Hybrid mean value on some Smarandache functions

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Abstract: The mean value properties of the Smarandache function acting on k-th roots sequences is studied by using the elementary method, an interesting asymptotic formula is obtained.

Key words: Smarandache function k-th roots mean value

1 Introduction and conclusion

For any positive integer n, the Smarandache function Sdf(n) is defined as following:

Sdf(n) = min{m : m ∈ N, m!! ≤ n, m!! ≤ n}

Where m!! = 2 · 4 · 6 · ... · m if m is an even; m!! = 1 · 3 · 5 · ... · m if m is an odd. The other function

a_k(n) = \lfloor n^{1/k} \rfloor

is the integer part of k-th root of n. That is the greatest integer less than or equal to real number x.

These two function were both proposed by professor F. Smarandache in reference[1], where he asked us to study the properties of these function.

About the relations between the sequence and the Smarandache function, it seems that none had studied it at least we have not seen any related papers before. However about the properties of Sdf(n) and a_k(n), many scholars showed great interest in reference[2-5].

In this paper we study the hybrid mean value properties of the Smarandache function acting on the k-th roots sequences and give an interesting asymptotic formula. That is, we shall prove the following conclusion:

Theorem 1 For any real number x ≥ 2 we have the asymptotic formula

\[ \sum_{n \leq x} Sdf(a_k(n)) = \frac{7\pi^2}{12(k+1)} \frac{x^{k+1/k}}{\ln x} + O\left( \frac{x^{k+1/k}}{\ln x} \right). \]

2 Some Lemmas

To complete the proof of the theorem, we need the following two simple Lemmas.

Lemma 1 If 2 \mid n and n = p_1^{a_1} p_2^{a_2} ... p_s^{a_s} is the factorization of n where
Where \( \pi(x) \) denotes all the numbers of prime which is not exceeding \( x \). Notice that \( \pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln x}\right) \).

Using the Euler summation formula, we get

\[
\text{Lemma 1} \quad \text{Forthefirstpart, weletthesets } A \text{ and } B \text{ following:}
\]

\[
S(n) = \min\{m : m \in \mathbb{N}, n \cdot m! \}
\]

\[
S_{df}(n) = \max(p^i \mid (p^i)!! = n, p^i \in \mathbb{N})
\]

Prove that \( m = P(n) \). We assume \( P(n) = p^i \).

Proof Let \( m = S_{df}(n) \). We get \( n \cdot m!! \leq x \). Similarly, from the Abel's identity

\[
\text{Lemma 2} \quad \text{let } n = P_1^{i_1} P_2^{i_2} \ldots \text{ is the prime powers factorization of } n \text{ and } P(n) = \max \{ m \} \text{. if there exists } P(n) \text{ satisfied with } P(n) > n \text{ then we have the identity } S_{df}(n) = P(n).
\]

Proof First we let \( S_{df}(n) = m \), then \( m \) is the smallest positive integer such that \( n \cdot m!! \leq x \). Now we will prove that \( m = P(n) \). We assume \( P(n) = P_1 \). From the definition of \( P(n) \) and Lemma 1, we know that \( S_{df}(n) = \max \{ P_1^{i_1}, (2a_i - 1) P_i \} \). Therefore we get

\[
(1) \quad \text{If } a_i = 1, \text{ then } S_{df}(n) = P_1^i \geq (2a_i - 1) P_i
\]

\[
(2) \quad \text{If } a_i > 1, \text{ then } S_{df}(n) = P_1^i > 2 \ln n^a > (2a_i - 1) P_i
\]

Combining (1) - (2) we can easily obtain \( S_{df}(n) = P(n) \). This proves Lemma 2.

\[
\text{Lemma 3} \quad \text{Let } x \geq 1 \text{ be any real number, we have the asymptotic formula}
\]

\[
\sum_{\nu \leq x} S(n) = \frac{x^2}{12} \frac{1}{\ln x} + O\left(\frac{x}{\ln x}\right).
\]

Where \( S(n) = m \in \mathbb{N} : n \cdot m! \).

Proof See reference [5].

\[
\text{Lemma 4} \quad \text{Let } x \geq 2 \text{ be any real number, we have the asymptotic formula}
\]

\[
\sum_{\nu \leq x} S_{df}(n) = \frac{7x^2}{24} \frac{1}{\ln x} + O\left(\frac{x}{\ln x}\right)
\]

Proof It is clear that

\[
\sum_{\nu \leq x} S_{df}(n) = \sum_{\nu \in (x^{1/2})} S_{df}(2u + 1) + \sum_{\nu \in 2} S_{df}(2u)
\]

For the first part we let the sets \( A \) and \( B \) as following

\[
A = \{ 2u + 1 : 2u + 1 \leq x, P(2u + 1) \leq \sqrt{2u + 1} \}
\]

and

\[
B = \{ 2u + 1 : 2u + 1 \leq x, P(2u + 1) > \sqrt{2u + 1} \}
\]

Using the Euler summation formula, we get

\[
\sum_{2u+1 \in A} S_{df}(2u+1) \ll \sum_{2u+1 \in x} \sqrt{2u+1} \ln(2u+1) \ll x^2 \ln x
\]

