphi(x) = 2^{s-1} \). By (4), if \( n \leq g(s) \), then \( sn \leq 2^{s-1}(s - 1) \), whence we get

\[
x^n \mid \varphi(x)^{\varphi(x)}.
\]

(6)

On the other hand, if \( h(s) < n \), then \( 2^{s-2}(s - 2) < sn \). It implies that

\[
x^n \mid m^m, \quad m = 2^t, \quad t = 0, 1, \ldots, s - 2.
\]

(7)

Therefore, by (1), we see from (6) and (7) that \( f(x^n) = \varphi(x) \) for \( x = 2^s \), namely, it is a positive integer solution of equation (2).

By the same method, we can prove that if \( x = 2^s \) is a solution of (2), then \( s \) satisfies (5). This proves Lemma 1.

**Proof of Theorem 1.** We see from (3) and (4) that \( g(r - 1) < n \leq g(r) \). If \( n \leq h(r) \), then we have \( g(r - 1) < h(r) \). However, since \( r \geq 2 \), by Lemma 1, it is impossible. So we have \( h(r) < n \leq g(r) \). Thus, by Lemma 2, \( x = 2^s \) is a solution of equation (2). This completes the proof of Theorem 1.
§2. Proof of the theorem 2

Lemma 3. For any fixed positive integer $n$, the equation (2) has at most two positive integer solutions with the form

$$x = 2^s, \quad s \in \mathbb{N}^+.$$  \hspace{1cm} (8)

Moreover, if (2) has exactly two solutions with the form (8), then $x = 2^r$ and $2^{r+1}$.

Proof. We now assume that equation (2) has three solutions $x = 2^{s_i}$ $(i = 1, 2, 3)$ with $s_1 < s_2 < s_3$. Then, by Lemma 2, we have $n \leq g(s_i)$ $(i = 1, 2, 3)$. Further, by (3), we get $r \leq s_1$ and $r+2 \leq s_3$. Furthermore, using Lemma 2 again, we have $h(r+2) \leq h(s_3) < n \leq g(r)$. However, by Lemma 1, it is impossible. Thus, equation (2) has at most two positive integer solutions with form (8).

In addition, by Theorem 1, using the same method, we can prove the second half of Lemma 3. This proves Lemma 3.

Lemma 4. If $x > 1$, then $\phi(x) > \frac{x}{4 \log x}$.

Proof. This is Lemma 2 of [5].

Proof of Theorem 2. By the result of [4], the theorem 2 holds for $n \leq 3$. So we can assume that $n > 3$. Since $\phi(1) = \phi(2) = 1$, the equation (2) has only the positive integer solution $x = 1$ with $x \leq 2$. Let $x$ be a solution of equation (2) with $x > 2$. Since $\phi(x)$ is even if $x > 2$, then $x$ must be even. By Lemma 2, if $x = 2^s$, then $h(s) < n \leq g(r)$. Hence, by (4), we get that $s > 3$ and

$$h(s) = 2^{s-2} \left(1 - \frac{2}{s}\right) = \frac{x}{4} \left(1 - \frac{2}{s}\right) < n,$$  \hspace{1cm} (9)

whence we obtain $x < 12n$.

If $x \neq 2^s$, then $x$ has the factorization

$$x = 2^{r_0} p_1^{r_1} \cdots p_k^{r_k},$$  \hspace{1cm} (10)

where $p_i$ $(i = 1, \cdots, k)$ are odd primes with $p_1 < \cdots < p_k$, $r_j$ $(j = 0, 1, \cdots, k)$ are positive integers. By (10), we have

$$\phi(x) = 2^{r_0-1} p_1^{r_1-1} \cdots p_k^{r_k-1} (p_1 - 1) \cdots (p_k - 1).$$  \hspace{1cm} (11)

Further, since $x$ is a positive integer solution of equation (2), we see from (1) and (11) that

$$\phi(x) = 2^{s_0} p_1^{s_1} \cdots p_k^{s_k}, \quad s_0 \geq r_0, \quad s_i \geq \max(1, r_i - 1), \quad i = 1, \cdots, k.$$  \hspace{1cm} (12)

Let

$$P = \{m \mid m \in \mathbb{N}^+, \quad m < \phi(x), \quad m = 2^{t_0} p_1^{t_1} \cdots p_k^{t_k}, \quad t_j \in \mathbb{N}^+, \quad j = 0, 1, \cdots, k\}.$$  \hspace{1cm} (13)

Since $f(x^n) = \phi(x)$, we see from (1) and (13) that

$$x^n \mid \phi(x)^{2^n}$$  \hspace{1cm} (14)
and
\[ x^n \nmid m^m, \quad m \in P. \]  
\hspace{1cm} \text{(15)}

If \( r_0 = 1, k = 1 \) and \( p_1 = 3 \), then \( x = 2 \times 3^{r_1} \) and \( \varphi(x) = 2 \times 3^{r_1-1} \). By (14), we get
\[ n = 2 \times 3^{r_1-1} \left( 1 - \frac{1}{r_1} \right). \]  
\hspace{1cm} \text{(16)}

Further, since \( n \geq 4 \), we see from (16) that \( r_1 \geq 3 \). On the other hand, since \( 2 \times 3^{r_1-2} \in P \) by (13), we get from (15) that either
\[ 2 \times 3^{r_1-2} < n \]  
\hspace{1cm} \text{(17)}
or
\[ 2 \times 3^{r_1-2} \left( 1 - \frac{2}{r_1} \right) < n. \]  
\hspace{1cm} \text{(18)}

Recall that \( x = 2 \times 3^{r_1} \) and \( r_1 \geq 3 \). We deduce from (17) and (18) that \( x < 27n \).

If \( x \neq 2 \times 3^{r_1} \), then either \( r_0 \geq 2, \ r_0 = k = 1 \) and \( p_1 \geq 5 \) or \( r_0 = 1 \) and \( k \geq 2 \). It implies that \( s_0 \geq 2 \) by (10) and (12). Therefore, we see from (12) and (13) that \( \frac{\varphi(x)}{2} \in P \). Further, by (10), (12) and (15), there exists a positive integer \( l \) such that \( 0 \leq l \leq k \) and
\[ r_1n > \begin{cases} 
\frac{1}{2} \varphi(x)(s_0 - 1), & \text{if } l = 0; \\
\frac{1}{2} \varphi(x)(s_l), & \text{if } l \neq 0.
\end{cases} \]  
\hspace{1cm} \text{(19)}

Furthermore, since \( s_0 \geq \max(2, r_0) \) and \( s_l \geq \max(1, r_l - 1) \) for \( l \neq 0 \), we get from (19) that
\[ n > \frac{1}{4} \varphi(x). \]  
\hspace{1cm} \text{(20)}

Since \( x > 2 \), by Lemma 4, we have
\[ \varphi(x) > \frac{x}{4 \log x} > \frac{\sqrt{x}}{4}. \]  
Hence, we see from (20) that
\[ x < 256n^2. \]  

To sum up, we obtain \( x < \max(12n, 27n, 256n^2) = 256n^2 \).

This completes the proof of Theorem 2.

References


On the F.Smarandache 3n-digital sequence

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Abstract The sequence \( \{a_n\} = \{13, 26, 39, 412, 515, 618, 721, \cdots \} \) is called the F.Smarandache 3n-digital sequence. That is, for any integer \( b \in \{a_n\} \), it can be partitioned into two groups such that the second is three times bigger than the first. The main purpose of this paper is to study the mean value properties of \( \frac{n}{a_n} \), and give an interesting mean value formula for it.

Keywords F.Smarandache 3n-digital sequence, mean value, asymptotic formula.

§1. Introduction and result

For any positive integer \( n \), the famous F.Smarandache 3n-digital sequence is defined as \( \{a_n\} = \{13, 26, 39, 412, 515, 618, 721, \cdots \} \). That is, for any integer \( b \in \{a_n\} \), it can be divided into two parts such that the second is three times bigger than the first. For example, \( a_{28} = 2884 \), \( a_{35} = 35105 \), \( a_{104} = 104312 \), \cdots . This sequence was first proposed by professor F.Smarandache in references [1] and [4], where he also asked us to study its elementary properties. About this problem, some scholars had studied it, and obtained some interesting conclusions. For example, professor Zhang Wenpeng proposed the following:

Conjecture. There does not exist any complete square number in the Smarandache 3n-digital sequence \( \{a_n\} \). That is, the equation \( a_n = m^2 \) has no positive integer solution.

Wu nan [3] had studied this problem, and proved that \( \{a_n\} \) is not a complete square number for some special positive integers \( n \), such as complete square numbers and square-free numbers. For general positive integer \( n \), whether Zhang’s conjecture is true is an open problem.

In this paper, we shall use the elementary method to study the estimate properties of the mean value \( \sum \frac{n}{a_n} \), and give a sharper asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. For any real number \( N > 1 \), we have the asymptotic formula

\[
\sum_{n \leq N} \frac{n}{a_n} = \frac{3}{10\ln 10} \cdot \ln N + O(1).
\]

§2. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of our theorem. First for any positive integer number \( n \), let \( 3n = b_k(n)b_{k(n)-1} \cdots b_2b_1 \), where \( 1 \leq b_k(n) \leq 9 \),
$0 \leq b_i \leq 9$, $i = 1, 2, \ldots, k(n) - 1$. According to the definition of $a_n$, we can write $a_n$ as

$$a_n = n \cdot 10^{k(n)} + 3n = n \cdot (10^{k(n)} + 3).$$

So we have

$$\sum_{n \leq N} \frac{n}{a_n} = \sum_{n \leq N} \frac{1}{10^{k(n)} + 3}.$$

It is clear that if $N \leq 3$, then

$$\sum_{n \leq N} \frac{n}{a_n} = \frac{3}{10} \log_{10} N$$

is a constant. So without loss of generality we can assume that $N > 3$. In this case, there exists a positive integer $M$ such that

$$\frac{33 \cdots 33}{M} < N \leq \frac{33 \cdots 33}{M+1}. \quad (1)$$

Note that for any positive integer $n$, if

$$\frac{33 \cdots 33}{u} \leq n \leq \frac{33 \cdots 33}{u-1},$$

then $3n = b_u b_{u-1} \cdots b_2 b_1$. So we have

$$\sum_{n \leq N} \frac{n}{a_n} = \sum_{n \leq 3} \frac{1}{10 + 3} + \sum_{3 < n \leq 33} \frac{1}{10^2 + 3} + \sum_{33 < n \leq 333} \frac{1}{10^3 + 3} + \cdots$$

$$+ \sum_{M-1 < n \leq M} \frac{1}{10^M + 3} + \sum_{M < n \leq N} \frac{1}{10^{M+1} + 3}$$

$$= \frac{3}{10 + 3} + \frac{30}{10^2 + 3} + \frac{300}{10^3 + 3} + \cdots + \frac{3 \cdot 10^{M-1}}{10^M + 3} + \frac{N - \frac{10^M - 1}{3}}{10^{M+1} + 3}$$

$$= \frac{3}{10} \left( \frac{10}{10 + 3} + \frac{10^2}{10^2 + 3} + \frac{10^3}{10^3 + 3} + \cdots + \frac{10^M}{10^M + 3} \right) + \frac{N - \frac{10^M - 1}{3}}{10^{M+1} + 3}$$

$$= \frac{3}{10} \left[ \left( 1 - \frac{3}{10 + 3} \right) + \left( 1 - \frac{3}{10^2 + 3} \right) + \left( 1 - \frac{3}{10^3 + 3} \right) + \cdots + \left( 1 - \frac{3}{10^M + 3} \right) \right]$$

$$+ \frac{N - \frac{10^M - 1}{3}}{10^{M+1} + 3}$$

$$= \frac{3}{10} \left[ M - \left( \frac{3}{10 + 3} + \frac{3}{10^2 + 3} + \frac{3}{10^3 + 3} + \cdots + \frac{3}{10^M + 3} \right) \right] + \frac{N - \frac{10^M - 1}{3}}{10^{M+1} + 3}$$

$$= \frac{3}{10} \cdot M - \frac{9}{10} \cdot \sum_{i=1}^{M} \frac{1}{10^i + 3} + \frac{N - \frac{10^M - 1}{3}}{10^{M+1} + 3}$$

$$= \frac{3}{10} \cdot M + O(1). \quad (2)$$

Now we estimate $M$, from inequality (1) we have

$$10^M - 1 < 3N \leq 10^{M+1} - 1$$
\[
M \ln 10 + \ln \left(1 - \frac{1}{10^M}\right) < \ln(3N) \leq (M + 1) \ln 10 + \ln \left(1 - \frac{1}{10^{M+1}}\right)
\]

\[
\ln \left(\frac{3N}{\ln 10}\right) - \ln \left(\frac{1}{10^M}\right) - 1 \leq M < \ln \left(\frac{3N}{\ln 10}\right) - \ln \left(\frac{1}{10^{M+1}}\right).
\]

Note that as \(N \to +\infty\), \(\ln \left(\frac{1}{10^M}\right) \sim \frac{1}{10^M}\), \(\ln \left(1 - \frac{1}{10^M}\right) \sim \frac{1}{10^M}\). So that

\[
\ln \left(\frac{3N}{\ln 10}\right) - 1 - O\left(\frac{1}{10^M}\right) \leq M < \ln \left(\frac{3N}{\ln 10}\right) - O\left(\frac{1}{10^M}\right).
\]

Combining this and (2) we may immediately deduce that

\[
\sum_{n \leq N} \frac{n}{a_n} = \frac{3}{10 \ln 10} \cdot \ln N + O(1).
\]

This completes the proof of Theorem.

**References**


An equation involving the Smarandache double factorial function and Euler function

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Abstract
For any positive integer \( n \), let \( Sd f(n) \) denotes the Smarandache double factorial function, and \( \varphi(n) \) is the Euler function. The main purpose of this paper is using the elementary method to study the solvability of the equation \( Sd f(n) = \varphi(n) \), and give its all positive integer solutions.

Keywords
Double factorial function, Euler function, equation, positive integer solutions.

§1. Introduction and results

For any positive integer \( n \), the famous Smarandache double factorial function \( Sd f(n) \) is defined as the smallest positive integer \( m \) such that \( m!! \) is divisible by \( n \), where the double factorial

\[
m!! = \begin{cases} 1 \cdot 3 \cdot 5 \cdots m, & \text{if } m \text{ is an odd number;} \\ 2 \cdot 4 \cdot 6 \cdots m, & \text{if } m \text{ is an even number.} \end{cases}
\]

That is, \( Sd f(n) = \min\{m : n|m!! \}, m \in N \}, \) where \( N \) denotes the set of all positive integers. For example, the first few values of \( Sd f(n) \) are: \( Sd f(1) = 1, Sd f(2) = 2, Sd f(3) = 3, Sd f(4) = 4, Sd f(5) = 5, Sd f(6) = 6, Sd f(7) = 7, Sd f(8) = 4, Sd f(9) = 9, Sd f(10) = 10, Sd f(11) = 11, Sd f(12) = 6, Sd f(13) = 13, Sd f(14) = 14, Sd f(15) = 5, Sd f(16) = 6, Sd f(17) = 17, Sd f(18) = 12, Sd f(19) = 19, Sd f(20) = 10, \ldots \). In reference [1] and [2], Professor F.Smarandache asked us to study the properties of \( Sd f(n) \). About this problem, some authors had studied it, and obtained some interesting results, see references [3], [4], [5] and [6]. For example, Maohua Le [4] discussed various problems and conjectures about \( Sd f(n) \), and obtained some useful results, one of them as follows: if \( 2|n \) and \( n = 2^{\alpha}n_1 \), where \( \alpha, n_1 \) are positive integers with \( 2 \nmid n_1 \), then

\[
Sd f(n) \leq \max\{Sd f(2^{\alpha}), 2Sd f(n_1)\}.
\]

Fuling Zhang and Jianghua Li [5] proved that for any real number \( x \geq 1 \), we have the asymptotic formula

\[
\sum_{n \leq x} Sd f(n) = \frac{x \ln x}{\ln \ln x} + O \left( \frac{x \ln x}{(\ln \ln x)^2} \right).
\]
Jianping Wang [6] proved that for any real number \( x \geq 1 \) and any fixed positive integer \( k \), we have the asymptotic formula

\[
\sum_{n \leq x} (\text{Sd f}(n) - S(n))^2 = \frac{\zeta(3)}{24} \frac{x^3}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^3}{\ln^i x} + O \left( \frac{x^3}{\ln^{k+1}x} \right),
\]

where \( \zeta(s) \) is the Riemann zeta-function, and \( c_i \) are constants.

In this paper, we shall use the elementary method to study the solvability of the equation \( \text{Sd f}(n) = \varphi(n) \), and give its all positive integer solutions. That is, we shall prove the following conclusion:

**Theorem.** The equation \( \text{Sd f}(n) = \varphi(n) \) have only 4 positive integer solutions, they are \( n = 1, 8, 36, 50 \).

§2. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of the theorem. It is easy to verify that \( n = 1 \) is a solution of the equation \( \text{Sd f}(n) = \varphi(n) \). In order to obtain the other positive integer solutions, we discuss the equation in the following several cases:

1. If \( n > 1 \) is an odd number. At this time, from the definition of the Smarandache double factorial function \( \text{Sd f}(n) \) we know that \( \text{Sd f}(n) \) is an odd number, but \( \varphi(n) \) is an even number, thus \( \text{Sd f}(n) \neq \varphi(n) \).

2. If \( n \) is an even number. We assume that \( n = 2^{\alpha_1} p_1^{\alpha_2} \cdots p_k^{\alpha_k} \), where \( p_1 < p_2 < \cdots < p_k \), \( p_i (1 \leq i \leq k) \) is an odd prime, \( \alpha_i \geq 0 \) (\( 1 \leq i \leq k \)), \( \alpha \geq 1 \).

   (1) If \( \alpha_i = 0 \) (\( 1 \leq i \leq k \)), then \( n = 2^{\alpha} \) (\( \alpha \geq 1 \)). It is easy to verify that \( n = 2, 2^2, 2^4 \) are not solutions of the equation \( \text{Sd f}(n) \neq \varphi(n) \), and \( n = 2^3 \) is a solution of the equation \( \text{Sd f}(n) = \varphi(n) \). If \( \alpha \geq 5 \), since \( n \mid \varphi(n) \), so we obtain \( 2^{\alpha} \mid 2 \cdot 2^2 \cdot 2^{\alpha-2} \), namely \( 2^{\alpha} \mid \left( \frac{\varphi(n)}{2} \right)!! \), which implies \( \text{Sd f}(n) < \frac{\varphi(n)}{2} < \varphi(n) \).

   (2) If \( \alpha_i \geq 1 \),

   (I) \( n = 2^{\alpha_1} p_1^{\alpha_2} \cdots p_k^{\alpha_k} \), where \( p_1 < p_2 < \cdots < p_k \), \( p_i (1 \leq i \leq k) \) is an odd prime, \( \alpha_i \geq 1 \) (\( 1 \leq i \leq k \)), \( \alpha \geq 2 \), \( k \geq 1 \). At this time,

   \[
   \varphi(n) = 2^{\alpha_1-1} p_1^{\alpha_1-1} p_2^{\alpha_2-1} \cdots p_k^{\alpha_k-1} (p_1 - 1)(p_2 - 1) \cdots (p_k - 1).
   \]

   If \( n \nmid \varphi(n) \), then from the definition of the Smarandache double factorial function \( \text{Sd f}(n) \) we know that \( \text{Sd f}(n) \neq \varphi(n) \). If \( n \mid \varphi(n) \), then

   (i) For \( 2^\alpha \mid n \), \( (\alpha \geq 2) \),

   \[
   \frac{\varphi(n)}{2} = 2^{\alpha_1-1} p_1^{\alpha_1-1} p_2^{\alpha_2-1} \cdots p_k^{\alpha_k-1} (p_1 - 1)(p_2 - 1) \cdots (p_k - 1) \left( \frac{p_k - 1}{2} \right).
   \]

   If \( \alpha_k = 1 \), \( p_k = 3 \), then \( n = 2^\alpha \cdot 3 \), \( \varphi(n) = 2^{\alpha} \). Since \( n \mid \varphi(n) \), so we have \( \alpha > 2 \). Thus

   \[ 2^{\alpha} \mid \left( \frac{\varphi(n)}{2} \right)!!. \]

   If \( \alpha_k > 1 \), \( p_k \geq 3 \) or \( \alpha_k = 1 \), \( p_k > 3 \), then \( 4 \leq p_k^{\alpha_k-1}(p_k - 1) \), so we have \( 2^{\alpha} \mid \left( \frac{\varphi(n)}{2} \right)!!. \)

   (ii) For \( p_i^{\alpha_i} \mid n \), \( (1 \leq i \leq k) \).
(a) If \( \alpha_i = 1, \ (1 \leq i \leq k) \), since \( n \mid \varphi(n)!! \), we can deduce that
\[
2p_k \leq 2^{\alpha - 1} (p_1 - 1)(p_2 - 1) \cdots (p_k - 1).
\]

(a') If \( k = 1 \), then \( \alpha > 2 \). If \( \alpha = 3 \), then \( n = 2^3 \cdot p_k \). According to the definition of the Smarandache double factorial function \( Sdf(n) \) we have
\[
Sdf(2^3 \cdot p_k) \leq \max\{Sdf(2^3), \ 2Sdf(p_k)\} \leq 2 \cdot p_k < 4(p_k - 1) = (2^3 \cdot p_k).
\]

At this time, the equation \( Sdf(n) = \varphi(n) \) have no positive integer solution.

If \( \alpha > 3 \), then from \( 4p_k \leq 2^{\alpha - 1}(p_1 - 1)(p_2 - 1) \), we have \( p_k \mid \left( \frac{\varphi(n)}{2} \right)!! \).

(b) If \( k = 2 \), then \( \alpha \geq 2 \). If \( \alpha = 2, p_1 = 3 \), then from the definition of the Smarandache double factorial function \( Sdf(n) \) we can deduce that
\[
Sdf(2^2 \cdot 3 \cdot p_k) \leq \max\{Sdf(2^2), \ 2Sdf(p_k)\} \leq 2 \cdot p_k < 4(p_k - 1) = (2^2 \cdot 3 \cdot p_k).
\]

At this time, the equation \( Sdf(n) = \varphi(n) \) have no positive integer solution.

If \( \alpha \geq 3 \) or \( \alpha = 2, p_1 > 3 \), then \( 4p_k \leq 2^{\alpha - 1}(p_1 - 1)(p_2 - 1) \), so we have \( p_k \mid \left( \frac{\varphi(n)}{2} \right)!! \).

(c) If \( k = 2 \), then \( \alpha \geq 2 \). From \( 4p_k \leq 2^{\alpha - 1}(p_1 - 1)(p_2 - 1) \cdot (p_k - 1) \), we can obtain
\[
p_k \mid \left( \frac{\varphi(n)}{2} \right)!!.
\]

(b) If \( \alpha_i \geq 2, \ (1 \leq i \leq k) \), then from \( n \mid \varphi(n)!! \) we have \( p_k^\alpha \mid \varphi(n)!! \), thus \( 2p_k < \varphi(n) \).

(a') If \( k = 1, \) then \( n = 2^\alpha \cdot p_k^\alpha, \ \varphi(2^\alpha \cdot p_k^\alpha) = 2^{\alpha - 1} \cdot p_k^\alpha - 1 \cdot (p_k - 1) \).

If \( \alpha = 2 \) and \( \alpha = 2 \), then \( n = 2^2 \cdot p_k^2 \). If \( p_k = 3 \), according to the definition of the Smarandache double factorial function \( Sdf(n) \) we can easily obtain \( Sdf(2^2 \cdot 3^2) = 12 = \varphi(2^2 \cdot 3^2) \), so \( n = 36 \) is another positive integer solution of the equation \( Sdf(n) = \varphi(n) \). If \( p_k > 3 \), then from the definition of the Smarandache double factorial function \( Sdf(n) \) we obtain
\[
Sdf(2^2 \cdot p_k^2) \leq \max\{Sdf(2^2), \ 2Sdf(p_k^2)\} = 2 \cdot 3 \cdot p_k < 2 \cdot p_k \cdot (p_k - 1) = (2^2 \cdot p_k^2).
\]

At this time, the equation \( Sdf(n) = \varphi(n) \) have no positive integer solution.

If \( \alpha = 2, \ \alpha_k > 2 \) or \( \alpha = 2, \ \alpha_k = 2 \) or \( \alpha > 2, \ \alpha_k > 2 \), then from \( 4p_k < 2^{\alpha - 1} \cdot p_k^\alpha - 1 \cdot (p_k - 1) \) we have \( 4p_k < \varphi(2^\alpha \cdot p_k^\alpha) \), thus \( p_k^\alpha \mid \left( \frac{\varphi(n)}{2} \right)!! \).

(b') If \( k \geq 2 \), then from \( n \mid \varphi(n)!! \) we have \( 4p_i < 2^{\alpha - 1} \cdot p_i^\alpha - 1 \cdot \cdots \cdot p_k^\alpha - 1 \cdot (p_i - 1) \cdots (p_k - 1) \), thus \( p_i^\alpha \mid \left( \frac{\varphi(n)}{2} \right)!! \).

Combining (i) and (ii) we can deduce that:

If \( n = 36 \), then \( Sdf(n) = \varphi(n) \).

If \( n = 2^3 \cdot p_k, \ n = 2^2 \cdot 3 \cdot p_k \) or \( n = 2^2 \cdot p_k^2 \) (\( p_k > 3 \)), then \( Sdf(n) < \varphi(n) \).

Otherwise, from \( n \mid \varphi(n)!! \), we have \( Sdf(n) \leq \frac{\varphi(n)}{2} < \varphi(n) \).

(II) \( n = 2^{p_1^\alpha_i} \cdot p_2^\alpha_i \cdots p_k^\alpha_i \), where \( p_1 < p_2 < \cdots < p_k, \ p_i \ (1 \leq i \leq k) \) is an odd prime, \( \alpha_i \geq 1 \ (1 \leq i \leq k), \ k \geq 1 \). At this time,
\[
\varphi(n) = p_1^{\alpha_1 - 1} \cdot \cdots \cdot p_k^{\alpha_k - 1} \cdot (p_1 - 1)(p_2 - 1) \cdots (p_k - 1).
\]
An equation involving the Smarandache double factorial function and Euler function

If \( n \nmid (\varphi(n))!! \), then from the definition of the Smarandache double factorial function \( Sdf(n) \) we know that \( Sdf(n) \neq \varphi(n) \).

If \( n \mid \varphi(n)!! \), then

(i) If \( k = 1 \), then \( n = 2p_k^{\alpha_k} \), \( \varphi(n) = p_k^{\alpha_k - 1}(p_k - 1) \). From \( 2p_k^{\alpha_k} \mid (p_k^{\alpha_k - 1}(p_k - 1))!! \), we have \( 2p_k < p_k^{\alpha_k - 1}(p_k - 1) \). Hence \( p_k = 3 \), \( \alpha_k \geq 3 \) or \( p_k \geq 5 \), \( \alpha_k \geq 2 \).

(a) If \( p_k = 3 \), \( \alpha_k \geq 3 \), then from \( 2 \cdot 3 < 2 \cdot 3^{\alpha_k - 1} \), so we have \( 4 \cdot 3 < 2 \cdot 3^{\alpha_k - 1} \). Thus \( 2p_k^{\alpha_k} \mid \left( \frac{\varphi(2p_k^{\alpha_k})}{2} \right)!! \).

(b) If \( p_k = 5 \), \( \alpha_k = 2 \), then \( n = 2 \cdot 5^2 \). According to the definition of the Smarandache double factorial function \( Sdf(n) \) we can easily obtain \( Sdf(2 \cdot 5^2) = 20 = \varphi(2 \cdot 5^2) \), so \( n = 50 \) is another solution of the equation \( Sdf(n) = \varphi(n) \).

(c) If \( p_k = 5 \), \( \alpha_k > 2 \) or \( p_k > 5 \), \( \alpha_k \geq 2 \), then we can deduce that \( 4p_k < p_k^{\alpha_k - 1}(p_k - 1) \). Thus \( 2p_k^{\alpha_k} \mid \left( \frac{\varphi(2p_k^{\alpha_k})}{2} \right)!! \).

(ii) If \( k \geq 2 \), then from \( p_i^{\alpha_i} \mid \varphi(n)!! \), so we have \( 2p_i < p_i^{\alpha_i - 1} \cdots p_i^{\alpha_i - 1}(p_i - 1) \cdots (p_i - 1)(p_k - 1) \), hence there at least exists a \( \alpha_i \) such that \( \alpha_i \geq 2 \), so we can deduce that \( 4p_i < p_i^{\alpha_i - 1} \cdots p_i^{\alpha_i - 1}(p_i - 1) \cdots (p_i - 1)(p_k - 1) \) which implies \( p_i^{\alpha_i} \mid \left( \frac{\varphi(n)}{2} \right)!! \).

Combining (i) and (ii), we can deduce that:

If \( n = 50 \), then \( Sdf(n) = \varphi(n) \).

Otherwise, from \( n \mid \varphi(n)!! \), we have \( n \mid \frac{\varphi(n)}{2}!! \) which implies \( Sdf(n) \leq \frac{\varphi(n)}{2} < \varphi(n) \).

From the above discussion (1) and (2), we know that if \( n \) is an even number, then \( Sdf(n) = \varphi(n) \) if and only if \( n = 8, 36, 50 \).

Combining the cases 1 and 2, we complete the proof of Theorem.

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S-perfect and completely S-perfect numbers

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Abstract This paper proves that the only Smarandache perfect numbers are \( n = 1, 6 \), and the only completely Smarandache perfect numbers are \( n = 1, 28 \), where the Smarandache function \( S(n) \) satisfies the condition that \( S(1) = 1 \).

Keywords The Smarandache function, the Smarandache perfect number, the completely Smarandache perfect number.

§1. Introduction

The Smarandache function is defined as follows.

Definition 1.1. For any integer \( n \geq 1 \), the Smarandache function \( S(n) \), is the smallest positive integer \( m \) such that \( 1 \cdot 2 \cdot \cdots \cdot m \equiv m! \) is divisible by \( n \). That is,

\[
S(n) = \min \{ m : m \in \mathbb{Z}^+, \ n \mid m! \}; \quad n \geq 1,
\]

where \( S(1) = 1 \) (and \( \mathbb{Z}^+ \) is the set of all positive integers).

In classical Theory of Numbers, an integer \( n \geq 1 \) is called perfect if it is the sum of its proper divisors. Pe [1] has extended the definition to \( f \)-perfect numbers: If \( f(n) \) is an arithmetical function, an integer \( n \) with proper divisors \( d_1 \equiv 1, d_2, \cdots, d_k \) is called \( f \)-perfect if

\[
n = \sum_{i=1}^{k} f(d_i).
\]

Further extension is the completely \( f \)-perfect number \( n \) where the condition of proper divisor is relaxed.

The S-perfect and completely S-perfect numbers are defined below.

Definition 1.2. Given an integer \( n \geq 1 \),

(1) \( n \) is called Smarandache perfect (or, simply S-perfect) if and only if \( n = \sum_{i=1}^{k} S(d_i) \),

(2) \( n \) is called completely S-perfect if and only if \( n = \sum_{d \mid k} S(d) \), (where the sum is over all divisors \( d \) of \( n \) (including \( n \))).

In this paper, we prove that the only S-perfect numbers are \( n = 1, 6 \), and the only completely S-perfect numbers are \( n = 1, 28 \). These are given in Theorem 2.1 and Theorem 2.2 in Section 2. We conclude this paper with some remarks in the final section, Section 3.
§2. S-perfect and completely S - perfect numbers

In this section, we derive the numbers that are S-perfect, and the numbers that are completely S-perfect. They are given in the following two theorems.

Theorem 2.1. The only S-perfect numbers are \( n = 1, 6 \).

Proof. Let \( n \) be an S-perfect number, so that, by Definition 1.2,

\[
    n = \sum_{\substack{d \mid n \\ 1 \leq d < n}} S(d). \tag{1}
\]

Clearly, no prime is a solution of (1). Thus, any solution of (1) must be composite a number. Now, for any divisor \( d \) of \( n \), \( S(d) \leq S(n) \), with strict inequality sign for at least one divisor of \( n \). Therefore, from (1), we get

\[
    n = \sum_{\substack{d \mid n \\ 1 \leq d < n}} S(d) < [d(n) - 1]S(n) \Rightarrow n < d(n)S(n). \tag{2}
\]

Thus, any solution \( n \) of (1) must satisfy the inequalities in (2).

We first consider some particular cases.

Case 1. When \( n \) is of the form \( n = p^\alpha \) (\( \alpha \geq 2 \)).

In this case, \( d(n) = \alpha + 1 \), so that (2) reads as

\[
    n = p^\alpha < \alpha S(p^\alpha) \leq \alpha \cdot (\alpha p) \Rightarrow p^{\alpha - 1} < \alpha^2. \tag{3}
\]

But (3) is impossible when (a) \( p = 2 \) and \( \alpha \geq 7 \), (b) \( p \geq 3 \) for any \( \alpha \geq 3 \).

The proof is as follows:

(a) \( \frac{2^{\alpha - 1}}{\alpha^2} \) is strictly increasing in \( \alpha \geq 7 \) with \( \frac{2^6}{7^2} > 1 \),

(b) for any \( p \geq 3 \), \( \frac{p^{\alpha - 1}}{\alpha^2} \) is strictly increasing in \( \alpha \geq 3 \) with \( \frac{3^2}{3^2} = 1 \).

Thus, in this case, the possible candidates for the equation (1) to hold true are

\[
    n = 2^\alpha, 2 \leq \alpha \leq 6; n = 3^2.
\]

But, when \( n = 2^\alpha \), the proper divisors of \( n \) are \( 1, 2, 2^2, \ldots 2^{\alpha - 1} \), with \( S(1) = 1 \), so that the r.h.s. of (1) is odd, while \( n \) is even. Again, when \( n = 3^2 \), r.h.s. of (1) is not divisible by 3.

Thus, in this case, there is no solution of the equation (1).

Case 2. When \( n \) is of the form \( n = pq \) with \( q > p \).

In this case, \( d(n) = 4 \), and so, from (2) we get

\[
    n = pq < 3q \Rightarrow p = 2.
\]

Now, if \( p = 2 \), then

\[
    \sum_{\substack{d \mid n \\ 1 \leq d < n}} S(d) = S(1) + S(p) + S(q) = 1 + p + q = 3 + q = n = 2q \Rightarrow q = 3.
\]

Thus, in this case, \( n = 2 \cdot 3 = 6 \) is the only solution of the equation (1).
Case 3. When \( n \) is of the form \( n = 2^\alpha q, q \geq 3, \alpha \geq 2 \).

First, let \( S(n) = S(2^\alpha) \) (so that \( S(2^\alpha) \geq q \)). Since \( d(n) = 2(\alpha + 1) \), from (2), we get

\[
n = 2^\alpha q < 2(\alpha + 1)S(2^\alpha) \leq 2^2 \alpha(\alpha + 1) \Rightarrow 2^{\alpha-2} q < \alpha(\alpha + 1),
\]

which is impossible if (i) \( q = 3 \) and \( \alpha \geq 6 \), or if (ii) \( q = 5 \) and also \( \alpha \geq 4 \), or if (iii) \( q \geq 7 \) and \( \alpha \geq 2 \). Thus, the possible candidates for the equation (1) to hold true are

\[
n = 3.2^\alpha, 2 \leq \alpha \leq 5; (n = 5.2^3 \text{ is excluded since } S(2^3) = 4 < q = 5).
\]

Now, when \( n = 2^2 q \), the proper divisors of \( n \) are 1, 2, 2\(^2\), and 2\(^2\)q, so that

\[
\sum_{\substack{d|n \\
1 \leq d < n}} S(d) = S(1) + S(2) + S(2^2) + S(q) + S(2q) = 1 + 2 + 4 + q + q = 7 + 2q,
\]

when \( n = 2^3 q \), the proper divisors of \( n \) are 1, 2, 2\(^2\), 2\(^3\), q, 2q, and 2\(^2\)q, and so

\[
\sum_{\substack{d|n \\
1 \leq d < n}} S(d) = S(1) + S(2) + S(2^2) + S(2^3) + S(q) + S(2q) + S(2^2q) + S(2^3q) = 1 + 2 + 4 + q + q + 4 + 4 = 15 + 2q,
\]

when \( n = 2^4 q \), the proper divisors of \( n \) are 1, 2, 2\(^2\), 2\(^3\), 2\(^4\), q, 2q, 2\(^2\)q, and 2\(^3\)q, so that

\[
\sum_{\substack{d|n \\
1 \leq d < n}} S(d) = S(1) + S(2) + S(2^2) + S(2^3) + S(2^4) + S(q) + S(2q) + S(2^2q) + S(2^3q) + S(2^4q) = 1 + 2 + 4 + 6 + q + q + 4 + 4 + 6 = 39 + 2q,
\]

the proper divisors of \( n = 2^\alpha q \) are 1, 2, 2\(^2\), 2\(^3\), 2\(^4\), q, 2q, 2\(^2\)q, 2\(^3\)q, and 2\(^4\)q, with

so that, in each case, the term on the right of (1) is odd, while \( n \) is even.

Next, let \( S(n) = q \). Then, from (2), we get \( n = 2^\alpha q < 2(\alpha + 1)q \Rightarrow 2^{\alpha-1} < \alpha + 1 \), which is possible only for \( \alpha = 2 \), and so, \( n = 2^2 q \). And we have just proved that \( n = 2^2 q \) is not a solution of (1). Thus, in this case, there is no solution of the equation (1).

Case 4. When \( n \) is of the form \( n = p^\alpha q^\beta (\alpha \geq 1, \beta \geq 1, \alpha/\beta \geq 2) \).

Let \( S(n) = S(p^\alpha) \). Now, since \( d(n) = (\alpha + 1)(\beta + 1) \), (2) takes the form

\[
n = p^\alpha q^\beta < (\alpha + 1)(\beta + 1)S(p^\alpha) \leq (\alpha + 1)(\beta + 1)\alpha p \Rightarrow \frac{p^{\alpha-1}}{\alpha(\alpha + 1)} \cdot \frac{q^\beta}{\beta + 1} < 1.
\]
Note that, since \( \frac{p^{\alpha - 1}}{\alpha(\alpha + 1)} > 1 \) for any \( p \geq 7 \) and any \( \alpha \geq 2 \) and \( \frac{q^\beta}{\beta + 1} \geq 1 \) for any \( q \geq 2 \) and any \( \beta \geq 1 \), (4) is impossible for any \( q \geq 7 \), any \( q \neq p \), any \( \alpha \geq 2 \) and any \( \beta \geq 1 \). Thus, it is sufficient to check with \( p = 2, 3, 5 \) only.

First, let \( p = 2 \) (so that \( q \geq 3, \alpha \geq \beta \geq 2 \)). Then, (4) reads as

\[
\frac{2^{\alpha - 1}}{\alpha(\alpha + 1)} \cdot \frac{q^\beta}{\beta + 1} < 1. \tag{5}
\]

If \( q \geq 3 \) and \( \beta \geq 2 \), then since \( \frac{q^\beta}{\beta + 1} \geq 3 \), we get from (5), \( \frac{3 \cdot 2^{\alpha - 1}}{\alpha(\alpha + 1)} < 1 \), which is impossible for any \( \alpha \geq 1 \). Thus, (1) has no solution corresponding to this case (by Case 3 above).

Next, let \( p = 3 \), so that from (4)

\[
\frac{3^{\alpha - 1}}{\alpha(\alpha + 1)} \cdot \frac{q^\beta}{\beta + 1} < 1. \tag{6}
\]

If \( q \geq 5 \) and \( \beta \geq 1 \), then since \( \frac{q^\beta}{\beta + 1} > 2 \), we get from (6), \( \frac{2 \cdot 3^{\alpha - 1}}{\alpha(\alpha + 1)} < 1 \), which is impossible for any \( \alpha \geq 1 \); if \( q = 2 \) and \( \beta \geq 3 \), then since \( \frac{2^\beta}{\beta + 1} \geq 2 \), from (6), \( \frac{2 \cdot 3^{\alpha - 1}}{\alpha(\alpha + 1)} < 1 \), which is impossible for \( \alpha \geq 4 \).

Thus, in this case, the only possible candidates for (1) to hold true are

\[ n = 2 \cdot 3^\alpha, 2^2 \cdot 3^\alpha \text{ where } 2 \leq \alpha \leq 3. \]

But, \( n = 2 \cdot 3^\alpha \) has the proper divisors \( 1, 2, 3, 2^2, \ldots, 3^\alpha, 2 \cdot 3, 2^2 \cdot 3, \ldots, 2 \cdot 3^{\alpha - 1} \), so that

\[
\sum_{d|n} S(d) = S(1) + S(2) + S(3) + S(3^2) + \cdots + S(3^\alpha) + S(2 \cdot 3) + S(2^2 \cdot 3) + \cdots + S(2 \cdot 3^{\alpha - 1})
\]

\[
= 1 + 2 + \sum_{k=1}^{\alpha} (3k) + \sum_{k=1}^{\alpha - 1} (3k) = 1 + 2 + 3\alpha(\alpha + 1) + 3\alpha(\alpha - 1) = 3(\alpha^2 + 1),
\]

and the equation \( 3(\alpha^2 + 1) = 2 \cdot 3^\alpha \) has no solution for \( \alpha \geq 2 \).

Again, since \( 1, 2, 2^2, 3, 3^2, \ldots, 3^\alpha, 2 \cdot 3, 2^2 \cdot 3, \ldots, 2 \cdot 3^{\alpha - 1} \) are the proper divisors of \( n = 2^2 \cdot 3^\alpha \), we get

\[
\sum_{d|n} S(d) = S(1) + S(2) + S(2^2) + S(3) + S(3^2) + \cdots + S(3^\alpha)
\]

\[
+ S(2 \cdot 3) + S(2^2 \cdot 3) + \cdots + S(2 \cdot 3^\alpha) + S(2^2 \cdot 3) + S(2^2 \cdot 3^2) + \cdots + S(2^2 \cdot 3^{\alpha - 1})
\]

\[
= 1 + 2 + 4 + 2 \sum_{k=1}^{\alpha} (3k) + 4 + \sum_{k=2}^{\alpha - 1} (3k),
\]

which is not divisible by 3.
Finally, let $p = 5$, so that (4) takes the form
\[
\frac{5^{\alpha - 1}}{\alpha (\alpha + 1)} \cdot q^3 < 1. \tag{7}
\]

If $q \geq 7$ and $\beta \geq 1$, then since $\frac{q^3}{\beta + 1} > 3$, we get from (7), $\frac{3.5^{\alpha - 1}}{\alpha (\alpha + 1)} < 1$, which is impossible for any $\alpha \geq 1$; if $q = 2$ and $\beta \geq 3$, then since $\frac{2^\beta}{\beta + 1} \geq 2$, from (7), $\frac{2 \cdot 5^{\alpha - 1}}{\alpha (\alpha + 1)} < 1$, which is impossible for any $\alpha \geq 1$; if $q = 3$ and $\beta \geq 2$, then since $\frac{3^\beta}{\beta + 1} \geq 3$, from (7), $\frac{3 \cdot 5^{\alpha - 1}}{\alpha (\alpha + 1)} < 1$, which is impossible for any $\alpha \geq 1$; moreover, from (7), $\frac{5^{\alpha - 1}}{\alpha (\alpha + 1)} < 1$, which is impossible for $\alpha \geq 3$.

Thus, in this case, the only possible candidates for (1) to hold true are $n = 3 \cdot 5^2, 2 \cdot 5^2, 2^2 \cdot 5^2$.

Note, however, that $n = 3 \cdot 5^2$ violates (7), since $\frac{5}{6} > \frac{3}{2} > 1$.

Now, when $n = 2 \cdot 5^2$, the proper divisors of $n$ are $1, 2, 5, 5^2$ and $2.5$, so that
\[
\sum_{d|n \atop 1 \leq d < n} S(d) = S(1) + S(2) + S(5) + S(5^2) + S(2.5) = 1 + 2 + 5 + 10 + 5 = 23,
\]

and the proper divisors of $n = 2^2 \cdot 5^2$ are $1, 2, 2^2, 5, 5^2, 2 \cdot 5, 2^2 \cdot 5$ and $2 \cdot 5^2$, with
\[
\sum_{d|n \atop 1 \leq d < n} S(d) = S(1) + S(2) + S(2^2) + S(5) + S(5^2) + S(2 \cdot 5) + S(2^2 \cdot 5) + S(2 \cdot 5^2) = 42.
\]

Thus, there is no solution of (1) of the form $n = p^\alpha q^3$.

Now, we consider the general case. So, let
\[
n = p^\alpha q^3 p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}
\]

be the representation of $n$ in terms of its distinct prime factors $p, q, p_1, p_2, \ldots, p_k (k \geq 1)$.

Let, for definiteness, $S(n) = S(p^\alpha)$. Then, from (2), we get
\[
n < d(n)S(n) \leq d(n)S(p^\alpha) \Rightarrow \frac{p^\alpha}{\alpha + 1} \cdot \frac{q^3}{\beta + 1} \cdot \frac{p_1^{\alpha_1}}{\alpha_1 + 1} \cdot \frac{p_2^{\alpha_2}}{\alpha_2 + 1} \cdots \frac{p_k^{\alpha_k}}{\alpha_k + 1} < S(p^\alpha) \leq \alpha p. \tag{8}
\]

Now, if $p = 2$, without loss of generality, $q \geq 5$, and so (8) takes the form
\[
\frac{2^\alpha}{\alpha (\alpha + 1)} \cdot \frac{p_1^{\alpha_1}}{\alpha_1 + 1} \cdot \frac{p_2^{\alpha_2}}{\alpha_2 + 1} \cdots \frac{p_k^{\alpha_k}}{\alpha_k + 1} < 1,
\]

which is impossible. On the other hand, if $p = 3$, with $q \geq 5$, (8) takes the form
\[
\frac{2 \cdot 3^{\alpha - 1}}{\alpha (\alpha + 1)} \cdot \frac{p_1^{\alpha_1}}{\alpha_1 + 1} \cdot \frac{p_2^{\alpha_2}}{\alpha_2 + 1} \cdots \frac{p_k^{\alpha_k}}{\alpha_k + 1} < 1,
\]

which is also impossible.

All these complete the proof of the theorem.

**Theorem 2.2.** The only completely $S$-perfect numbers are $n = 1, 28$. 
**Proof.** Let \( n \) be a completely \( S \)-perfect number, so that, by Definition 2.1,

\[
    n = \sum_{d|n, 1 \leq d \leq n} S(d). \tag{9}
\]

Clearly, no prime is a solution of (9). Thus, any solution of (9) must be composite a number.

In this case, any solution of (9) must satisfy the following condition:

\[
    n = \sum_{d|n, 1 \leq d \leq n} S(d) < d(n)S(n). \tag{10}
\]

As in the proof of Theorem 2.1, in this case also, we consider the following case:

**Case 1.** When \( n \) is of the form \( n = p^\alpha (\alpha \geq 2) \).

In this case, however, there is no solution of the equation (9)(by an argument similar to that given in the proof of Theorem 2.1).

**Case 2.** When \( n \) is of the form \( n = pq \) with \( q \geq p \).

In this case,

\[
    \sum_{d | n, 1 \leq d \leq n} S(d) = S(1) + S(p) + S(q) + S(pq) = 1 + p + q + q = 1 + p + 2q,
\]

which is odd only if \( p = 2 \). But, then

\[
    \sum_{d|n, 1 \leq d \leq n} S(d) = 3 + 2q > 2q.
\]

Thus, there is no solution in this case.

**Case 3.** When \( n \) is of the form \( n = 2^\alpha q, q \geq 3, \alpha \geq 2 \).

Letting \( S(n) = S(2^\alpha) \), the possible candidates for (9) to hold true are \( n = 3.2^\alpha, 2 \leq \alpha \leq 5 \).

But,

(a) When \( n = 2^2q \), \[
    \sum_{d|n, 1 \leq d \leq n} S(d) = \sum_{d|n, 1 \leq d < n} S(d) + S(2^2q) = 11 + 2q,
\]

(b) When \( n = 2^3q \), \[
    \sum_{d|n, 1 \leq d \leq n} S(d) = \sum_{d|n, 1 \leq d < n} S(d) + S(2^3q) = 19 + 2q,
\]

(c) When \( n = 2^4q \), \[
    \sum_{d|n, 1 \leq d \leq n} S(d) = \sum_{d|n, 1 \leq d < n} S(d) + S(2^4q) = 31 + 2q,
\]

(d) When \( n = 2^5q \), \[
    \sum_{d|n, 1 \leq d \leq n} S(d) = \sum_{d|n, 1 \leq d < n} S(d) + S(2^5q) = 47 + 2q.
\]
so that in all the cases, the r.h.s. of (9) is odd.
Next, assuming that \( S(n) = q \), we get from (10),
\[
n = 2^\alpha q < 2(\alpha + 1)q \quad \Rightarrow \quad 2^{\alpha - 1} < \alpha + 1,
\]
which is possible only for \( \alpha = 2 \), and so, \( n = 2^2 q \). Then,
\[
\sum_{\substack{d \mid n \\ 1 \leq d < n}} S(d) = S(1) + S(2) + S(2^2) + S(q) + S(2q) + S(2^2q)
\]
\[
= 1 + 2 + 4 + q + q = 3q + 7 = 2^2q
\]
\[
\Rightarrow q = 7.
\]
Thus, in this case, \( n = 2^2 \cdot 7 = 28 \) is the only solution of the equation (9).

**Case 4.** When \( n \) is of the form \( n = p^\alpha q^\beta \).

Letting \( S(n) = S(p^\alpha) \) with \( p = 3 \), the only possible candidates for (9) to hold true are
\( n = 2 \cdot 3^\alpha \), \( 2 \cdot 3^\alpha \) where \( 2 \leq \alpha \leq 3 \). But, \( n = 2 \cdot 3^\alpha \) has the divisors \( 1, 2, 3, 3^2, \cdots, 3^\alpha \), \( 2 \cdot 3, 2 \cdot 3^2, \cdots, 2 \cdot 3^\alpha \), so that
\[
\sum_{\substack{d \mid n \\ 1 \leq d < n}} S(d) = S(1) + S(2) + S(3) + S(3^2) + \cdots + S(3^\alpha) + S(2 \cdot 3) + S(2 \cdot 3^2) + \cdots + S(2 \cdot 3^\alpha)
\]
\[
= 1 + 2 + 3 \alpha + \sum_{k=1}^{\alpha} (3k) = 1 + 2 + 3 \alpha + \frac{\alpha(\alpha + 1)}{2} = 3(\alpha^2 + \alpha + 1),
\]
and the equation \( 3(\alpha^2 + \alpha + 1) = 2 \cdot 3^k \) has no solution for \( \alpha \geq 2 \).

Again, since \( 1, 2, 2^2, 3, 3^2, \cdots, 3^\alpha, 2 \cdot 3, 2 \cdot 3^2, \cdots, 2 \cdot 3^\alpha, 2^2 \cdot 3, 2^2 \cdot 3^2, \cdots, 2^2 \cdot 3^\alpha \) are the divisors
of \( n = 2^2 \cdot 3^\alpha \), we get
\[
\sum_{\substack{d \mid n \\ 1 \leq d < n}} S(d) = S(1) + S(2) + S(2^2) + S(3) + S(3^2) + \cdots + S(3^\alpha)
\]
\[
+ S(2 \cdot 3) + S(2 \cdot 3^2) + \cdots + S(2 \cdot 3^\alpha) + S(2^2 \cdot 3) + S(2^2 \cdot 3^2) + \cdots + S(2^2 \cdot 3^\alpha)
\]
\[
= 1 + 2 + 4 + 2 \sum_{k=1}^{\alpha} (3k) + 4 + \sum_{k=2}^{\alpha} (3k),
\]
which is not divisible by 3.

When \( p = 5 \), the only possible candidates for (9) to hold true are \( n = 5 \cdot 2^2, 2 \cdot 5^2, 2^2 \cdot 5^2 \). But, when \( n = 2 \cdot 5^2 \), the divisors of \( n \) are \( 1, 2, 5, 5^2, 2 \cdot 5 \) and \( 2 \cdot 5^2 \), so that
\[
\sum_{\substack{d \mid n \\ 1 \leq d < n}} S(d) = S(1) + S(2) + S(5) + S(5^2) + S(2 \cdot 5) + S(2 \cdot 5^2) = 33,
\]
and the divisors of \( n = 2^2 \cdot 5^2 \) are \( 1, 2, 2^2, 5, 5^2, 2 \cdot 5, 2^2 \cdot 5, 2 \cdot 5^2 \) and \( 2^2 \cdot 5^2 \) with
\[
\sum_{\substack{d \mid n \\ 1 \leq d < n}} S(d) = S(1) + S(2) + S(2^2) + S(5) + S(5^2)
\]
\[
+ S(2 \cdot 5) + S(2^2 \cdot 5) + S(2 \cdot 5^2) + S(2^2 \cdot 5^2) = 52.
\]
Thus, there is no solution of (9) of the form \(n = p^\alpha q^\beta\).
In the general case when \(n = p^\alpha q^\beta p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}\) with \(k \geq 1\), an analysis similar to the proof of Theorem 2.1 shows that the equation (9) has no solution.

\section*{§3. Remarks}

Sandor [2] has considered the problem of finding the S-perfect and completely S-perfect numbers, but his proof is not complete. He has proved that the only S-perfect of the form \(n = pq\) is \(n = 6\) and there is no S-perfect number of the form \(n = 2^k q\) where \(k \geq 2\) is an integer and \(q\) is an odd prime. On the other hand, Theorem 2.1 gives all the S-perfect numbers. Again, Sandor only proved that, the only completely S-perfect number of the form \(n = p^2 q\) is \(n = 28\), and all completely S-perfect numbers are given by Theorem 2.2.

Theorem 2.1 and Theorem 2.2 find respectively the S-perfect and completely S-perfect numbers when \(S(1) = 1\). The situation is quite different if one adopts the convention that \(S(0) = 1\). In the latter case, as has been proved by Gronas [3], all completely S-perfect numbers are \(n = p\) (prime), 9, 16, 24. All that is known about the S-perfect numbers is that, among the first \(10^6\) numbers, \(n = 12\) is the only S-perfect number (see Ashbacher [4]).

In exactly the same way, the Z-perfect and completely Z-perfect numbers may be defined. Thus, given an integer \(n \geq 1\),

(1) \(n\) is called Z-perfect if and only if

\[
    n = \sum_{d|n, 1 \leq d \leq n} Z(d);
\]

(2) \(n\) is called completely Z-perfect if and only if

\[
    n = \sum_{d|n, 1 \leq d \leq n} Z(d);
\]

where \(Z(n)\) is the pseudo-Smarandache function, defined as follows :

\[
    Z(n) = \min \left\{ m : m \in \mathbb{Z}^+, n|\frac{m(m+1)}{2} \right\}, n \geq 1.
\]

**Open Problem.** To find all the Z-perfect and completely Z-perfect numbers.

Ashbacher [4] reports, on the basis of computer findings, that the only Z-perfect number less than \(10^6\) are \(n = 4, 6\) (see also Remark 3.5 in Majumdar [5]).

**References**


Cyclic dualizing elements in Girard quantales

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Abstract In this paper, we study the interior structures of Girard quantale and the cyclic dualizing elements of Girard quantale. Some equivalent descriptions for Girard quantale are given and an example which shows that the cyclic dualizing element is not unique is given.

Keywords Complete lattice, quantale, Girard quantale, cyclic dualizing element.

§1. Preliminaries

Quantales were introduced by C.J. Mulvey in [1] with the purpose of studying the spectrum of $C^*$-algebras and the foundations of quantum mechanics. The study of such partially ordered algebraic structures goes back to a series of papers by Ward and Dilworth [2, 3] in the 1930s. It has become a useful tool in studying noncommutative topology, linear logic and $C^*$-algebra theory [4-6]. Following Mulvey, various types and aspects of quantales have been considered by many researchers [7-9]. The importance of quantales for linear logic is revealed in Yetter’s work [10]. Yetter has clarified the use of quantales in linear logic and he has introduced the term “Girard quantale”. In [11], J. Paseka and D. Kruml have shown that any quantale can be embedded into a unital quantale. In [12], K.I. Rosenthal has proved that every quantale can be embedded into a Girard quantale. Thus, it is important to study Girard quantale. This is the motivation for us to investigate Girard quantale. In the note, we shall study the interior structures of Girard quantale and the cyclic dualizing element in Girard quantales.

We use 1 to denote the top element and 0 the bottom element in a complete lattice. For notions and concepts, but not explained, please to refer to [12].

Definition 1.1. A quantale is a complete lattice $Q$ with an associative binary operation “&” satisfying:

$$a \& (\bigvee b_\alpha) = \bigvee (a \& b_\alpha)$$

and

$$\bigvee (b_\alpha) \& a = \bigvee (b_\alpha \& a)$$

for all $a \in Q$, $\{b_\alpha\} \subseteq Q$.

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An element \( e \in Q \) is called a unit if \( a \& e = e \& a = a \) for all \( a \in Q \). \( Q \) is called unital if \( Q \) has the unit \( e \).

Since \( a \& \_ \) and \( \_ \& a \) preserve arbitrary sups for all \( a \in Q \), they have right adjoints and we shall denote them by \( \longrightarrow_r \) and \( \longrightarrow_l \) respectively.

**Proposition 1.2.** Let \( Q \) be a quantale, \( a, b, c \in Q \). Then

1. \( a \& (a \longrightarrow_r b) \leq b \);
2. \( a \longrightarrow_r (b \longrightarrow_r c) = b \& a \longrightarrow_r c \);

Again, analogous results hold upon replacing \( \longrightarrow_r \) by \( \longrightarrow_l \).

**Definition 1.3.** Let \( Q \) be a quantale. An element \( c \) of \( Q \) is called cyclic, if \( a \longrightarrow_r c = a \longrightarrow_l c \) for all \( a \in Q \). \( d \in Q \) is called a dualizing element, if \( a = (a \longrightarrow_l d) \longrightarrow_r d = (a \longrightarrow_r d) \longrightarrow_l d \) for all \( a \in Q \).

**Definition 1.4.** A quantale \( Q \) is called a Girard quantale if it has a cyclic dualizing element \( d \).

Let \( Q \) be a Girard quantale with cyclic dualizing element \( d \) and \( a, b \in Q \), define the binary operation \( \| \) by \( \| b = (b^+ \& b^-)^\perp \), then we can prove that \( \| \) and \( \| a \) preserve arbitrary infs for all \( a \in Q \), hence they have left adjoints and we shall denote them by \( \longrightarrow_r \) and \( \longrightarrow_l \) respectively. If \( a \longrightarrow_r d = a \longrightarrow_l d \), we shall denote it by \( a \longrightarrow d \), or more frequently by \( a^\perp \) if \( d \) is a cyclic dualizing element.

## §2. The equivalent descriptions for Girard quantale

In this section, we shall study the interior structures of Girard quantale and give some equivalent descriptions for Girard quantale. According to the above, we know that there are six binary operations on a Girard quantale such as \( \& \longrightarrow_r \longrightarrow_l \| \longrightarrow_r \longrightarrow_l \) respectively. If \( a \longrightarrow_r c = a \longrightarrow_l c \) for all \( a \in Q \), \( d \in Q \) is called a dualizing element, if \( a = (a \longrightarrow_l d) \longrightarrow_r d = (a \longrightarrow_r d) \longrightarrow_l d \) for all \( a \in Q \).

**Theorem 2.1.** Let \( Q \) be a unital quantale, \( ^\perp : Q \longrightarrow Q \) an unary operation on \( Q \). Then \( Q \) is a Girard quantale if and only if

1. \( a \longrightarrow_l b = (a \& b^\perp)^\perp \);
2. \( a \longrightarrow_r b = (b^\perp \& a)^\perp \).

**Proof.** The necessity is obvious. Sufficiency: suppose (1) and (2) hold, \( a \in Q \). Denote the unit element by \( e \) on \( Q \), then \( a = e \longrightarrow_l a = (e \& a^\perp)^\perp = (a^\perp)^\perp \). Thus \( a = a^\perp \), hence

\[
a \longrightarrow_l e^\perp = (a \& e^\perp)^\perp = (a \& e)^\perp = a^\perp.
\]

Similarly we get \( a^\perp = a \longrightarrow_r e^\perp \). Take \( d = e^\perp \), thus \( d \) is a cyclic element of \( Q \). Again \( \forall a \in Q, (a \longrightarrow e^\perp) \longrightarrow e^\perp = a^\perp \longrightarrow e^\perp = (a^\perp)^\perp = a \). This proves \( d = e^\perp \) is a dualizing element on \( Q \). Thus the proof is completed.

**Theorem 2.2.** Let \( Q \) be a complete lattice. \( \longrightarrow_r : Q \times Q \longrightarrow Q \) is a binary operation on \( Q \), \( a \longrightarrow_r : Q \longrightarrow Q \) and \( \longrightarrow_r : a : Q \longrightarrow Q^{op} \) preserve arbitrary sups for all \( a \in Q \).

\( ^\perp : Q \longrightarrow Q \) is a unary operation on \( Q \), \( e \in Q \). For all \( a, b, c \in Q \),

1. \( e \longrightarrow_r a = a; \ a \longrightarrow_r e^\perp = a^\perp \);
2. \( (a^\perp)^\perp = a; \ a \leq b \Longrightarrow b^\perp \leq a^\perp \);
3. \( (a \longrightarrow_r b^\perp)^\perp \longrightarrow_r c = a \longrightarrow_r (b \longrightarrow_r e) \);
Then $Q$ is a Girard quantale and $\rightarrow_r$ is the right implication operation.

**Proof.** Define the binary operation $a \& b = (b \rightarrow_r a^\perp)^\perp$ for all $a, b \in Q$. & satisfies associative law: In fact, for all $a, b, c \in Q$,

$$(a \& b) \& c = (b \rightarrow_r a^\perp)^\perp \& c = (c \rightarrow_r (b \rightarrow_r a^\perp))^\perp.$$  

Using the condition (3), we have $(c \rightarrow_r (b \rightarrow_r a^\perp))^\perp = ((c \rightarrow_r b^\perp) \rightarrow_r a^\perp)^\perp$. By the definition of the binary operation & we can get

$$a \& (b \& c) = a \& (c \rightarrow_r b^\perp) = ((c \rightarrow_r b^\perp) \rightarrow_r a^\perp)^\perp.$$  

So $(a \& b) \& c = a \& (b \& c)$.

Using the condition (4), we have $a \& b \leq c \iff (b \rightarrow_r a^\perp)^\perp \leq c \iff c^\perp \leq b \rightarrow_r a^\perp \iff b \leq a \rightarrow_r c$ for all $a, b, c \in Q$.

For any $a \in Q$, $(b_i)_{i \in I} \subseteq Q$. If $I = \emptyset$, then $a \& 0 = (a \rightarrow_r 0^\perp)^\perp = (a \rightarrow_r 1)^\perp$, since again $(a \rightarrow_r 1)^\perp \leq 0 \iff 1 \leq a \rightarrow_r 1 \iff a \& 1 \leq 1$, the last inequality obviously holds. So $a \& 0 = 0$. Thus $a \&$ preserves empty-supers. If $I \neq \emptyset$, then

$$a \& (\bigvee_{i \in I} b_i) = (\bigvee_{i \in I} b_i) \rightarrow_r a^\perp$$
$$= (\bigwedge_{i \in I} (b_i \rightarrow_r a^\perp))^\perp$$
$$= \bigvee_{i \in I} (b_i \rightarrow_r a^\perp)^\perp$$
$$= \bigvee_{i \in I} (a \& b_i).$$

Hence $a \&$ preserves arbitrary sups for all $a \in Q$. Similarly, we can prove $\& a$ preserves arbitrary sups for all $a \in Q$. Thus $(Q, \&)$ is a quantale.

In accordance with the condition (1), we know $e$ is the unit element corresponding to $\&$ on $Q$ and $a \rightarrow_r e^\perp = a^\perp$. Denote by $a \rightarrow_r$ the right adjoint of $\& a$. Then

$$a \rightarrow_r e^\perp = \bigvee\{x \in Q | x \leq a \rightarrow_r e^\perp\}$$
$$= \bigvee\{x \in Q | x \& a \leq e^\perp\}$$
$$= \bigvee\{x \in Q | (a \rightarrow_r x^\perp)^\perp \leq e^\perp\}$$
$$= \bigvee\{x \in Q | e \leq a \rightarrow_r x^\perp\}$$
$$= \bigvee\{x \in Q | a \& e \leq x^\perp\}$$
$$= \bigvee\{x \in Q | a \leq x^\perp\}$$
$$= \bigvee\{x \in Q | x \leq a^\perp\}$$
$$= a^\perp.$$  

This show $e^\perp$ is a cyclic element in $Q$. Using conditions (1) and (2) we know $e^\perp$ is also a dualizing element on $Q$. Hence $(Q, \&, \perp)$ is a Girard quantale. We can easily prove $\perp \rightarrow_r \perp$ is the right implication operation on $Q$ by the above consideration.

**Theorem 2.3.** Let $Q$ be a complete lattice. $\rightarrow_r : Q \times Q \rightarrow Q$ is a binary operation in $Q$, $a \mapsto_r : Q \rightarrow Q$ and $\rightarrow_r a : Q^{op} \rightarrow Q$ preserve arbitrary sups for all $a \in Q$. $\perp : Q \rightarrow Q$ is an unary operation in $Q, d \in Q$. For all $a, b, c \in Q$,
(1) \(d \dashv \rightarrow_r a = a; \ a \dashv \rightarrow_r d_\perp = a_\perp;\)

(2) \((a_\perp)_\perp = a; \ a \leq b \Rightarrow b_\perp \leq a_\perp;\)

(3) \((a \dashv \rightarrow_r b)_\perp \dashv \rightarrow_r c = a \dashv \rightarrow_r (b_\perp \dashv \rightarrow_r c);\)

(4) \(c \geq a \dashv \rightarrow_r b \iff b_\perp \geq c \dashv \rightarrow_r a_\perp.\)

Then \(Q\) is a Girard quantale and \(\_ \dashv \rightarrow_r \_\) is the dual right implication operation.

**Proof.** Define binary operation \& is associative: Since \(\forall a, b, c \in Q,\)

(i) The binary operation \& is associative: Since \(\forall a, b, c \in Q,\)

\[\begin{align*}
(a \& b) \& c &= (b_\perp \dashv \rightarrow_r a) \& c \\
&= c_\perp \dashv \rightarrow_r (b_\perp \dashv \rightarrow_r a) \\
&= (c_\perp \dashv \rightarrow_r b)_\perp \dashv \rightarrow_r a \\
&= a \& (c_\perp \dashv \rightarrow_r b) \\
&= a \& (b \& c).
\end{align*}\]

(ii) Using the condition (2), we can prove

\[(\bigvee_{i \in I} a_i)_\perp = \bigwedge_{i \in I} (a_i)_\perp; \quad (\bigwedge_{i \in I} a_i)_\perp = \bigvee_{i \in I} (a_i)_\perp\]

for any set \(I\) and \(\{a_i\}_{i \in I} \subseteq Q.\)

(iii) For all \(a \in Q; \{b_i\}_{i \in I} \subseteq Q,\) we have

\[a \& (\bigvee_{i \in I} b_i) = (\bigvee_{i \in I} b_i)_\perp \dashv \rightarrow_r a = \bigwedge_{i \in I} (b_i)_\perp \dashv \rightarrow_r a = \bigvee_{i \in I} (b_i)_\perp \dashv \rightarrow_r a = \bigvee_{i \in I} (a \& b_i).\]

Similarly we have \((\bigvee_{i \in I} b_i) \& a = \bigvee_{i \in I} (b_i \& a).\) Hence \((Q, \&)\) is a quantale. Since \(a \& d_\perp = (d_\perp)_\perp \dashv \rightarrow_r a = d \dashv \rightarrow_r a = a; \ d_\perp \& b = b_\perp \dashv \rightarrow_r d_\perp = b,\) thus \((Q, \&)\) is a unit quantale with unit element \(d_\perp.\)

(iv) If \(a \in Q,\) we have

\[a \dashv \rightarrow_l d = \bigvee \{x \in Q | x \leq a \dashv \rightarrow_l d\} \]

\[= \bigvee \{x \in Q | x \& a \leq d\} \]

\[= \bigvee \{x \in Q | a_\perp \dashv \rightarrow_r x \leq d\} \]

\[= \bigvee \{x \in Q | d \dashv \rightarrow_r a \leq x_\perp\} \]

\[= \bigvee \{x \in Q | x \leq a_\perp\} \]

\[= a_\perp.\]

Similarly, \(a \dashv \rightarrow_r d = a_\perp,\) hence \(d\) is a cyclic element in \(Q.\) \(d\) is also a dual element in \(Q\) by condition (2). Thus \((Q, \&)\) is a Girard quantale with cyclic dual element \(d.\) We easily know \(\_ \dashv \rightarrow_r \_\) is the dual right implication operation in \(Q\) by the definition of \&.

Obviously, Theorem 2.2 and Theorem 2.3 also hold if \(\dashv \rightarrow_r \) and \(\dashv \rightarrow_r \) are substituted by \(\dashv \rightarrow_l \) and \(\dashv \rightarrow_l \) respectively, “right” and “left” replace each other.

**Theorem 2.4.** Let \(Q\) be a unital quantale with a unary operation \(^\perp\) satisfying the condition

\[\text{CN : } (a_\perp)_\perp = a \quad \text{and} \quad a \dashv \rightarrow_r b = b_\perp \dashv \rightarrow_l a_\perp\]

for all \(a, b \in Q.\) Then \(Q\) is a Girard quantale.
§3. The cyclic dualizing element of Girard quantale

According to the definition of Girard quantale, we know that the cyclic dualizing element plays an important role in Girard quantale, so we shall discuss the cyclic dualizing element in this section. We shall account for whether the cyclic dualizing element is unique in a Girard quantale; when it is unique; whether these Girard quantales determined by different cyclic dualizing elements are different. Let us see the following example

**Example 3.1.** Let $Q = \{0, a, b, c, 1\}$, the partial order on $Q$ be defined as Fig 1, the operator $\&$ on $Q$ be defined by Table 1. Then we can prove that $Q$ is a commutative Girard quantale. And we can prove that $a$, $b$ and $c$ are cyclic dualizing elements of $Q$.

Let $Q = \{0, a, b, c, 1\}$, the partial order on $Q$ be defined as Fig 1, the operator $\&$ on $Q$ be defined by Table 1. Then we can prove that $Q$ is a commutative Girard quantale. And we can prove that $a$, $b$ and $c$ are cyclic dualizing elements of $Q$.

**Proposition 3.2.** Let $Q$ be a unital quantale with the unit element $e$. $\perp_1$ and $\perp_2$ satisfy the condition CN in Theorem 2.4. Then $e^{\perp_1} = e^{\perp_2}$ if and only if $\perp_1 = \perp_2$.

**Proposition 3.3.** Let $Q$ be a quantale, $d_1, d_2$ are cyclic dualizing elements of $Q$, $\perp_1, \perp_2$ are unary operations on $Q$ induced by $d_1, d_2$ respectively. Then $d_1 = d_2$ if and only if $\perp_1 = \perp_2$.

**Theorem 3.4.** Let $Q$ be a Girard quantale. Then there is a one-to-one correspondence between the set of cyclic dualizing elements in $Q$ and the set of unary operations satisfying the condition CN in Theorem 2.4.

**Proposition 3.5.** Let $Q$ be a Girard quantale. If 0 is a cyclic dualizing element of $Q$, then $Q$ is strictly two-sided.

**Proof.** Assume 0 is a cyclic dualizing element in $Q$. Then $0^\perp = 0 \rightarrow 0 = 1$ is the unit of $Q$, hence $\forall a \in Q, a \& 1 = 1 \& a = a$, this finished the proof.

**Proposition 3.6.** If $Q$ is a two-sided Girard quantale, then the unique cyclic dualizing element is the least element 0.

**Proof.** If $Q$ is a two-sided Girard quantale, then we have $a = a \& c \leq a \& 1 \leq a$ for all $a \in Q$. Similarly, we have $1 \& a = a$. Thus $Q$ is strictly two-sided. Suppose $d$ is a cyclic dual element in $Q$, $^d$ is the unary operation induced by $d$, then we have $d = 1 \rightarrow d = 1^d = 0$. the proof is finished.

**Corollary 3.7.** Let $Q$ be a Girard quantale with cyclic dualizing element 0. Then the cyclic dualizing element of $Q$ is unique.

**Theorem 3.8.** Any complete lattice implication algebra is a Girard quantale with unique cyclic dualizing element 0.

According the above conclusions, we have a question: Whether the cyclic dualizing element must be the least element 0 if a Girard quantale has an unique cyclic dualizing element. The
answer is negative. Let us see the following example.

**Example 3.9.** Let $Q = \{0, e, 1\}$, the partial order on $Q$ be defined by $0 < e < 1$, the binary operation $\&$ be defined by Table 2

<table>
<thead>
<tr>
<th>$&amp;$</th>
<th>0</th>
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<td>1</td>
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</table>

Table 2

It is immediate to verify $Q$ being a Girard quantale with the unique cyclic dualizing element $e$.

**Question 3.10.** What is the necessary condition when the cyclic dualizing element of Girard quantale is unique?

**References**

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On an equation involving the Smarandache function and the Dirichlet divisor function

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Abstract For any positive integer \( n \), the famous F.Smarandache function \( S(n) \) is defined as the smallest positive integer \( m \) such that \( n \mid m! \). That is to say, \( S(n) = \min\{m : n \mid m!, n \in N\} \). \( d(n) \) denotes the Dirichlet divisor function. The main purpose of this paper is using the elementary methods to study the solvability of the equation \( S(n) = d(n) \), and give its all positive integer solutions.

Keywords The Smarandache function, Dirichlet divisor function, equation, positive integer-solution.

§1. Introduction and result

For any positive integer \( n \), the famous F.Smarandache function \( S(n) \) is defined as the smallest positive integer \( m \) such that \( n \mid m! \). That is to say, \( S(n) = \min\{m : n \mid m!, n \in N\} \). From this definition we know that if \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \) be the prime powers factorization of \( n \), then \( S(n) = \max_{1 \leq i \leq r} \{S(p_i^{\alpha_i})\} \). About the properties of \( S(n) \), many people had studied it, and obtained a series results, see references [4], [5] and [6]. On the other hand, for any positive integer \( n \), the famous Dirichlet divisor function \( d(n) \) is defined as the number of all distinct divisors of \( n \). If \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \) be the prime power factorization of \( n \), then \( d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1) \). About function \( d(n) \), some scholars had studied it, and obtained some deeply conclusions, see references [7] and [8].

In this paper, we shall use the elementary methods to study the solvability of the equation \( S(n) = d(n) \), and prove that this equation has infinite positive integer solutions. That is, we shall prove the following main conclusion:

Theorem. For any positive integer \( n \), the equation

\[ S(n) = d(n) \]

holds if and only if \( n = 2^{2^n-1} \), \( n = 0, 1, 2 \cdots \), and \( n = p^\alpha \cdot m \), where \( m > 1 \) and \( m \mid [(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1)]! \), if \( \alpha \neq 1 \), \( p \mid \alpha + 1, 1 < s < \log \frac{n}{\log 2} \).

§2. Some preliminary lemmas

In this section, we shall give several simple Lemmas which are necessary in the proof of our theorem. They are stated as follows:
Lemma 1. For any prime \( p \), we have the following inequality:
\[
(p - 1) \cdot \alpha + 1 \leq S(p^n) \leq (p - 1) \cdot [\alpha + 1 + \log p^\alpha] + 1.
\]

Proof. (See reference [3]).

Lemma 2. Let \( a \) and \( b \) be two positive integers, \( S = \{1, 2, 3, \ldots, a\} \). Then the number of all positive integers in \( S \) which can be divided by \( b \) is \( \left\lfloor \frac{a}{b} \right\rfloor \), where \( \lfloor x \rfloor \) denotes the largest integer less than or equal to \( x \).

Proof. (See reference [1]).

Lemma 3. For any prime \( p \), we have:
\[
S(p^k) = k \left( \phi(p^\alpha) + \frac{1}{k} \right) p.
\]
where \( 1 \leq k \leq p \), and \( \phi(n) \) denotes the Euler function.

Proof. (See reference [3]).

§3. Proof of the theorem

In this section, we shall complete the proof of our theorem in five cases:

1. If \( n = 1 \), then \( S(1) = 1 \), \( d(1) = 1 \), so that \( S(1) = d(1) \).
2. If \( n = 2 \), then we can get \( S(2) = 2 \), \( d(2) = 2 \), so that \( S(2) = d(2) \).
3. If \( n = p \), where \( p > 2 \), then we have \( S(p) = p \), \( d(p) = 2 \), so \( S(p) > d(p) \).
4. If \( n = p^\alpha \), and \( n \neq 2 \), when \( \alpha \leq p \), \( S(p^n) = \alpha p \), \( d(p^n) = \alpha + 1 \), so that \( S(p^n) > d(p^n) \); when \( n = p^\alpha \), \( \alpha > p \), we discuss it in two ways:
   a. If \( p \geq 3 \) and \( \alpha > 3 \), then from Lemma 1 we can easily get the inequality \( S(p^n) \geq (p - 1) \cdot \alpha + 1 > \alpha + 1 = d(p^n) \). That is, \( S(p^n) > d(p^n) \).
   b. If \( p = 2 \) and \( \alpha \geq 2 \). It is obvious that \( S(2^n) \) is an even number, and \( d(2^n) = \alpha + 1 \), that is to say, equality sign holds if and only if \( \alpha \) is an odd number. when \( \alpha = 2^n - 1 \), then from the definition of \( d(n) \) we may immediately get \( d(2^{2^n - 1}) = 2^n \). But we also have \( S(2^{2^n - 1}) = 2^n \).

In fact from Lemma 3 with \( k = 1 \) and \( p = 2 \) we can get
\[
S(2^n) = (\phi(2^n) + 1) \cdot 2 = 2^n + 2.
\]

Since \( S(2^n) = 2^n + 2 \), according to the definition of \( S(n) \) we have \( 2^{2^n} \mid (2^n + 2)! \), so that \( 2^{2^n - 1} \mid 2^n!(2^n - 1)! \), because \( 2^n - 1 \) is an odd number, we can deduce that \( 2^{2^n - 1} \mid 2^n! \) and \( 2^n \mid 2^n! \). So \( 2^{2^n - 1} \mid (2^n - 1)! \cdot 2^{2^n - 1} \), or \( 2^{2^n - 1} \mid (2^n - 2)! \). That is to say, \( S(2^{2^n - 1}) = 2^n = d(n) \).

When \( \alpha = k \cdot 2^n - 1 \), \( k \geq 3 \), and \( k \) is an odd number, we have
\[
S(2^{k \cdot 2^n - 1}) - d(2^{k \cdot 2^n - 1}) = 2.
\]

In fact from Lemma 1 we can get the inequality
\[
S(2^{k \cdot 2^n - 1}) \geq k \cdot 2^n - 1 + 1 = k \cdot 2^n.
\]

Especially the equality holds if and only if \( k = 1 \).
If \( k \geq 3 \) is an odd number, then we have
\[
\sum_{j=1}^{\infty} \left\lfloor \frac{k \cdot 2^n}{2^j} \right\rfloor = k \cdot 2^{n-1} + k \cdot 2^{n-2} + \ldots + \sum_{j=1}^{\infty} \left\lfloor \frac{k}{2^j} \right\rfloor
\]
\[
= k \cdot 2^n - k + \sum_{j=1}^{\infty} \left\lfloor \frac{k}{2^j} \right\rfloor.
\]
(1)

It is obvious that
\[
k \cdot 2^n - k + \sum_{j=1}^{\infty} \left\lfloor \frac{k}{2^j} \right\rfloor - k \cdot 2^n + 1 < 0.
\]

So \( S(2^{k-2^n-1}) > k \cdot 2^n \). That is to say, \( 2^{k-2^n-1} \mid k \cdot 2^n. \)

We also have
\[
\sum_{j=1}^{\infty} \left\lfloor \frac{k \cdot 2^n + 2}{2^j} \right\rfloor = k(2^{n-1} + 2^{n-2} + \ldots + 1) + \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}}\right) + \sum_{j=n+1}^{\infty} \left\lfloor \frac{k \cdot 2^n + 2}{2^j} \right\rfloor
\]
\[
= k \cdot 2^n - k + 2 - 2 \cdot \left(\frac{1}{2}\right)^n + \sum_{j=n+1}^{\infty} \left\lfloor \frac{k \cdot 2^n + 2}{2^j} \right\rfloor.
\]
(2)

It is clear that
\[
k \cdot 2^n - k + 2 - 2 \cdot \left(\frac{1}{2}\right)^n + \sum_{j=n+1}^{\infty} \left\lfloor \frac{k \cdot 2^n + 2}{2^j} \right\rfloor - k \cdot 2^n + 1 \geq 0.
\]

That means \( 2^{k-2^n-1} \mid (k \cdot 2^n + 2) \) and \( S(2^{k-2^n-1}) = k \cdot 2^n + 2 = d(n) + 2. \)

(5). If \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \), then we have
\[
d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1).
\]
\[
S(n) = \max_{1 \leq i \leq r} \{ S(p_i^{\alpha_i}) \} = S(p^\alpha).
\]

Since we have the inequality
\[
(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1) - \alpha p \geq 2^{s-1}(\alpha + 1) - \alpha p
\]
and \( S(p^\alpha) \leq \alpha p \), so if \( 2^{s-1}(\alpha + 1) - \alpha p > 0 \), we can get \( s > \log 2^{\frac{\alpha p}{\alpha p - 1}}. \) In this case \( S(n) \neq d(n). \)

If \( 1 < s \leq \log 2^{\frac{\alpha p}{\alpha p - 1}} \), we suppose that
\[
S(n) = d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1).
\]

We can write \( n = p^\alpha \cdot m. \) \( (p^\alpha, m) = 1. \) From the definition of \( S(n) \) we have \( n \mid (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1)! \). So \( p^\alpha \cdot m \mid (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1)! \).

Now we discuss it in following three cases:

(i). If \( \alpha = 1 \), then \( n = p \cdot m, \) \( p \neq 2, \) then, \( S(n) = p, d(n) = 2d(m), \) so \( S(n) \neq d(n). \)

(ii). If \( 1 < \alpha \leq p, p \neq 2, \) then \( n = p^\alpha \cdot m, \) so that \( S(n) = \alpha \cdot p = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1). \)

If \( p \mid \alpha + 1, \) then \( (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1) \geq (\alpha + 1) \cdot p > \alpha \cdot p, \) so \( S(n) \neq d(n). \) If \( p \mid (\alpha + 1, \) we can deduce \( \alpha = p - 1, \) so \( n = p^{p-1} \cdot m, \) therefore, \( m \mid \frac{((p-1) \cdot p)!}{p^{p-1}}. \)
(iii). If $\alpha > p$, then $n = p^\alpha \cdot m$, so that $S(n) = t \cdot p = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1) < \alpha \cdot p$. If $p \nmid \alpha + 1$, then $(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1) \geq (\alpha + 1) \cdot p > \alpha \cdot p$, so $S(n) \neq d(n)$. If $p \mid \alpha + 1$, we have $n = p^\alpha \cdot m$, therefore, $m \mid \frac{[(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1)]!}{p^\alpha}$.

This completes the proof of our theorem.

References

An equation involving the Euler function and the Smarandache $m$-th power residues function

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Abstract For any positive integer $n$, let $\phi(n)$ be the Euler function, and $a_m(n)$ denotes the Smarandache $m$-th power residues function of $n$. The main purpose of this paper is using the elementary method to study the solvability of the equation $\phi(\phi(n)) = a_m(n)$, and give its all positive integer solutions.

Keywords Smarandache $m$-th power residues function, Euler function, equation, positive integer solution.

§1. Introduction and result

Let $m$ be a fixed positive integer with $m \geq 2$. For any positive integer $n$, the Smarandache $m$-th power residues function $a_m(n)$ is defined as $a_m(1) = 1$, if $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of $n$ into prime powers, then $a_m(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, where $\beta_i = \min\{m - 1, \alpha_i\}$, $i = 1, 2, \cdots, k$. This function was introduced by professor F.Smarandache in his book “Only problems, not solutions ”, where he also asked us to study the properties of $a_m(n)$. About this problem and related contents, many authors had studied it, and obtained a series interesting results, see references [1], [2], [3] and [4]. For example, Zhang Tianping [4] studied the value distribution problem of $a_3(n)b_k(n)$, and proved the following conclusion:

$$\sum_{n \leq x} a_3(n)b_k(n) = \frac{6x^{k+1}}{(k+1)^2} \cdot R(k+1) + O\left(x^{\frac{k+\frac{2}{7}+\epsilon}}\right),$$

where $k \geq 2$ be a positive integer, $\epsilon$ denotes any fixed positive number, $b_k(n)$ is the Smarandache $k$-power complement function,

$$R(k+1) = \prod_p \left(1 + \frac{p^3 + p}{p^2 + p^k - p - 1}\right), \ k = 2$$

and

$$R(k+1) = \prod_p \left(1 + \sum_{j=2}^{k} \frac{p^{k-j+3}}{(p+1)p^{(k+1)j}} + \sum_{j=1}^{k} \frac{p^{k-j+3}}{(p+1)(p^{(k+1)(k+j)} - p^{(k+1)j})}\right), \ k > 2.$$
In this paper, we shall study the solvability of the equation

$$\phi(\phi(n)) = a_m(n),$$

(1)

where $\phi(n)$ is the famous Euler function. About this content, Zhang Wenpeng [5] studied the solvability of the equation $\phi(n) = a_m(n)$, and obtained its all positive integer solution. But for equation (1), it seems that none had studied it yet, at least we have not seen any related papers before. In this paper, we use the elementary method to obtain all positive integer solutions of the equation (1). That is, we shall prove the following conclusion:

**Theorem.** Let $m$ be a fixed positive integer with $m \geq 2$. Then the equation (1) have $m + 1$ positive integer solutions, they are:

$$n = 1, 2^{m+1}, 2^\alpha \cdot 3^{m+1}, \alpha = 1, 2, 3, \ldots, m - 1.$$

It is clear that using the method of proving our Theorem we can also obtain all positive integer solutions of the equation $\phi(\phi(n))) = a_m(n)$.

§2. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of our Theorem directly. About the properties of the Euler function, it can be found in references [8] and [9].

It is clear that $n = 1$ is a solution of (1). Now we assume that $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denote the factorization of $n$ into prime powers, then from the definition of $\phi(n)$ and $a_m(n)$ we have

$$a_m(n) = p_1^{\beta_1}p_2^{\beta_2} \cdots p_k^{\beta_k}, \beta_i = \min\{m - 1, \alpha_i\}, i = 1, 2, \ldots, k$$

(2)

and

$$\phi(n) = p_1^{\alpha_1 - 1}(p_1 - 1)p_2^{\alpha_2 - 1}(p_2 - 1)\cdots p_k^{\alpha_k - 1}(p_k - 1).$$

(3)

Now we discuss the problem in following four cases:

(A) If $\exists \alpha_i > m + 1$, then $p_i^{m+1} \mid \phi(n)$, and $p_i^m \mid \phi(\phi(n))$. But from (2) we know that all $\beta_i \leq m - 1$, so $\phi(\phi(n)) \neq a_m(n)$. That is, the equation (1) does not hold.

(B) If $\alpha_1 = \alpha_2 = \cdots = \alpha_k = m + 1$, then from (2) and (3) we have

$$a_m(n) = p_1^{m-1}p_2^{m-1}p_3^{m-1} \cdots p_k^{m-1}$$

and

$$\phi(n) = p_1^m(p_1 - 1)p_2^m(p_2 - 1)p_3^m(p_3 - 1)\cdots p_k^m(p_k - 1).$$

If $p_i$ divide one of $(p_{i+j} - 1), \text{ where } (i = 1, 2, \ldots, k - 1; j = 1, 2, \ldots, k-i)$, then $p_i^m \mid \phi(\phi(n))$, so that $\phi(\phi(n)) \neq a_m(n)$. Note that if $\phi(\phi(n)) = a_m(n)$, then $\phi(\phi(n))$ and $a_m(n)$ have the same prime divisors. So in this case, the equation (1) holds only if $n = 2^{m+1}$.

(C) If $\max\{\alpha_i\} < m + 1$, then $p_i^{m-1} \mid \phi(n)$ and $p_i^m \mid \phi(\phi(n))$, but $k - 2 < m - 1$ and $\beta_k = \min\{m - 1, \alpha_k\} > \alpha_k - 2$. So in this case, the equation (1) has no solution.
(D) If \( \exists \ i, \ j \) such that \( \alpha_i = m + 1, \ \alpha_j < m + 1 \). In this time, if \( n \) satisfy the equation (1), then \( n \) must be an even number, and \( p_1 = 2 \). In this case, we can easily deduce that \( k \leq 2 \). In fact if \( k \geq 3 \), then note that \( p_1 = 2, 2 \mid p_2 - 1, 4 \mid p_3 - 1 \) or \( 2 \mid \phi(p_3 - 1) \), from \( \phi(n) = p_1^{\alpha_1 - 1}(p_1 - 1)p_2^{\alpha_2 - 1}(p_2 - 1) \cdots p_k^{\alpha_k - 1}(p_k - 1) \) we may immediately deduce that \( 2^{\alpha + 1} \mid \phi(n) \), but \( 2^{\alpha + 1} \nmid a_\alpha(n) \), so \( \phi(\phi(n)) \neq a_\alpha(n) \). So if \( n \) satisfy the equation (1), then \( k \leq 2 \). Since \( \exists \ i, \ j \) such that \( \alpha_i = m + 1, \ \alpha_j < m + 1 \). So we have \( k = 2 \) and \( n = 2^\alpha p^{m + 1} \) or \( n = 2^m p^\alpha \), where \( \alpha < m + 1 \). If \( n = 2^\alpha p^{m + 1} \), then \( \phi(n) = 2\alpha \cdot \frac{p - 1}{2} \cdot p^m \) and \( \phi(\phi(n)) = \phi\left(2\alpha \cdot \frac{p - 1}{2}\right) \cdot p^{m - 1} \cdot (p - 1) \). But \( a_\alpha(n) = 2\beta p^{m - 1} \), where \( \beta = \min\{m - 1, \ \alpha\} \). So \( \phi(\phi(n)) = a_\alpha(n) \) if and only if \( p - 1 = 2, \) or \( p = 3, \ \alpha \leq m - 1 \). That is to say, \( n = 2^\alpha \cdot 3^{m + 1} \) satisfy the equation (1) for all \( \alpha \leq m - 1 \). If \( n = 2^m p^{\alpha} \), then \( 2^{m + 1} \) divide \( \phi(\phi(n)) \), but \( 2^{m + 1} \nmid a_\alpha(n) \), so this time, \( n \) does not satisfy the equation (1).

Combining (A), (B), (C) and (D) we may immediately deduce that all positive integer solutions of the equation (1) are \( n = 1, \ 2^{m + 1}, \ 2^\alpha \cdot 3^{m + 1} \), where \( \alpha = 1, \ 2, \ \cdots, \ m - 1 \). This completes the proof of Theorem.

References

On numerical values of \( Z(n) \)

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Abstract  The pseudo Smarandache function, \( Z(n) \), is defined as the minimum positive integer \( m \) such that \( 1 + 2 + \cdots + m \) is divisible by \( n \). The main purpose of this paper is to reproduce the values of \( Z(n) \) for \( n = 1 \) (1)5000.

Keywords  Pseudo Smarandache function, unsolved problems.

§1. Introduction

The pseudo Smarandache function, denoted by \( Z(n) \), has been defined by Kashihara [1] as follows.

**Definition 1.1.** For any integer \( n \geq 1 \), the pseudo Smarandache function, \( Z(n) \) is the minimum integer \( m \) such that \( 1 + 2 + \cdots + m \) is divisible by \( n \), that is

\[
Z(n) = \min \left\{ m : m \geq 1, n \mid \frac{m(m+1)}{2} \right\}, n \geq 1.
\]

Soon after introduction, the pseudo Smarandache function has attracted the attention of many researchers. However, till date, only little is known about the function. The following lemma, due to Kashihara [1], Ibstedt [2] and Ashbacher [3], summarizes the properties of \( Z(n) \) known so far.

**Lemma 1.1.** For any integer \( n \geq 1 \),

1. \( 1 \leq Z(n) \leq 2n - 1 \),
2. \( Z(n) \leq n - 1 \) for any odd integer \( n \geq 3 \),

with

(a) \( Z(n) = 1 \) if and only if \( n = 1 \),
(b) \( Z(n) = 2 \) if and only if \( n = 3 \),
(c)  \( Z(n) \geq \max \{ Z(d) : d \mid n \} \),
(d)  \( Z(n) \geq \frac{1}{2}(\sqrt{1 + 8n} - 1) \).

As for the expressions for \( Z(n) \), we have the following results, due to Kashihara [1] and Ashbacher [3].

**Lemma 1.2.** For any integer \( k \geq 1 \),

1. \( Z\left(\frac{k(k+1)}{2}\right) = k \),
2. \( Z(2^k) = 2^{k+1} - 1 \),
On numerical values of $Z(n)$

(3) $Z(p^k) = p^k - 1$.

In addition to the above expressions, we have explicit expressions for $Z(3 \cdot 2^k)$ and $Z(5 \cdot 2^k)$, due to Ashbacher [2], and $Z(7 \cdot 2^k)$, due to Ibstedt [4]. Majumdar [5] gives explicit expressions for $Z(4p)$, $Z(5p)$, $Z(6p)$, $Z(7p)$, $Z(11p)$ and $Z(pq)$.

On the other hand, questions and conjectures involving $Z(n)$ are quite plenty, some of which have been settled but many still remain unsolved. Some of the unsolved problems are given below.

Problem 1. Given the integers $k$ and $m$, find all integers $n$ such that $Z^k(n) = m$, (where $Z^k(\cdot)$ denotes the $k$-fold composition of $z$ with itself).

Problem 2. Find relationships between each of $Z(m + n)$ and $Z(mn)$ with $Z(m)$ and $Z(n)$.

Problem 4. Are there solutions to $(1) Z(n + 2) = Z(n + 1) + Z(n)$?

(2) $Z(n + 2) = Z(n + 1) + Z(n) + 2$?

(3) $Z(n + 2) = Z(n + 1) + Z(n)$?

(4) $Z(n) = Z(n + 1) + Z(n + 2)$?

(5) $Z(n + 2) = Z(n + 1) + 2Z(n + 1)$?

(6) $Z(n + 2)Z(n) = [Z(n + 1)]^2$?

Problem 5. For a given $m$, how many $n$ are there such that $Z(n) = m$? Moreover, if $Z(n) = m$ has only one solution, what are the conditions on $m$?

Problem 6. Are there infinitely many instances of 3 consecutive increasing or decreasing terms in the sequence $\{Z(n)\}_{n=1}^{\infty}$?

Problem 7. Find all solutions of $(1) Z(n) = S(n)$, (2) $Z(n) + 1 = S(n)$.

Recently, Wengpeng Zhang and Ling Li [7] have made the following conjecture:

Conjecture 1. For any integer $n \geq 1$, the equation $Z(n) + S_{c}(n) = 2n$ has the only solutions $n = 1, 3^k, p^{2m-1}$, where $k \geq 2$ is an integer such that $3^k$ and $3^k + 2$ are twin primes, and $p \geq 5$ is a prime and $m \geq 1$ is an integer such that $p^{2m-1}$ and $p^{2m-1} + 2$ are twin primes.

Recall that $S_{c}(n)$ is the Smarandache reciprocal function, $S_{c}(n)$, is defined as follows: $S_{c}(n) = x$ if and only if $x + 1$ is the smallest prime greater than $n$.

To address such problems, it seems that the existing expressions of $Z(n)$ are not sufficient. With this in mind, we give a table of values of $Z(n)$ for $n = 1(1)5000$. It may be mentioned here that, Ibstedt [6] provides a table of values of $S(n)$ over the same range, and our table would supplement that.

§2. Numerical values of $Z(n)$

The values of $Z(n)$ have been calculated on a computer, using Definition 1.1. In the tables that follow, we reproduce these values.
Table 1: Values of $Z(n)$ for $n=1(1)273$

| n  | $Z(n)$ | n  | $Z(n)$ | n  | $Z(n)$ | n  | $Z(n)$ | n  | $Z(n)$ | n  | $Z(n)$ | n  | $Z(n)$ | n  | $Z(n)$ | n  | $Z(n)$ | n  | $Z(n)$ | n  | $Z(n)$ | n  | $Z(n)$ | n  | $Z(n)$ | n  | $Z(n)$ |
|-----|--------|-----|--------|-----|--------|-----|--------|-----|--------|-----|--------|-----|--------|-----|--------|-----|--------|-----|--------|-----|--------|-----|--------|-----|--------|-----|--------|-----|--------|
On numerical values of $Z(n)$

**Table 2: Values of $Z(n)$ for $n=274(1)546$**

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§3. Some remarks

The following conjecture has been made by Wengpeng Zhang and Ling Li [7]:

**Conjecture 2.** For any integer \( n \geq 1 \), the equation \( S_c(n) = Z_c(n) + n \) has the only solutions

\[
n = p^{2m-1},
\]

where \( p \geq 5 \) is a prime and \( m \geq 1 \) is an integer such that \( p^{2m-1} \) and \( p^{2m-1} + 2 \) are twin primes.

Though the numerical values given in the tables support Conjecture 1 of Wengpeng Zhang and Ling Li [7], we found several counter-examples to the “only if” part of Conjecture 2. For examples,

\[
S_c(35) = 36 = Z_c(35) + 35, \quad S_c(65) = 66 = Z_c(65) + 65, \quad S_c(77) = 78 = Z_c(77) + 77.
\]

References


Table 19: Values of Z(n) for n=4330(1)4563

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Table 21: Values of $Z(n)$ for $n=4798(1)5000$
m-graphoidal path covers of a graph

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Abstract The concept of graphoidal cover was introduced by B. D. Acharya and E. Sampathkumar and 2-graphoidal path cover of a graph was introduced by K. Nagarajan, A. Nagarajan and S. Somasundaram. In this paper, we define a m-graphoidal path cover of a graph. A m-graphoidal path cover of a graph $G$ is a partition of edge set of $G$ into paths in which every vertex is an internal vertex of at most $m$ paths and the minimum cardinality of it is a m-graphoidal path covering number $\eta_m(G)$. In this paper we initiate a study of the parameter $\eta_m(G)$ and we find $\eta_m(G)$ for some standard graphs. Also we find for which $m$, the parameters $\eta_m(G)$ and the path partition number $\pi(G)$ are the same for wheels, complete graphs and stars.

Keywords Graphoidal cover, 2-graphoidal path cover, m-graphoidal path cover, m-graphoidal path covering number, Path partition number.

§1. Introduction

By a graph, we mean a finite, undirected, non-trivial, connected graph without loops and multiple edges. The order and size of a graph are denoted by $p$ and $q$ respectively. For terms not defined here we refer to Harary [3].

Let $P = (v_1, v_2, \cdots, v_n)$ be a path in a graph $G = (V, E)$. The vertices $v_2, v_3, \cdots, v_n-1$ are called internal vertices of $P$ and $v_1$ and $v_n$ are called external vertices of $P$.

A decomposition of a graph $G$ is a collection of edge-disjoint subgraphs $H_1, H_2, \cdots, H_r$ of $G$ such that every edge of $G$ belongs to exactly one $H_i$.

Definition 1.1. If each $H_i$ is a path, then $\psi$ is called a path partition or path cover of $G$.

The minimum cardinality of a path partition of $G$ is called the path partition number of $G$ and is denoted by $\pi(G)$ and any path partition $\psi$ of $G$ for which $|\psi| = \pi(G)$ is called a minimum path partition or $\pi$-cover of $G$.

The parameter $\pi$ was studied by Harary and Schwenk [4], Stanton et.al., [8] and Arumugam and Suresh Suseela [2]. Path partition numbers of some standard graphs are given in the following theorem.

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Theorem 1.2. [8] 
(a) \( \pi(K_p) = \left\lceil \frac{p}{2} \right\rceil \) 
(b) \( \pi(W_n) = \left\lfloor \frac{n}{2} \right\rfloor \) 
and 
(c) \( \pi(K_{1,n}) = \left\lceil \frac{n}{2} \right\rceil \).

Definition 1.3. [7] A graphoidal cover \( \psi \) of a graph \( G \) is a partition of \( E(G) \) into non-trivial (not necessarily open) paths in \( G \) such that every vertex of \( G \) is an internal vertex of at most one path in \( \psi \).

The minimum cardinality of a graphoidal cover of \( G \) is called the graphoidal covering number of \( G \). C. Pakkiam and S. Arumugam [6] have determined the graphoidal covering number of several families of graphs. B. D. Acharya [1] studied the results on the graphoidal covering number of a graph. Further S. Arumugam and J. Suresh Suseela [2] introduced the concept of acyclic graphoidal cover.

Definition 1.4. [2] An acyclic graphoidal cover of \( G \) is a graphoidal cover \( \psi \) of \( G \) such that every element of \( \psi \) is a path in \( G \). The minimum cardinality of an acyclic graphoidal cover of \( G \) is called the acyclic graphoidal covering number of \( G \) and is denoted by \( \eta_a(G) \).


Definition 1.5. [5] A 2-graphoidal path cover \( \psi \) is a collection of non-trivial paths in \( G \) such that
(i) Every edge is in exactly one path in \( \psi \).
(ii) Every vertex is an internal vertex of at most two paths in \( \psi \).

Let \( g_2 \) denote the collection of all 2-graphoidal path covers of \( G \). Since \( E(G) \) is a 2-graphoidal path cover, we have \( g_2 \neq \phi \). The minimum cardinality of a 2-graphoidal path cover of \( G \) is called the 2-graphoidal path covering number of \( G \) and is denoted by \( \eta_2(G) \).

That is \( \eta_2(G) = \min\{ |\psi| : \psi \in g_2(G) \} \).

§2. Main results

The concept of graphoidal cover was introduced by B. D. Acharya and E. Sampathkumar [7] and K. Nagarajan, A. Nagarajan and S. Somasundaram [5] introduced the concept of 2-graphoidal path cover. It motivates us to define the generalized graphoidal path cover called an m-graphoidal path cover of \( G \), which is defined as follows.

Definition 2.1. An m-graphoidal path cover \( \psi \) is a collection of non-trivial paths in \( G \) such that
(i) Every edge is in exactly one path in \( \psi \).
(ii) Every vertex is an internal vertex of at most \( m \) paths in \( \psi \).

Let \( g_m \) denote the collection of all m-graphoidal path covers of \( G \). Since \( E(G) \) is an m-graphoidal path cover, we have \( g_m \neq \phi \). The minimum cardinality of an m-graphoidal path cover of \( G \) is called the m-graphoidal path covering number of \( G \) and is denoted by \( \eta_m(G) \).

Thus \( \eta_m(G) = \min\{ |\psi| : \psi \in g_m(G) \} \).
Remark 2.2. (1) We also call acyclic graphoidal cover as 1-graphoidal path cover. Hereafter we denote the acyclic graphoidal covering number of \( G \) by \( \eta_1(G) \) or simply \( \eta_1 \) when there is no possibility of confusion.

(2) If \( m = 2 \), then we get a 2-graphoidal path cover [5]. The 2-graphoidal path covering number \( \eta_2 \) was studied in [5].

(3) If \( l \leq m \), then every \( l \)-graphoidal path cover is an \( m \)-graphoidal path cover and consequently, \( \eta_m \leq \eta_l \).

From the definition, we derive the following observations.

Observation 2.3. For any graph \( G \), \( 1 \leq \eta_m(G) \leq q \). Also \( \eta_m(G) = 1 \) if and only if \( G \) is a path and \( \eta_m(G) = q \) if and only if \( G \cong K_2 \).

Observation 2.4. For any graph \( G \), \( \eta_m(G) \geq \Delta - m \).

Observation 2.5. Since there exists a vertex in \( G \) which is internal in at most \( \left\lfloor \frac{\Delta}{2} \right\rfloor \) paths, we have \( m \leq \left\lfloor \frac{\Delta}{2} \right\rfloor \).

Next we give some notations, which will be used to simplify the proofs of the theorems.

Notations 2.6. Let \( \Psi \) be an \( m \)-graphoidal path cover of \( G \).

\( i_{\Psi}(P) \)-number of internal vertices of the path \( P \) in \( \Psi \).

\( t_i(\Psi) \)-number of internal vertices which appear exactly \( i \) times in the paths of \( \Psi \) of \( G \) for \( i = 1, 2, \cdots, m \).

\( t_{\Psi} \)-number of vertices which are not internal in any path of \( \Psi \).

If an \( m \)-graphoidal path cover \( \Psi \) of \( G \) is minimum, then clearly \( t_i(\Psi) \) should be maximum and \( t_{\Psi} \) should be minimum. So we define the following:

\( t_i = \max t_i(\Psi) \) where the maximum is taken from all \( m \)-graphoidal path covers \( \Psi \) of \( G \) for \( i = 1, 2, \cdots, m \) and \( t = \min t_{\Psi} \) where the minimum is taken from all \( m \)-graphoidal path covers \( \Psi \) of \( G \).

The following theorem gives the lower bound for the parameter \( \eta_m \).

Theorem 2.7. Let \( G \) be a \((p, q)\) graph. Then \( q - mp \leq \eta_m(G) \).

Proof. Suppose \( \{P_1, P_2, \cdots, P_k\} \) is a minimum \( m \)-graphoidal path cover of \( G \). Then

\[
|E(G)| = \sum_{j=1}^{k} |E(P_j)| = \sum_{j=1}^{k} (i_{\Psi}(P_j) + 1) = k + \sum_{j=1}^{k} i_{\Psi}(P_j).
\]
Since every vertex of $G$ is an internal vertex of at most $m$ paths, we have

$$\sum_{j=1}^{k} i_{\psi}(P_j) \leq m |V(G)|$$

Thus $|E(G)| \leq k + m |V(G)| = \eta_m(G) + m |V(G)|$

$$|E(G)| - m |V(G)| \leq \eta_m(G)$$

$$q - mp \leq \eta_m(G).$$

Next we present a general result which is useful in determining the value of $\eta_m$ for some standard graphs.

**Theorem 2.8.** For any graph $G$, $\eta_m(G) = q - p - \sum_{k=2}^{m} (k - 1)t_k + t$

**Proof.** For any $\psi \in \mathfrak{g}_m$, we have

$$q = \sum |E(P)| = \sum (i_{\psi}(P) + 1) = \sum i_{\psi}(P) + |\psi| = \sum_{k=1}^{m} kt_k(\psi) + |\psi| = \sum_{k=2}^{m} (k - 1)t_k(\psi) + \sum_{k=1}^{m} t_k(\psi) + |\psi| = \sum_{k=2}^{m} (k - 1)t_k(\psi) + (p - t_\psi) + |\psi|$$

$$|\psi| = q - p - \sum_{k=2}^{m} (k - 1)t_k(\psi) + t_\psi.$$

Thus, $\eta_m(G) = q - p - \sum_{k=2}^{m} (k - 1)t_k + t$.

**Corollary 2.9.** [2] For any $(p, q)$ graph $G$, $\eta_1(G) = q - p + t$.

**Corollary 2.10.** [5] For any $(p, q)$ graph $G$, $\eta_2(G) = q - p - t_2 + t$.

**Corollary 2.11.** For any $(p, q)$ graph $G$, $\eta_m(G) \geq q - p - \sum_{k=2}^{m} (k - 1)t_k$.

**Corollary 2.12.** For any $(p, q)$ graph $G$, the following are equivalent.

(a) $\eta_m(G) = q - p - \sum_{k=2}^{m} (k - 1)t_k$ (b) There exists an $m$-graphoidal path cover in which every vertex is an internal vertex of a path in $\psi$.

**Corollary 2.13.** There exists a $m$-graphoidal path cover $\psi$ of $G$ in which every vertex is an internal vertex of exactly $m$ paths in $\psi$ of $G$ if and only if $\eta_m(G) = q - mp$.

**Proof.** Since every vertex is an internal vertex of exactly $m$ paths in $\psi$ of $G$, $t_2 = t_3 = \cdots = t_{m-1} = 0$ and $t_m = p$. Then the result follows from Corollary 2.12.
Corollary 2.14. For a graph $G$ with $\Delta(G) \leq 3$, $\eta_m(G) = \eta_l(G)$ for $l = 1, 2, \ldots, m - 1$.

Proof. Since $\Delta \leq 3$, $t_k = 0$ for all $k = 2, \ldots, m - 1$. Then the result follows from Theorem 2.8 and Corollary 2.12.

Corollary 2.15. For a graph $G$ with $\Delta(G) \leq 2m - 1$, $\eta_m(G) = \eta_{m-1}(G)$.

Proof. Since $\Delta \leq 2m - 1$, $t_m = 0$. Then from Theorem 2.8, it follows that $\eta_m(G) = q - p - \sum_{k=2}^{m-1} (k-1)t_k + t = \eta_{m-1}(G)$.

Corollary 2.16. Let $G$ be any $(p, q)$ graph such that $\eta_m(G) = q - mp$. Then $\delta(G) \geq 2m$.

Proof. By the Corollary 2.13, there exists an $m$-graphoidal path cover $\Psi$ of $G$ in which every vertex is an internal vertex of exactly $m$ paths in $\psi$ and so $\delta \geq 2m$.

Corollary 2.17. If $G$ is a $(p, q)$ graph with $\eta_m(G) = q - mp$, then $\Delta(G) \geq 2m + 1$.

Proof. By Corollary 2.16, $d(v) \geq 2m$ for all $v \in V(G)$. Suppose there is no vertex of degree exceeding $2m$, $G$ must be $2m$-regular so, we have $q = mp$ and $\eta_m(G) = mp - mp = 0$, which is a contradiction. Hence $\Delta(G) \geq 2m + 1$.

The following theorem gives that the necessary condition for the equality of the parameters $\eta_m$ and $\pi$.

Theorem 2.18. If $G$ is a $(p, q)$ graph with $\Delta \leq 2m + 1$, then $\eta_m(G) = \pi(G)$.

Proof. Since every $m$-graphoidal path cover is a path cover we have $\pi(G) \leq \eta_m(G)$. If $\Delta \leq 2m + 1$, then every path cover is an $m$-graphoidal path cover of $G$ and hence $\eta_m(G) = \pi(G)$.

We give the sufficient condition as an open problem.

Problem 2.19. If $G$ is a $(p, q)$ graph with $\eta_m(G) = \pi(G)$, then $\Delta \leq 2m + 1$.

In the following theorems we determine the value of $\eta_m$ of several classes of graphs such as wheel and complete graphs and also investigate for which $m$ the parameters $\eta_m$ and $\pi$ are equal.

Theorem 2.20. For a wheel $W_n = K_1 + C_{n-1}(n \geq 4)$,

$$
\eta_m(W_n) = \begin{cases} 
  n - (m + 1) & \text{if } n \geq 2m + 1 \\
  n - m & \text{if } n \leq 2m.
\end{cases}
$$

Proof. Let $V(W_n) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ and let $E(W_n) = \{v_i v_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_1 v_{n-1}\} \cup \{v_n v_i : 1 \leq i \leq n - 1\}$. Since $d(v_i) = 3$ ($i = 1, 2, \ldots, n - 1$) and $d(v_n) = n - 1$, the vertices $v_1, v_2, \ldots, v_{n-1}$ are internal in at most one path and the vertex $v_n$ is internal in at most $m$ paths and so $t_k = 0$ for $k = 2, 3, \ldots, m - 1$ and $t_m \leq 1$. From Corollary 2.11 it follows that

$$
\eta_m(W_n) \geq q - p - (m - 1)t_m \\
\geq 2n - 2 - n - (m - 1) \\
= n - (m + 1)
$$

Suppose $n > 2m + 1$. Now let

$P = (v_1, v_2, \ldots, v_{n-1}, v_n)$

$P_1 = (v_{n-1}, v_1, v_n, v_2)$
Then the paths $P_i$, $P_1$, $P_2$, $\ldots$, $P_m$ together with the remaining edges form an $m$-graphoidal path cover $\psi$ of $W_n$ and so

$$|\psi| = m + 1 + |E(W_n) - \{E(P) \cup E(P_1) \cup E(P_2) \cup E(P_3) \cup \ldots E(P_m)\}|$$

$$= m + 1 + (2n - 2) - (n - 1 + 3 + 2 + 2 + \ldots 2)$$

$$(m-1)\text{-times}$$

$$= n - (m + 1).$$

and $\eta_m(W_n) \leq n - (m + 1)$

Hence $\eta_m(W_n) = n - (m + 1)$ for $n > 2m + 1$.

Now suppose $n = 2m + 1$. Let

$P_i = (v_{i+1}, v_i, v_{2m+1}, v_{m+1}, v_{m+1+i})$, $1 \leq i \leq m - 1$ and

$P_m = (v_{m+1}, v_m, v_{2m+1}, v_{2m}, v_1)$.

Then the paths $P_1, P_2, \ldots, P_{m-1}$ and $P_m$ form an $m$-graphoidal path cover $\psi$ of $W_n$ and so $\eta_m(W_n) \leq |\psi| = m = n - (m + 1)$ as $n = 2m + 1$.

Hence $\eta_m(W_n) = n - (m + 1)$ for $n = 2m + 1$.

If $n \leq 2m$, then $n - 1 \leq 2m - 1$ and so $\Delta(W_n) \leq 2m - 1$. Then from Corollary 2.15, it follows that $\eta_m(W_n) = n - m$.

**Corollary 2.21.** $\eta_{\left\lfloor \frac{n-1}{2} \right\rfloor}(W_n) = \pi(W_n)$.

**Proof.** From Theorem 2.20, it follows that

$$\eta_{\left\lfloor \frac{n-1}{2} \right\rfloor}(W_n) = n - \left\lfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 1 \right\rfloor$$

$$= n - \left\lfloor \frac{n}{2} \right\rfloor$$

$$= \left\lfloor \frac{n}{2} \right\rfloor$$

$$= \pi(W_n).$$

**Theorem 2.22.** For a complete graph $K_p$ ($p \geq 4$),

$$\eta_m(K_p) = \begin{cases} p(p - (2m + 1)), & \text{if } p \geq 2m + 2 \\ p + 1, & \text{if } p = 2m + 1 \\ p(p - (2m - 1))/2, & \text{if } p \leq 2m. \end{cases}$$

**Proof.** Let $V(K_p) = \{v_1, v_2, \ldots, v_p\}$. Then $q = \frac{p(p-1)}{2}$.

Case (i) $p > 2m + 1$.

If $p$ is odd, then let $p = 2n + 1$ and consider the paths

$P_i = (v_{1}, v_{2n+1}, v_2, v_{2n}, v_3, v_{2n-1}, \ldots, v_{n+2}, v_{n+1})$. 

removing an edge from each cycle and the path $P_\eta P P$.

Now, consider the paths $P_1, P_2, \ldots, P_m, P_{m+1}$ form an m-graphoidal path cover of $K_p$ in which every vertex is an internal vertex of exactly $m$ paths. Then from Corollary 2.13, it follows that

$$
\eta_m(K_p) = \frac{q - mp}{2} - mp
= \frac{p(p - (2m + 1))}{2}
$$

**Case(ii) $p = 2m + 1$.**

Now, consider the paths

$P_1 = (v_2, v_3, v_2, v_3, v_2, v_3, \ldots, v_m, v_m+1, v_{m+1}, v_{m+2}, v_1, v_2)$

$P_2 = (v_2, v_3, v_2, v_3, v_2, v_3, \ldots, v_m, v_m+1, v_{m+1}, v_{m+2}, v_1, v_3)$

$P_3 = (v_3, v_3, v_3, v_3, v_3, v_3, \ldots, v_m, v_m+1, v_{m+1}, v_{m+2}, v_1, v_4)$

$\ldots$

$P_{m-1} = (v_{m-1}, v_{m+1}, v_{m-2}, v_{m+2}, v_{m-3}, \ldots, v_{2m-2}, v_{2m+1}, v_{2m-2}, v_1, v_{m})$

$P_m = (v_m, v_{m+1}, v_{m-2}, v_{m+2}, v_{m-3}, \ldots, v_{2m-2}, v_{2m+1}, v_{2m-1}, v_2, v_{m})$

$P_{m+1} = (v_{2m+1}, v_2, v_3, v_4, v_5, \ldots, v_{m-1}, v_m, v_{m+1})$.

The paths $P_i$ ($1 \leq i \leq m$) can be obtained from $m$ hamiltonian cycles of $K_{2m+1}$ by removing an edge from each cycle and the path $P_{m+1}$ is obtained by joining the removed edges and so $\eta_m(K_p) \leq m + 1$. Also, $\eta_m(K_p) \geq \frac{q}{p - 1} = \frac{2m + 1}{2} = m + \frac{1}{2}$ and hence $\eta_m(K_p) = m + 1 = \frac{p + 1}{2}$.

If $p \leq 2m$, then $p - 1 \leq 2m - 1$ and so $\Delta(K_p) \leq 2m - 1$. Then from Corollary 2.15, it follows that $\eta_m(K_p) = \frac{p(p - (2m - 1))}{2}$.

**Corollary 2.23.** $\eta(\frac{2}{p+1})(K_p) = \pi(K_p)$.

**Proof.** From Theorem 2.22, it follows that

$\eta(\frac{2}{p+1})(K_p) = \frac{p + 1}{2}$, if $p$ is odd and $\eta(\frac{2}{p+1})(K_p) = \frac{p}{2}$, if $p$ is even. Thus $\eta(\frac{2}{p+1})(K_p) = \frac{p}{2}$.
\[ \left\lceil \frac{p}{2} \right\rceil = \pi(K_p). \]

Next we find \( \eta_m \) for tree, unicyclic graph and star.

**Theorem 2.24.** For any tree \( T \), there exists an m-graphoidal cover \( \psi \) of \( T \) such that every vertex of degree greater than one is an internal vertex of some \( \psi \) - path.

**Proof.** The proof is given by induction on \( p \). We can assume that \( p > 1 \) and the conclusion holds for any tree whose number of vertices is less than \( p \). Let \( u \) be a pendant vertex and let \( v \) be a vertex which is adjacent to \( u \). Let \( \psi_1 \) be an m-graphoidal path cover of \( T - u \), satisfying the conclusion. If \( v \) is a pendant vertex of \( T - u \), then replacing the path \( P \) in \( \psi_1 \), which ends at \( v \) by the path \( P \cup \{uv\} \), we get a m-graphoidal path cover of \( T \). This satisfies the requirement; otherwise \( \psi_1 \cup \{uv\} \) is the required m-graphoidal path cover of \( T \).

**Corollary 2.25.** Let \( n \) be the number of pendant vertices of a tree \( T \). Then \( \eta_m(T) = n - 1 - \sum_{k=2}^{m} (k-1)t_k \)

**Proof.** From Theorem 2.24, it follows that \( t = n \).

By Theorem 2.8,

\[ \eta_m(T) = q - p - \sum_{k=2}^{m} (k-1)t_k + t \]
\[ = p - 1 - p - \sum_{k=2}^{m} (k-1)t_k + n \]
\[ = n - 1 - \sum_{k=2}^{m} (k-1)t_k. \]

**Corollary 2.26.** For any tree \( T, \eta_m(T) \geq \Delta - 1 - \sum_{k=2}^{m} (k-1)t_k \)

**Proof.** \( T \) has at least \( \Delta \) vertices of degree one and the result follows from Corollary 2.25.

**Corollary 2.27.** Let \( T \) be a tree with \( \Delta \geq 2m + 1 \). Let \( v \) be a vertex in \( T \) such that \( d(v) = \Delta \). Then \( \eta_m(T) = \Delta - m \) if \( d(w) = 1 \) or \( 2 \) for all other vertices \( w \neq v \).

**Proof.** \( T \) has exactly \( \Delta \) vertices of degree one if and only if \( d(w) = 1 \) or \( 2 \) for all other vertices \( w \neq v \). Then \( t_k = 0 \) for \( k = 2, 3, \cdots, m - 1 \) and \( t_m = 1 \) and \( n = \Delta \). The result follows from Corollary 2.25.

**Theorem 2.28.** Let \( G \) be an unicycle graph with \( n \) pendant vertices. Let \( C \) be the unique cycle in \( G \). Let \( l \) be the number of vertices of degree greater than \( 2 \) on \( C \). Then

\[ \eta_m(G) = \begin{cases} 
2 & \text{if } l = 0, \\
n + 1 - \sum_{k=2}^{m} (k-1)t_k & \text{if } l = 1, \\
n - \sum_{k=2}^{m} (k-1)t_k & \text{otherwise.}
\end{cases} \]

**Proof.** Case (i) : \( l = 0 \). Then \( G = C \) and \( \eta_m(G) = 2 \).

Case (ii) : \( l = 1 \).
Let \( v \) be the unique vertex of degree greater than 2 on \( C \). Let \( e = uv \) be an edge on \( C \) incident at \( v \). Then \( G - e \) is a tree with \( n + 1 \) pendant vertices and it follows from Corollary 2.25 that \( \eta_m(G - e) = n - \sum_{k=2}^{m} (k - 1) t_k \).

Let \( \psi_1 \) be a minimum \( m \)-graphoidal path cover of \( G - e \). Then \( \psi_1 \cup \{ P \} \), where \( P \) is a path of length one consisting of the edge \( e \) is a \( m \)-graphoidal path cover of \( G \) so that \( \eta_m(G) \leq n + 1 - \sum_{k=2}^{m} (k - 1) t_k \). Further for any \( m \)-graphoidal path cover \( \psi \) of \( G \), all the \( n \) pendant vertices and at least one vertex on \( C \) are not internal in any path in \( \psi \). Hence \( t \geq n + 1 \) and by the Theorem 2.8, \( \eta_m(G) \geq n + 1 - \sum_{k=2}^{m} (k - 1) t_k \). Hence \( \eta_m(G) = n + 1 - \sum_{k=2}^{m} (k - 1) t_k \).

Case (iii) : \( l > 1 \).

Let \( v, w \) be vertices of degree greater than 2 on \( C \) such that all vertices in a \( (v, w) \) -section of \( C \) other than \( v, w \) have degree 2. Let \( P \) denote this \( (v, w) \)-section. If \( P \) has length 1, let \( G_1 = G - e \) where \( e \) is the edge \( vw \). Otherwise let \( G_1 \) be the subgraph obtained by deleting all the internal vertices of \( P \). Clearly \( G_1 \) is a tree with \( n \) pendant vertices and hence \( \eta_m(G_1) = n - 1 - \sum_{k=2}^{m} (k - 1) t_k \). If \( \psi_1 \) is a minimum \( m \)-graphoidal path cover of \( G_1 \), then \( \psi_1 \cup \{ P \} \) is a \( m \)-graphoidal path cover of \( G \) and hence \( \eta_m(G) \leq n - \sum_{k=2}^{m} (k - 1) t_k \). Since \( G \) has \( n \) pendant vertices, \( t \geq n \) and again by the Theorem 2.8, \( \eta_m(G) \geq n - \sum_{k=2}^{m} (k - 1) t_k \). Hence \( \eta_m(G) = n - \sum_{k=2}^{m} (k - 1) t_k \).

**Theorem 2.29.** For a star \( K_{1,n} \) \((n \geq 2)\),

\[
\eta_m(K_{1,n}) = \begin{cases} 
  n - m & \text{if } n \geq 2m \\
  n - (m - 1) & \text{if } n \leq 2m - 1.
\end{cases}
\]

**Proof.** Now, let \( X = \{ x_1 \} \) and \( Y = \{ y_1, y_2, \ldots, y_n \} \). Suppose \( n \geq 2m \). Then \( d(x_1) \geq 2m \) and so \( t_m \leq 1 \). Also \( t_k = 0 \) for \( k = 2, 3, \ldots, m - 1 \). Since \( K_{1,n} \) is a tree with \( n \) pendant vertices, it follows from Corollary 2.26 that \( \eta_m(K_{1,n}) \geq n - 1 - (m - 1) = n - m \). Now, the paths \( P_i = (y_{2i-1}, x_1, y_{2i}) \) \((1 \leq i \leq m)\) together with the remaining edges form a \( m \)-graphoidal path cover \( \psi \) of \( G \) such that

\[
\eta_m(K_{1,n}) \leq |\psi| = m + |E(K_{1,n}) - \{ E(P_1) \cup E(P_2) \cdots E(P_m) \}| = m + (n - 2m) = n - m.
\]

Hence \( \eta_m(K_{1,n}) = n - m \).

Now, suppose \( n \leq 2m - 1 \) and so \( \Delta(K_{1,n}) \leq 2m - 1 \). Then from Corollary 2.15, it follows that \( \eta_m(K_{1,n}) = n - (m - 1) \).
Corollary 2.30. $\eta_{[\frac{n}{2}]}(K_{1,n}) = \pi(K_{1,n})$.

Proof. From Theorem 2.29, it follows that
$\eta_{[\frac{n}{2}]}(K_{1,n}) = n - (\lfloor \frac{n}{2} \rfloor) = \lceil \frac{n}{2} \rceil = \pi(K_{1,n})$.

References


On the cubic Gauss sums and its fourth power mean\footnote{This work is supported by N.S.F. (10671155) of P.R.China and P.E.D.F. (08JK398) of Shaanxi}

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Abstract The main purpose of this paper is to study the calculating problem of the fourth power mean of the cubic Gauss sums, and give an exact calculating formula for it.

Keywords Cubic Gauss sums, fourth power mean, calculating formula.

§1. Introduction

Let \( q \geq 3 \) be a positive integer. For any fixed positive integer \( r \) and integer \( n \), we define the \( r \)-th Gauss sums \( G(n, r, q) \) as follows:

\[
G(n, r, q) = \sum_{a=1}^{q} e \left( \frac{na^r}{q} \right),
\]

where \( e(y) = e^{2\pi iy} \). This summation is more important, because it is a generalization of the quadratic Gauss sums \( G(n; q) = \sum_{a=1}^{q} e \left( \frac{na^2}{q} \right) \). The various properties of \( G(n; q) \) were investigated by many authors (see [1] and [2]). For any positive integer \( r \geq 3 \), using the elementary method we can easily prove that (see [3])

\[
|G(n, r, q)| \leq \sqrt{q} \prod_{p^\alpha \parallel q \atop p^\alpha | q} (r, \phi(p^\alpha)),
\]

where \( \phi(q) \) is the Euler function, \( n \) is any integer with \( (n, q) = 1 \), and \( \prod_{p^\alpha \parallel q \atop p^\alpha | q} \) denotes the product over all prime divisors of \( q \) with \( p^\alpha | q \) and \( p^\alpha \nmid q \).

But about the other properties of \( G(n, r, q) \), we know very little at present. For \( r = 2 \), we have studied its properties and given two accurate calculation formulas for the \( k \)-th power mean of this sum(see [4]).

The main purpose of this paper is to study the fourth power mean properties of \( \sum_{n=1}^{q} |G(n, 3, q)|^4 \), and give an accurate calculating formula for it. That is, we shall prove the following main conclusion:
Theorem. Let \( q = 3^β q_1 \) be a positive integer, where \( β \geq 0 \) and \( 3 \nmid q_1 \). Then we have the calculating formula
\[
\sum_{a=1}^{q} \left| \sum_{b=1}^{q} e \left( \frac{ab^3}{q} \right) \right|^4 = q^4 \left[ 3 - \frac{-7}{3} \right] = q^4 \left[ 3 - \frac{-1}{3} \right],
\]
\[
\prod_{\substack{p^\alpha || q \leq 1 \pmod{3}} \atop p || q} \left[ 1 + \left( \frac{7}{p} + \frac{1}{p^2} \right) \left( 1 - \frac{1}{p^{\alpha+1}} \right) - \frac{1}{p^{\alpha+1}} \left( 1 - \frac{1}{p^{\alpha+2}} \right) \right],
\]
where \( \prod_{p || q} \) denotes the product over all \( p \) such that \( p^α \mid q \) and \( p^{α+1} \nmid q \).

For general positive integer \( m \geq 3 \) and \( r \) with \( r \mid q - 1 \), whether there exists an calculating formula for \( \sum_{n=1}^{q} \left| G(n, r, q)^{2m} \right| \) is an open problem.

§2. Some Lemmas

To complete the proof of the theorem, we need the following Lemmas.

Lemma 1. Let \( p \) be a prime with \( p \equiv 1 \pmod{3} \). Then we have the identity
\[
\sum_{a=1}^{p^α} \left| \sum_{b=1}^{p^α} e \left( \frac{ab^3}{p^α} \right) \right|^4 = \begin{cases} 
\frac{p^{8k} \phi(p^{3k})}{6}, & \text{if } α = 3k; \\
\frac{p^{8k+2} \phi(p^{3k+1})}{2}, & \text{if } α = 3k + 1; \\
\frac{p^{8k+4} \phi(p^{3k+2})}{6}, & \text{if } α = 3k + 2,
\end{cases}
\]
where \( \phi(d) \) is the Euler function.

Proof. Let \( χ_3(b) \) be a cubic character modulo \( p \), then we have
\[
\begin{align*}
\sum_{a=1}^{p} \left| \sum_{b=1}^{p} e \left( \frac{ab^3}{p} \right) \right|^4 &= \sum_{a=1}^{p-1} \left| 1 + \sum_{b=1}^{p-1} \left( 1 + χ_3(b) + χ_3^2(b) \right) e \left( \frac{ab}{p} \right) \right|^4 \\
&= \sum_{a=1}^{p-1} \left| 1 + \sum_{b=1}^{p-1} e \left( \frac{ab}{p} \right) + \sum_{b=1}^{p-1} χ_3(b) e \left( \frac{ab}{p} \right) + \sum_{b=1}^{p-1} χ_3^2(b) e \left( \frac{ab}{p} \right) \right|^4 \\
&= \sum_{a=1}^{p-1} \left| \chi_3(a) τ(χ_3) + \chi_3^2(a) τ(χ_3) \right|^4 \\
&= \sum_{a=1}^{p-1} \left| \chi_3(a) τ(χ_3) + χ_3(a) τ(χ_3) \right|^4 \\
&= 6(p - 1)p^2.
\end{align*}
\]
If \( α > 1 \) and \( (a, p) = 1 \) (see reference [4]), then we have
\[
\sum_{b=1}^{p^α} \left( 1 + χ_3(b) + χ_3^2(b) \right) e \left( \frac{ab}{p^α} \right) = \mu(p^α) + \chi_3(a) τ(χ_3) + \chi_3^2(a) τ(χ_3) = 0.
\]
Let $S_1(p^\alpha) = \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} e \left( \frac{ab^3}{p^\alpha} \right)^4$, then from (2) we obtain

$$S_1(p^2) = \sum_{a=1}^{p^2} \sum_{b=1}^{p^2} e \left( \frac{ab^3}{p^2} \right)^4 = \sum_{a=1}^{p^2} \sum_{b=1}^{p^2} e \left( \frac{ab^3}{p^2} \right) + p^4 = p^4 \phi(p^2),$$

(3)

$$S_1(p^3) = \sum_{a=1}^{p^3} \sum_{b=1}^{p^3} e \left( \frac{ab^3}{p^3} \right)^4 = \sum_{a=1}^{p^3} \sum_{b=1}^{p^3} e \left( \frac{ab^3}{p^3} \right) + p^2 = p^8 \phi(p^3),$$

(4)

and

$$S_1(p^\alpha) = \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} e \left( \frac{ab^3}{p^\alpha} \right)^4 = \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} e \left( \frac{ab^3}{p^\alpha} \right) + \sum_{b=1}^{p^{\alpha-1}} e \left( \frac{ab^3}{p^{\alpha-3}} \right)^4$$

$$= \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \left( 1 + \chi_3(b) + \chi_3^2(b) \right) e \left( \frac{ab}{p^a} \right) + p^2 \sum_{b=1}^{p^{\alpha-3}} e \left( \frac{ab^3}{p^{\alpha-3}} \right)^4$$

$$= p^{11} \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} e \left( \frac{ab^3}{p^{\alpha-3}} \right)^4$$

$$= p^{11} S_1(p^{\alpha-3}).$$

If $\alpha = 3k$, combining (4) and (5) we have

$$S_1(p^\alpha) = p^{11} S_1(p^{\alpha-3}) = p^{22} S_1(p^{\alpha-6}) = \cdots = p^{11(k-1)} S_1(p^3) = p^{8k} \phi(p^{3k}).$$

If $\alpha = 3k + 1$, combining (1) and (5) we have

$$S_1(p^\alpha) = p^{11} S_1(p^{\alpha-3}) = p^{22} S_1(p^{\alpha-6}) = \cdots = p^{11k} S_1(p) = 6p^{8k+2} \phi(p^{3k+1}).$$

If $\alpha = 3k + 2$, combining (3) and (5) we have

$$S_1(p^\alpha) = p^{11} S_1(p^{\alpha-3}) = p^{22} S_1(p^{\alpha-6}) = \cdots = p^{11k} S_1(p^2) = p^{8k+4} \phi(p^{3k+2}).$$

This completes the proof of Lemma 1.

**Lemma 2.** Let $p$ be a prime with $p \equiv 1 \pmod{3}$. Then we have the identity

$$\sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} e \left( \frac{ab^3}{p^\alpha} \right)^4 = p^{4\alpha} \left[ 1 + \left( \frac{7}{p} + \frac{1}{p^2} \right) \left( 1 - \frac{1}{p^{\alpha-1}} \right) - \frac{1}{p^{\alpha-1}} \left( 1 - \frac{1}{p^{\alpha-1}} \right) \right].$$
Proof. Let $S(p^\alpha) = \sum_{a=1}^{p^\alpha} \left| \sum_{b=1}^{p^\alpha} e \left( \frac{ab^3}{p^\alpha} \right) \right|^4$, we have

$$S(p^\alpha) = \sum_{a=1}^{p^\alpha} \left| \sum_{b=1}^{p^\alpha} e \left( \frac{ab^3}{p^\alpha} \right) \right|^4 + \sum_{a=1}^{p^{\alpha-1}} \left| \sum_{b=1}^{p^\alpha} e \left( \frac{ab^3}{p^{\alpha-1}} \right) \right|^4$$

(6)

$$= S_1(p^\alpha) + p^4 \sum_{a=1}^{p^{\alpha-1}} \left| \sum_{b=1}^{p^\alpha} e \left( \frac{ab^3}{p^{\alpha-1}} \right) \right|^4$$

$$= S_1(p^\alpha) + p^4 S(p^{\alpha-1})$$

$$= S_1(p^\alpha) + p^4 S_1(p^{\alpha-1}) + p^8 S(p^{\alpha-2})$$

$$= S_1(p^\alpha) + p^4 S_1(p^{\alpha-1} + p^8 S_1(p^{\alpha-2}) + \cdots + p^{4(\alpha-2)} S_1(p^2) + p^{4(\alpha-1)} S(p).$$

If $\alpha = 3k$, note that $S(p) = \sum_{a=1}^{p} \left| \sum_{b=1}^{p} e \left( \frac{ab^3}{p} \right) \right|^4 = p^4 + S_1(p) = p^4 + 6p^2(p-1)$, (6) and Lemma 1 we have

$$S(p^\alpha) = \phi(p^\alpha) p^{\frac{5}{2} \alpha} + p^4 \phi(p^{\alpha-1}) p^{\frac{7}{2} (\alpha-1) - \frac{3}{4}} + 6p^8 \phi(p^{\alpha-2}) p^{\frac{7}{2} (\alpha-2) - \frac{3}{4}}$$

$$+ \cdots + p^{4(\alpha-2)} p^4 \phi(p^2) + p^{4(\alpha-1)} \left( p^4 + 6p^2(p-1) \right)$$

$$= p^{4\alpha} \left[ 7 \left( \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^{\frac{7}{2}} + \frac{3}{4}} \right) + \left( \frac{1}{p^3} + \frac{1}{p^4} + \cdots + \frac{1}{p^{\frac{7}{2}} + 2} \right) \right] \left( p-1 \right) + p^{4\alpha}$$

$$= p^{4\alpha} \left[ 1 + \left( \frac{7}{p} + \frac{1}{p^2} \right) \left( 1 - \frac{1}{p^{\frac{7}{2}}} \right) \right].$$

Applying the same methods, we also have

$$S(p^\alpha) = p^{4\alpha} \left[ 1 + \left( \frac{7}{p} + \frac{1}{p^2} \right) \left( 1 - \frac{1}{p^{\frac{7}{2}}} \right) - \frac{1}{p^{\frac{7}{2}}} \left( 1 - \frac{1}{p} \right) \right], \text{ if } \alpha = 3k + 2;$$

and

$$S(p^\alpha) = p^{4\alpha} \left[ 1 + \left( \frac{7}{p} + \frac{1}{p^2} \right) \left( 1 - \frac{1}{p^{\frac{7}{2}}} \right) - \frac{1}{p^{\frac{7}{2}}} \left( 1 - \frac{1}{p^2} \right) \right], \text{ if } \alpha = 3k + 1.$$

This completes the proof of Lemma 2.

Lemma 3. For any integer $\alpha \geq 0$, we have the identity

$$\sum_{a=1}^{3^\alpha} \sum_{b=1}^{3^\alpha} e \left( \frac{ab^3}{3^\alpha} \right)^4 = 3^{4\alpha} \left( 3 - \frac{7(-1)^{\frac{\alpha+1}{2}} - [\frac{\alpha}{2}]}{3^{1+\frac{\alpha}{2}}} \right).$$
Applying the method of proving (5), we may get

\[ S_1(3^2) = \sum_{a=1}^{3^2} \sum_{b=1}^{3^2} e \left( \frac{ab^3}{3^2} \right) = 31^2 S_1(3^{\alpha-3}). \] (7)

Note that \( S_1(3) = 3^{11(k)} S_1(3) = 0 \), if \( \alpha = 3k + 1 \).

If \( \alpha = 3k \), from (8) we have

\[ S_1(3^\alpha) = 3^{11(k-1)} S_1(3^3) = 3^{11(k-1)} \sum_{a=1}^{3^3} \sum_{b=1}^{3^3} e \left( \frac{ab^3}{3^3} \right) = 3^{11(k-1)} \cdot 3^8 S_1(3^3) = 3^{8k} \phi(3^{3k}). \]

If \( \alpha = 3k + 2 \), combining (7) and (8) we have

\[ S_1(3^\alpha) = 3^{11k} S_1(3^2) = 15 \cdot 3^{8k+2} \phi(3^{3k+2}). \]

Applying the method of proving Lemma 2, we have

\[ S(3^\alpha) = S_1(3^\alpha) + 3^4 S_1(3^{\alpha-1}) + 3^8 S_1(3^{\alpha-2}) + \cdots + 3^{4(\alpha-2)} S_1(3^2) + 3^{4(\alpha-1)} S(3) = \phi(3^\alpha)3^{2^\alpha} + 15 \cdot 3^4 \phi(3^{\alpha-1})3^{2(\alpha-1)-2} + 0 + \cdots + 15 \cdot 3^{4(\alpha-2)}3^4 \phi(3^2) + 3^{4(\alpha-1)}3^4(3-1) + 3^{4\alpha} \]

\[ = 3^{4\alpha} \left[ 1 + \left( \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{3+\alpha}} \right) \right] (3-1) + 3^{4\alpha} \]

\[ = 3^{4\alpha} \left[ 3 - \frac{1}{3^3} \right], \text{ if } \alpha = 3k. \]
Similarly, we may obtain $S(3^\alpha) = 3^{4\alpha} \left( 3 - 7 - (-1) \left[ \frac{\alpha + 1}{3} \right] - \left[ \frac{\alpha}{3} \right] \right)$. This proves Lemma 3.

§3. Proof of the theorem

From Lemma 2 and Lemma 3 on the above section, we can complete the proof of the theorem. Let $q$ has the prime power decomposition $q = \prod p_i^{\alpha_i}$. It is clear that if $a_i$ pass through a complete residue system modulo $p_i^{\alpha_i}$, then $a = \sum a_i \frac{q}{p_i^{\alpha_i}} \equiv \sum a_i M_i$ pass through a complete residue system modulo $q$. Note that

\[
\sum_{a=1}^{p_i^{\alpha_i}} \sum_{b=1}^{p_i^{\alpha_i}} e \left( \frac{ab^3}{p_i^{\alpha_i}} \right) = \sum_{a=1}^{p_i^{\alpha_i}} \sum_{b=1}^{p_i^{\alpha_i}} e \left( \frac{ab^3}{p_i^{\alpha_i}} \right) = p^{4\alpha},
\]

if $p \equiv 2(\text{mod}3)$ and the multiplicative property of

\[
\left| \sum_{b=1}^{p_i^{\alpha_i}} e \left( \frac{ab^3}{p_i^{\alpha_i}} \right) \right|^4 = \prod_{p_i^{\alpha_i}} \left( 1 + \left( \frac{7}{p_i^{\alpha_i}} + \frac{1}{p_i^{3\alpha_i}} \right) \left( 1 - \frac{1}{p_i} \right) - \frac{1}{p_i^{\alpha_i}} \right) \left( 1 - \frac{1}{p_i} \right) \left( 1 - \frac{1}{p_i^{\alpha_i}} \right).
\]

This completes the proof of Theorem.

References

On the dual functions $Z_*(n)$ and $S_*(n)$

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Abstract  The pseudo Smarandache dual function, denoted by $Z_*(n)$, is defined as the maximum positive integer $m$ such that $\frac{m(m+1)}{2}$ divides $n$. The Smarandache dual function, denoted by $S_*(n)$, is defined as the maximum positive integer $m$ such that $m!$ divides $n$. This paper derives the explicit expressions for $Z_*(2^p k), Z_*(3^p k), Z_*(4^p k)$ and $Z_*(5^p k)$, where $p$ is an odd prime, as well as an inequality involving $S_*((2n+1)!(2n+3)!)$.

Keywords  Pseudo Smarandache dual function, Smarandache dual function.

§1. Introduction

The pseudo Smarandache dual function, denoted by $Z_*(n)$, introduced by Sandor [1], is defined as follows (where $\mathbb{Z}^+$ is the set of all positive integers).

Definition 1.1. For any integer $n \geq 1$, $Z_*(n) = \max \{m : m \in \mathbb{Z}^+, \frac{m(m+1)}{2} | n \}$.

Sandor [1] has also studied some elementary properties of the function $Z_*(n)$. They are given in the following lemmas.

Lemma 1.1. For any integer $n \geq 1$, $1 \leq Z_*(n) \leq \frac{1}{2}(\sqrt{1 + 8n} - 1)$.

Lemma 1.2. $Z_*(\frac{k(k+1)}{2}) = k$ for any integer $k \geq 1$.

Lemma 1.3. For any integers $a, b \geq 1$, $Z_*(ab) \geq \max\{Z_*(a), Z_*(b)\}$.

Lemma 1.4. Let $p \geq 3$ be a prime. Then, for any integer $k \geq 1$,

$$Z_*(p^k) = \begin{cases} 2, & \text{if } p = 3 \\ 1, & \text{if } p \neq 3 \end{cases}$$

Lemma 1.5. Any solution of the equation $Z(n) = Z_*(n)$ is of the form $n = \frac{k(k+1)}{2}$, $k \geq 1$.

The Smarandache dual function, denoted by $S_*(n)$, has been defined by Sandor [2] as follows.

Definition 1.2. For any integer $n \geq 1$, $S_*(n) = \max \{m : m \in \mathbb{Z}^+, m! | n \}$.

Some elementary properties of the function $S_*(n)$, studied by Sandor [2], are given in the following lemmas.

Lemma 1.6. For any prime $p \geq 2$, and any integer $n \geq p$, $S_*(n! + (p - 1)!) = p - 1$. 
Lemma 1.7. For any integer \( n \geq 1 \),
\[
S_*((2n)!(2n+2)!) \begin{cases} 
2n+2, & \text{if } 2n+3 \text{ is a prime} \\
\geq 2n+3, & \text{if } 2n+3 \text{ is not a prime}
\end{cases}
\]

It may be mentioned here that, there is some mistake in Proposition 8 in Sandor [2]; the correct form is given in Lemma 1.7 above.

In this paper, we derive the expressions for \( Z_* (2p^k) \), \( Z_* (3p^k) \), \( Z_* (4p^k) \) and \( Z_* (5p^k) \). These are given in the next Section 2. In Section 3, we give a new inequality involving \( S_* (n) \). Some concluding remarks are given in the final Section 4.

§2. The pseudo Smarandache dual function \( Z_* (n) \)

In this section, we derive the explicit expressions for \( Z_* (2p^k) \), \( Z_* (3p^k) \), \( Z_* (4p^k) \) and \( Z_* (5p^k) \), where \( p \) is a prime and \( k \geq 1 \) is an integer. They are given in the following lemmas.

In what follows, we shall denote by \( T_m \) the \( m \)-th triangular number, that is,
\[
T_m = \frac{m(m+1)}{2}; \quad m = 1, 2, \ldots
\]
Note that \( T_m \) is strictly increasing in \( m \).

Lemma 2.1. Let \( p \geq 3 \) be a prime. Then, for any integer \( k \geq 1 \),
\[
Z_* (2p^k) = \begin{cases} 
3, & \text{if } p = 3 \\
4, & \text{if } p = 5 \\
1, & \text{if } p \geq 7
\end{cases}
\]

Proof. By definition,
\[
Z_* (2p^k) = \max \{ m : m \in \mathbb{Z}^+, \ T_m \mid 2p^k \} = \max \{ m : m \in \mathbb{Z}^+, \ m(m+1) \mid 4p^k \}.
\]

Then, one of \( m \) and \( m + 1 \) is \( p \), and the other one must be 4. Now,
\[
m + 1 = 4, m = p \implies p = 3, \quad m + 1 = p, m = 4 \implies p = 5.
\]
Thus, if \( p \geq 7 \), we must have \( m = 1 \).

Lemma 2.2. Let \( p \geq 5 \) be a prime. Then, for any integer \( k \geq 1 \),
\[
Z_* (3p^k) = \begin{cases} 
5, & \text{if } p = 5 \\
6, & \text{if } p = 7 \\
2, & \text{if } p \geq 11
\end{cases}
\]

Proof. In this case, by definition,
\[
Z_* (3p^k) = \max \{ m : m \in \mathbb{Z}^+, \ T_m \mid 3p^k \} = \max \{ m : m \in \mathbb{Z}^+, \ m(m+1) \mid 6p^k \}.
\]
Now,
\[ m + 1 = 6, m = p \implies p = 5, \quad m + 1 = p, m = 6 \implies p = 7. \]
If \( p \geq 11 \), since \( T_2 \) divides 3 and \( T_3 > 3 \), it follows that \( m = 2 \).

**Lemma 2.3.** Let \( p \geq 3 \) be a prime. Then, for any integer \( k \geq 1 \),
\[
Z_* (4p^k) = \begin{cases} 
7, & \text{if } p = 7 \\
1, & \text{otherwise}
\end{cases}
\]

**Proof.** Here,
\[
Z_* (4p^k) = \max \{ m : m \in \mathbb{Z}^+, T_m | 4p^k \} = \max \{ m : m \in \mathbb{Z}^+, m(m + 1) | 8p^k \}.
\]
Now,
\[
m + 1 = 8, m = p \implies p = 7, \quad m + 1 = p, m = 8 \implies p = 9.
\]
Thus, the second case cannot occur, and for \( p \neq 7 \), we must have \( m = 1 \).

**Lemma 2.4.** Let \( p \neq 5 \) be a prime. Then,
\[
Z_* (5k) = \begin{cases} 
10, & \text{if } p = 11 \\
1, & \text{otherwise}
\end{cases}
\]

**Proof.** In this case, since
\[
Z_* (5p) = \max \{ m : m \in \mathbb{Z}^+, T_m | 5p \} = \max \{ m : m \in \mathbb{Z}^+, m(m + 1) | 10p \},
\]
and since,
\[
m + 1 = 10, m = p \implies p = 9, \quad m + 1 = p, m = 10 \implies p = 11,
\]
it follows that the first case cannot occur. Thus, for \( p \neq 11 \), we must have \( m = 1 \).

**Lemma 2.5.** Let \( p \neq 5 \) be a prime. Then, for any integer \( k \geq 2 \),
\[
Z_* (5p^k) = \begin{cases} 
p^2, & \text{if } p = 3 \\
10, & \text{if } p = 11 \\
1, & \text{otherwise}
\end{cases}
\]

**Proof.** Here,
\[
Z_* (5p^k) = \max \{ m : m \in \mathbb{Z}^+, T_m | 5p^k \} = \max \{ m : m \in \mathbb{Z}^+, m(m + 1) | 10p^k \}
\]
When \( p = 3 \), then \( p^2 + 1 = 10 \) divides 10. Therefore, in this case, \( m = p^2 \). Now,
\[
m + 1 = 10, m = p \implies p = 9, \quad m + 1 = p, m = 10 \implies p = 11.
\]
Thus, the first case cannot occur.
§3. The Smarandache dual function $S_*(n)$

In this section, we prove the following result.

**Lemma 3.1.** For any integer $n \geq 1$, $S_*((2n + 1)!(2n + 3)!) \geq 2(n + 2)$.

**Proof.** We first prove that

$2(n + 2)$ divides $(2n + 1)!$ for any integer $n \geq 1$.

The proof is as follows: If $n \geq 2$, then $2n + 1 \geq n + 2$ and so

$2(n + 2)$ divides $(2n + 1)! = (n + 2)!(n + 3) \cdots (2n)(2n + 1)$.

Since $2 \times 3$ divides $3!$, the result is true for $n = 1$ as well.

Now,

$$2(n + 2) \mid (2n + 1)! \Rightarrow [2(n + 2)]! = (2n + 3)! [2(n + 2)] \mid (2n + 1)! (2n + 3)!$$

This proves the lemma.

§4. Some remarks

The values of $S_*((2n + 1)!(2n + 3)!)$ for some small $n$ are given in Sandor [2]. Motivated by these values, he makes the following conjecture:

Conjecture: $S_*((2n + 1)!(2n + 3)!) = q_n - 1$, where $q_n$ is the first prime following $2n + 3$.

It may be mentioned that the case $n = 2$ is a violation to the conjecture, since $S_*(5!7!) = 8$, though other values of $n$ supports the conjecture. Le [3] claims to have proved the conjecture, using the inequality

$$ord_p((2n + 1)!) + ord_p((2n + 3)!) < ordp((q_n - 1)!),$$

where

$$ord_p(n!) = \sum_{k=1}^{\infty} \left[ \frac{n}{p^k} \right]$$

([x] being the Gauss function of $x$),

but the inequality remains to be proved.

References


Smarandache sequence of Ulam numbers

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Abstract In this article, we present the results of investigation of Smarandache Concatenate Sequence formed from the sequence of Ulam Numbers, Ulam primes and report some primes and other results found from the sequence.

Keywords Ulam numbers, U-sequence, Smarandache U-sequence, reversed Smarandache U-sequence, prime, Ulam Prime numbers, UP-sequence, Smarandache UP-sequence.

§1. Introduction

The standard Ulam sequence starts with $U_1 = 1$ and $U_2 = 2$ being the first two Ulam numbers. Then for $n > 2$, $U_n$ is defined to be the smallest integer that is the sum of two distinct earlier terms in exactly one way [1]. So, for example, 11 is an Ulam number because it is the sum of the pair of smaller Ulam numbers 8 and 3, and no other pair, while 13 is also an Ulam number because it is the sum of 11 and 2, and no other pair. 12 is not an Ulam number because it is the sum of 1 and 11, and of 4 and 8. The first few terms of sequence of Ulam numbers [3] are:


Let us denote the sequence of Ulam numbers as U-sequence. So, the sequence of Ulam numbers,

\[ U = 1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, \ldots \]

The Ulam numbers that are also prime numbers can be termed as Ulam Prime numbers. The first few terms of sequence of Ulam Prime numbers [4] are:

2, 3, 11, 13, 47, 53, 97, 131, 197, 241, 409, 431, 607, 673, 739, 751, 983, 991, 1103, 1433, 1489, 1531, 1553, 1709, 1721, 2371, 2393, 2447, 2633, 2789, 2833, 2897, 3041, 3109, 3217, 3371, 3373, 3527, 3547, 3593, 3671, 3691, 4057, 4153, 4211, 4297, 4363, 4409, 4451, 4517, 4519, 4729,
4903, 4969, 5081, 5053, 6029, 6481, 6833, 6911, 7043, 7297, 7459, 7559, 7583, 7603, 7691, 7727, 8011, 8101, 8167, 8539, 8573, 8969, 8971, 9013, 9137, 9311, 9377, 9511, 9619, 9643, 9721, 9743, 9851, 9941, ···

Let us denote the sequence of Ulam Prime numbers as UP-sequence. So, the sequence of Ulam Prime numbers,

\[ UP = 2, 3, 11, 13, 47, 53, 97, 131, 197, 241, 409, 431, 607, 673, \cdots \]

§2. Smarandache sequence

Let \( S_1, S_2, S_3, \cdots, S_n, \cdots \) be an infinite integer sequence (termed as S-sequence), then the Smarandache sequence [5] or Smarandache Concatenated sequence [2] or Smarandache S-sequence is given by

\[ S_1, \overline{S_1 S_2}, \overline{S_1 S_2 S_3}, \cdots, \overline{S_1 S_2 S_3 \cdots S_n}, \cdots \]

Also Smarandache Back Concatenated sequence or Reversed Smarandache S-sequence is

\[ S_1, \overline{S_2 S_1}, \overline{S_3 S_2 S_1}, \cdots, \overline{S_n \cdots S_3 S_1}, \cdots \]

§3. Smarandache U-Sequence

Smarandache sequence of Ulam numbers or Smarandache U-sequence is the sequence formed from concatenation of numbers in U-sequence (Note that U-sequence is the sequence of Ulam numbers). So, Smarandache U-sequence is

1, 12, 123, 1234, 12346, 123468, 12346811, 1234681113, 12346811131618, 1234681113161826, 123468111316182628, 12346811131618262836, 1234681113161826283638, \cdots

Let us denote the \( n \)th term of the Smarandache U-sequence by \( SU(n) \). So,

\[ SU(1) = 1 \]
\[ SU(2) = 12 \]
\[ SU(3) = 123 \]
\[ SU(4) = 1234 \]
\[ SU(5) = 12346 \] and so on.

§4. Observations on Smarandache U-sequence

We have investigated Smarandache U-sequence for the following two problems

(i) How many terms of Smarandache U-sequence are primes?
(ii) How many terms of Smarandache U-sequence belong to the initial U-sequence?

In search of answer to these problems, we find that

(a) There are only 2 primes in the first 3200 terms of Smarandache U-sequence. These are

\[ SU(22) = 12346811131618262836384748535762697277. \]

\[ SU(237) = 1234681113161826283638474853576269727782879799102106114126131138145 \]
\[ 1481551751771801821891972062092192212362382412432532582602732823093 \]
\[ 1631932433934135635836337038239040040240941241442943143444445145648 \]
\[ 348549750252252454546566568585602605607612624627646666673685686896 \]
\[ 69572072273237397517817837988082084784986184866868918939059279499 \]
\[ 8398699110181020102310251030103210351037105210791081110111101125115 \]
\[ 51157116411671169118611911208112301125112711296113081131113131335133813 \]
\[ 4013551360137713871389140414061428143114331462146514701472148914921 \]
\[ 50915141516151315315381550155315941602160416161641643164616481660 \]
\[ 1682170717091721172417481765177017901792181218141834183618531856185 \]
\[ 819001902191919419441946196619681985201020122032203420542056209020 \]
\[ 93209521122115211721342156217822472249225222522882327 \]

It may be noted that \( SU(3200) \) consists of 15016 digits.

(b) Other than the trivial 1, no other Ulam numbers have been found in the Smarandache U-sequence.

**Open Problem:**

(i) Can you find more primes in Smarandache U-sequence and are there infinitely many such primes?

(ii) Can you find more Ulam numbers in Smarandache U-sequence and are there infinitely many such Ulam numbers?

§4. Reversed Smarandache U-Sequence

It is defined as the sequence formed from the concatenation of Ulam numbers (U-sequence) written backward i.e. in reverse order. So, Reversed Smarandache U-sequence is

\[ \text{RSU}(1) = 1 \]
\[
\begin{align*}
RSU(2) &= 21 \\
RSU(3) &= 321 \\
RSU(4) &= 4321 \\
RSU(5) &= 64321 \quad \text{and so on.}
\end{align*}
\]

§5. Observations on reversed Smarandache U-sequence

(a) As against only 2 prime in Smarandache U-sequence, no primes in first 3200 terms of Reversed Smarandache U-sequence have been found.

(b) Other than the trivial 1, no Ulam number is known in the Reversed Smarandache U-sequence.

**Open Problem:**

(i) Can you find primes in Reversed Smarandache U-sequence and are there infinitely many such primes?

(ii) Can you find Ulam numbers in Reversed Smarandache U-sequence and are there infinitely many such Ulam numbers?

§6. Smarandache UP-Sequence

Smarandache sequence of Ulam Prime numbers or Smarandache UP-sequence is the sequence formed from concatenation of numbers in UP-sequence ( Note that UP-sequence is the sequence of Ulam Prime numbers). So, Smarandache UP-sequence is

\[
\begin{align*}
2, \ 23, \ 2311, \ 231113, \ 23111347, \ 2311134753, \ 231113475397, \ 231113475397131, \ 231113475397131197, \\
231113475397131197241, \ 231113475397131197241409, \ 231113475397131197241409431, \\
231113475397131197241409431607, \ 231113475397131197241409431607673, \ \ldots.
\end{align*}
\]

Let us denote the nth term of the Smarandache UP-sequence by SUP(n). So, 

\[
\begin{align*}
SUP(1) &= 2 \\
SUP(2) &= 23 \\
SUP(3) &= 2311 \\
SUP(4) &= 231113 \\
SUP(5) &= 23111347, \quad \text{and so on.}
\end{align*}
\]
§7. Observations on Smarandache UP-sequence

We have investigated Smarandache UP-sequence to find as to how many terms of Smarandache UP-sequence are primes?

We find that there are 5 primes in the first 245 terms of Smarandache UP-sequence. These are

\[ SUP(1) = 2, \quad SUP(2) = 23, \quad SUP(3) = 2311, \quad SUP(14) = 23113475397131197241409431607673, \]

and

\[ SUP(106) = 231134753971311972414094316076737397519839911103143341891531155317091721237123932447263327828328973041310932173713373352735473593367136914057415342114297436344094451451745194729490349695059081553160296481656968336911704372197297745975377559758376037691772780118101816785398573896989719013913793119377951196196343972197439851994110247102711036910391104571045910501105671065710691108311090911087111971127311779. \]

This prime consists of 413 digits.

It may be noted that SUP(245) consists of 1108 digits.

Can you find more primes in Smarandache UP-sequence and are there infinitely many such primes?

References

A new Smarandache multiplicative
function and its arithmetical properties

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Abstract In this paper, we introduce a new number theory function \( D(n) \), then we use the elementary method to study the arithmetical properties of \( D(n) \), and give several interesting conclusions for it.

Keywords Number theory function; Divisor function; Arithmetical properties.

§1. Introduction and results

For any positive integer \( n \), the famous Dirichlet divisor function \( d(n) \) is defined as the number of all distinct positive divisors of \( n \). If \( n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r} \) be the prime power factorization of \( n \), then from the definition and properties of \( d(n) \) we may get

\[
d(n) = (\alpha_1 + 1) \cdot (\alpha_2 + 1) \cdots (\alpha_r + 1).
\]

From this formula we may immediately deduce that the first few values of \( d(n) \) are

\[
\begin{align*}
d(1) &= 1, \\
d(2) &= 2, \\
d(3) &= 2, \\
d(4) &= 3, \\
d(5) &= 2, \\
d(6) &= 4, \\
d(7) &= 1, \\
d(8) &= 4 \cdots
\end{align*}
\]

Now for any positive integer \( n \), we use divisor function \( d(n) \) to define a new number theory function \( D(n) \) as follows:

\[ D(n) \text{ denotes the smallest positive integer } m \text{ such that } n \text{ divide product } d(1) d(2) \cdots d(m). \]

That is,

\[ D(n) = \min \left\{ m : \ n \mid \prod_{i=1}^{m} d(i) \right\}. \]

For example, \( D(1) = 1, \ D(2) = 2, \ D(3) = 4, \ D(4) = 3, \ D(5) = 2, \ D(6) = 4, \ D(7) = 2^6, \ D(8) = 4, \ D(9) = 9, \ D(10) = 16, \ D(11) = 2^{10}, \ D(12) = 4, \ D(13) = 2^{12}, \ D(14) = 64, \ D(15) = 16, \ D(16) = 6, \ D(17) = 2^{16}, \ D(18) = 9, \ D(19) = 2^{18}, \ D(20) = 16 \cdots \cdots \) 

Recently, Professor Zhang Wenpeng asked us to study the arithmetical properties and the mean value properties of \( D(n) \). About these problems, it seems that none had studied it yet, at least we have not seen any related papers before. I think these problems are interesting, because there are some close relations between \( D(n) \) and the Dirichlet divisor function \( d(n) \), so it can help us to find more information of \( d(n) \). The main purpose of this paper is using the elementary method to study the calculating problem of \( D(n) \), and give several interesting calculating formulae. That is, we shall prove the following:

**Theorem 1.** Let \( p \) be a prime, then we have the calculating formulae

A. \( D(p) = 2^{p-1} \), for all prime \( p \).
B. \( D(3^2) = 3^2, \ D(p^2) = 2^{p-1} \cdot 3, \ D(3^3) = 2^2 \cdot 3, \ D(p^3) = 2^{p-1} \cdot 5, \) if prime \( p \geq 5; \)
\( D(3^4) = 3^2 \cdot 2, \ D(5^4) = 3^4, \ D(p^4) = 2^{p-1} \cdot 7, \) if \( p \geq 7; \ D(3^5) = 2^2 \cdot 5, \ D(5^5) = 2^4 \cdot 7, \)
$D(p^5) = 2^{p-1} \cdot 9$, $D(3^6) = 2^{1} \cdot 9$, $D(p^6) = 2^{p-1} \cdot 11$, if prime $p \geq 7$; $D(3^7) = 2^5$, $D(5^7) = 3^{1} \cdot 2$, $D(7^7) = 3^9$, $D(p^7) = 2^{p-1} \cdot13$, if prime $p \geq 11$; $D(3^8) = 2^{1} \cdot 3^2$, $D(5^8) = 2^{1} \cdot 11$, $D(7^8) = 2^6 \cdot 13$, $D(p^8) = 2^{p-1} \cdot15$, if prime $p \geq 11$.

**Theorem 2.** If $n$ be a square-free number (i.e., $p|n$ if and only if $p^2 \nmid n$), then $D(n) = 2^{P(n)-1}$, where $P(n)$ denotes the largest prime divisor of $n$.

**Theorem 3.** $D(n)$ is neither an additive function nor a multiplicative function, but $D(n)$ is a Smarandache multiplicative function.

§2. Proof of the theorems

In this section, we shall prove our Theorems directly. First we prove Theorem 1. We discuss the value distribution of $D(n)$ in the following two cases:

(i). If $n = p$ be a prime, let $D(p) = m$, then from the definition of $D(n)$ we have

$$p | \prod_{i=1}^{m} d(i), \quad p | \prod_{i=1}^{j} d(i), \quad 0 < j < m.$$ 

So that $p | d(m)$. Let $m = p_{1}^{\alpha_{1}} \cdot p_{2}^{\alpha_{2}} \cdots p_{r}^{\alpha_{r}}$ be the prime power factorization of $m$, then $p | d(m) = (\alpha_{1} + 1)(\alpha_{2} + 1) \cdots (\alpha_{r} + 1)$ implies that $p$ divide one of $\alpha_{i} + 1$, where $1 \leq i \leq r$. So $m = 2^{p-1}$ be the smallest positive integer such that $p | d(m)$.

(ii). If $n = p^2$, where $p \geq 5$ be a prime, let $D(p^2) = m$. Then from the definition of $D(p^2)$ we can deduce that $p^2 | \prod_{i=1}^{m} d(i)$. So $p^2$ divide one of $d(1), d(2), \cdots, d(m)$, or $p$ divide two of $d(1), d(2), \cdots, d(m)$. If $p^2$ divide one of $d(1), d(2), \cdots, d(m)$, then $m = 2^{p^2-1}$ or $m = 2^{p-1} \cdot 3^{p-1}$; If $p$ divide two of $d(1), d(2), \cdots, d(m)$, then $m = 2^{p-1} \cdot 3$. So the smallest positive integer $m$ such that $p^2 | d(1)d(2)\cdots d(m)$ is $m = 2^{p-1} \cdot 3$, since $2^{p-1} \cdot 3 < 3^{p-1} < 2^{p-1}$.

If $p \geq 5$, then it is easy check the inequality

$$2^{p-1} < 3^{p-1} < 2^{p-1} < 5^{p-1} < 7^{p-1} < 2^{3p-1} < 3^{2p-1},$$

so using the same method we can also obtain the calculating formula $D(3^2) = 3^2$, $D(p^2) = 2^{p-1} \cdot 3$, $D(3^3) = 2^{1} \cdot 3$, $D(p^3) = 2^{1} \cdot 5$, if prime $p \geq 5$; $D(3^4) = 2^{1} \cdot 2$, $D(5^4) = 3^{4}$, $D(p^4) = 2^{p-1} \cdot 7$, if $p \geq 7$; $D(3^5) = 2^{2} \cdot 5$, $D(5^5) = 2^{4} \cdot 7$, $D(p^5) = 2^{p-1} \cdot 9$, $D(3^6) = 2^{2} \cdot 7$, $D(5^6) = 2^{4} \cdot 9$, $D(p^6) = 2^{p-1} \cdot11$, if prime $p \geq 7$; $D(3^7) = 2^{5}$, $D(5^7) = 3^{4} \cdot 2$, $D(7^7) = 3^{6}$, $D(p^7) = 2^{p-1} \cdot13$, if prime $p \geq 11$; $D(3^8) = 2^{1} \cdot 3^{2}$, $D(5^8) = 2^{1} \cdot 11$, $D(7^8) = 2^{6} \cdot 13$, $D(p^8) = 2^{p-1} \cdot15$, if prime $p \geq 11$.

This proves Theorem 1.

Now we prove Theorem 2. If $n$ be a square-free number, let $n = p_1p_2 \cdots p_k$, where $p_1 < p_2 < \cdots < p_k$ are $k$ deferent primes. Suppose that $D(n) = m$, then it is clear that

$$n = p_1 \cdot p_2 \cdots p_k | \prod_{i=1}^{2^{p_k-1}} d(i) = d(1)d(2) \cdots d(2^{p_k-1})d(2^{p_k-1})\cdots d(2^{p_k-1}),$$

and $n$ does not divide $d(1)d(2)d(3)\cdots d(m)$, if $m < 2^{p_k-1}$. So the smallest positive integer $m$ such that $n = p_1 \cdot p_2 \cdots p_k$ divide $d(1)d(2)\cdots d(m)$ is $m = 2^{p_k-1}$. This proves Theorem 2.
Finally, we prove Theorem 3. It is clear that $D(1) + D(2) = 3$, $D(1 + 2) = 4$, $D(1) + D(2) \neq D(1 + 2)$; $D(1) + D(2) + D(3) = 7$, $D(1 + 2 + 3) = 4$, $D(1) + D(2) + D(3) \neq D(1 + 2 + 3)$. So $D(n)$ is not a additive function.

On the other hand, $D(1) \cdot D(2) \cdot D(3) = 8$, $D(1 \times 2 \times 3) = D(6) = 4$, $D(1) \cdot D(2) \cdot D(3) \neq D(1 \times 2 \times 3)$; $D(1) \cdot D(2) \cdot D(3) \cdot D(4) = 8$, $D(1 \times 2 \times 3 \times 4) = D(24) = 5$, $D(1) \cdot D(2) \cdot D(3) \cdot D(4) \neq D(1 \times 2 \times 3 \times 4)$. That is, $D(n)$ is not a multiplicative function. So $D(n)$ is neither an additive function nor a multiplicity function.

A number theory function $f(n)$ is called the Smarandache multiplicative function, if $f(n) = \max\{f(p_1^{\alpha_1}), f(p_2^{\alpha_2}), \cdots, f(p_k^{\alpha_k})\}$, where $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime power factorization of $n$. Now we prove that $D(n)$ be a Smarandache multiplicative function. In fact, let $m = \max\{D(p_1^{\alpha_1}), D(p_2^{\alpha_2}), \cdots, D(p_k^{\alpha_k})\} = D(p_i^{\alpha_i})$ for all $i = 1, 2, \cdots, k$. So $p_i^{\alpha_i}$ divide the product $d(1)d(2) \cdots d(m)$ for all $i = 1, 2, \cdots, k$. Note that ($p_i^{\alpha_i}, p_j^{\alpha_j}$) = 1, if $i \neq j$. So $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ also divide the product $d(1)d(2) \cdots d(m)$. That is means, $D(n)$ is a Smarandache multiplicative function. This completes the proof of Theorem 3.

References

A predator-prey epidemic model with infected predator

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Abstract A predator-prey epidemic model with infected predator is studied, and the effects of the disease on predation for the infected predator are not considered. A basic reproductive number which determines the outcome of the disease is given and the existence of the endemic equilibrium is discussed. By limiting system theory and Liapunov’s stability, the necessary and sufficient conditions ensuring the global asymptotical stability of the disease-free equilibrium and the locally asymptotically stable of the endemic equilibrium is obtained.

Keywords Predator-prey, epidemic model, equilibrium, stability

§1. Introduction

The study which includes ecology and epidemiology is now termed as eco-epidemiology. Quite a number of studies have already been performed in eco-epidemilogical systems [1, 2, 3]. It is practical and significant to study the combined model when one part of the recovered individuals can acquire permanent immunity while the other part has no immunity [4]. However, there are many disease with latent period [5, 6]. Thus, in this paper, we will consider a kind of the combination of SEIR and SEIS models in predator species. This paper is organized as follows: In section 2, the mathematical model is formulated. In section 3, the basic reproduction is obtained and the existence and stability of the endemic equilibrium is investigated.

§2. Model formulation

Let $X(t), Y(t)$ be the numbers of individuals in prey population and predator population at time $t$, respectively. The following is a predator-prey model:

$$
\begin{aligned}
X' &= X(a - bX - cY), \\
Y' &= Y(hX - \mu).
\end{aligned}
$$

(1)

where $a, b, c, h, \mu$ are positive constants.

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For system (1), the following lemma is well known [7].

**Lemma 1.** System (1) always has two equilibria $E_0(0,0)$ and $E_1\left(\frac{a}{b},0\right)$. Equilibrium $E_0$ is always unstable, equilibrium $E_1$ is unstable if $Q = \frac{b\mu}{ah} < 1$ and is globally asymptotically stable (GAS) if $Q = \frac{b\mu}{ah} > 1$; When $Q = \frac{b\mu}{ah} < 1$, system (1) also has a unique positive equilibrium $E_2\left(\mu h, a(1 - Q)/c\right)$ which is GAS.

Assume that the disease only spreads among the predator species, and that the effects of the disease on predation for the infected predator are not considered. The total population of predator $Y(t)$ is divided into four subpopulation: the susceptible $S(t)$, the exposed $E(t)$, the infectious $I(t)$, and the recovered $R(t)$. Assume that the disease has the vertical transmission for the exposed and the infectious individuals, and that the corresponding proportions are $q, p(0 < q, p < 1)$ respectively. So we will consider the combination of SEIR and SEIS model with bilinear infection rate $\beta SI$ as follows:

$$
\begin{align*}
X' &= X(a - bX) - cXS - cXI - cXE - cXR, \\
Y' &= Y(hX - \mu), \\
S' &= hXS + (1-p)hXI + (1-q)hXE + hXR - \beta SI - \mu S + \delta I, \\
E' &= phXI + qhXE + \beta SI - \epsilon E - \mu E, \\
I' &= \epsilon E - \gamma I - \mu I - \delta I, \\
R' &= \gamma I - \mu R.
\end{align*}
$$

(2)

Here $\epsilon$ is the transfer rate constant from the exposed subpopulation to the infectious subpopulation, $\delta$ is the transfer rate constant from the infective subpopulation to the susceptible subpopulation, and $\gamma$ is the transfer rate constant from the infective subpopulation to the recovered subpopulation.

Substituting $R = Y - S - E - I$ into the first two equations in (2) gives the following equations

$$
\begin{align*}
X' &= X(a - bX) - cXY, \\
Y' &= Y(hX - \mu), \\
S' &= hXS + (1-p)hXI + (1-q)hXE + hX(Y - S - E - I) - \beta SI - \mu S + \delta I, \\
E' &= phXI + qhXE + \beta SI - \epsilon E - \mu E, \\
I' &= \epsilon E - \gamma I - \mu I - \delta I.
\end{align*}
$$

(3)

Obviously, it follows from Lemma 1 that, when $Q > 1$, $\lim_{t \to \infty} S(t) = \lim_{t \to \infty} E(t) = \lim_{t \to \infty} I(t) = \lim_{t \to \infty} R(t) = 0$ and $\lim_{t \to \infty} X(t) = \frac{a}{b}$. Further, when $Q < 1$, the system consisting of the last three equations in (3) has the following limiting system

$$
\begin{align*}
S' &= \mu K - \beta SI - p\mu I - q\mu E - \mu S + \delta I, \\
E' &= p\mu I + q\mu E + \beta SI - \epsilon E - \mu E, \\
I' &= \epsilon E - \gamma I - \mu I - \delta I.
\end{align*}
$$

(4)

where $K = \frac{a(1 - Q)}{c}$.
In the following, we consider the dynamical behavior of (4), which is equivalent with that of (3).

§3. The existence and the stability of equilibria

Denote \( N = S + E + I \), it follows from (4) that \( N' = \mu K - \mu N - \gamma I \leq \mu(K - N) \), so \( \limsup_{t \to \infty} N(t) \leq K \). Therefore, the set \( \Omega = \{(S, E, I) \in \mathbb{R}^3_+|S + E + I \leq K\} \) is positively invariant to (4).

By straightforward calculation, there are the following result on the existence of equilibrium for (4).

**Theorem 1.** Denote \( R_0 = \frac{\beta \epsilon K}{(\mu + \gamma + \delta)(\mu + \epsilon - q \mu) - p \epsilon \mu} \). System (4) always exists the disease-free equilibrium \( E_0^3(K,0,0) \in \Omega \). And system also has a unique endemic equilibrium \( E_2^3(S^*, E^*, I^*) \in \Omega \) if \( R_0 > 1 \), where

\[
S^* = \frac{\mu K - (\mu + \epsilon)(\mu + \gamma + \delta)I^*}{\mu}, \quad E^* = \frac{\mu + \gamma + \delta}{\epsilon} I^*, \quad I^* = \frac{\mu((\mu + \gamma + \delta)(\mu + \epsilon - q \mu) - p \epsilon \mu)}{\beta(\mu + \gamma + \delta)(\mu + \gamma)}(R_0 - 1).
\]

On the stability of equilibria of (4), we have

**Theorem 2.** The disease-free equilibrium \( E_0^3 \) is GAS in \( \Omega \) if \( R_0 < 1 \) and it is unstable if \( R_0 > 1 \). The endemic equilibrium \( E_2^3 \) is locally asymptotically stable (LAS) if it exists.

**Proof.** Let \( V = \epsilon E + (\mu + \epsilon - q \mu)I \), then straightforward calculation shows that \( V' < 0 \) if \( R_0 < 1 \). Therefore, it follows from Liapunov stability theorem that the equilibrium \( E_2^3 \) is GAS if \( R_0 < 1 \).

Jacobian matrix of (4) at \( E_2^3 \) is

\[
J(E_2^3) = \begin{pmatrix}
-\mu & -q \mu & -\beta K - p \mu - \delta \\
\beta I & q \mu - \mu - \epsilon & \beta K + p \mu \\
0 & \epsilon & -(\mu + \gamma + \delta)
\end{pmatrix},
\]

and \( \det(\lambda I - J(E_2^3)) = (\lambda + \mu)(\lambda^2 + \lambda(2\mu + \epsilon + \gamma - q \mu + \delta) + (\mu + \gamma + \delta)(\mu + \epsilon - q \mu) - \epsilon(p \mu + \beta K)) = 0 \).

Since \((2\mu + \epsilon + \gamma - q \mu + \delta) > 0 \) for \( 0 < q < 1 \), then all eigenvalues have negative real parts if and only if \((\mu + \gamma + \delta)(\mu + \epsilon - q \mu) - \epsilon(p \mu + \beta K) > 0 \), that is \( R_0 < 1 \). Therefore \( E_2^3 \) is unstable if \( R_0 > 1 \).

Jacobian matrix of (4) at \( E_2^3 \) is

\[
J(E_2^3) = \begin{pmatrix}
-\beta I^* - \mu & -q \mu & \beta S^* - p \mu - \delta \\
\beta I^* & q \mu - (\mu + \epsilon) & \beta S^* + p \mu \\
0 & \epsilon & -(\mu + \gamma + \delta)
\end{pmatrix},
\]

and \( \det(\lambda I - J(E_2^3)) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 \), where \( a_1 = \beta I^* + (3 - q)\mu + \epsilon + \gamma + \delta > 0 \), \( a_2 = (\mu + \gamma + \delta)(\mu + \beta I^*) + \mu(\mu + \epsilon - q \mu) + \beta I^*(\mu + \epsilon) > 0 \), \( a_3 = \beta I^*(\mu + \epsilon)(\mu + \gamma + \delta) + \beta I^* \epsilon \delta > 0 \).
0. Straightforward calculation shows that $a_1a_2 - a_3 > 0$. It follows from Routh-Hurwitz Criterion[8] that the endemic equilibrium $E_2^*$ is LAS when $R_0 > 1$.

This completes the proof.

References

An equation involving the Lucas numbers and Smarandache primitive function

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Abstract For any positive integer \( n \), let \( S_p(n) \) denotes the Smarandache primitive function, \( L_n \) denotes the Lucas numbers. The main purpose of this paper is using the elementary methods to study the number of the solutions of the equation \( S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = S_p(L_{n+2} - 3) \), and give all positive integer solutions for this equation.

Keywords Lucas numbers, Smarandache primitive function, equation, solutions.

§1. Introduction

As usual, the Lucas sequence \( L_n \) is defined by the second-order linear recurrence sequence \( L_{n+2} = L_{n+1} + L_n \) for \( n \geq 1 \), \( L_1 = 1 \), \( L_2 = 3 \). This sequence plays a very important role in the studied of theory and application of mathematics. Therefore, the various properties of \( L_n \) was investigated by many authors. For example, J.L.Brown [1] studied the unique representation of integers as sums of Distinct Lucas Numbers. Wenpeng Zhang [2] obtained some identities involving the Lucas numbers.

Let \( p \) be a prime, \( n \) be any positive integer. The Smarandache primitive function \( S_p(n) \) is defined as the smallest positive integer such that \( S_p(n)! \) is divisible by \( p^n \). For example, \( S_2(1) = 2, S_2(2) = S_2(3) = 4, S_2(4) = 6, \cdots \). In problem 47, 48 and 49 of book [3], the famous Rumanian-born American number theorist, Professor F.Smarandache asked us to study the properties of the \( S_p(n) \). There are closely relations between the Smarandache primitive function \( S_p(n) \) and the famous function \( S(n) \), where

\[
S(n) = \min\{m : m \in N, n \mid m!\}.
\]

From the definition of \( S(n) \), obviously we have \( S(p) = p \), and if \( n \neq 4, n \neq p \), then \( S(n) < n \). So we have

\[
\pi(x) = -1 + \sum_{i=2}^{[x]} \left\lfloor \frac{S(n)}{n} \right\rfloor,
\]

where \( \pi(x) \) denotes the number of primes which less than \( x \).

The research on Smarandache function \( S(n) \), Smarandache primitive function \( S_p(n) \) and the equations involving Smarandache primitive function \( S_p(n) \) is an significant and important problem in Number Theory. Therefore, many scholars and researchers have studied them before,
see reference [4-6]. Professor Zhang [7] have obtained an interesting asymptotic formula. That is, for any fixed prime $p$ and any positive integer $n$, we have

$$S_p(n) = (p - 1)n + O \left( \frac{p}{\ln p} \cdot \ln n \right).$$

Li Jie [8] studied the solvability of the equation $S_p(1) + S_p(2) + \cdots + S_p(n) = S_p(n \cdot (n + 1) \cdot 2)$, and gave its all positive integer solutions. But it seems that no one knows the relationship between the Lucas numbers and the Smarandache primitive function. In this paper, we use the elementary methods to study the solvability of the equation

$$S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = S_p(L_{n+2} - 3),$$

and give all positive integer solutions for this equation. That is, we will prove the following:

**Theorem.** Let $p$ be a given prime, $n$ be any positive integer, then the equation

$$S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = S_p(L_{n+2} - 3) \quad (1)$$

has finite solutions. They are $n = 1, 2, \cdots, n_p$, where

$$n_p = \left\lfloor \frac{\log \left( \frac{(p + 3) + \sqrt{(p + 3)^2 + 4}}{2} \right) - \log 2}{\log(1 + \sqrt{5}) - \log 2} - 2 \right\rfloor,$$

[$x$] denotes the biggest integer $\leq x$.

Especially, taking $p = 3, 5, 11$, we may immediately deduce the following:

**Corollary 1.** All of the positive integer solutions for the equation

$$S_3(L_1) + S_3(L_2) + \cdots + S_3(L_n) = S_3(L_{n+2} - 3)$$

are $n = 1$.

**Corollary 2.** All of the positive integer solutions for the equation

$$S_5(L_1) + S_5(L_2) + \cdots + S_5(L_n) = S_5(L_{n+2} - 3)$$

are $n = 1, 2$.

**Corollary 3.** All of the positive integer solutions for the equation

$$S_{11}(L_1) + S_{11}(L_2) + \cdots + S_{11}(L_n) = S_{11}(L_{n+2} - 3)$$

are $n = 1, 2, 3$.

**§2. Several lemmas**

To complete the proof of the theorem, we need the following several simple lemmas.

**Lemma 1.** Let $p$ be a prime, $n$ be any positive integer, $S_p(n)$ denote the Smarandache primitive function, then we have

$$S_p(k) \begin{cases} = pk, & \text{if } k \leq p, \\ < pk, & \text{if } k > p. \end{cases}$$
Proof. (See reference [9]).

Lemma 2. Let \( L_n \) be the Lucas sequence with \( L_1 = 1 \) and \( L_2 = 3 \), then we have the identity
\[
L_1 + L_2 + \cdots + L_n = L_{n+2} - 3.
\]

Proof. From the second-order linear recurrence sequence \( L_{n+2} = L_{n+1} + L_n \) we can easily get the identity of Lemma 2.

Lemma 3. Let \( p \) be a prime, \( n \) be any positive integer, if \( n \) and \( p \) satisfying \( p^\alpha \parallel n! \), then
\[
\alpha = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor .
\]

Proof. (See reference [10]).

Lemma 4. Let \( p \) be a prime, \( n \) be any positive integer. Then there must exist a positive integer \( M_k \) with \( 1 \leq M_k \leq L_k (k=1,2,\ldots,n) \) such that
\[
S_p(L_1) = M_1p, \quad S_p(L_2) = M_2p, \quad \cdots, \quad S_p(L_n) = M_np.
\]

and
\[
L_k \leq \sum_{i=1}^{\infty} \left\lfloor \frac{M_k p}{p^i} \right\rfloor .
\]

Proof. From the definition of \( S_p(n) \), Lemma 1 and Lemma 3, we can easily get the conclusions of Lemma 4.

§3. Proof of the theorem

In this section, we will complete the proof of Theorem.

First, if \( p = 2 \), then the equation (1) is \( S_2(L_1) + S_2(L_2) + \cdots + S_2(L_n) = S_2(L_{n+2} - 3) \).

(i) If \( n = 1 \), \( S_2(L_1) = 2 = S_2(L_3 - 3) \), so \( n = 1 \) is the solution of the equation (1).

(ii) If \( n = 2 \), \( S_2(L_1) + S_2(L_2) = 2 + 2 \times 2 = S_2(L_4 - 3) \), so \( n = 2 \) is the solution of the equation (1).

(iii) If \( n = 3 \), \( S_2(L_1) + S_2(L_2) + S_2(L_3) = 2 + 2 \times 2 + 3 \times 2 = 12 \), but \( S_2(L_5 - 3) = S_2(8) = 10 \), so \( n = 3 \) is not the solution of the equation (1).

(iv) If \( n > 3 \), then from Lemma 3 we know that there must exist a positive integer \( M_k \) with \( 1 \leq M_k \leq L_k (k=1,2,\ldots,n) \) such that
\[
S_2(L_1) = 2M_1, S_2(L_2) = 2M_2, \cdots, S_2(L_n) = 2M_n.
\]

So we have \( S_2(L_1) + S_2(L_2) + \cdots + S_2(L_n) = 2(M_1 + M_2 + \cdots + M_n) \).
On the other hand, notice that $M_1 = 1$, $M_2 = 2$, $M_3 = 3$, then from Lemma 3 we have

\[
\sum_{i=1}^{\infty} \left[ \frac{2(M_1 + M_2 + \cdots + M_n) - 1}{2^i} \right]
= \sum_{i=1}^{\infty} \left[ \frac{2(M_1 + M_2 + \cdots + M_n - 1) + 1}{2^i} \right]
= M_1 + M_2 + \cdots + M_n - 1 + \sum_{i=2}^{\infty} \left[ \frac{2(M_1 + M_2 + M_3 - 1) + 1}{2^i} \right]
\geq M_1 + M_2 + \cdots + M_n - 1 + \sum_{i=2}^{\infty} \left[ \frac{2(M_1 + M_2 + M_3 - 1) + 1}{2^i} \right]
+ \sum_{i=1}^{\infty} \left[ \frac{M_4 + M_5 + \cdots + M_n}{2^i} \right]
\geq M_1 + (M_2 + 1) + (M_3 + 1) + \left( M_4 + \sum_{i=1}^{\infty} \left[ \frac{M_i}{2^i} \right] \right)
+ \cdots + \left( M_n + \sum_{i=1}^{\infty} \left[ \frac{M_i}{2^i} \right] \right)
\geq L_1 + L_2 + L_3 + \sum_{i=1}^{\infty} \left[ \frac{2M_i}{2^i} \right] + \cdots + \sum_{i=1}^{\infty} \left[ \frac{2M_i}{2^i} \right]
\geq L_1 + L_2 + L_3 + L_4 + \cdots + L_n
= L_{n+2} - 3
\]
then from Lemma 2 we can get

\[
2^{L_{n+2} - 3} \mid (2(M_1 + M_2 + \cdots + M_n) - 1)!
\]

Therefore,

\[
S_2(L_{n+2} - 3) \leq 2(M_1 + M_2 + \cdots + M_n) - 1
\]
\[
< 2(M_1 + M_2 + \cdots + M_n)
\]
\[
= S_2(L_1) + S_2(L_2) + \cdots + S_2(L_n).
\]

so there is no solutions for the equation (1) in this case.

If $p \geq 3$ we will discuss the solutions of the equation $S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = S_p(L_{n+2} - 3)$ in the following three cases:

(i) If $L_{n+2} - 3 \leq p$, solving this inequality we can get $1 \leq n \leq n_p$, where

\[
n_p = \left[ \frac{\log (p + 3) + \sqrt{p^2 + 6p + 10 - \log 2}}{\log(1 + \sqrt{5}) - \log 2} \right] - 2
\]

[2] denotes the biggest integer $\leq x$.

Then from Lemma 1 we have

\[
S_p(L_{n+2} - 3) = p(L_{n+2} - 3).
\]
Noting that $1 \leq n \leq n_p$, that is $L_n \leq L_{n+2} - 1 \leq p$, from Lemma 1 and Lemma 2 we can get

$$S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = pL_1 + pL_2 + \cdots + pL_n = p(L_{n+2} - 3)$$

Combining above two formulae, we may easily get $n = 1, 2, \cdots, n_p$ are the solutions of the equation $S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = S_p(L_{n+2} - 3)$.

(ii) If $L_n < p < L_{n+2} - 3$, solving this inequality we can get $n_p < n \leq N_p$, where

$$N_p = \left\lfloor \frac{\log \left(\sqrt{5}p + \sqrt{5p^2 + 4}\right) - \log 2}{\log(1 + \sqrt{5}) - \log 2} \right\rfloor,$$

[3] denotes the biggest integer $\leq x$.

Then from Lemma 1 we have

$$S_p(L_{n+2} - 3) < p(L_{n+2} - 3).$$

But

$$S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = pL_1 + pL_2 + \cdots + pL_n = p(L_{n+2} - 3).$$

Hence the equation (1) has no solution in this case.

(iii) If $n \geq N_p + 1$, that is $p < L_n < L_{n+2} - 1$.

Now from Lemma 4 there must exist a positive integer $M_k$ with $1 \leq M_k \leq L_k (k = 1, 2, \cdots, n)$ such that

$$S_p(L_1) = M_1 p, \quad S_p(L_2) = M_2 p, \quad \cdots, \quad S_p(L_n) = M_n p.$$

Then we have

$$S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = p(M_1 + M_2 + \cdots + M_n).$$

(a) If $L_{N_p+1} = p + 1$, then $L_{N_p+2} - 4 = L_{N_p+1} + L_{N_p} - 4 = L_{N_p} + p - 3 \geq p$. 

Notice that \( M_1 = L_1, M_2 = L_2, \ldots, M_{N_p} = L_{N_p} \), from Lemma 4 we also have

\[
\sum_{i=1}^{\infty} \left[ \frac{\left( M_1 + M_2 + \cdots + M_n \right)p - 1}{p^i} \right] = \sum_{i=1}^{\infty} \left[ \frac{p(M_1 + M_2 + \cdots + M_n - 1) + p - 1}{p^i} \right]
\]

\[
= M_1 + M_2 + \cdots + M_n - 1 + \sum_{i=2}^{\infty} \left[ \frac{p(M_1 + M_2 + \cdots + M_n - 1) + p - 1}{p^i} \right]
\]

\[
\geq M_1 + M_2 + \cdots + M_n - 1 + \sum_{i=2}^{\infty} \left[ \frac{p(M_1 + M_2 + \cdots + M_{N_p} - 1) + p - 1}{p^i} \right]
\]

\[
+ \sum_{i=2}^{\infty} \left[ \frac{p(M_{N_p+1} + M_{N_p+2} + \cdots + M_n)}{p^i} \right]
\]

\[
= M_1 + M_2 + \cdots + M_n - 1 + \sum_{i=2}^{\infty} \left[ \frac{p(L_{N_p+2} - 4) + p - 1}{p^i} \right] + \sum_{i=1}^{\infty} \left[ \frac{M_{N_p+1} + \cdots + M_n}{p^i} \right]
\]

\[
\geq L_1 + L_2 + \cdots + L_{N_p} + \sum_{i=1}^{\infty} \left[ \frac{M_{N_p+1}p}{p^i} \right] + \cdots + \sum_{i=1}^{\infty} \left[ \frac{M_n p}{p^i} \right]
\]

\[
\geq L_1 + L_2 + \cdots + L_n
\]

\[
= L_{n+2} - 3.
\]

Then from Lemma 3 we can get

\[
p^{L_{n+2}-3} \mid (p(M_1 + M_2 + \cdots + M_n) - 1)!. \]

Therefore,

\[
S_p(L_{n+2} - 3) \leq p(M_1 + M_2 + \cdots + M_n) - 1 < p(M_1 + M_2 + \cdots + M_n) = S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n).
\]

Hence the equation (1) has no solution in this case.

(b) If \( L_{N_p+1} > p + 1 \), then \( p < M_{N_p+1} \leq L_{N_p+1} = L_{N_p} + L_{N_p-1} < 2p \).

Notice that \( M_1 = L_1, M_2 = L_2, \ldots, M_{N_p} = L_{N_p} \) and \( p < M_{N_p+1} < 2p \). from Lemma 4 we
also have
\[
\sum_{i=1}^{\infty} \left[ \frac{(M_1 + M_2 + \cdots + M_n)p - 1}{p^i} \right] = \sum_{i=2}^{\infty} \left[ \frac{p(M_1 + M_2 + \cdots + M_n - 1) + p - 1}{p^i} \right] = M_1 + M_2 + \cdots + M_n - 1 + \sum_{i=2}^{\infty} \left[ \frac{p(M_1 + M_2 + \cdots + M_n - 1) + p - 1}{p^i} \right] + \sum_{i=2}^{\infty} \left[ \frac{p(M_{N_p+1} + M_{N_p+2} + \cdots + M_n - 1)}{p^i} \right] \\
\geq M_1 + M_2 + \cdots + M_n - 1 + \sum_{i=2}^{\infty} \left[ \frac{p(L_{N_p+2} - 3) + p - 1}{p^i} \right] + \sum_{i=1}^{\infty} \left[ \frac{M_{N_p+1} + \cdots + M_n - 1}{p^i} \right] + \cdots + \left( M_n + \sum_{i=1}^{\infty} \left[ \frac{M_n}{p^i} \right] \right) \\
\geq L_1 + L_2 + \cdots + L_{N_p} + \sum_{i=1}^{\infty} \left[ \frac{M_{N_p+1}p}{p^i} \right] + \cdots + \sum_{i=1}^{\infty} \left[ \frac{M_n p}{p^i} \right] \\
= L_{n+2} - 3.
\]

Then from Lemma 3 we can get
\[
p^{L_{n+2}-3} | (p(M_1 + M_2 + \cdots + M_n) - 1)!.
\]

Therefore,
\[
S_p(L_{n+2} - 3) \leq p(M_1 + M_2 + \cdots + M_n) - 1 \\
< p(M_1 + M_2 + \cdots + M_n) \\
= S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n).
\]

Hence \( n \geq N_p + 1 \) the equation (1) has no solution.
Now the Theorem follows from (i), (ii), and (iii).

References


Two rings in Is-algebras

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Abstract
Let X be a IS-algebra(BCI-Semigroup), R(X) = \{x \in X|0 \ast x = x\} be called ring part of X, AR(X) = \{x \in X|0 \ast (0 \ast x) = x\} be called adjoint ring part of X. They are subalgebras and rings in X, and some equivalent conditions of R(X) and AR(X) being ideals are shown.

Keywords IS-algebra ideal, subalgebra, ring part.

§1. Basic concepts

Mathematician K.Iseki of Japan introduced the BCI–algebra in1966. Mathematician Y.B.Jun in Korea introduced the BCI-semigroup( IS-algebra) in 1993. We show some relational definitions and conclusions for convenience of discussion.

Definition 1[1]. An algebra \((X, \ast, 0)\) of \((2, 0)\) is said to be a BCI-algebra if it satisfies:
(1) \(((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0.\)
(2) \((x \ast (x \ast y)) \ast y = 0.\)
(3) \(x \ast x = 0.\)
(4) \(x \ast y = 0\) and \(y \ast x = 0\) imply \(x = y.\)

In BCI–algebra, we have the following basic formulas:
(1) \((x \ast y) \ast z = (x \ast z) \ast y.\)
(2) \(x \ast 0 = x.\)
(3) \(0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y).\)
(4) \(0 \ast (0 \ast (0 \ast x)) = 0 \ast x.\)

Definition 2[2]. A IS-algebra \((X, \ast, \cdot, 0)\) is a non-empty set X with two operation “\(\ast\)” and “\(\cdot\)” , and with a constant element 0 such that following axioms are satisfied: for all \(x, y, z \in X,\) we have
(1)\((X, \ast, 0)\) is BCI-algebra.
(2) \((X, \cdot)\) is semigroup.
(3) Distributive law: \(x \cdot (y \ast z) = (x \cdot y) \ast (x \cdot z), (x \ast y) \cdot z = (x \cdot z) \ast (y \cdot z).\)
\(x \cdot y\) is usual to be written \(xy\) and IS-algebra \((X, \ast, \cdot, 0)\) is usual to be written \(X\) for short.

In IS-algebra \(X,\) we have \(0x = x0 = 0[2].\)

Let \(Y\) be the non-empty subset of IS-algebra \(X,\) if “\(\ast\)” and “\(\cdot\)” are closed in \(Y,\) then \((Y, \ast, \cdot, 0)\) is IS-algebra too, we call it is a subalgebra of \(X[2].\)
Definition 3. [3] I is non-empty subset of IS-algebra X, It is said to be ideal of X, if
(1) \( \forall x \in X, \forall a \in I, \) we have \( xa, ax \in I. \)
(2) \( \forall x, y \in X, \) if \( x \cdot y \in I \) and \( y \in I, \) then \( x \in I. \)
It is easy to prove that \((R, - , 0)\) is IS-algebra in Ring \((R, + , \cdot)\), the ideal of \( R \) as Ring is agree
with the ideal of \( R \) as IS-algebra.

§2. Ring part of Is-algebra

Similar with BCI-G part of BCI-algebra, we introduce some relational definition.

Definition 4. Let \((X, \ast , \cdot , 0)\) be IS-algebra, \( R(X) = \{ x \in X | 0 \ast x = x \} \) is said to be ring
part of X.

Clearly, \( Y \) is ring part of IS-algebra \((X, \ast , \cdot , 0)\) if and only if \( Y \) is G part of BCI-algebra X.
We call \( R(X) \) is ring part of X, because the following conclusion:

Theorem 1. Ring part \( R(X) \) of IS-algebra X is subalgebra of X and ring that character
is 2. Also, \( R(X) \) is maximal ring about operation “\( \ast \)” and “\( \cdot \)”.

Proof. \( \forall x, y \in R(X) \), We have
\[
0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y) = x \ast y, 0 \ast (xy) = (0y) \ast (xy) = (0 \ast x)y = xy.
\]
So \( x \ast y, xy \in R(X), \) in other words, \( R(X) \) is subalgebra of X.
Since \( R(X) \) is combination part of BCI-algebra, then \((R(X), \ast , 0)\) certainly be combination BCI-
algebra[6], so it is Abel group that 0 is identity element. Also, since multiplication is closed in
\( R(X) \) and \( x \ast x = 0, \) then \((R(X), \ast , \cdot )\) is a ring which character is 2.

Let \( Y \) be ring in X about operation “\( \ast \)” and “\( \cdot \)”, \( \forall y \in Y \subseteq X, \) because \( 0 \ast y = y \ast 0 = y, \)
Then \( y \in R(X), \) and finally \( Y \subseteq R(X). \)

But generally speaking, \( R(X) \) may not be ideal of X.

Example 1. Let \( X = 0, a, b, \) put \( x \cdot y = 0, \) and operation “\( \ast \)” is following:

\[
\begin{array}{ccc}
  \ast & 0 & a & b \\
  0 & 0 & 0 & b \\
  a & 0 & 0 & b \\
  b & b & b & 0 \\
\end{array}
\]

Then \((X, \cdot , 0)\) is IS-algebra and \( R(X) = \{0, b\}. \) However, \( R(X) \) is not the ideal of X.

Following, we discuss the condition of \( R(X) \) to be ideal.

Lemma 1. \( R(X) \) is ideal of IS-algebra \((X, \ast , \cdot , 0)\) if and only if \( R(X) \) is ideal of BCI-algebra
\((X, \ast , 0)\).

Proof. The necessity is clearly, we discuss sufficient part as follow.

Let \( R(X) \) be ideal of BCI-algebra \((X, \ast , \cdot ), \) for every \( x \in X, a \in R(X), \) we have
\[
0 \ast (ax) = (0x) \ast (ax) = (0 \ast a)x = ax,
\]

Therefore \( ax \in R(X), \) by the same reasoning \( xa \in R(X), \) thus, \( R(X) \) is ideal of BCI-
algebra. Put \((X, \ast , 0)\) is BCI-algebra, \( a \) is a element in X, in reference [5], it gives us the
following mapping:

\[ a_r : x \rightarrow x \ast a (\forall x \in X) \]

and the following conclusion:

Lemma 2.\cite{5} Let \( G(X) \) be G-part (connection part) of BCI-algebra \( X \), The following conditions are equivalent:

1. \( G(X) \) is ideal of \( X \).
2. \( \forall a \in X, a_r : x \rightarrow x \ast a \) is injective.
3. \( \forall a, b \in X, a_r b_r = (a \ast b)_r = (b \ast a)_r \).
4. \( \forall x, y \in X, \forall a \in R(X) \), if \( x \ast a = y \ast a \), then \( x = y \).

We obtain the following conclusion by lemma 1 and lemma 2.

Theorem 2. Let \( R(X) \) be ring part of IS-algebra \((X, \ast, \cdot, 0)\), then the following conditions are equivalent:

1. \( R(X) \) is ideal of IS-algebra \((X, \ast, \cdot, 0)\).
2. \( \forall a \in X, a_r : x \rightarrow x \ast a \) is injective.
3. \( \forall a, b \in X, a_r b_r = (a \ast b)_r = (b \ast a)_r \).
4. \( \forall x, y \in X, \forall a \in R(X) \), if \( x \ast a = y \ast a \), then \( x = y \).

§3. Adjoint ring part of IS-algebra

Similar with p-semi simple part\cite{4} (generalized connection part\cite{6}), we take relational definitions to IS-algebra.

Definition 5. In IS-algebra \((X, \ast, \cdot, 0)\), it is said \( AR(X) = \{ x \in X | 0 \ast (0 \ast x) = x \} \) to be adjoint ring part of \( X \).

Example 2. Let \( X = \{0, a, b, c\}, xy = 0 \), and operation “\( \ast \)” is following:

<table>
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<tr>
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<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>c</td>
<td>0</td>
<td>a</td>
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<tr>
<td>a</td>
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<td>b</td>
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<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>0</td>
<td>c</td>
<td>a</td>
</tr>
</tbody>
</table>

Then \((X, \ast, \cdot, 0)\) is IS-algebra and \( R(X) = \{0\}, AR(X) = \{0, a, c\} \). Clearly \( Y \) is adjoint ring part of IS algebra \((X, \ast, \cdot, 0)\) if and only if \( Y \) is p-half single part of BCI-algebra \((X, \ast, 0)\). In IS-algebra \( X \), put \( x + y = x \ast (0 \ast y), \forall x, y \in AR(X) \), we have

\[ 0 \ast (0 \ast (x + y)) = 0 \ast (0 \ast (x \ast (0 \ast y))) = (0 \ast (0 \ast x)) \ast (0 \ast (0 \ast y)) = x \ast (0 \ast y) = x + y. \]

Therefore, \( x + y \in AR(X) \), then this addition can be operation of \( AR(X) \).

Theorem 3. \cite{6}In generalized connection BCI-algebra \((X, \ast, 0)\), let \( x + y = x \ast (0 \ast y) \), then \((X, +)\) is Abel group that 0 is the zero element.
Theorem 4. In IS-algebra $(X, \ast, 0)$, we have

1. $AR(X)$ is subalgebra of $(X, \ast, 0)$
2. $AR(X)$ is maximal ring of $X$ about $\ast$ and $\cdot$.

Proof. (1) $\forall x, y \in AR(X)$, we have

$$0 \ast (0 \ast (x \ast y)) = (0 \ast (0 \ast x)) \ast (0 \ast (0 \ast y)) = x \ast y,$$

$$0 \ast (0 \ast (xy)) = (0y) \ast (0y) \ast (xy)) = (0 \ast (0 \ast x))y = xy,$$

That is $x \ast y, xy \in AR(X)$, so $AR(X)$ is subalgebra of $(X, \ast, \cdot, 0)$.

(2) $(AR(X), \ast, 0)$ is BCI-algebra by (1), and is generalized connection. For lemma 3, $(AR(X), +)$ is Abel group that $0$ is zero element. Also

$$(x + y)z = (x \ast (0 \ast y))z = (xz) \ast ((0z) \ast (yz)) = (xz) \ast (0 \ast (yz)) = xz + yz$$

For same reasoning, $x(y + z) = xy + xz$, thus $(AR(X), +, \cdot)$ is ring. Let $(Y, +, \cdot)$ be ring in $X$, $\forall y \in Y, y + 0 = y \ast (0 \ast 0) = y$, hence $0 + y = y$, that is $0 \ast (0 \ast y) = y$, therefore $y \in AR(X)$, finally $Y \subseteq AR(X)$. Next, we discuss the condition that $R(X)$ to be ideal.

Similar with lemma 1, we have the following conclusion:

Lemma 4. AR(X) is ideal of IS-algebra $(X, \ast, \cdot, 0)$ if and only if AR(X) is ideal of BCI-algebra $(X, \ast, 0)$.

Lemma 5. [4] The following conditions are equivalent in BCI-algebra $(X, \ast, 0)$:

1. p-half single part $SP(X)$ is ideal of $X$.
2. $\forall a \in SP(X), a_\ast : x \rightarrow x \ast a$ is injective.
3. $\forall v \in X, \nu \in SP(X)$, have $(v \ast \nu) \ast (0 \ast \nu) = v$.

We can obtain the following conclusion by lemma 4 and lemma 5.

Theorem 5. In IS-algebra $(X, \ast, \cdot, 0)$, The following conditions are equivalent:

1. $AR(X)$ is ideal of $X$.
2. $\forall a \in AR(X), a_\ast : x \rightarrow x \ast a$ is injective.
3. $\forall v \in X, \nu \in SP(X), have (v \ast \nu) \ast (0 \ast \nu) = v$.

References

An identity related to Dedekind sums

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Abstract The main purpose of this paper is using the analytic methods and the properties of the Dirichlet $L$-function to study a summation problem involving the Dedekind sums, and give an interesting identity for it.

Keywords Dedekind sums, Dirichlet series, identity.

§1. Introduction

For any integer $q \geq 2$ and integer $h$, the classical Dedekind sums is defined by

$$S(h, q) = \sum_{a=1}^{q} \left( \left( \frac{a}{q} \right) \left( \frac{ah}{q} \right) \right),$$

where

$$\left( \left( x \right) \right) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

About the various properties of $S(h, q)$, many authors had studied it, and obtained a series results. For example, L.Carlitz [1] obtained a reciprocity theorem of $S(h, k)$. J. B. Conrey et al. [2] studied the mean value distribution of $S(h, k)$, and proved the following important asymptotic formula

$$\sum_{h=1}^{k} |S(h, k)|^{2m} = f_m(k) \left( \frac{k}{12} \right)^{2m} + O \left( \left( k^2 + k^{2m-1+\frac{1}{m+1}} \right) \log^3 k \right),$$

where $\sum'_h$ denotes the summation over all $h$ such that $(k, h) = 1$, and

$$\sum_{h=1}^{\infty} \frac{f_m(n)}{n^s} = 2 \frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s + 4m - 1)}{\zeta^2(s + 2m)} \cdot \zeta(s).$$

Jia Chaohua [3] improved the error term in (1) as $O \left( k^{2m-1} \right)$, provide $m \geq 2$. Zhang Wenpeng [4] improved the error term of (1) for $m = 1$. That is, he proved the following

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1This work is supported by the N.S.F. (10671155) of P.R.China
An identity related to Dedekind sums

Asymptotic formula

\[ \sum_{h=1}^{k} |S(h, k)|^2 = \frac{5}{144} k \phi(k) \prod_{p \mid k} \left( \frac{1}{1 + \frac{1}{p}} - \frac{1}{p^{\alpha+1}} \right) + O \left( k \exp \left( \frac{4 \ln k}{\ln \ln k} \right) \right), \]

where \( p^n \mid k \) denotes that \( p^n \mid k \) but \( p^{n+1} \nmid k \).

In this paper, we consider following Dirichlet series involving the Dedekind sums:

\[ \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{\mu(a) \mu(b)}{ab} S(ab, q), \quad (1) \]

where \( \sum' \) denotes the summation over all \( a \) with \( (a, q) = 1 \), \( \mu(a) \) be the Möbius function, and \( \overline{b} \) denotes the solution of the congruent equation \( xb \equiv 1 \pmod{q} \).

About the properties of (1), it seems that none had study it yet, at least we have not seen any related result before. In this paper, we use the analytic methods and the properties of Dirichlet \( L \)-function to study the calculating problem of (1), and give an interesting identity. That is, we shall prove the following:

**Theorem.** Let \( \alpha \) be any positive integer. Then for any odd prime \( p \), we have the identity

\[ \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{\mu(a) \mu(b)}{ab} S(ab, p^\alpha) = \frac{p^\alpha}{2\pi^2} \left( 1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots + \frac{1}{p^{2\alpha-2}} \right). \]

From this Theorem we may immediately deduce the following:

**Corollary.** For any odd prime \( p \), we have the identity

\[ \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{\mu(a) \mu(b)}{ab} S(ab, p) = \frac{p}{2\pi^2}. \]

For general positive integer \( q \geq 3 \), whether there exists a calculating formula for (1) is an open problem.

§2. Several Lemmas

To complete the proof of our Theorem, we need the following two simple Lemmas.

**Lemma 1.** Let \( q \geq 3 \) be a positive integer. Then for any integer \( c \) with \( (c, q) = 1 \), we have the identity

\[ S(c, q) = \frac{1}{\pi q} \sum_{d \mid q} \frac{d^2}{\phi(d)} \sum_{\chi \mod{d} \chi(-1) = -1} \chi(c) \mid L(1, \chi) \mid^2, \]

where \( \phi(n) \) is the Euler function, \( \sum_{\chi \mod{d} \chi(-1) = -1} \) denotes the summation over all Dirichlet character modulo \( d \) with \( \chi(-1) = -1 \).
Proof. See Lemma 2 of reference [6].

Lemma 2. Let $\alpha$ be a positive integer, $p$ be any odd prime. Then for any divisor $p^\beta > 1$ of $p^\alpha$ and any Dirichlet non-principal character $\chi$ modulo $p^\beta$, we have

$$L(1, \chi \chi_0^0) = L(1, \chi),$$

where $\chi_0^0$ denotes the principal character modulo $p^\alpha$.

Proof. Let $\chi$ be any non-principal character modulo $p^\beta$. For any integer $n$ with $(n, p^\beta) = 1$, it is clear that $(n, p^\alpha) = 1$. So from the Euler product formula (see reference [5]) we have

$$L(1, \chi \chi_0^0) = \prod_q \left( 1 - \frac{\chi(q) \chi_0^0(q)}{q} \right) = \prod_q \left( 1 - \frac{\chi(q)}{q} \right) = L(1, \chi),$$

where $\prod_q$ denotes the product over all prime. This proves Lemma 2.

§3. Proof of Theorem

In this section, we shall use these two simple Lemmas to complete the proof of our Theorem. For any odd prime $p$ and positive integer $\alpha$, let $p^\beta | p^\alpha$ and $\beta \geq 1$, $\chi$ denotes any non-principal character modulo $p^\beta$. Note that

$$\frac{1}{L(1, \chi)} = \sum_{n=1}^{\infty} \frac{\mu(n) \chi(n)}{n},$$

where $\mu(n)$ denotes the M"obius function. From Lemma 1 and Lemma 2 we have

$$\sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{\mu(a) \mu(b)}{ab} S(ab, p^\alpha)$$

$$= \sum_{d | p^\alpha} \frac{d^2}{\phi(d)} \sum_{\chi \mod d, \chi(-1)=-1} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{\mu(a) \mu(b) \chi(a) \chi(b)}{ab} \left| L(1, \chi) \right|^2$$

$$= \frac{1}{\pi^2 p^\alpha} \sum_{d | p^\alpha} \frac{d^2}{\phi(d)} \sum_{\chi \mod d, \chi(-1)=-1} \left| L(1, \chi) \right|^2$$

$$= \frac{1}{\pi^2 p^\alpha} \sum_{d | p^\alpha} \frac{d^2}{\phi(d)} \sum_{\chi \mod d, \chi(-1)=-1} 1 = \frac{1}{\pi^2 p^\alpha} \sum_{d | p^\alpha} \frac{d^2}{\phi(d)} \frac{\phi(d)}{2}$$

$$= \frac{p^\alpha}{2\pi^2} \left( 1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots + \frac{1}{p^{2\alpha-2}} \right).$$

This completes the proof of our Theorem.
For general $q \geq 3$, from Lemma 1 and Lemma 2 we can deduce that

$$\sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{\mu(a)\mu(b)}{ab} S(\alpha a, \beta b, q) = \frac{1}{\pi^2 q} \sum_{\chi \mod d} \frac{d^2}{\phi(d)} \sum_{\chi \equiv -1 \mod d} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{\mu(a)\mu(b)\chi(a)\chi(b)}{ab} |L(1, \chi)|^2$$

$$= \frac{1}{\pi^2 q} \sum_{\chi \mod d} \frac{d^2}{\phi(d)} \sum_{\chi \equiv -1 \mod d} \sum_{r|q/d} \frac{|L(1, \chi)|^2}{|L(1, \chi)|^2} \cdot |L(1, \chi)|^2$$

$$= \frac{1}{\pi^2 q} \sum_{\chi \mod d} \frac{d^2}{\phi(d)} \sum_{\chi \equiv -1 \mod d} \prod_{\chi(p) \equiv 1} \left|1 - \frac{\chi(p)}{p}\right|^{-2}.$$ 

References


An equation involving the Smarandache dual function and Smarandache ceil function

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Abstract For any positive integer \( n \), the Smarandache dual function \( S^*(n) \) is defined as
\[
S^*(n) = \max \{ m : m \in \mathbb{N}, m! \mid n \}.
\]
For any fixed positive integer \( k \) and any positive integer \( n \), the Smarandache ceil function \( S_k(n) \) is defined as
\[
S_k(n) = \min \{ m : m \in \mathbb{N}, n \mid m^k \}.
\]
The main purpose of this paper is using the elementary methods to study the solvability of the equation \( S^*(n) = S_k(n) \), and give its all positive integer solutions.

Keywords Smarandache dual function, Smarandache ceil function, equation, solution.

§1. Introduction and Results

For any positive integer \( n \), the Smarandache dual function \( S^*(n) \) is defined as the greatest positive \( m \) such that \( m! \) divides \( n \). That is,
\[
S^*(n) = \max \{ m : m \in \mathbb{N}, m! \mid n \}.
\]

This function was introduced by J.Sandor in [1], where he studied the elementary properties of \( S^*(n) \), and obtained a series interesting results.

On the other hand, for any fixed positive integer \( k \) and any positive integer \( n \), the Smarandache ceil function \( S_k(n) \) is defined as follows:
\[
S_k(n) = \min \{ m : m \in \mathbb{N}, n \mid m^k \}.
\]
This function was introduced by F.Smarandache who proposed many problems in [2]. There are many papers on the Smarandache ceil function. For example, Ibstedt [3] and [4] studied these functions both theoretically and computationally, and got the following conclusions:
\[
(\forall a, b \in \mathbb{N}^*) (a, b) = 1 \Rightarrow S_k(ab) = S_k(a)S_k(b).
\]
\[
S_k(p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_r^{\alpha_r}) = S_k(p_1^{\alpha_1})S_k(p_2^{\alpha_2}) \ldots S_k(p_r^{\alpha_r}).
\]

It is easily to show that \( S_k(p^a) = p^\lceil \frac{a}{k} \rceil \), where \( p \) be a prime and \( \lceil x \rceil \) denotes the least integer \( \geq x \). So if \( n = p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_r^{\alpha_r} \) is the factorization of \( n \) into prime power, then the following identity is obviously:
\[
S_k(n) = S_k(p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_r^{\alpha_r}) = p_1^{\lceil \frac{\alpha_1}{k} \rceil}p_2^{\lceil \frac{\alpha_2}{k} \rceil} \ldots p_r^{\lceil \frac{\alpha_r}{k} \rceil}.
\]
In this paper, we use the elementary methods to study the solvability of the equation $S^*(n) = S_k(n)$, and give its all positive integer solutions. That is, we will prove the following:

**Theorem.** Let $k$ be a fixed positive integer and $n$ be any positive integer, then every positive integer solution of the equation $S^*(n) = S_k(n)$ can be expressed as $n = 1$ or $(n, k) = (2^\alpha, k)$, where $\alpha$ be any positive integer, $k$ be any integer $\geq \alpha$.

§2. Proof of the theorem

In this section, we will complete the proof of our Theorem. It is clear that $n = 1$ is a solution of the equation $S^*(n) = S_k(n)$ for any fixed positive integer $k$. Now we suppose that $n > 1$, we discuss the solutions of the equation $S^*(n) = S_k(n)$ in following several cases:

(I). If $n > 1$ be an odd integer, then from the definition of $S^*(n) = \max \{ m : m \in N, \: m! \mid n \}$ and $2 \nmid n$ we have $S^*(n) = 1$. Now we discuss the function $S_k(n)$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ be the factorization of $n$ into prime power, then we have

$$S_k(n) = S_k(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}) = p_1^{\lceil \frac{\alpha_1}{k} \rceil} p_2^{\lceil \frac{\alpha_2}{k} \rceil} \ldots p_r^{\lceil \frac{\alpha_r}{k} \rceil}.$$

Because $n$ is an odd integer $> 1$, in the factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, we have

(a). $\forall p_i$ is a prime $\geq 3$;

(b). $\forall \alpha_i$ is an integer $\geq 0$ and there exists at least an integer $\alpha_i$ satisfying $\alpha_i \geq 1$.

Then $\frac{\alpha_i}{k} \geq 0$. Hence $\lceil \frac{\alpha_i}{k} \rceil \geq 0$ and there exists at least an integer $\lceil \frac{\alpha_i}{k} \rceil \geq 1$. Therefore, $S_k(n)$ is an odd integer $\geq 3$.

So in this case, $S^*(n) < S_k(n)$. This means that the equation $S^*(n) = S_k(n)$ has no odd positive integer solution if $n > 1$.

(II). If $n$ be an even integer.

(a). If $n = 2^\alpha$, where $\alpha$ be any positive integer, then

$$S^*(n) = S^*(2^\alpha) = \max \{ m : m \in N, \: m! \mid 2^\alpha \} = 2$$

and

$$S_k(n) = S_k(2^\alpha) = 2^{\lceil \frac{\alpha}{k} \rceil}.$$

It is obvious that $S^*(2^\alpha) = S_k(2^\alpha)$ if and only if $k \geq \alpha$, we have $0 < \frac{\alpha}{k} \leq 1$, then $\lceil \frac{\alpha}{k} \rceil = 1$. Only in this case it satisfy $S^*(n) = S_k(n) = 2$. So all integers $(n, k) = (2^\alpha, k)$ (where $\alpha$ be any positive integer and $k$ be any integer $\geq \alpha$) are the solutions of the equation $S^*(n) = S_k(n)$.

(b). If $n = 2^\alpha \cdot m$, where $2 \nmid m$ and $\alpha$ be any positive integer, then we have

$$S_k(n) = S_k(2^\alpha \cdot m) = 2^{\lceil \frac{\alpha}{k} \rceil} \cdot S_k(m).$$

Let

$$u = S^*(n),$$
because $m$ is an odd number, as is verified in (I), $S_k(m)$ is an odd integer $\geq 3$. So $S_k(n) = 2^{\left\lceil \frac{m}{2} \right\rceil} \cdot S_k(m)$ is even and it is impossible that $S_k(m) = 2^i$. Hence if $u = S^*(n) = S_k(n)$, $u$ is not odd and $u \neq 2^i$. It just left one case, that is $u = 2^i \cdot l$, where $2 \nmid l$ and $i$ be any positive integer.

If $u = 2^i \cdot l$ satisfying the equation $u = S^*(n) = S_k(n)$, from the definition of $S^*(n)$ we have $u! | n$. At the same time, from the definition of $S_k(n)$ we also have $n | u^k$.

So $u! | u^k$.

It is a contradiction. Because if $u! | u^k$ is true, from $u \geq 1$ we get $u - 1 | u!$.

It is not true. Because $u = 2^i \cdot l \geq 6$, for any integer $u \geq 1$ we have $(u - 1, u) = 1$.

Then $u - 1 \nmid u^k$.

Therefore there is no solutions in this case.

Associating (I) and (II), we complete the proof of Theorem.

References

On the quadratic mean value of the Smarandache dual function $S^{**}(n)$

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Abstract For any positive integer $n$, the Smarandache dual function $S^*(n)$ is defined as the greatest positive $m$ such that $m!$ divides $n$. The Smarandache dual function $S^{**}(n)$ is defined as follows: $S^{**}(n)$ is the greatest positive $2m - 1$ such that $(2m - 1)!$ divides $n$ if $n$ is an odd integer; $S^{**}(n)$ is the greatest positive $2m$ such that $(2m)!$ divides $n$ if $n$ is an even integer.

The main purpose of this paper is using the elementary methods to study the quadratic mean value of $S^{**}(n)$, and gives a more sharper asymptotic formulas for it.

Keywords Smarandache dual function, mean value, asymptotic formula.

§1. Introduction

For any positive integer $n$, the Smarandache dual function $S^*(n)$ is defined as the greatest positive $m$ such that $m!$ divides $n$. That is,

$$S^*(n) = \max\{m : \ m \in N, \ m! \mid n\}.$$

Many people had studied the properties of $S^*(n)$, and obtained some interesting results, see references [1]-[4]. For example, J. Sandor [1] gave following propose: For any positive integer $k$, if $q$ is the first prime following $2^k + 1$, then we have the identity

$$S^*((2k - 1)!(2k + 1)! = q - 1.$$ $$Maohua Le [2] proved that this propose holds. Jie Li [3] studied the calculating problem of the series $\sum_{n=1}^{\infty} \frac{S^*(n)}{n^s}$, and proved following conclusions: For any real number $s > 1$, $\sum_{n=1}^{\infty} \frac{S^*(n)}{n^s}$ is absolutely convergent, and

$$\sum_{n=1}^{\infty} \frac{S^*(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{1}{(n!)^s},$$

where $\zeta(s)$ is the Riemann zeta-function.

Shejiao Xue [4] studied the property of $\sum_{n=1}^{\infty} \frac{1}{SL^*(n)n^s}$, and gave the following identity

$$\sum_{n=1}^{\infty} \frac{1}{SL^*(n)n^s} = \zeta(s) \left(1 - \sum_{n=1}^{\infty} \frac{1}{n(n + 1)((n + 1)!)^s}\right).$$
where real number $s > 1$.

Su Gou and Xiaoying Du [5] defined the Smarandache LCM dual function $S^{**}(n)$ as follows:

$$S^{**}(n) = \begin{cases} 
\max\{2m - 1 : m \in \mathbb{N}, (2m - 1)!! \mid n\}, & 2 \nmid n, \\
\max\{2m : m \in \mathbb{N}, (2m)!! \mid n\}, & 2 \mid n.
\end{cases}$$

and proved that: For any real number $s > 1$, \(\sum_{n=1}^{\infty} \frac{S^{**}(n)}{n^s}\) is absolutely convergent, and

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)}{n^s} = \zeta(s) \left(1 - \frac{1}{2^s}\right) \left(1 + \sum_{m=1}^{\infty} \frac{2}{((2m + 1)!!)^s}\right) + \zeta(s) \sum_{m=1}^{\infty} \frac{2}{((2m)!!)^s}.$$

The main purpose of this paper is using the elementary methods to study the quadratic mean value properties of $S^{**}(n)$, and give a sharper asymptotic formulas for it. That is, we shall prove the following:

**Theorem** For any real integer $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} (S^{**}(n))^2 = \frac{13x}{2} + O \left(\frac{\ln x}{\ln \ln x}\right)^3.$$

§2. Some Lemmas

In this section, we shall give two lemmas which are necessary in the proof of our theorem.

**Lemma 1.** Let $x \geq 1$ be a real number, for any positive integer $k$, if $(2k)!! \leq x < (2k+2)!!$, then we have the asymptotic formula

$$k = \frac{\ln x}{\ln \ln x} + O \left(\frac{\ln x \ln \ln x}{(\ln x)^2}\right).$$

**Proof.** For any positive integer $k$, we have

$$(2k)!! = 2^k k!, \quad (2k+2)!! = 2^{k+1} (k+1)!!.$$

If $(2k)!! \leq x < (2k+2)!!$, we know that

$$k \ln 2 + \sum_{i=1}^{k} \ln i \leq \ln x < (k+1) \ln 2 + \sum_{i=1}^{k+1} \ln i.$$

According to [6], we have

$$\sum_{i=1}^{k} \ln i = \left(k + \frac{1}{2}\right) \ln (k + 1) - k - 1 + C + O \left(\frac{1}{k}\right) = k \ln k - k + O(\ln k).$$

$$\sum_{i=1}^{k+1} \ln i = \left(k + 1 + \frac{1}{2}\right) \ln (k + 2) - k - 2 + C + O \left(\frac{1}{k}\right) = k \ln k - k + O(\ln k).$$

Thus

$$\ln x = k \ln k + k \ln 2 - k + O(\ln k).$$
Then
\[
\ln \ln x = \ln [k \ln k + k \ln 2 - k + O(\ln k)] \\
= \ln k + \ln \left[ \ln k + \ln 2 - 1 + O \left( \frac{\ln k}{k} \right) \right] \\
= \ln k + \ln \left\{ \ln k \left[ 1 - \frac{1 - \ln 2}{\ln k} + O \left( \frac{1}{k} \right) \right] \right\} \\
= \ln k + \ln k + \ln \left[ 1 - \frac{1 - \ln 2}{\ln k} + O \left( \frac{1}{k} \right) \right] \\
= \ln k - \ln \ln k + O \left( \frac{1}{\ln k} \right).
\]

Thus
\[
\ln k = \ln \ln x + O(\ln \ln k),
\]

\[
\ln \ln k = \ln \left\{ \ln \ln x \left[ 1 + O \left( \frac{\ln \ln k}{\ln \ln x} \right) \right] \right\} = \ln \ln \ln x + O(1).
\]

From above the asymptotic formula we may immediately deduce that
\[
k = \frac{\ln x}{\ln k - 1 + \ln 2} + O \left( \frac{\ln k}{\ln k - 1 + \ln 2} \right) \\
= \frac{\ln x}{\ln \ln x - \ln \ln k + O \left( \frac{1}{\ln \ln x} \right) - 1 + \ln 2} + O \left( \frac{\ln k}{\ln k - 1 + \ln 2} \right) \\
= \frac{\ln x}{\ln \ln x} \cdot \frac{1}{1 - \frac{\ln \ln k + 1 - \ln 2 + O \left( \frac{1}{\ln \ln x} \right)}{\ln \ln x}} + O \left( \frac{\ln k}{\ln k - 1 + \ln 2} \right) \\
= \frac{\ln x}{\ln \ln x} \cdot \left[ 1 + \frac{\ln \ln k + 1 - \ln 2 + O \left( \frac{1}{\ln \ln x} \right)}{\ln \ln x} \right] + O(1) \\
= \frac{\ln x}{\ln \ln x} + O \left( \frac{\ln x \ln \ln \ln x}{(\ln \ln x)^2} \right).
\]

This completes the proof of Lemma 1.

**Lemma 2.** Let \( x \geq 1 \) be a real number, for any positive integer \( k \), if \( (2k - 1)!! \leq x < (2k + 1)!! \), then we have the asymptotic formula
\[
k = \frac{\ln x}{\ln \ln x} + O \left( \frac{\ln x \ln \ln \ln x}{(\ln \ln x)^2} \right).
\]

**Proof.** For any positive integer \( k \), we know that
\[
(2k - 1)!! = \frac{(2k)!}{(2k)!!} = \frac{(2k)!}{2^k k!},
\]
\[
(2k + 1)!! = \frac{(2k + 2)!}{2^{k+1} (k + 1)!}.
\]
Then

\[ \ln (2k-1)!! = \sum_{i=1}^{2k} \ln i - k \ln 2 - \sum_{i=1}^{k} \ln i = \left( 2k + \frac{1}{2} \right) \ln (2k + 1) - 2k - 1 + C + O \left( \frac{1}{k} \right) - k \ln 2 \]

Meanwhile

\[ \ln (2k+1)!! = \sum_{i=1}^{2k+2} \ln i - (k+1) \ln 2 - \sum_{i=1}^{k+1} \ln i = \left( 2k + 2 + \frac{1}{2} \right) \ln (2k + 3) - 2k - 3 + C + O \left( \frac{1}{k} \right) - (k+1) \ln 2 \]

If \((2k-1)!! \leq x < (2k+1)!!\), we can get

\[ \ln (2k-1)!! \leq \ln(x) < \ln (2k+1)!! \]

Then we deduce that

\[ \ln x = k \ln k + k \ln 2 - k + O(\ln k) \]

Now Lemma 2 follows from Lemma 1.

\[ \S 3. \text{Proof of the theorem} \]

In this section, we shall complete the proof of the theorem. By the definition of \(S^{**}(n)\), we know that: If \(S^{**}(n) = 2m - 1, n \leq x\), then \((2m-1)!!|n\). For \(n = (2m-1)!!u, 2m + 1 \nmid u\)
and $2 \nmid u$, $(2m-1)!! \leq x < (2m+1)!!$. By Lemma 1, we have

$$\sum_{n \leq x \atop 2 \nmid n} (S^{**}(n))^2 = \sum_{(2m-1)!! \leq x \atop 2 \nmid u} (2m-1)^2$$

$$= \sum_{(2m-1)!! \leq x} (2m-1)^2 \sum_{u \leq (2m-1)!! \atop 2 \nmid u} 1$$

$$= \sum_{(2m-1)!! \leq x} \left[ \frac{x}{2(2m-1)!!} - \frac{x}{2(2m+1)!!} \right]$$

$$+ O \left( \sum_{(2m-1)!! \leq x} (2m-1)^2 \right)$$

$$= x \sum_{(2m-1)!! \leq x} \left[ \frac{(2m-1)^2}{(2m-1)!!} - \frac{(2m-1)^2}{(2m+1)!!} \right] + O \left( \sum_{(2m-1)!! \leq x} (2m-1)^2 \right)$$

$$= x \sum_{m \leq \frac{\ln x}{2m-1}} \left[ \frac{(2m-1)^2}{(2m-1)!!} - \frac{(2m-1)^2}{(2m+1)!!} \right] + O \left( \left( \frac{\ln x}{\ln \ln x} \right)^3 \right)$$

Then

$$\sum_{m=1}^{\infty} \left[ \frac{(2m-1)^2}{(2m-1)!!} - \frac{(2m-1)^2}{(2m+1)!!} \right] = \sum_{m=1}^{\infty} \frac{(2m-1)^2}{(2m-1)!!} - \sum_{m=1}^{\infty} \frac{(2m-1)^2}{(2m+1)!!}$$

$$= 1 + \sum_{m=1}^{\infty} \frac{(2m-1)^2 - (2m+1)^2}{(2m+1)!!}$$

$$= 1 + 4 \sum_{m=1}^{\infty} \frac{2m}{(2m+1)!!}$$

$$= 1 + 4 \sum_{m=1}^{\infty} \left[ \frac{1}{(2m-1)!!} - \frac{1}{(2m+1)!!} \right]$$

$$= 5.$$
So we obtain
\[ \sum_{n \leq x \atop 2 | n} (S^{**}(n))^2 = \frac{5x}{2} + O \left( \left( \frac{\ln x}{\ln \ln x} \right)^3 \right). \]

By the definition of $S^{**}(n)$, we know that: if $S^{**}(n) = 2m$, $n \leq x$, then $(2m)!! | n$. let $n = (2m)!! v$, $2m + 2 \nmid v$, then $(2m)!! \leq x < (2m + 2)!!$, By use Lemma 2, we have
\[
\sum_{n \leq x \atop 2 | n} (S^{**}(n))^2 = \sum_{(2m)!! \leq x \atop 2m + 2 | v} (2m)^2
\]
\[
= \sum_{(2m)!! \leq x} (2m)^2 \sum_{n \leq \frac{x}{2m+2} \atop 2m+2 | v} 1
\]
\[
= \sum_{(2m)!! \leq x} (2m)^2 \left[ \frac{x}{(2m)!!} - \frac{x}{(2m+2)!!} \right] + O \left( \sum_{(2m)!! \leq x} (2m)^2 \right)
\]
\[
= x \sum_{(2m)!! \leq x} \left[ \frac{(2m)^2}{(2m)!!} - \frac{(2m)^2}{(2m+2)!!} \right] + O \left( \sum_{(2m)!! \leq x} (2m)^2 \right)
\]
\[
= x \sum_{m \leq \frac{x}{2m+2} \atop 2m+2 | v} \left[ \frac{(2m)^2}{(2m)!!} - \frac{(2m)^2}{(2m+2)!!} \right] + O \left( \sum_{m \leq \frac{x}{2m+2} \atop 2m+2 | v} (2m)^2 \right)
\]
\[
+ O \left( \sum_{m \leq \frac{x}{2m+2} \atop 2m+2 | v} (2m)^2 \right)
\]
\[
= x \sum_{m=1}^{\infty} \left[ \frac{(2m)^2}{(2m)!!} - \frac{(2m)^2}{(2m+2)!!} \right] + O \left( x \sum_{(2m)!! > x} (2m)^2 \right)
\]
\[
+ O \left( x \sum_{(2m)!! > x} \frac{(2m)^2}{(2m+2)!!} \right) + O \left( \frac{\ln x}{\ln \ln x} \right)^3
\]
\[
= x \sum_{m=1}^{\infty} \left[ \frac{(2m)^2}{(2m)!!} - \frac{(2m)^2}{(2m+2)!!} \right] + O \left( \frac{\ln x}{\ln \ln x} \right)^3.
\]

Since
\[
\sum_{m=1}^{\infty} \left[ \frac{(2m)^2}{(2m)!!} - \frac{(2m)^2}{(2m+2)!!} \right] = \sum_{m=1}^{\infty} \frac{(2m)^2}{(2m)!!} - \sum_{m=1}^{\infty} \frac{(2m)^2}{(2m+2)!!}
\]
\[
= 2 + \sum_{m=1}^{\infty} \frac{(2m+2)^2 - (2m)^2}{(2m+1)!!}
\]
\[
= 2 + 4 \sum_{m=1}^{\infty} \frac{2m+1}{(2m+2)!!}
\]
\[
= 2 + 4 \sum_{m=1}^{\infty} \left[ \frac{1}{(2m)!!} - \frac{1}{(2m+2)!!} \right]
\]
\[
= 4.
\]
We have
\[ \sum_{n \leq x} (S^{**}(n))^2 = 4x + O\left(\left(\frac{\ln x}{\ln \ln x}\right)^3\right). \]

By the above two estimates and the definition of \(S^{**}(n)\), we have
\[ \sum_{n \leq x} (S^{**}(n))^2 = \sum_{n \leq x} (S^{**}(n))^2 + \sum_{n \leq x} (S^{**}(n))^2 \]
\[ = \frac{13x}{2} + O\left(\left(\frac{\ln x}{\ln \ln x}\right)^3\right). \]

This completes the proof of Theorem.

References

Distribution of quadratic residues over short intervals

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Abstract Let \( p \) be an odd prime, \( S \) be the set of all quadratic residues mod \( p \), \( n \) be a fixed positive integer with \( (n, p) = 1 \) and \( \ell | n \). Suppose \( A \) and \( B \) consist of consecutive integers coprime to \( p \). Define

\[
N(A, B, n, \ell; p) = \sum_{a \in A} \sum_{b \in B} 1.
\]

The main purpose of this paper is using the elementary methods to study the asymptotic properties of \( N(A, B, n, \ell; p) \) and give a sharp asymptotic formula for it.

Keywords quadratic residue, short interval.

§1. Introduction

Let \( p \) be an odd prime. Suppose \( A \) and \( B \) consist of consecutive integers which are coprime to \( p \), that is

\[
A = \{m : M < m \leq M + A, (m, p) = 1\},
\]

\[
B = \{m : N < m \leq N + B, (m, p) = 1\},
\]

where \( A, B > 0 \). Let \( S \) be the set of all quadratic residues mod \( p \), \( n \) be a fixed positive integer with \( (n, p) = 1 \) and \( \ell | n \). We consider the distribution of quadratic residues,

\[
N(A, B, n, \ell; p) = \sum_{a \in A} \sum_{b \in B} 1.
\]

In this paper, we shall prove

Theorem 1.

\[
N(A, B, n, \ell; p) = \frac{|A| |B|}{2n} \phi \left( \frac{n}{\ell} \right) + E(A, B, n, \ell; p),
\]

and

\[
E(A, B, n, \ell; p) \ll \min \left( \left( |A| |B| \right)^{1/2}, p \log^2 p \right),
\]

where \( \phi(n) \) is the Euler function, \( |A| \) denotes the cardinality of \( A \), and the \( O \)-constant only depends on \( n \) and \( \ell \).

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Taking \( n = 2 \) and \( \ell = 1 \), we have

**Corollary 1.** The number of quadratic-tuples \((a, b)\) in \( A \times B \) such that \( a, b \) are of opposite parity and \( a \cdot b \) is a quadratic residue mod \( p \)

\[
= \frac{1}{4} |A| |B| + O \left( \min \left( (|A||B|)^{1/2}, p \log^2 p \right) \right).
\]

Further, observing

\[
\sum_{\ell|n} \sum_{(m,n)=\ell} = \sum_{m},
\]

one can obtain that

**Corollary 1.** The number of quadratic-tuples \((a, b)\) in \( A \times B \) such that \( a \cdot b \) is a quadratic residue mod \( p \)

\[
= \sum_{\ell|n} N(A, B, n, \ell; p) = \frac{1}{2} |A| |B| + O \left( \min \left( (|A||B|)^{1/2}, p \log^2 p \right) \right).
\]

§2. Proof of the theorem

In order to prove the theorem, we require a preliminary estimate, see reference [1], §5.1 Lemma 3.

**Lemma 1.** Let \( x \) be a real number, we denote \( \|x\| = \min_{n\in\mathbb{Z}} |x - n| \). Assume \( U \) is a positive real number, \( K_0 \) is an integer, \( K \) is a positive integer, \( \alpha \) and \( \beta \) are two arbitrary real numbers. If \( \alpha \) can be written in the form

\[
\alpha = \frac{h}{q} + \frac{\theta}{q^2}, \quad (q, h) = 1, \; q \geq 1, \; |\theta| \leq 1,
\]

we will have

\[
\sum_{k=K_0+1}^{K_0+K} \min \left( U, \frac{1}{\|\alpha k + \beta\|} \right) \ll \left( \frac{K}{q} + 1 \right) (U + q \log q).
\]

Now we turn to prove the theorem. Obviously we have

\[
N(A, B, n, \ell; p) = \frac{1}{2} \sum_{a \in A} \sum_{b \in B} \left( 1 + \frac{ab}{p} \right)
\]

\[
= \frac{1}{2} \sum_{a \in A} \sum_{b \in B} 1 + \frac{1}{2} \sum_{a \in A} \sum_{b \in B} \frac{ab}{p}
\]

\[
:= I_1 + I_2.
\]
Firstly,

\[ I_1 = \frac{1}{2} \sum_{a \in A} \sum_{b \in B} (a + b, n) = \frac{1}{2} \sum_{a \in A} \sum_{b \in B} \sum_{r | a+b, n} \mu(r) \]

\[ = \frac{1}{2} \sum_{a \in A} \sum_{b \in B} \mu(r) \sum_{r | a+b, n=1} 1 \]

\[ = \frac{1}{2} \sum_{a \in A} \sum_{b \in B} \mu(r) \left( \frac{|B|}{r \ell} + O(1) \right) \]

\[ = \frac{|B|}{2 \ell} \sum_{a \in A} \sum_{b \in B} \mu(r) + O(|A|) \]

\[ = \frac{|A| |B|}{2 n} \phi \left( \frac{n}{\ell} \right) + O(\min(|A|, |B|)) \quad (2) \]

and

\[ I_2 = \frac{1}{2} \sum_{a \in A} \sum_{b \in B} \left( \frac{ab}{p} \right) = \frac{1}{2} \sum_{r | a+b, n=1} \mu(r) \sum_{a \in A} \sum_{b \in B} \left( \frac{ab}{p} \right) \]

\[ = \frac{1}{2 \ell} \sum_{r | a+b, n=1} \sum_{m=1}^{r \ell} \sum_{a \in A} \sum_{b \in B} \left( \frac{ma+b}{p} \right) \left( \frac{ab}{r \ell} \right) \]

\[ = \frac{1}{2 \ell} \sum_{r | a+b, n=1} \left( \sum_{a \in A} \left( \frac{a}{p} \right) e \left( \frac{ma}{r \ell} \right) \right) \left( \sum_{b \in B} \left( \frac{b}{p} \right) e \left( \frac{mb}{r \ell} \right) \right) \quad (3) \]

On one hand, we define

\[ T = \sum_{m=1}^{r \ell} \left( \sum_{a \in A} \left( \frac{a}{p} \right) e \left( \frac{ma}{r \ell} \right) \right) \left( \sum_{b \in B} \left( \frac{b}{p} \right) e \left( \frac{mb}{r \ell} \right) \right) \]

Applying Cauchy Inequality, we have

\[ T^2 \leq \sum_{m=1}^{r \ell} \left( \sum_{a \in A} \left( \frac{a}{p} \right) e \left( \frac{ma}{r \ell} \right) \right)^2 \times \sum_{n=1}^{r \ell} \left( \sum_{b \in B} \left( \frac{b}{p} \right) e \left( \frac{mb}{r \ell} \right) \right)^2 \]

\[ = \ell^2 |A| |B|. \]

Thus

\[ I_2 \ll_{\ell, \ell} (|A| |B|)^{1/2} \quad (4) \]

On the other hand, we define

\[ G_1 = \sum_{a \in A} \left( \frac{a}{p} \right) e \left( \frac{ma}{r \ell} \right), G_2 = \sum_{b \in B} \left( \frac{b}{p} \right) e \left( \frac{mb}{r \ell} \right). \]

It suffices to estimate \( G_1 \) and \( G_2 \) separately.
Notice that

\[
\left( \frac{a}{p} \right) = \frac{1}{p} \sum_{s=1}^{p-1} G(s) e \left( -\frac{as}{p} \right),
\]

where \( G(s) = \sum_{n=1}^{p-1} \left( \frac{n}{p} \right) e \left( \frac{ns}{p} \right) \), and also notice that \( \frac{m}{r\ell} = \frac{s}{p} \neq 0 \) for \( 1 \leq m \leq r\ell, 1 \leq s \leq p-1 \) and \((r\ell, p) = 1\), thus

\[
G_1 = \frac{1}{p} \sum_{a \in A} \sum_{s=1}^{p-1} G(s) e \left( \left( \frac{m}{r\ell} - \frac{s}{p} \right) a \right) = \frac{1}{p} \sum_{s=1}^{p-1} G(s) f(A, m, s; r, \ell, p) e \left( \frac{s}{p} - \frac{m}{r\ell} \right) - 1
\]

\[
\leq \frac{1}{\sqrt{p}} \sum_{s=1}^{p-1} \left| e \left( \frac{s}{p} - \frac{m}{r\ell} \right) - 1 \right| \leq \frac{1}{\sqrt{p}} \sum_{s=1}^{p-1} \left\| \frac{s}{p} - \frac{m}{r\ell} \right\|
\]

Since

\[
\left\| \frac{s}{p} - \frac{m}{r\ell} \right\| = \left\| \frac{sr\ell - mp}{p r\ell} \right\| \geq \frac{1}{p r\ell},
\]

Lemma 1 yields that

\[
G_1 \ll \frac{1}{\sqrt{p}} \sum_{s \leq p-1} \min \left( pr\ell, \frac{1}{\left\| \frac{s}{p} - \frac{m}{r\ell} \right\|} \right) \ll \sqrt{p} \log p. \tag{5}
\]

Similarly,

\[
G_2 \ll \sqrt{p} \log p. \tag{6}
\]

Combining (3), (5) and (6), we have

\[
I_2 \ll p \log^2 p, \tag{7}
\]

where the \( O \) constant depends on \( n \) and \( \ell \).

The theorem follows immediately from (1), (2), (4) and (7).

References

An equation involving the Smarandache power function

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Abstract For any positive integer \( n \), the famous Smarandache power function \( SP(n) \) is defined as the smallest positive integer \( m \) such that \( m^m \) is divided by \( n \). That is, \( SP(n) = \min\{m : n \mid m^m, \ m \in \mathbb{N} \} \). The main purpose of this paper is using the elementary and analytic methods to study the solvability of the equation \( SP(1) + SP(2) + \cdots + SP(n) = SP\left(\frac{2(n+1)}{3}\right) \), and give its all positive integer solutions \( n = 1, 2, 3 \).

Keywords The Smarandache power function, Möbius function, positive integer solutions.

§1. Introduction

For any positive integer \( n \), the famous Smarandache power function \( SP(n) \) is defined as the smallest positive integer \( m \) such that \( m^m \) is divided by \( n \). That is, \( SP(n) = \min\{m : n \mid m^m, \ m \in \mathbb{N} \} \). For example, the first few values of \( SP(n) \) are: \( SP(1) = 1, SP(2) = 2, SP(3) = 3, SP(4) = 2, SP(5) = 5, SP(6) = 6, SP(7) = 7, SP(8) = 4, SP(9) = 3, SP(10) = 10, SP(11) = 11, SP(12) = 6, SP(13) = 13, SP(14) = 14, SP(15) = 15, SP(16) = 4, SP(17) = 17, SP(18) = 6, SP(19) = 19, SP(20) = 10, \cdots \).

In reference [1], Professor F.Smarandache asked us to study the properties of \( SP(n) \). From the definition of \( SP(n) \) we can easily get the following conclusions: If \( n = p^\alpha \), then

\[
SP(n) = \begin{cases} p, & 1 \leq \alpha \leq p; \\ p^2, & p + 1 \leq \alpha \leq 2p^2; \\ p^3, & 2p^2 + 1 \leq \alpha \leq 3p^3; \\ \cdots \cdots \\ p^\alpha, & (\alpha - 1)p^{\alpha-1} + 1 \leq \alpha \leq \alpha p^\alpha. \end{cases}
\]

Let \( n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) denotes the factorization of \( n \) into prime powers. If \( \alpha_i \leq p_i \) for all \( \alpha_i \) \((i = 1, 2, \cdots, r)\), then we have \( SP(n) = U(n) \), where \( U(n) = \prod_{p \mid n} p \prod_{p \mid n} \) denotes the product over all different prime divisors of \( n \). It is clear that \( SP(n) \) is not a multiplicative function. For example, \( SP(3) = 3, SP(8) = 4, SP(24) = 6 \neq SP(3) \times SP(8) \). But for most \( n \) we have \( SP(n) = U(n) \).

About the deeply arithmetical properties of \( SP(n) \), many scholars had studied it, and obtained a series results. See [2], [3] and [4]. For example, Chenglian Tian [2] used the elementary
method to study the solvability of the equation $SP(n^k) = \phi(n)$, and give its all positive integer solutions for $k = 1, 2, 3$. Zhefeng Xu [3] studied the mean value properties of $SP(n)$, and obtained an interesting asymptotic formula for $\sum_{n \leq x} SP(n)$.

The main purpose of this paper is using the elementary and analytic methods to study the solvability of the equation $SP(1) + SP(2) + \cdots, + SP(n) = SP\left(\frac{n(n+1)}{2}\right)$, and give its all positive solutions. That is, we shall prove the following:

**Theorem.** For any positive integer $n$, the equation

$$SP(1) + SP(2) + \cdots + SP(n) = SP\left(\frac{n(n+1)}{2}\right)$$

holds if and only if $n = 1, 2, 3$.

§2. A Lemma.

To complete the proof of Theorem, we need the following:

**Lemma.** For any positive integer $n \geq 1500$, we have the inequality

$$SP(1) + SP(2) + \cdots + SP(n) \leq \frac{n(n+1)}{4}.$$  

**Proof.** Let $A$ denotes the set of all square-free numbers. That is, $A = \{n : n = p_1p_2 \cdots p_r, \text{where } p_i \text{ are deferent primes}\}$. Note that $|\mu(n)| = \sum_{d \mid n} \mu(d)$, then from the properties of the Möbius function we have

$$SP(1) + SP(2) + \cdots + SP(n) \geq \sum_{a \leq n} a = \sum_{a \leq n} a | \mu(a) | = \sum_{a \leq n} \sum_{a \in A} \mu(d)$$

$$= \sum_{d \leq \sqrt{n}} d^2 \mu(d) \left(\left\lfloor \frac{n}{d^2} \right\rfloor + 1\right)$$

$$= \sum_{d \leq \sqrt{n}} d^2 \mu(d) \left\{\frac{n^2}{2d^2} - \frac{n}{d^2} \left\lfloor \frac{n}{d^2} \right\rfloor + \frac{n}{2d^2} + \frac{1}{2} \left\{\frac{n}{d^2}\right\}^2 - \frac{1}{2} \left\{\frac{n}{d^2}\right\}\right\}$$

$$= \frac{n^2}{2} \sum_{d \leq \sqrt{n}} \frac{\mu(d)}{d^2} - n \sum_{d \leq \sqrt{n}} \mu(d) \left\lfloor \frac{n}{d^2} \right\rfloor + \frac{n}{2} \sum_{d \leq \sqrt{n}} \mu(d)$$

$$+ \frac{1}{2} \sum_{d \leq \sqrt{n}} d^2 \mu(d) \left\{\frac{n}{d^2}\right\}^2 - \frac{1}{2} \sum_{d \leq \sqrt{n}} d^2 \mu(d) \left\{\frac{n}{d^2}\right\}$$

$$\geq \frac{n^2}{2} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \frac{n^2}{2} \sum_{d > \sqrt{n}} \frac{\mu(d)}{d^2} - n \sqrt{n} - \frac{n \sqrt{n}}{2} = \frac{n \sqrt{n}}{2} - \frac{n \sqrt{n}}{2} = \frac{n \sqrt{n}}{2} - \frac{n \sqrt{n}}{2}$$

$$= \frac{n^2}{2} \zeta(2) - \frac{5}{2} n^2 = \frac{n^2}{2} \frac{6}{\pi^2} - \frac{5}{2} n^2 = \frac{3n^2}{2} \frac{1}{\pi^2} - \frac{5}{2} n^2.$$
For any positive integer \( n \geq 1500 \), we can easily get the inequality

\[
\frac{3n^2}{\pi^2} - \frac{5}{2} n^2 > \frac{n^2 + n^2}{4} > \frac{n(n+1)}{4}.
\]

This completes the proof of Lemma.

\section{Proof of the theorem}

In this section, we shall use the above Lemma to complete the proof of our Theorem. In fact from the definition of the function \( SP(n) \) we can easily deduce that \( n = 1, 2, 3 \) are the positive integer solutions of the equation (2). Now we shall prove that for any positive integer \( n \geq 4 \), the equation (2) does not hold. To prove this, we consider following two cases:

(A). For any positive integer \( n \geq 4 \), if \( \frac{n(n+1)}{2} = p_1p_2\cdots p_r \), where \( p_i \), \( (i = 1, 2, \cdots, r) \) are different primes, then we have

\[
SP(1) + SP(2) + \cdots + SP(n) < \frac{n(n+1)}{2} = p_1p_2\cdots p_r = SP\left( \frac{n(n+1)}{2} \right).
\]

(B). For any positive integer \( n \geq 4 \), if \( \frac{n(n+1)}{2} = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r} \), let \( \alpha = \max\{\alpha_1, \alpha_2, \cdots \alpha_r\} \) and \( \alpha \geq 2 \), then we have

\[
SP\left( \frac{n(n+1)}{2} \right) \leq \frac{n(n+1)}{4}.
\]

Combining the above section Lemma and (3) we may immediately deduce the inequality

\[
SP(1) + SP(2) + \cdots + SP(n) > \frac{n(n+1)}{4} > SP\left( \frac{n(n+1)}{2} \right),
\]

if \( n \geq 1500 \). So from this inequality we know that there is no positive integer \( n \) satisfying the equation (2), if \( n \geq 1500 \).

Combining the above all conclusions we know that the equation (2) has and only has three positive integer solutions \( n = 1, 2, 3 \). This completes the proof of our Theorem.

\textbf{References}

A new Smarandache multiplicative function and its mean value formula

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Abstract For any positive integer \( n \), we define the Smarandache multiplicative function \( D(n) \) as the smallest positive integer \( m \) such that \( n \) divide \( d(1)d(2)\cdots d(m) \), where \( d(n) \) is the Dirichlet divisor function. The main purpose of this paper is using the elementary and analytic method to study the mean value properties of \( \ln(D(n)) \), and give two interesting asymptotic formula.

Keywords Smarandache multiplicative function, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer \( n \), the famous Dirichlet divisor function \( d(n) \) is defined as the number of all distinct positive divisors of \( n \). If \( n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) be the prime power factorization of \( n \), then from the definition and properties of \( d(n) \) we may get

\[
d(n) = (\alpha_1 + 1) \cdot (\alpha_2 + 1) \cdots (\alpha_r + 1).
\]

From this formula we may immediately deduce that the first few values of \( d(n) \) are \( d(1) = 1, d(2) = 2, d(3) = 2, d(4) = 3, d(5) = 2, d(6) = 4, d(7) = 2, d(8) = 4, d(9) = 3, d(10) = 4, d(11) = 2, d(12) = 6, d(13) = 2, d(14) = 4, d(15) = 4, d(16) = 5, d(17) = 2, d(18) = 6, d(19) = 2, d(20) = 6, \cdots \). Now for any positive integer \( n \), we use divisor function \( d(n) \) to define a new number theory function \( D(n) \) as follows:

\[
D(n) = \min \left\{ m : n \prod_{i=1}^{m} d(i) \right\}.
\]

For example, \( D(1) = 1, D(2) = 2, D(3) = 4, D(4) = 4, D(5) = 2^4, D(6) = 4, D(7) = 2^6, D(8) = 5, D(9) = 9, D(10) = 16, D(11) = 2^{10}, D(12) = 4, D(13) = 2^{12}, D(14) = 64, D(15) = 16, D(16) = 6, D(17) = 2^{16}, D(18) = 9 \cdots \). Recently, Professor Zhang Wenpeng asked us to study the mean value properties of \( \ln(D(n)) \). About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. I think this
problem is interesting, because there are some close relations between $D(n)$ and the Dirichlet divisor function $d(n)$, so it can help us to find more information about $d(n)$. The main purpose of this paper is using the elementary method to study the mean value properties of $\frac{\ln(D(n))}{n}$, and give two interesting asymptotic formulae for them. That is, we shall prove the following:

**Theorem 1.** For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n \leq x} \ln(D(n)) = \frac{\pi^2 \cdot \ln 2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} c_i \cdot \frac{x^2}{i \cdot \ln i x} + O\left(\frac{x^2}{\ln^k i + 1} x\right),$$

where $c_i$ ($i = 2, 3, \cdots, k$) are computable constants.

**Theorem 2.** For any real number $x \geq 2$, we also have the asymptotic formula

$$\sum_{n \leq x} \frac{\ln(D(n))}{n} = \frac{\pi^2 \cdot \ln 2}{6} \cdot \frac{x}{\ln x} + \sum_{i=2}^{k} d_i \cdot \frac{x}{i \cdot \ln i x} + O\left(\frac{x}{\ln^{k+1} i x}\right),$$

where $d_i$ ($i = 2, 3, \cdots, k$) are computable constants.

§2. Some Lemmas.

In this Section, we shall give two simple Lemmas which are necessary in the proof of our Theorems. First we have the following:

**Lemma 1.** For any prime $p$, we have the identity $D(p) = 2^{p-1}$.

Proof. In fact for any prime $p$, let $D(p) = m$, from the definition of $D(n)$ we have $p$ divide $d(1)d(2)\cdots d(m)$, so $p$ divide one of $d(1)$, $d(2)$, $\cdots$, $d(m)$. Since $m$ is the smallest positive integer such that $p$ divide $d(1)d(2)\cdots d(m)$, so $p \mid d(m)$ and $m = 2^{p-1}$.

**Lemma 2.** For any positive integer $n \geq 2$, $D(n)$ is a Smarandache multiplicative function.

Proof. From the definition of $D(n)$ we can easily deduce this conclusion.

**Lemma 3.** Let $p$ denotes a prime, then we have the asymptotic formula

$$\sum_{2 \leq p \leq \frac{x}{m}} p = \frac{x^2}{2m^2 \cdot \ln x \frac{x}{m}} + \sum_{i=2}^{k} b_i \cdot \frac{x^2}{m^2 \cdot \ln^i x \frac{x}{m}} + O\left(\frac{x^2}{m^2 \cdot \ln^{k+1} x \frac{x}{m}}\right),$$

where $b_i$ ($i = 2, 3, \ldots, k$) are computable constants.

Proof. Let $\pi(x)$ denotes the number of all primes not to exceeding $x$, note that for any positive integer $k$, we have the asymptotic formula (see reference [7])

$$\pi(x) = \sum_{i=1}^{k} a_i \cdot \frac{x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where $a_i$ ($i = 1, 2, \cdots, k$) are constants and $a_1 = 1$.

Using the above asymptotic formula, the Able’s Summation formula (see reference [6]) and
Using Lemma 3 we have the asymptotic formula

\[
\sum_{2 \leq p \leq x} p = \pi \left( \frac{x}{m} \right) \cdot \frac{x}{m} - \pi(2) \cdot 2 - \int_{2}^{x} \pi(t) dt
\]

\[
= \frac{x^2}{2m^2 \cdot \ln \frac{x}{m}} + \sum_{i=2}^{k} \frac{b_i \cdot x^2}{m^2 \cdot \ln^{i+1} \frac{x}{m}} + O \left( \frac{x^2}{m^2 \cdot \ln^{k+1} \frac{x}{m}} \right),
\]

where \( b_i \) (\( i = 2, 3 \ldots, k \)) are computable constants.

\section{Proof of the theorems}

In this section, we shall complete the proof of our Theorems. First we prove Theorem 1. We define two sets \( A \) and \( B \) as following:

\[
A = \{ n : n \leq x, P(n) < \sqrt{n} \}
\]

and

\[
B = \{ n : n \leq x, P(n) \geq \sqrt{n} \},
\]

where \( P(n) \) denotes the biggest prime divisor of \( n \). By Lemma 1 and Lemma 2 we have

\[
\sum_{n \in A} \ln D(n) \leq \sum_{n \leq x} \ln 2^{\sqrt{n}} \ln n = \ln 2 \cdot \sum_{n \leq x} \sqrt{n} \ln n \leq \ln 2 \cdot x^2 \ln x \ll x^2 \ln x. \quad (1)
\]

\[
\sum_{n \in B} \ln D(n) = \sum_{n \leq x, P(n) \geq \sqrt{n}} \ln D(n) = \sum_{mp \leq x} \ln(D(p)) = \sum_{mp \leq x} \ln 2^{p-1} = \ln 2 \cdot \sum_{p \leq x} \ln 2^{p-1} - \sum_{m \leq \sqrt{x}} \sum_{p \leq \frac{x}{m}} (p-1)
\]

\[
= \ln 2 \cdot \sum_{m \leq \sqrt{x}} \sum_{m<p \leq \frac{x}{m}} (p-1) = \ln 2 \cdot \left( \sum_{m \leq \sqrt{x}} \sum_{m<p \leq \frac{x}{m}} p - \sum_{m \leq \sqrt{x}} \sum_{m<p \leq \frac{x}{m}} 1 \right). \quad (2)
\]

It is clear that

\[
\sum_{m \leq \sqrt{x}} \sum_{m<p \leq \frac{x}{m}} 1 = \sum_{m \leq \sqrt{x}} \frac{x}{m} = x \cdot \sum_{m \leq \sqrt{x}} \frac{1}{m} = x \cdot \left( \log \sqrt{x} + c + O \left( \frac{1}{\sqrt{x}} \right) \right) < x^2. \quad (3)
\]

Using Lemma 3 we have the asymptotic formula

\[
\sum_{m \leq \sqrt{x}} \sum_{m<p \leq \frac{x}{m}} p = \sum_{m \leq \sqrt{x}} \left( \frac{x^2}{2m^2 \cdot \ln \frac{x}{m}} + \sum_{i=2}^{k} \frac{b_i \cdot x^2}{m^2 \cdot \ln^{i+1} \frac{x}{m}} + O \left( \frac{x^2}{m^2 \cdot \ln^{k+1} \frac{x}{m}} \right) \right)
\]

\[
= \sum_{m \leq \sqrt{x}} \left( \frac{x^2}{2m^2 \cdot \ln \frac{x}{m}} + \sum_{i=2}^{k} \frac{b_i \cdot x^2}{m^2 \cdot \ln^{i+1} \frac{x}{m}} + O \left( \frac{x^2}{m^2 \cdot \ln^{k+1} \frac{x}{m}} \right) \right)
\]

\[
+ \sum_{m \leq \sqrt{x}} \left( \frac{x^2}{2m^2 \cdot \ln \frac{x}{m}} + \sum_{i=2}^{k} \frac{b_i \cdot x^2}{m^2 \cdot \ln^{i+1} \frac{x}{m}} + O \left( \frac{x^2}{m^2 \cdot \ln^{k+1} \frac{x}{m}} \right) \right)
\]

\[
= \frac{x^2 \cdot \ln 2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^{i+1} x} + O \left( \frac{x^2}{\ln^{k+1} x} \right). \quad (4)
\]
Combining (1), (2), (3) and (4) we may immediately deduce that
\[
\sum_{n \leq x} \ln(D(n)) = \frac{\pi^2 \ln 2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O \left( \frac{x^2}{\ln^{k+1} x} \right),
\]
where \(c_i (i = 2, 3, \ldots, k)\) are computable constants. This proves Theorem 1.

Now we prove Theorem 2. From the Abel’s summation formula and Theorem 1 we have
\[
\sum_{n \leq x} \frac{\ln(D(n))}{n} = \frac{1}{x} \sum_{n \leq x} \ln(D(n)) + \int_2^x \frac{1}{t^2} \sum_{n \leq t} \ln(D(t)) dt
\]
\[
= \frac{1}{x} \left( \frac{\pi^2 \ln 2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O \left( \frac{x^2}{\ln^{k+1} x} \right) \right)
\]
\[
+ \int_2^x \frac{1}{t^2} \left( \frac{\pi^2 \ln 2}{12} \cdot \frac{t^2}{\ln t} + \sum_{i=2}^{k} \frac{c_i \cdot t^2}{\ln^i t} + O \left( \frac{t^2}{\ln^{k+1} t} \right) \right) dt
\]
\[
= \frac{\pi^2 \ln 2}{12} \cdot \frac{x}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x}{\ln^i x} + O \left( \frac{x}{\ln^{k+1} x} \right)
\]
\[
+ \frac{\pi^2 \ln 2}{12} \int_2^x \frac{1}{\ln t} dt + \sum_{i=2}^{k} \frac{c_i \cdot x}{\ln^i t} + O \left( \frac{x}{\ln^{k+1} x} \right) dt
\]
\[
= \frac{\pi^2 \ln 2}{6} \cdot \frac{x}{\ln x} + \sum_{i=2}^{k} \frac{d_i \cdot x}{\ln^i x} + O \left( \frac{x}{\ln^{k+1} x} \right),
\]
where \(d_i (i = 2, 3, \ldots, k)\) are computable constants.

This completes the proof of Theorem 2.

References


The mean value of the $k$-th Smarandache dual function

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Abstract The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of the $k$-th Smarandache dual function, and give a sharper asymptotic formula for it.

Keywords Riemann zeta-function, Perron’s formula, the $k$-th Smarandache dual function.

§1. Introduction

Let $k$ be a fixed positive integer. For any positive integer $n$, the F.Smarandache ceil function of order $k$ is defined as the smallest positive integer $m$ such that $n$ divide $m^k$. That is, $S_k(n) = \min\{m : m \in \mathbb{N}, n|m^k\}$. The dual function of this function $S_k(n)$ is defined as the largest positive integer $m$ such that $m^k$ divide $n$. That is, $\overline{S}_k(n) = \max\{m : m \in \mathbb{N}, m^k|n\}$.

For examples, if $k = 2$, then the first few value of $S_2(n)$ are $S_2(1) = 1$, $S_2(2) = 2$, $S_2(3) = 3$, $S_2(4) = 2$, $S_2(5) = 5$, $S_2(6) = 6$, $S_2(7) = 7$, $S_2(8) = 4$, $S_2(9) = 3$, ···. The first few value of $\overline{S}_2(n)$ are $\overline{S}_2(1) = 1$, $\overline{S}_2(2) = 1$, $\overline{S}_2(3) = 1$, $\overline{S}_2(4) = 2$, $\overline{S}_2(5) = 1$, $\overline{S}_2(6) = 1$, $\overline{S}_2(7) = 1$, $\overline{S}_2(8) = 2$, $\overline{S}_2(9) = 3$, ···.

About the properties of functions $S_k(n)$ and $\overline{S}_k(n)$, many scholars have studied them, and obtained some interesting conclusions, see references [2], [3], [4] and [5]. For example, Wang Yongxing [2] proved that for any positive integers $a$ and $b$ with $(a, b) = 1$, we have

$$\overline{S}_k(ab) = \max\{m : m \in \mathbb{N}, m^k|a\} \cdot \max\{m : m \in \mathbb{N}, m^k|b\} = \overline{S}_k(a) \cdot \overline{S}_k(b)$$

and

$$\overline{S}_k(p^\alpha) = p^{\frac{\alpha}{k}} - 1,$$

where $[x]$ denotes the smallest positive integer $\geq x$. For any positive integer $n$, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of $n$ into prime powers, then we can deduce the identity

$$\overline{S}_k(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = p_1^{[\frac{\alpha_1}{k}]} p_2^{[\frac{\alpha_2}{k}]} \cdots p_r^{[\frac{\alpha_r}{k}]} = \overline{S}_k(p_1^{\alpha_1}) \overline{S}_k(p_2^{\alpha_2}) \cdots \overline{S}_k(p_r^{\alpha_r}).$$

From this properties we know that $\overline{S}_k(n)$ is a multiplicative function, so we can use the Euler product formula and the analytic method to study the mean value properties of $\overline{S}_k(n)$, and obtain an interesting mean value formula for it. The main purpose of this paper is using the elementary and analytic methods to study this problem, and prove the following conclusion:
The mean value of the $k$-th Smarandache dual function

**Theorem.** For any real number $x \geq 1$, we have the asymptotic formula

$$
\sum_{n \leq x} S_k(n) = \begin{cases} 
\frac{3}{\pi^2} \left( \ln x + 3\gamma - 1 - \frac{12}{\pi^2} \zeta'(2) \right) \cdot x + O \left( x^{\frac{3}{4}} \ln x \right), & \text{if } k = 2; \\
\frac{\zeta(k-1)}{\zeta(k)} x + O \left( \min \{ x^{\frac{3}{4}}, x^{\frac{3}{5}} \} \right), & \text{if } k > 2.
\end{cases}
$$

where $\zeta(s)$ is the Riemann zeta-function, $\epsilon$ denotes any fixed positive number, $\gamma$ is the Euler constant and $\zeta'(2) = -\sum_{n=1}^\infty \frac{\ln n}{n^2}$.

§2. Proof of the theorem

In this section, we shall complete the proof of our Theorem. First we give three simple Lemmas which are necessary in the proof of our Theorem. The proofs of these Lemmas can be found in reference [8].

**Lemma 1.** For any real number $x \geq 1$ and $\alpha > 0$, we have the asymptotic formula

$$
\sum_{n \leq x} n^\alpha = \frac{x^{1+\alpha}}{1+\alpha} + O \left( x^\alpha \right).
$$

**Lemma 2.** For any real number $x \geq 1$, we have the asymptotic formula

$$
\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O \left( \frac{1}{x} \right),
$$

where $\gamma$ is the Euler constant.

**Lemma 3.** For any real number $x \geq 1$ and $1 \neq s > 0$, we have the asymptotic formula

$$
\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O \left( x^{-s} \right),
$$

where $\zeta(s)$ is the Riemann zeta-function.

Now we use these Lemmas to prove our conclusion. First we use the elementary method to obtain an asymptotic formula. From the definition of $S_k(n)$ we can assume that $S_k(n) = m$ and $n = m^k h$, where $h$ is a $k$-th power-free number. That is, for any prime $p$, $p^k$ does not divide $h$. Note that if $h$ is a $k$-th power free number, then $\sum_{d^k|h} \mu(d) = 1$, and if $h$ is not a $k$-th power free number, then $\sum_{d^k|h} \mu(d) = 0$, from this properties of the M"obius function and Lemma 2 we have

$$
\sum_{n \leq x} S_k(n) = \sum_{m \leq \sqrt[2k]{x}} \sum_{h \leq x/m^k} \sum_{d^k|h} \mu(d) m
$$

$$
= \sum_{m \leq \sqrt[2k]{x}} m \sum_{h \leq x/m^k} \sum_{d^k|h} \mu(d) + \sum_{h \leq \sqrt[2k]{x}} \sum_{m \leq \sqrt[2k]{x}} \sum_{d^k|h} \mu(d)
$$

$$
- \left( \sum_{m \leq \sqrt[2k]{x}} m \right) \left( \sum_{h \leq \sqrt[2k]{x}} \sum_{d^k|h} \mu(d) \right).
$$

(2)
Now we estimate the three terms in the right hand side of formula (2). by Lemma 2 we have

\[
\sum_{m \leq \sqrt{x}} \sum_{h \leq x/m^2} \sum_{d^2 \leq x/m^2} \mu(d) = \sum_{m \leq \sqrt{x}} \sum_{h \leq x/m^2} \mu(d) = \sum_{m \leq \sqrt{x}} \sum_{d^2 \leq x/m^2} 1
\]

\[
= \sum_{m \leq \sqrt{x}} \sum_{d^2 \leq x/m^2} \mu(d) \left[ \frac{x}{m^2 d^2} + O(1) \right]
\]

\[
= \frac{x}{\zeta(2)} \sum_{m \leq \sqrt{x}} \frac{1}{m} + O \left( x^{\frac{3}{4}} \right)
\]

\[
= \frac{x}{\zeta(2)} \left[ \frac{1}{4} \ln x + \gamma + O \left( x^{-\frac{1}{4}} \right) \right] + O \left( x^{\frac{3}{4}} \right)
\]

\[
= \frac{3}{\pi^2} \left( \frac{1}{2} \ln x + 2\gamma \right) \cdot x + O \left( x^{\frac{3}{4}} \right),
\]  

(3)

where we have used the identity \( \zeta(2) = \frac{\pi^2}{6} \) and \( \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} \).

\[
\sum_{h \leq \sqrt{x}} \sum_{m \leq \sqrt{x}} \sum_{d^2 \leq x/m^2} \mu(d) = \sum_{m \leq \sqrt{x}} \sum_{d^2 \leq x/m^2} \mu(d) \sum_{h \leq \sqrt{x}} m
\]

\[
= \sum_{d^2 \leq \sqrt{x}} \mu(d) \left[ \frac{x}{2d^2 h} + O \left( \frac{\sqrt{x}}{d h} \right) \right]
\]

\[
= \frac{x}{2} \sum_{d^2 \leq \sqrt{x}} \mu(d) \frac{1}{d^2 h} + O \left( x^{\frac{3}{4}} \ln x \right)
\]

\[
= \frac{x}{2} \sum_{d^2 \leq \sqrt{x}} \mu(d) \frac{1}{d^2} \sum_{h \leq \sqrt{x}} \frac{1}{h} + O \left( x^{\frac{3}{4}} \ln x \right)
\]

\[
= \frac{x}{2} \sum_{d^2 \leq \sqrt{x}} \mu(d) \frac{1}{d^2} \left[ \ln \frac{\sqrt{x}}{d^2} + \gamma + O \left( \frac{\sqrt{x}}{d^2} \right) \right] + O \left( x^{\frac{3}{4}} \ln x \right)
\]

\[
= \frac{x}{2\zeta(2)} \left[ \frac{1}{2} \ln x + \gamma \right] - \sum_{n=1}^{\infty} \frac{\mu(n) \ln n}{n^2} + O \left( x^{\frac{3}{4}} \ln x \right)
\]

\[
= \frac{x}{2\zeta(2)} \left[ \frac{1}{2} \ln x + \gamma \right] - \frac{\zeta'(2)}{\zeta(2)} \cdot x + O \left( x^{\frac{3}{4}} \ln x \right),
\]  

(4)

where we have used the identity \(- \frac{\zeta'(2)}{\zeta(2)} = - \sum_{n=1}^{\infty} \frac{\mu(n) \ln n}{n^2} \).

Similarly, we also have the asymptotic formulæ

\[
\sum_{h \leq \sqrt{x}} \sum_{d^2 \leq x/m^2} \mu(d) = \sum_{d^2 \leq \sqrt{x}} \mu(d) \sum_{h \leq \frac{x}{d^2}} 1 = \sum_{d^2 \leq \sqrt{x}} \mu(d) \left[ \frac{\sqrt{x}}{d^2} + O(1) \right] = \frac{\sqrt{x}}{\zeta(2)} + O \left( x^{\frac{3}{4}} \right)
\]  

(5)

and

\[
\sum_{m \leq \sqrt{x}} m = \frac{\sqrt{x}}{2} + O \left( x^{\frac{3}{4}} \right).
\]  

(6)
From (2), (3), (4), (5) and (6) we deduce the asymptotic formula
\[ \sum_{n \leq x} \overline{S}_k(n) = \frac{3}{\pi^2} \left( \ln x + 3\gamma - 1 - \frac{12}{\pi^2} \zeta'(2) \right) \cdot x + O \left( x^\frac{3}{2} \ln x \right). \]

If \( k > 2 \), then from Lemma 1 and Lemma 3 we have
\[
\sum_{n \leq x} \overline{S}_k(n) = \sum_{m^k h \leq x} \sum_{d^k | h} \mu(d)m = \sum_{m \leq \sqrt{x}} m \sum_{h \leq x/m^k} \sum_{d \leq x/h} \mu(d) = \sum_{m \leq \sqrt{x}} m \sum_{h \leq x/m^k} \mu(d) \left[ \frac{x}{m^k d^k} + O(1) \right]
\]
\[
= \sum_{m \leq \sqrt{x}} m \sum_{h \leq x/m^k} \mu(d) \frac{x}{m^k d^k} + O \left( x^\frac{3}{2} \right)
\]
\[
= \frac{\zeta(k-1)}{\zeta(k)} x + O \left( x^\frac{3}{2} \right).
\]
Combining (6) and (7) we may immediately deduce the asymptotic formula
\[
\sum_{n \leq x} \overline{S}_k(n) = \begin{cases} 
\frac{3}{\pi^2} \left( \ln x + 3\gamma - 1 - \frac{12}{\pi^2} \zeta'(2) \right) \cdot x + O \left( x^\frac{3}{2} \ln x \right), & \text{if } k = 2; \\
\frac{\zeta(k-1)}{\zeta(k)} x + O \left( x^\frac{3}{2} \right), & \text{if } k > 2.
\end{cases}
\]
where \( \zeta(s) \) is the Riemann zeta-function.

Now we use the analytic method to prove our Theorem. It is clear that \( \overline{S}_k(n) \) is a multiplicative function, so for any real number \( s > 2 \), from the Euler product formula we have
\[
f(s) = \sum_{n=1}^\infty \frac{\overline{S}_k(n)}{n^s} = \prod_p \left( 1 + \frac{\overline{S}_k(p)}{p^s} + \frac{\overline{S}_k(p^2)}{p^{2s}} + \cdots + \frac{\overline{S}_k(p^k)}{p^{ks}} + \cdots \right)
\]
\[
= \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{(k-1)s}} + \frac{p}{p^{ks}} + \frac{p}{p^{(k+1)s}} + \cdots + \frac{p}{p^{(2k-1)s}} + \cdots \right)
\]
\[
= \prod_p \left\{ \frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p}} \left( 1 + \frac{p}{p^{ks}} + \frac{p^2}{p^{2ks}} + \cdots \right) \right\}
\]
\[
= \prod_p \left\{ \frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p}} \left( 1 + \frac{1}{p^{ks-1}} + \frac{1}{p^{2(ks-1)}} + \cdots \right) \right\}
\]
\[
= \frac{\zeta(s) \zeta(ks - 1)}{\zeta(ks)},
\]
where \( \zeta(s) \) is the Riemann zeta-function.

It is clear that if \( k = 2 \), then function \( G(s) = \frac{\zeta(s) \zeta(2s-1)}{\zeta(2s)} \frac{x^s}{s} \) has a pole point at \( s = 1 \) with order 2. The residue of the function \( G(s) \) at point \( s = 1 \) is
\[
\frac{3}{\pi^2} \left( x \ln x + 3\gamma - 1 - \frac{12}{\pi^2} \zeta'(2) \right).
\]
If \( k > 2 \), then the function 
\[
G(s) = \frac{\zeta(s) \zeta(ks - 1)}{\zeta(ks)} x^s
\]
has a simple pole point at \( s = 1 \) with residue \( \frac{\zeta(k-1)}{\zeta(k)} x^s \). Then by the Perron’s formula (see [6]) we have
\[
\sum_{n \leq x} S_k(n) = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{\zeta(s) \zeta(ks - 1)}{\zeta(ks)} \frac{x^s}{s} ds + O\left(\frac{x^{2+\epsilon}}{T}\right). 
\] (9)

Moving the integral line in (9) from \( s = 2 \pm iT \) to \( s = \frac{1}{2} \pm T \), we may immediately deduce the asymptotic formula
\[
\sum_{n \leq x} S_k(n) = \begin{cases} 
\frac{3}{\pi^2} \left( \ln x + 3\gamma - 1 - \frac{12}{\pi^2} \zeta'(2) \right) \cdot x + O\left(x^{\frac{2}{2}+\epsilon}\right), & \text{if } k = 2; \\
\frac{\zeta(k-1)}{\zeta(k)} x + O\left(x^{\frac{k}{2}+\epsilon}\right), & \text{if } k > 2. 
\end{cases}
\]

where \( \epsilon \) denotes any fixed positive number, \( \zeta'(2) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^2} \).

Now combining two methods we may immediately deduce our Theorem.

References

This book contains 23 papers, most of which were written by participants to the fifth International Conference on Number Theory and Smarandache Notions held in Shangluo University, China, in March, 2009. In this Conference, several professors gave a talk on Smarandache Notions and many participants lectured on them both extensively and intensively.

All these papers are original and have been refereed. The themes of these papers range from the mean value or hybrid mean value of Smarandache type functions, the mean value of some famous number theroretic functions acting on the Smarandache sequences, to the convergence property of some infinite series involving the Smarandache type sequences.

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