Similarly from the Abel's identity and Lemma 2, we also get

\[
\sum_{2u+1 \in B} S_{df}(2u+1) = \sum_{2u+1 \in B} P(2u+1) =
\]

\[
\sum_{2u+1 \in B} \left( \frac{x}{2} - \frac{x}{2} \right) - \frac{x}{2} \ln \frac{x}{2} + \frac{1}{2} \left( \frac{x}{2} \right)^2 + O\left(\frac{x^2}{\ln x}\right)
\]

where \( \pi(x) \) denotes all the numbers of prime which is not exceeding \( x \). Notice that \( \pi(x) = x/\ln x + O(\sqrt{x} \ln x) \)

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and
\[
\sum_{\ell \leq \sqrt{x}} \left( \frac{\ell}{2} \right) = (2 \ell + 1) \frac{\pi}{2} (\ell + 1) - \int_0^{\pi(1-\ell)} \frac{\pi(x)}{x} \, dx
\]
Hence
\[
\sum_{\ell \leq \sqrt{x}} \left( \frac{\ell}{2} \right) = (2 \ell + 1) \frac{\pi}{2} (\ell + 1) + \left( \frac{\pi}{2} \right) \left( \frac{\pi}{\ell(x(2\ell + 1))} - \pi^{(2\ell + 1)} \right)
\]
(4)

Therefore, we can obtain the asymptotic formula:

\[
B(y) = \frac{\pi^2}{8} \ln x + O(\frac{\pi}{\ell(x(2\ell + 1))})
\]
(5)

Combining (2), (3), (4) and (5) we obtain

\[
\sum_{y \leq \sqrt{x}} Sdf(x) = \frac{\pi^2}{8} \ln x + O\left( \frac{\pi}{\ell(x(2\ell + 1))} \right)
\]
(6)

For the second part we notice that \(y = 2 \eta \) where \( \eta \) are positive integers with \( 2 \leq \eta \). Let \( Sdf(x) = m \ln m | 2 \eta | m! \), from the definition of \( Sdf \) and Lemma 3 we have

\[
\sum_{2 \leq x} Sdf(2 \eta) = \sum_{2 \leq x} \sum_{\ell \leq \sqrt{x}} \ell \ln x
\]
(7)

and

\[
\sum_{2 \leq x} Sdf(2 \eta) = \frac{\pi^2}{6} \ln x + O\left( \frac{\pi}{\ell(x(2\ell + 1))} \right)
\]
(8)

Combining (7) and (8) we obtain

\[
\sum_{2 \leq x} Sdf(2 \eta) = \frac{\pi^2}{6} \ln x + O\left( \frac{\pi}{\ell(x(2\ell + 1))} \right)
\]
(9)

From (1), (6) and (9) we can get the result of Lemma 4.

### 3 Proof of the Theorem 1

For any real number \( \eta \geq 1 \) let \( M \) be a fixed positive integer such that \( M \leq x < (M+1)^k \). From the definition of \( Sdf \) we have

\[
\sum_{y \leq x} Sdf(\ell(\eta)) = \sum_{\ell \leq \sqrt{x}} \sum_{\ell \leq \ell^{(k+1)}} Sdf(\ell(\eta)) + \sum_{\ell \leq \sqrt{x}} \sum_{M \leq \ell} Sdf(\ell(\eta)) = \sum_{\ell \leq \sqrt{x}} \ell^{(k+1)} Sdf(\ell) + O(x^{k+1})
\]

Let \( B(y) = \sum_{y \leq x} Sdf(\eta) \) by the Abel's identity and Lemma 4 we can easily deduce that

\[
\sum_{k=1}^M \ell^{(k)} Sdf(\eta) = M^{k+1} B(M) - (k-1) \int_0^{\pi^2} \frac{y^{k+1} B(y) \, dy}{24} = M^{k+1} \frac{\pi^2}{24} B(M) - (k-1) \int_0^{\pi^2} \frac{y^{k+1} B(y) \, dy}{24} = \sum_{k=1}^M \frac{\pi^2}{24} M^{k+1} B(M) - \frac{k-1}{24} \pi^2 M^{k+1} B(M) + O\left( \frac{M^{k+1}}{\ell(M)} \right)
\]

Therefore we can obtain the asymptotic formula.
\[ \sum_{n \leq x} Sdf(a_k(n)) = \frac{7\pi^2}{12(k+1)} \frac{M^{k+1}}{\ln M} + O\left( \frac{M^{k+1}}{\ln M} \right). \]

On the other hand, we also have the estimate
\[ 0 \leq x-M^k < (M+1)^k - M^k \leq x^{(k-1)/k} \cdot \]

Now combining the above, we may immediately obtain the asymptotic formula
\[ \sum_{n \leq x} Sdf(a_k(n)) = \frac{7\pi^2}{12(k+1)} \frac{x^{(k+1)/k}}{\ln x} + O\left( \frac{x^{(k+1)/k}}{\ln x} \right). \]

This completes the proof of Theorem 1.

References: