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# DSm models and Non-Archimedean Reasoning

Published in:
Florentin Smarandache & Jean Dezert (Editors)
**Advances and Applications of DSmT for Information Fusion**
(Collected works), Vol. II
American Research Press, Rehoboth, 2006
ISBN: 1-59973-000-6
Chapter VII, pp. 183 - 204

**Abstract:** *The Dezert-Smarandache theory of plausible and paradoxical reasoning is based on the premise that some elements $\theta_i$ of a frame $\Theta$ have a non-empty intersection. These elements are called exhaustive. In number theory, this property is observed only in non-Archimedean number systems, for example, in the ring $\mathbf{Z}_p$ of p-adic integers, in the field $^*\mathbf{Q}$ of hyperrational numbers, in the field $^*\mathbf{R}$ of hyperreal numbers, etc. In this chapter, I show that non-Archimedean structures are infinite DSm models in that each positive exhaustive element is greater (or less) than each positive exclusive element. Then I consider three principal versions of the non-Archimedean logic: p-adic valued logic $\mathfrak{M}_{\mathbf{Z}_p}$, hyperrational valued logic $\mathfrak{M}_{^*\mathbf{Q}}$, hyperreal valued logic $\mathfrak{M}_{^*\mathbf{R}}$, and their applications to plausible reasoning. These logics are constructed for the first time.*

## 7.1 Introduction

The development of fuzzy logic and fuzziness was motivated in large measure by the need for a conceptual framework which can address the issue of uncertainty and lexical imprecision. Recall that fuzzy logic was introduced by Lofti Zadeh in 1965 (see [20]) to represent data and information possessing nonstatistical uncertainties. Florentin Smarandache had generalized fuzzy logic and introduced two new concepts (see [16], [18], [17]):

1. neutrosophy as study of neutralities;

2. neutrosophic logic and neutrosophic probability as a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, etc.

Neutrosophy is a new branch of philosophy, which studies the nature of neutralities, as well as their logical applications. This branch represents a version of paradoxism studies. The essence of paradoxism studies is that there is a neutrality for any two extremes. For example, denote by $A$ an idea (or proposition, event, concept), by *Anti*-$A$ the opposite to $A$. Then there exists a neutrality *Neut*-$A$ and this means that something is neither $A$ nor *Anti*-$A$. It is readily seen that the paradoxical reasoning can be modeled if some elements $\theta_i$ of a frame $\Theta$ are not exclusive, but exhaustive, i. e., here $\theta_i$ have a non-empty intersection. A mathematical model that has such a property is called the Dezert-Smarandache model (DSm model). A theory of plausible and paradoxical reasoning that studies DSm models is called the Dezert-Smarandache theory (DSmT). It is totally different from those of all existing approaches managing uncertainties and fuzziness. In this chapter, I consider plausible reasoning on the base of particular case of infinite DSm models, namely, on the base of non-Archimedean structures.

Let us remember that Archimedes' axiom is the formula of infinite length that has one of two following notations:

- for any $\varepsilon$ that belongs to the interval $[0, 1]$, we have

$$(\varepsilon > 0) \supset [(\varepsilon \geq 1) \vee (\varepsilon + \varepsilon \geq 1) \vee (\varepsilon + \varepsilon + \varepsilon \geq 1) \vee \ldots], \tag{7.1}$$

- for any positive integer $\varepsilon$, we have

$$[(1 \geq \varepsilon) \vee (1 + 1 \geq \varepsilon) \vee (1 + 1 + 1 \geq \varepsilon) \vee \ldots]. \tag{7.2}$$

Formulas (7.1) and (7.2) are valid in the field $\mathbf{Q}$ of rational numbers and as well as in the field $\mathbf{R}$ of real numbers. In the ring $\mathbf{Z}$ of integers, only formula (7.2) has a nontrivial sense, because $\mathbf{Z}$ doesn't contain numbers of the open interval $(0, 1)$.

Also, Archimedes' axiom affirms the existence of an integer multiple of the smaller of two numbers which exceeds the greater: for any positive real or rational number $\varepsilon$, there exists a positive integer $n$ such that $\varepsilon \geq \frac{1}{n}$ or $n \cdot \varepsilon \geq 1$.

The negation of Archimedes' axiom has one of two following forms:

- there exists $\varepsilon$ that belongs to the interval $[0, 1]$ such that

$$(\varepsilon > 0) \wedge [(\varepsilon < 1) \wedge (\varepsilon + \varepsilon < 1) \wedge (\varepsilon + \varepsilon + \varepsilon < 1) \wedge \ldots], \tag{7.3}$$

- there exists a positive integer $\varepsilon$ such that

$$[(1 < \varepsilon) \wedge (1 + 1 < \varepsilon) \wedge (1 + 1 + 1 < \varepsilon) \wedge \ldots]. \tag{7.4}$$

Let us show that (7.3) is the negation of (7.1). Indeed,

$$\begin{aligned}
\neg\forall\varepsilon\, [(\varepsilon > 0) \supset [(\varepsilon \geq 1) \vee (\varepsilon + \varepsilon \geq 1) \vee (\varepsilon + \varepsilon + \varepsilon \geq 1) \vee \ldots]] &\equiv \\
\exists\varepsilon\, \neg\neg[(\varepsilon > 0) \wedge \neg[(\varepsilon \geq 1) \vee (\varepsilon + \varepsilon \geq 1) \vee (\varepsilon + \varepsilon + \varepsilon \geq 1) \vee \ldots]] &\equiv \\
\exists\varepsilon\, (\varepsilon > 0) \wedge [\neg(\varepsilon \geq 1) \wedge \neg(\varepsilon + \varepsilon \geq 1) \wedge \neg(\varepsilon + \varepsilon + \varepsilon \geq 1) \wedge \ldots]] &\equiv \\
\exists\varepsilon\, (\varepsilon > 0) \wedge [(\varepsilon < 1) \wedge (\varepsilon + \varepsilon < 1) \wedge (\varepsilon + \varepsilon + \varepsilon < 1) \wedge \ldots]]
\end{aligned}$$

It is obvious that formula (7.3) says that there exist *infinitely small numbers* (or *infinitesimals*), i. e., numbers that are smaller than all real or rational numbers of the open interval $(0, 1)$. In other words, $\varepsilon$ is said to be an infinitesimal if and only if, for all positive integers $n$, we have $|\varepsilon| < \frac{1}{n}$. Further, formula (7.4) says that there exist *infinitely large integers* that are greater than all positive integers. Infinitesimals and infinitely large integers are called *nonstandard numbers* or *actual infinities*.

The field that satisfies all properties of $\mathbf{R}$ without Archimedes' axiom is called the field of *hyperreal numbers* and it is denoted by $^*\mathbf{R}$. The field that satisfies all properties of $\mathbf{Q}$ without Archimedes' axiom is called the field of hyperrational numbers and it is denoted by $^*\mathbf{Q}$. By definition of field, if $\varepsilon \in \mathbf{R}$ (respectively $\varepsilon \in \mathbf{Q}$), then $1/\varepsilon \in \mathbf{R}$ (respectively $1/\varepsilon \in \mathbf{Q}$). Therefore $^*\mathbf{R}$ and $^*\mathbf{Q}$ contain simultaneously infinitesimals and infinitely large integers: for an infinitesimal $\varepsilon$, we have $N = \frac{1}{\varepsilon}$, where $N$ is an infinitely large integer.

The ring that satisfies all properties of $\mathbf{Z}$ without Archimedes' axiom is called the ring of hyperintegers and it is denoted by $^*\mathbf{Z}$. This ring includes infinitely large integers. Notice that there exists a version of $^*\mathbf{Z}$ that is called the ring of $p$-adic integers and is denoted by $\mathbf{Z}_p$.

I shall show in this chapter that nonstandard numbers (actual infinities) are exhaustive elements (see section 7.3). This means that their intersection isn't empty with some other elements. Therefore non-Archimedean structures of the form $^*\mathbf{S}$ (where we obtain $^*\mathbf{S}$ on the base of the set $\mathbf{S}$ of exclusive elements) are particular case of the DSm model. These structures satisfy the properties:

1. all members of $\mathbf{S}$ are exclusive and $\mathbf{S} \subset {}^*\mathbf{S}$,
2. all members of $^*\mathbf{S}\backslash\mathbf{S}$ are exhaustive,
3. if a member $a$ is exhaustive, then there exists a exclusive member $b$ such that $a \cap b \neq \emptyset$,
4. there exist exhaustive members $a$, $b$ such that $a \cap b \neq \emptyset$,
5. each positive exhaustive member is greater (or less) than each positive exclusive member.

I shall consider three principal versions of the logic on non-Archimedean structures: hyperrational valued logic $\mathfrak{M}_{^*\mathbf{Q}}$, hyperreal valued logic $\mathfrak{M}_{^*\mathbf{R}}$, $p$-adic valued logic $\mathfrak{M}_{\mathbf{Z}_p}$, and their applications to plausible and fuzzy reasoning.

## 7.2 Standard many-valued logics

Let us remember that a *first-order logical language* $\mathcal{L}$ consists of the following symbols:

1. Variables:

   *(i)* Free variables: $a_0, a_1, a_2, \ldots, a_j, \ldots$ $(j \in \omega)$

   *(ii)* Bound variables: $x_0, x_1, x_2, \ldots, x_j, \ldots$ $(j \in \omega)$

2. Constants:

   *(i)* Function symbols of arity $i$ ($i \in \omega$): $F^i_0, F^i_1, F^i_2, \ldots, F^i_j, \ldots$ ($j \in \omega$). Nullary function symbols are called constants.

   *(ii)* Predicate symbols of arity $i$ ($i \in \omega$): $P^i_0, P^i_1, P^i_2, \ldots, P^i_j, \ldots$ ($j \in \omega$).

3. Logical symbols:

   *(i)* Propositional connectives of arity $n_j$ : $\square^{n_0}_0, \square^{n_1}_1, \ldots, \square^{n_r}_r$, which are built by superposition of negation $\neg$ and implication $\supset$.

   *(ii)* Quantifiers: $\mathrm{Q}_0, \mathrm{Q}_1, ..., \mathrm{Q}_q$.

4. Auxiliary symbols: (, ), and , (comma).

*Terms* are inductively defined as follows:

1. Every individual constant is a term.

2. Every free variable (and every bound variable) is a term.

3. If $F^n$ is a function symbol of arity $n$, and $t_1, \ldots, t_n$ are terms, then $F^n(t_1, \ldots, t_n)$ is a term.

*Formulas* are inductively defined as follows:

1. If $P^n$ is a predicate symbol of arity $n$, and $t_1, \ldots, t_n$ are terms, then $P^n(t_1, \ldots, t_n)$ is a formula. It is called atomic or an atom. It has no outermost logical symbol.

2. If $\varphi_1, \varphi_2, \ldots, \varphi_n$ are formulas and $\square^n$ is a propositional connective of arity $n$, then $\square^n(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is a formula with outermost logical symbol $\square^n$.

3. If $\varphi$ is a formula not containing the bound variable $x$, $a$ is a free variable and Q is a quantifier, then $\mathrm{Q}x\varphi(x)$, where $\varphi(x)$ is obtained from $\varphi$ by replacing $a$ by $x$ at every occurrence of $a$ in $\varphi$, is a formula. Its outermost logical symbol is Q.

A formula is called *open* if it contains free variables, and *closed* otherwise. A formula without quantifiers is called *quantifier-free*. We denote the set of formulas of a language $\mathcal{L}$ by $L$. We will write $\varphi(x)$ for a formula possibly containing the bound variable $x$, and $\varphi(a)$ respectively $\varphi(t)$ for the formula obtained from $\varphi$ by replacing every occurrence of the variable $x$ by the free variable $a$ respectively the term $t$. Hence, we shall need meta-variables for the symbols of a language $\mathcal{L}$. As a notational convention we use letters $\varphi, \phi, \psi, \ldots$ to denote formulas.

A *matrix*, or *matrix logic*, $\mathfrak{M}$ for a language $\mathcal{L}$ is given by:

1. a non-empty set of truth values $V$ of cardinality $|V| = m$,

2. a subset $D \subseteq V$ of designated truth values,

3. an algebra with domain $V$ of appropriate type: for every $n$-place connective $\square$ of $\mathcal{L}$ there is an associated truth function $f$: $V^n \mapsto V$ , and

4. for every quantifier Q, an associated truth function $\widetilde{\mathrm{Q}}$: $\wp(V)\backslash\emptyset \mapsto V$

Notice that a truth function for quantifiers is a mapping from non-empty sets of truth values to truth values: for a non-empty set $M \subseteq V$, a quantified formula $\mathrm{Q}x\varphi(x)$ takes the truth value $\widetilde{\mathrm{Q}}(M)$ if, for every truth value $v \in V$, it holds that $v \in M$ iff there is a domain element $d$ such that the truth value of $\varphi$ in this point $d$ is $v$ (all relative to some interpretation). The set $M$ is called the distribution of $\varphi$. For example, suppose that there are only the universal quantifier $\forall$ and the existential quantifier $\exists$ in $\mathcal{L}$. Further, we have the set of truth values $V = \{\top, \bot\}$, where $\bot$ is false and $\top$ is true, i. e., the set of designated truth values $D = \{\top\}$. Then we define the truth functions for the quantifiers $\forall$ and $\exists$ as follows:

1. $\widetilde{\forall}(\{\top\}) = \top$
2. $\widetilde{\forall}(\{\top, \bot\}) = \widetilde{\forall}(\{\bot\}) = \bot$
3. $\widetilde{\exists}(\{\bot\}) = \bot$
4. $\widetilde{\exists}(\{\top, \bot\}) = \widetilde{\exists}(\{\top\}) = \top$

Also, a matrix logic $\mathfrak{M}$ for a language $\mathcal{L}$ is an algebraic system denoted

$$\mathfrak{M} = <V, f_0, f_1, \ldots, f_r, \widetilde{\mathrm{Q}}_0, \widetilde{\mathrm{Q}}_1, \ldots, \widetilde{\mathrm{Q}}_q, D>$$

where

1. $V$ is a non-empty set of truth values for well-formed formulas of $\mathcal{L}$,
2. $f_0, f_1, \ldots, f_r$ are a set of matrix operations defined on the set $V$ and assigned to corresponding propositional connectives $\Box_0^{n_0}, \Box_1^{n_1}, \ldots, \Box_r^{n_r}$ of $\mathcal{L}$,
3. $\widetilde{\mathrm{Q}}_0, \widetilde{\mathrm{Q}}_1, \ldots, \widetilde{\mathrm{Q}}_q$ are a set of matrix operations defined on the set $V$ and assigned to corresponding quantifiers $\mathrm{Q}_0, \mathrm{Q}_1, ..., \mathrm{Q}_q$ of $\mathcal{L}$,
4. $D$ is a set of designated truth values such that $D \subseteq V$.

Now consider $(n + 1)$*-valued Łukasiewicz's matrix logic* $\mathfrak{M}_{n+1}$ defined as the ordered system $<V_{n+1}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{n\}>$ for any $n \geqslant 2$, $n \in \mathbf{N}$, where

1. $V_{n+1} = \{0, 1, \ldots, n\}$,
2. for all $x \in V_{n+1}$, $\neg x = n - x$,
3. for all $x, y \in V_{n+1}$, $x \supset y = \min(n, n - x + y)$,
4. for all $x, y \in V_{n+1}$, $x \vee y = (x \supset y) \supset y = \max(x, y)$,
5. for all $x, y \in V_{n+1}$, $x \wedge y = \neg(\neg x \vee \neg y) = \min(x, y)$,
6. for a subset $M \subseteq V_{n+1}$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of $M$,
7. for a subset $M \subseteq V_{n+1}$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of $M$,
8. $\{n\}$ is the set of designated truth values.

The truth value $0 \in V_{n+1}$ is false, the truth value $n \in V_{n+1}$ is true, and other truth values $x \in V_{n+1}$ are neutral.

The ordered system $<V_{\mathbf{Q}}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{1\}>$ is called *rational valued Łukasiewicz's matrix logic* $\mathfrak{M}_{\mathbf{Q}}$, where

1. $V_{\mathbf{Q}} = \{x \colon x \in \mathbf{Q}\} \cap [0, 1]$,
2. for all $x \in V_{\mathbf{Q}}$, $\neg x = 1 - x$,
3. for all $x, y \in V_{\mathbf{Q}}$, $x \supset y = \min(1, 1 - x + y)$,
4. for all $x, y \in V_{\mathbf{Q}}$, $x \vee y = (x \supset y) \supset y = \max(x, y)$,
5. for all $x, y \in V_{\mathbf{Q}}$, $x \wedge y = \neg(\neg x \vee \neg y) = \min(x, y)$,
6. for a subset $M \subseteq V_{\mathbf{Q}}$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of $M$,
7. for a subset $M \subseteq V_{\mathbf{Q}}$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of $M$,
8. $\{1\}$ is the set of designated truth values.

The truth value $0 \in V_{\mathbf{Q}}$ is false, the truth value $1 \in V_{\mathbf{Q}}$ is true, and other truth values $x \in V_{\mathbf{Q}}$ are neutral.

*Real valued Łukasiewicz's matrix logic* $\mathfrak{M}_{\mathbf{R}}$ is the ordered system $<V_{\mathbf{R}}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{1\}>$, where

1. $V_{\mathbf{R}} = \{x \colon x \in \mathbf{R}\} \cap [0, 1]$,
2. for all $x \in V_{\mathbf{R}}$, $\neg x = 1 - x$,
3. for all $x, y \in V_{\mathbf{R}}$, $x \supset y = \min(1, 1 - x + y)$,
4. for all $x, y \in V_{\mathbf{R}}$, $x \vee y = (x \supset y) \supset y = \max(x, y)$,
5. for all $x, y \in V_{\mathbf{R}}$, $x \wedge y = \neg(\neg x \vee \neg y) = \min(x, y)$,
6. for a subset $M \subseteq V_{\mathbf{R}}$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of $M$,
7. for a subset $M \subseteq V_{\mathbf{R}}$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of $M$,
8. $\{1\}$ is the set of designated truth values.

The truth value $0 \in V_{\mathbf{R}}$ is false, the truth value $1 \in V_{\mathbf{R}}$ is true, and other truth values $x \in V_{\mathbf{R}}$ are neutral.

Notice that the elements of truth value sets $V_{n+1}$, $V_{\mathbf{Q}}$, and $V_{\mathbf{R}}$ are exclusive: for any members $x, y$ we have $x \cap y = \emptyset$. Therefore Łukasiewicz's logics are based on the premise of existence *Shafer's model*. In other words, these logics are built on the families of exclusive elements (see [15], [14]).

However, for a wide class of fusion problems, "the intrinsic nature of hypotheses can be only vague and imprecise in such a way that precise refinement is just impossible to obtain in reality so that the exclusive elements $\theta_i$ cannot be properly identified and precisely separated" (see [19]). This means that if some elements $\theta_i$ of a frame $\Theta$ have non-empty intersection, then sources of evidence don't provide their beliefs with the same absolute interpretation of elements of the same frame $\Theta$ and the conflict between sources arises not only because of the possible unreliability of sources, but also because of possible different and relative interpretation of $\Theta$ (see [3], [4]).

## 7.3 Many-valued logics on DSm models

**Definition 1.** *A many-valued logic is said to be a many-valued logic on DSm model if some elements of its set $V$ of truth values are not exclusive, but exhaustive.*

Recall that a DSm model (*Dezert-Smarandache model*) is formed as a hyper-power set. Let $\Theta = \{\theta_1, \dots, \theta_n\}$ be a finite set (called frame) of $n$ exhaustive elements. The *hyper-power set* $D^\Theta$ is defined as the set of all composite propositions built from elements of $\Theta$ with $\cap$ and $\cup$ operators such that:

1. $\emptyset, \theta_1, \dots, \theta_n \in D^\Theta$;
2. if $A, B \in D^\Theta$, then $A \cap B \in D^\Theta$ and $A \cup B \in D^\Theta$;
3. no other elements belong to $D^\Theta$, except those obtained by using rules 1 or 2.

The cardinality of $D^\Theta$ is majored by $2^{2^n}$ when the cardinality of $\Theta$ equals $n$, i. e. $|\Theta| = n$. Since for any given finite set $\Theta$, $|D^\Theta| \geq |2^\Theta|$, we call $D^\Theta$ the hyper-power set of $\Theta$. Also, $D^\Theta$ constitutes what is called the *DSm model* $\mathcal{M}^f(\Theta)$. However elements $\theta_i$ can be truly exclusive. In such case, the hyper-power set $D^\Theta$ reduces naturally to the classical power set $2^\Theta$ and this constitutes the most restricted hybrid DSm model, denoted by $\mathcal{M}^0(\Theta)$, coinciding with Shafer's model. As an example, suppose that $\Theta = \{\theta_1, \theta_2\}$ with $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$, where $\theta_1$ and $\theta_2$ are truly exclusive (i. e., Shafer's model $\mathcal{M}^0$ holds), then because $\theta_1 \cap \theta_2 =_{\mathcal{M}^0} \emptyset$, one gets $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2 =_{\mathcal{M}^0} \emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\} = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\} = 2^\Theta$.

Now let us show that every non-Archimedean structure is an infinite DSm model, but no vice versa. The most easy way of setting non-Archimedean structures was proposed by Abraham Robinson in [13]. Consider a set $\Theta$ consisting only of exclusive members. Let $I$ be any infinite index set. Then we can construct an indexed family $\Theta^I$, i. e., we can obtain the set of all functions: $f\colon I \mapsto \Theta$ such that $f(\alpha) \in \Theta$ for any $\alpha \in I$.

A *filter* $\mathcal{F}$ on the index set $I$ is a family of sets $\mathcal{F} \subset \wp(I)$ for which:

1. $A \in \mathcal{F}$, $A \subset B \Rightarrow B \in \mathcal{F}$;
2. $A_1, \dots, A_n \in \mathcal{F} \Rightarrow \bigcap\limits_{k=1}^{n} A_k \in \mathcal{F}$;
3. $\emptyset \notin \mathcal{F}$.

The set of all complements for finite subsets of $I$ is a filter and it is called a *Frechet filter*. A maximal filter (ultrafilter) that contains a Frechet filter is called a *Frechet ultrafilter* and it is denoted by $\mathcal{U}$.

Let $\mathcal{U}$ be a Frechet ultrafilter on $I$. Define a new relation $\backsim$ on the set $\Theta^I$ by

$$f \backsim g \equiv \{\alpha \in I\colon f(\alpha) = g(\alpha)\} \in \mathcal{U}. \tag{7.5}$$

It is easily be proved that the relation $\backsim$ is an equivalence. Notice that formula (7.5) means that $f$ and $g$ are equivalent iff $f$ and $g$ are equal on an infinite index subset. For each $f \in \Theta^I$ let $[f]$ denote the equivalence class of $f$ under $\backsim$. The *ultrapower* $\Theta^I/\mathcal{U}$ is then defined to be the set of all equivalence classes $[f]$ as $f$ ranges over $\Theta^I$:

$$\Theta^I/\mathcal{U} \triangleq \{[f]\colon f \in \Theta^I\}.$$

Also, Robinson has proved that each non-empty set $\Theta$ has an ultrapower with respect to a Frechet ultrafilter $\mathcal{U}$. This ultrapower $\Theta^I/\mathcal{U}$ is said to be a *proper nonstandard extension* of $\Theta$ and it is denoted by ${}^*\Theta$.

**Proposition 1.** *Let $X$ be a non-empty set. A nonstandard extension of $X$ consists of a mapping that assigns a set ${}^*A$ to each $A \subseteq X^m$ for all $m \geq 0$, such that ${}^*X$ is non-empty and the following conditions are satisfied for all $m, n \geq 0$:*

1. *The mapping preserves Boolean operations on subsets of $X^m$: if $A \subseteq X^m$, then ${}^*A \subseteq ({}^*X)^m$; if $A, B \subseteq X^m$, then ${}^*(A \cap B) = ({}^*A \cap {}^*B), {}^*(A \cup B) = ({}^*A \cup {}^*B)$, and ${}^*(A \backslash B) = ({}^*A) \backslash ({}^*B)$.*
2. *The mapping preserves Cartesian products: if $A \subseteq X^m$ and $B \subseteq X^n$, then ${}^*(A \times B) = {}^*A \times {}^*B$, where $A \times B \subseteq X^{m+n}$.* □

This proposition is proved in [5].

Recall that each element of ${}^*\Theta$ is an equivalence class $[f]$ as $f\colon I \mapsto \Theta$. There exist two groups of members of ${}^*\Theta$ (see Fig. 7.1):

1. functions that are constant, e. g., $f(\alpha) = m \in \Theta$ for infinite index subset $\{\alpha \in I\}$. A constant function $[f = m]$ is denoted by ${}^*m$,
2. functions that aren't constant.

The set of all constant functions of ${}^*\Theta$ is called *standard set* and it is denoted by ${}^\sigma\Theta$. The members of ${}^\sigma\Theta$ are called *standard*. It is readily seen that ${}^\sigma\Theta$ and $\Theta$ are isomorphic: ${}^\sigma\Theta \simeq \Theta$.

The following proposition can be easily proved:

**Proposition 2.** *For any set $\Theta$ such that $|\Theta| \geq 2$, there exists a proper nonstandard extension ${}^*\Theta$ for which ${}^*\Theta \backslash {}^\sigma\Theta \neq \emptyset$.*

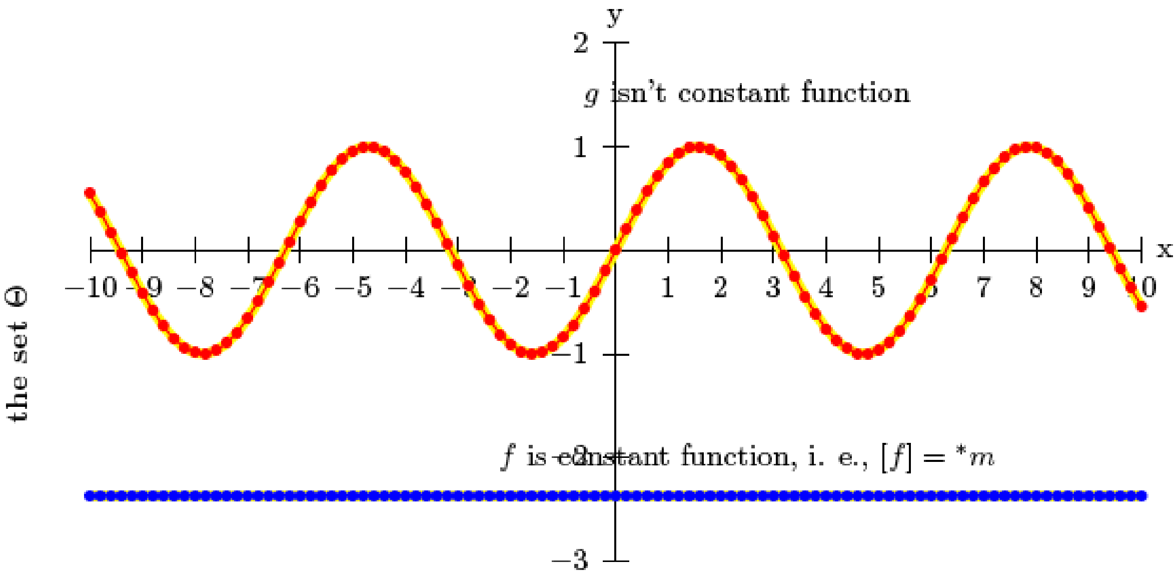


Figure 7.1: The members of $^*\Theta$: constant and non-constant functions.

*Proof.* Let $I_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\} \subset I$ be an infinite set and let $\mathcal{U}$ be a Frechet ultrafilter. Suppose that $\Theta_1 = \{m_1, \ldots, m_n\}$ such that $|\Theta_1| \geq 1$ is the subset of $\Theta$ and there is a mapping:

$$f(\alpha) = \begin{cases} m_k & \text{if } \alpha = \alpha_k; \\ m_0 \in \Theta & \text{if } \alpha \in I \backslash I_1 \end{cases}$$

and $f(\alpha) \neq m_k$ if $\alpha = \alpha_k \mod (n+1)$, $k = 1, \ldots, n$ and $\alpha \neq \alpha_k$.

Show that $[f] \in {}^*\Theta \backslash {}^\sigma\Theta$. The proof is by reductio ad absurdum. Suppose there is $m \in \Theta$ such that $m \in [f(\alpha)]$. Consider the set:

$$I_2 = \{\alpha \in I \colon f(\alpha) = m\} = \begin{cases} \{\alpha_k\} & \text{if } m = m_k,\ k = 1, \ldots, n; \\ I \backslash I_1 & \text{if } m = m_0. \\ \emptyset & \text{if } m \notin \{m_0, m_1, \ldots, m_n\}. \end{cases}$$

In any case $I_2 \notin \mathcal{U}$, because $\{\alpha_k\} \notin \mathcal{U}$, $\emptyset \notin \mathcal{U}$, $I \backslash I_1 \notin \mathcal{U}$. Thus, $[f] \in {}^*\Theta \backslash {}^\sigma\Theta$. □

The standard members of $^*\Theta$ are exclusive, because their intersections are empty. Indeed, the members of $\Theta$ were exclusive, therefore the members of $^\sigma\Theta$ are exclusive too. However the members of $^*\Theta \backslash {}^\sigma\Theta$ are exhaustive. By definition, if a member $a \in {}^*\Theta$ is nonstandard, then there exists a standard member $b \in {}^*\Theta$ such that $a \cap b \neq \emptyset$ (for example, see the proof of proposition 2). We can also prove that there exist exhaustive members $a \in {}^*\Theta$, $b \in {}^*\Theta$ such that $a \cap b \neq \emptyset$.

**Proposition 3.** *There exist two inconstant functions $f_1, f_2$ such that the intersection of $f_1, f_2$ isn't empty.*

*Proof.* Let $f_1 \colon I \mapsto \Theta$ and $f_2 \colon I \mapsto \Theta$. Suppose that $[f_i \neq n]$, $\forall n \in \Theta$, $i = 1, 2$, i. e., $f_1, f_2$ aren't constant. By proposition 2, these functions are nonstandard members of $^*\Theta$. Further consider an indexed family $F(\alpha)$ for all $\alpha \in I$ such that $\{\alpha \in I \colon f_i(\alpha) \in F(\alpha)\} \in \mathcal{U} \equiv [f_i] \in B$ as $i = 1, 2$.

Consequently it is possible that, for some $\alpha_j \in I$, $f_1(\alpha_j) \cap f_2(\alpha_j) = n_j$ and $n_j \in F(\alpha_j)$. □

Thus, **non-Archimedean structures are infinite DSm-models, because these contain exhaustive members**. In next sections, we shall consider the following non-Archimedean structures:

1. the nonstandard extension ${}^*\mathbf{Q}$ (called the field of hyperrational numbers),
2. the nonstandard extension ${}^*\mathbf{R}$ (called the field of hyperreal numbers),
3. the nonstandard extension $\mathbf{Z}_p$ (called the ring of $p$-adic integers) that we obtain as follows. Let the set $\mathbf{N}$ of natural numbers be the index set and let $\Theta = \{0, \dots, p-1\}$. Then the nonstandard extension $\Theta^{\mathbf{N}} \backslash \mathcal{U} = \mathbf{Z}_p$.

Further, we shall set the following logics on non-Archimedean structures: hyperrational valued logic $\mathfrak{M}_{^*\mathbf{Q}}$, hyperreal valued logic $\mathfrak{M}_{^*\mathbf{R}}$, $p$-adic valued logic $\mathfrak{M}_{\mathbf{Z}_p}$. Note that these many-valued logics are the particular cases of logics on DSm models.

## 7.4 Hyper-valued Reasoning

### 7.4.1 Hyper-valued matrix logics

Assume that ${}^*\mathbf{Q}_{[0,1]} = \mathbf{Q}_{[0,1]}^{\mathbf{N}}/\mathcal{U}$ is a nonstandard extension of the subset $\mathbf{Q}_{[0,1]} = \mathbf{Q} \cap [0,1]$ of rational numbers and ${}^\sigma\mathbf{Q}_{[0,1]} \subset {}^*\mathbf{Q}_{[0,1]}$ is the subset of standard members. We can extend the usual order structure on $\mathbf{Q}_{[0,1]}$ to a partial order structure on ${}^*\mathbf{Q}_{[0,1]}$:

1. for rational numbers $x$, $y \in \mathbf{Q}_{[0,1]}$ we have $x \leq y$ in $\mathbf{Q}_{[0,1]}$ iff $[f] \leq [g]$ in ${}^*\mathbf{Q}_{[0,1]}$, where $\{\alpha \in \mathbf{N} \colon f(\alpha) = x\} \in \mathcal{U}$ and $\{\alpha \in \mathbf{N} \colon g(\alpha) = y\} \in \mathcal{U}$,

   i. e., $f$ and $g$ are constant functions such that $[f] = {}^*x$ and $[g] = {}^*y$,
2. each positive rational number ${}^*x \in {}^\sigma\mathbf{Q}_{[0,1]}$ is greater than any number $[f] \in {}^*\mathbf{Q}_{[0,1]} \backslash {}^\sigma\mathbf{Q}_{[0,1]}$,

   i. e., ${}^*x > [f]$ for any positive $x \in \mathbf{Q}_{[0,1]}$ and $[f] \in {}^*\mathbf{Q}_{[0,1]}$, where $[f]$ isn't constant function.

These conditions have the following informal sense:

1. The sets ${}^\sigma\mathbf{Q}_{[0,1]}$ and $\mathbf{Q}_{[0,1]}$ have isomorphic order structure.
2. The set ${}^*\mathbf{Q}_{[0,1]}$ contains actual infinities that are less than any positive rational number of ${}^\sigma\mathbf{Q}_{[0,1]}$.

Define this partial order structure on ${}^*\mathbf{Q}_{[0,1]}$ as follows:

$\mathcal{O}_{^*\mathbf{Q}}$ 1. For any hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, we set $[f] \leq [g]$ if

$$\{\alpha \in \mathbf{N} \colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}.$$

2. For any hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, we set $[f] < [g]$ if

$$\{\alpha \in \mathbf{N} \colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}$$

and $[f] \neq [g]$, i. e., $\{\alpha \in \mathbf{N} \colon f(\alpha) \neq g(\alpha)\} \in \mathcal{U}$.

3. For any hyperrational numbers $[f],[g] \in {}^*\mathbf{Q}_{[0,1]}$, we set $[f]=[g]$ if $f \in [g]$.

This ordering relation is not linear, but partial, because there exist elements $[f],[g] \in {}^*\mathbf{Q}_{[0,1]}$, which are incompatible.

Introduce two operations max, min in the partial order structure $\mathcal{O}_{^*\mathbf{Q}}$:

1. for all hyperrational numbers $[f],[g] \in {}^*\mathbf{Q}_{[0,1]}$, $\min([f],[g]) = [f]$ if and only if $[f] \leq [g]$ under condition $\mathcal{O}_{^*\mathbf{Q}}$,
2. for all hyperrational numbers $[f],[g] \in {}^*\mathbf{Q}_{[0,1]}$, $\max([f],[g]) = [g]$ if and only if $[f] \leq [g]$ under condition $\mathcal{O}_{^*\mathbf{Q}}$,
3. for all hyperrational numbers $[f],[g] \in {}^*\mathbf{Q}_{[0,1]}$, $\min([f],[g]) = \max([f],[g]) = [f] = [g]$ if and only if $[f]=[g]$ under condition $\mathcal{O}_{^*\mathbf{Q}}$,
4. for all hyperrational numbers $[f],[g] \in {}^*\mathbf{Q}_{[0,1]}$, if $[f],[g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{Q}}$, then $\min([f],[g]) = [h]$ iff there exists $[h] \in {}^*\mathbf{Q}_{[0,1]}$ such that
$$\{\alpha \in \mathbf{N}\colon \min(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$
5. for all hyperrational numbers $[f],[g] \in {}^*\mathbf{Q}_{[0,1]}$, if $[f],[g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{Q}}$, then $\max([f],[g]) = [h]$ iff there exists $[h] \in {}^*\mathbf{Q}_{[0,1]}$ such that
$$\{\alpha \in \mathbf{N}\colon \max(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$

Note there exist the maximal number ${}^*1 \in {}^*\mathbf{Q}_{[0,1]}$ and the minimal number ${}^*0 \in {}^*\mathbf{Q}_{[0,1]}$ under condition $\mathcal{O}_{^*\mathbf{Q}}$. Therefore, for any $[f] \in {}^*\mathbf{Q}_{[0,1]}$, we have: $\max({}^*1,[f]) = {}^*1$, $\max({}^*0,[f]) = [f]$, $\min({}^*1,[f]) = [f]$ and $\min({}^*0,[f]) = {}^*0$.

Now define *hyperrational-valued matrix logic* $\mathfrak{M}_{^*\mathbf{Q}}$:

**Definition 2.** *The ordered system* $<V_{^*\mathbf{Q}}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{{}^*1\}>$ *is called hyperrational valued matrix logic* $\mathfrak{M}_{^*\mathbf{Q}}$, *where*

1. $V_{^*\mathbf{Q}} = {}^*\mathbf{Q}_{[0,1]}$ *is the subset of hyperrational numbers,*
2. *for all* $x \in V_{^*\mathbf{Q}}$, $\neg x = {}^*1 - x$,
3. *for all* $x,y \in V_{^*\mathbf{Q}}$, $x \supset y = \min({}^*1, {}^*1 - x + y)$,
4. *for all* $x,y \in V_{^*\mathbf{Q}}$, $x \vee y = (x \supset y) \supset y = \max(x,y)$,
5. *for all* $x,y \in V_{^*\mathbf{Q}}$, $x \wedge y = \neg(\neg x \vee \neg y) = \min(x,y)$,
6. *for a subset* $M \subseteq V_{^*\mathbf{Q}}$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* $M$,
7. *for a subset* $M \subseteq V_{^*\mathbf{Q}}$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* $M$,
8. $\{{}^*1\}$ *is the set of designated truth values.*

The truth value $^*0 \in V_{^*\mathbf{Q}}$ is false, the truth value $^*1 \in V_{^*\mathbf{Q}}$ is true, and other truth values $x \in V_{^*\mathbf{Q}}$ are neutral.

Let us consider a nonstandard extension $^*\mathbf{R}_{[0,1]} = \mathbf{R}^{\mathbf{N}}_{[0,1]}/\mathcal{U}$ for the subset $\mathbf{R}_{[0,1]} = \mathbf{R} \cap [0,1]$ of real numbers. Let $^\sigma\mathbf{R}_{[0,1]} \subset {}^*\mathbf{R}_{[0,1]}$ be the subset of standard members. We can extend the usual order structure on $\mathbf{R}_{[0,1]}$ to a partial order structure on $^*\mathbf{R}_{[0,1]}$:

1. for real numbers $x$, $y \in \mathbf{R}_{[0,1]}$ we have $x \leq y$ in $\mathbf{R}_{[0,1]}$ iff $[f] \leq [g]$ in $^*\mathbf{R}_{[0,1]}$, where $\{\alpha \in \mathbf{N} \colon f(\alpha) = x\} \in \mathcal{U}$ and $\{\alpha \in \mathbf{N} \colon g(\alpha) = y\} \in \mathcal{U}$,
2. each positive real number $^*x \in {}^\sigma\mathbf{R}_{[0,1]}$ is greater than any number $[f] \in {}^*\mathbf{R}_{[0,1]} \backslash {}^\sigma\mathbf{R}_{[0,1]}$,

As before, these conditions have the following informal sense:

1. The sets $^\sigma\mathbf{R}_{[0,1]}$ and $\mathbf{R}_{[0,1]}$ have isomorphic order structure.
2. The set $^*\mathbf{R}_{[0,1]}$ contains actual infinities that are less than any positive real number of $^\sigma\mathbf{R}_{[0,1]}$.

Define this partial order structure on $^*\mathbf{R}_{[0,1]}$ as follows:

$\mathcal{O}_{^*\mathbf{R}}$ 1. For any hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, we set $[f] \leq [g]$ if

$$\{\alpha \in \mathbf{N} \colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}.$$

2. For any hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, we set $[f] < [g]$ if

$$\{\alpha \in \mathbf{N} \colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}$$

and $[f] \neq [g], i.e.,\{\alpha \in \mathbf{N} \colon f(\alpha) \neq g(\alpha)\} \in \mathcal{U}$.

3. For any hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, we set $[f] = [g]$ if $f \in [g]$.

Introduce two operations max, min in the partial order structure $\mathcal{O}_{^*\mathbf{R}}$:

1. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, $\min([f],[g]) = [f]$ if and only if $[f] \leq [g]$ under condition $\mathcal{O}_{^*\mathbf{R}}$,
2. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, $\max([f],[g]) = [g]$ if and only if $[f] \leq [g]$ under condition $\mathcal{O}_{^*\mathbf{R}}$,
3. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, $\min([f],[g]) = \max([f],[g]) = [f] = [g]$ if and only if $[f] = [g]$ under condition $\mathcal{O}_{^*\mathbf{R}}$,
4. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, if $[f], [g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{R}}$, then $\min([f],[g]) = [h]$ iff there exists $[h] \in {}^*\mathbf{R}_{[0,1]}$ such that

$$\{\alpha \in \mathbf{N} \colon \min(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$

5. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, if $[f], [g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{R}}$, then $\max([f],[g]) = [h]$ iff there exists $[h] \in {}^*\mathbf{R}_{[0,1]}$ such that

$$\{\alpha \in \mathbf{N} \colon \max(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$

Note there exist the maximal number $^*1 \in {}^*\mathbf{R}_{[0,1]}$ and the minimal number $^*0 \in {}^*\mathbf{R}_{[0,1]}$ under condition $\mathcal{O}_{^*\mathbf{R}}$.

As before, define *hyperreal valued matrix logic* $\mathfrak{M}_{^*\mathbf{R}}$:

**Definition 3.** *The ordered system* $<V_{^*\mathbf{R}}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{^*1\}>$ *is called hyperreal valued matrix logic* $\mathfrak{M}_{^*\mathbf{R}}$*, where*

1. $V_{^*\mathbf{R}} = {}^*\mathbf{R}_{[0,1]}$ *is the subset of hyperreal numbers,*
2. *for all* $x \in V_{^*\mathbf{R}}$, $\neg x = {}^*1 - x$,
3. *for all* $x, y \in V_{^*\mathbf{R}}$, $x \supset y = \min({}^*1, {}^*1 - x + y)$,
4. *for all* $x, y \in V_{^*\mathbf{R}}$, $x \vee y = (x \supset y) \supset y = \max(x, y)$,
5. *for all* $x, y \in V_{^*\mathbf{R}}$, $x \wedge y = \neg(\neg x \vee \neg y) = \min(x, y)$,
6. *for a subset* $M \subseteq V_{^*\mathbf{R}}$, $\widetilde{\exists}(M) = \max(M)$*, where* $\max(M)$ *is a maximal element of* $M$,
7. *for a subset* $M \subseteq V_{^*\mathbf{R}}$, $\widetilde{\forall}(M) = \min(M)$*, where* $\min(M)$ *is a minimal element of* $M$,
8. $\{^*1\}$ *is the set of designated truth values.*

The truth value $^*0 \in V_{^*\mathbf{R}}$ is false, the truth value $^*1 \in V_{^*\mathbf{R}}$ is true, and other truth values $x \in V_{^*\mathbf{R}}$ are neutral.

### 7.4.2 Hyper-valued probability theory and hyper-valued fuzzy logic

Let $X$ be an arbitrary set and let $\mathcal{A}$ be an *algebra of subsets* $A \subset X$, i. e.

1. union, intersection, and difference of two subsets of $X$ also belong to $\mathcal{A}$;
2. $\emptyset, X$ belong to $\mathcal{A}$.

Recall that a *finitely additive probability measure* is a nonnegative set function $\mathbf{P}(\cdot)$ defined for sets $A \in \mathcal{A}$ that satisfies the following properties:

1. $\mathbf{P}(A) \geq 0$ for all $A \in \mathcal{A}$,
2. $\mathbf{P}(X) = 1$ and $\mathbf{P}(\emptyset) = 0$,
3. if $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are disjoint, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$. In particular $\mathbf{P}(\neg A) = 1 - \mathbf{P}(A)$ for all $A \in \mathcal{A}$.

The algebra $\mathcal{A}$ is called a *σ-algebra* if it is assumed to be closed under countable union (or equivalently, countable intersection), i. e. if for every $n$, $A_n \in \mathcal{A}$ causes $A = \bigcup\limits_n A_n \in \mathcal{A}$.

A set function $\mathbf{P}(\cdot)$ defined on a σ-algebra is called a *countable additive probability measure* (or a *σ-additive probability measure*) if in addition to satisfying equations of the definition of finitely additive probability measure, it satisfies the following countable additivity property: for any sequence of pairwise disjoint sets $A_n$, $\mathbf{P}(A) = \sum\limits_n \mathbf{P}(A_n)$. The ordered system $(X, \mathcal{A}, \mathbf{P})$ is called a *probability space.*

Now consider hyper-valued probabilities. Let $I$ be an arbitrary set, let $\mathcal{A}$ be an algebra of subsets $A \subset I$, and let $\mathcal{U}$ be a Frechet ultrafilter on $I$. Set for $A \in \mathcal{A}$:

$$\mu_{\mathcal{U}}(A) = \begin{cases} 1, & A \in \mathcal{U}; \\ 0, & A \notin \mathcal{U}. \end{cases}$$

Hence, there is a mapping $\mu_{\mathcal{U}} \colon \mathcal{A} \mapsto \{0, 1\}$ satisfying the following properties:

1. $\mu_{\mathcal{U}}(\emptyset) = 0$, $\mu_{\mathcal{U}}(I) = 1$;
2. if $\mu_{\mathcal{U}}(A_1) = \mu_{\mathcal{U}}(A_2) = 0$, then $\mu_{\mathcal{U}}(A_1 \cup A_2) = 0$;
3. if $A_1 \cap A_2 = \emptyset$, then $\mu_{\mathcal{U}}(A_1 \cup A_2) = \mu_{\mathcal{U}}(A_1) + \mu_{\mathcal{U}}(A_2)$.

This implies that $\mu_{\mathcal{U}}$ is a probability measure. Notice that $\mu_{\mathcal{U}}$ isn't $\sigma$-additive. As an example, if $A$ is the set of even numbers and $B$ is the set of odd numbers, then $A \in \mathcal{U}$ implies $B \notin \mathcal{U}$, because the filter $\mathcal{U}$ is maximal. Thus, $\mu_{\mathcal{U}}(A) = 1$ and $\mu_{\mathcal{U}}(B) = 0$, although the cardinalities of $A$ and $B$ are equal.

**Definition 4.** *The ordered system $(I, \mathcal{A}, \mu_{\mathcal{U}})$ is called a probability space.*

Let's consider a mapping: $f \colon I \ni \alpha \mapsto f(\alpha) \in M$. Two mappings $f$, $g$ are equivalent: $f \backsim g$ if $\mu_{\mathcal{U}}(\{\alpha \in I \colon f(\alpha) = g(\alpha)\}) = 1$. An equivalence class of $f$ is called a probabilistic events and is denoted by $[f]$. The set ${}^*M$ is the *set of all probabilistic events* of $M$. This ${}^*M$ is a proper nonstandard extension defined above.

Under condition 1 of proposition 1, we can obtain a nonstandard extension of an algebra $\mathcal{A}$ denoted by ${}^*\mathcal{A}$. Let ${}^*X$ be an arbitrary nonstandard extension. Then the nonstandard algebra ${}^*\mathcal{A}$ is an algebra of subsets $A \subset {}^*X$ if the following conditions hold:

1. union, intersection, and difference of two subsets of ${}^*X$ also belong to ${}^*\mathcal{A}$;
2. $\emptyset, {}^*X$ belong to ${}^*\mathcal{A}$.

**Definition 5.** *A hyperrational (respectively hyperreal) valued finitely additive probability measure is a nonnegative set function ${}^*\mathbf{P} \colon {}^*\mathcal{A} \mapsto V_{{}^*\mathbf{Q}}$ (respectively ${}^*\mathbf{P} \colon {}^*\mathcal{A} \mapsto V_{{}^*\mathbf{R}}$) that satisfies the following properties:*

1. ${}^*\mathbf{P}(A) \geq {}^*0$ *for all* $A \in {}^*\mathcal{A}$,
2. ${}^*\mathbf{P}({}^*X) = {}^*1$ *and* ${}^*\mathbf{P}(\emptyset) = {}^*0$,
3. *if* $A \in {}^*\mathcal{A}$ *and* $B \in {}^*\mathcal{A}$ *are disjoint, then* ${}^*\mathbf{P}(A \cup B) = {}^*\mathbf{P}(A) + {}^*\mathbf{P}(B)$. *In particular* ${}^*\mathbf{P}(\neg A) = {}^*1 - {}^*\mathbf{P}(A)$ *for all* $A \in {}^*\mathcal{A}$.

Now consider hyper-valued fuzzy logic.

**Definition 6.** *Suppose ${}^*X$ is a nonstandard extension. Then a hyperrational (respectively hyperreal) valued fuzzy set $A$ in ${}^*X$ is a set defined by means of the membership function ${}^*\mu_A$: ${}^*X \mapsto V_{{}^*\mathbf{Q}}$ (respectively by means of the membership function ${}^*\mu_A$: ${}^*X \mapsto V_{{}^*\mathbf{R}}$).*

A set $A \subset {}^*X$ is called *crisp* if ${}^*\mu_A(u) = {}^*1$ or ${}^*\mu_A(u) = {}^*0$ for any $u \in {}^*X$.

The logical operations on hyper-valued fuzzy sets are defined as follows:

1. ${}^*\mu_{A\cap B}(x) = \min({}^*\mu_A(x), {}^*\mu_B(x))$;
2. ${}^*\mu_{A\cup B}(x) = \max({}^*\mu_A(x), {}^*\mu_B(x))$;
3. ${}^*\mu_{A+B}(x) = {}^*\mu_A(x) + {}^*\mu_B(x) - {}^*\mu_A(x) \cdot {}^*\mu_B(x)$;
4. ${}^*\mu_{\neg A}(x) = \neg{}^*\mu_A(x) = {}^*1 - {}^*\mu_A(x)$.

## 7.5 $p$-Adic Valued Reasoning

Let us remember that the expansion

$$n = \alpha_{-N} \cdot p^{-N} + \alpha_{-N+1} \cdot p^{-N+1} + \ldots + \alpha_{-1} \cdot p^{-1} + \alpha_0 + \alpha_1 \cdot p + \ldots + \alpha_k \cdot p^k + \ldots = \sum_{k=-N}^{+\infty} \alpha_k \cdot p^k,$$

where $\alpha_k \in \{0, 1, \ldots, p-1\}$, $\forall k \in \mathbf{Z}$, and $\alpha_{-N} \neq 0$, is called the *canonical expansion of $p$-adic number $n$* (or $p$-adic expansion for $n$). The number $n$ is called $p$-adic. This number can be identified with sequences of digits: $n = \ldots \alpha_2\alpha_1\alpha_0, \alpha_{-1}\alpha_{-2} \ldots \alpha_{-N}$. We denote the set of such numbers by $\mathbf{Q}_p$.

The expansion $n = \alpha_0 + \alpha_1 \cdot p + \ldots + \alpha_k \cdot p^k + \ldots = \sum_{k=0}^{\infty} \alpha_k \cdot p^k$, where $\alpha_k \in \{0, 1, \ldots, p-1\}$, $\forall k \in \mathbf{N} \cup \{0\}$, is called the *expansion of $p$-adic integer $n$*. The integer $n$ is called $p$-adic. This number sometimes has the following notation: $n = \ldots \alpha_3\alpha_2\alpha_1\alpha_0$. We denote the set of such numbers by $\mathbf{Z}_p$.

If $n \in \mathbf{Z}_p$, $n \neq 0$, and its canonical expansion contains only a finite number of nonzero digits $\alpha_j$, then $n$ is natural number (and vice versa). But if $n \in \mathbf{Z}_p$ and its expansion contains an infinite number of nonzero digits $\alpha_j$, then $n$ is an infinitely large natural number. Thus the set of $p$-adic integers contains actual infinities $n \in \mathbf{Z}_p \backslash \mathbf{N}$, $n \neq 0$. This is one of the most important features of non-Archimedean number systems, therefore it is natural to compare $\mathbf{Z}_p$ with the set of nonstandard numbers ${}^*\mathbf{Z}$. Also, the set $\mathbf{Z}_p$ contains *exhaustive elements.*

### 7.5.1 $p$-Adic valued matrix logic

Extend the standard order structure on $\{0, \ldots, p-1\}$ to a partial order structure on $\mathbf{Z}_p$. Define this partial order structure on $\mathbf{Z}_p$ as follows:

$\mathcal{O}_{\mathbf{Z}_p}$ Let $x = \ldots x_n \ldots x_1x_0$ and $y = \ldots y_n \ldots y_1y_0$ be the canonical expansions of two $p$-adic integers $x, y \in \mathbf{Z}_p$.

1. We set $x \leq y$ if we have $x_n \leq y_n$ for each $n = 0, 1, \ldots$
2. We set $x < y$ if we have $x_n \leq y_n$ for each $n = 0, 1, \ldots$ and there exists $n_0$ such that $x_{n_0} < y_{n_0}$.
3. We set $x = y$ if $x_n = y_n$ for each $n = 0, 1, \ldots$

Now introduce two operations max, min in the partial order structure on $\mathbf{Z}_p$:

**1** for all $p$-adic integers $x, y \in \mathbf{Z}_p$, $\min(x, y) = x$ if and only if $x \leq y$ under condition $\mathcal{O}_{\mathbf{Z}_p}$,

**2** for all $p$-adic integers $x, y \in \mathbf{Z}_p$, $\max(x, y) = y$ if and only if $x \leq y$ under condition $\mathcal{O}_{\mathbf{Z}_p}$,

**3** for all $p$-adic integers $x, y \in \mathbf{Z}_p$, $\max(x, y) = \min(x, y) = x = y$ if and only if $x = y$ under condition $\mathcal{O}_{\mathbf{Z}_p}$.

The ordering relation $\mathcal{O}_{\mathbf{Z}_p}$ is not linear, but partial, because there exist elements $x, z \in \mathbf{Z}_p$, which are incompatible. As an example, let $p = 2$ and let $x = -\frac{1}{3} = \ldots 10101 \ldots 101$, $z = -\frac{2}{3} = \ldots 01010 \ldots 010$. Then the numbers $x$ and $z$ are incompatible.

Thus,

**4** Let $x = \ldots x_n \ldots x_1 x_0$ and $y = \ldots y_n \ldots y_1 y_0$ be the canonical expansions of two $p$-adic integers $x, y \in \mathbf{Z}_p$ and $x$, $y$ are incompatible under condition $\mathcal{O}_{\mathbf{Z}_p}$. We get $\min(x, y) = z = \ldots z_n \ldots z_1 z_0$, where, for each $n = 0, 1, \ldots$, we set

1. $z_n = y_n$ if $x_n \geq y_n$,
2. $z_n = x_n$ if $x_n \leq y_n$,
3. $z_n = x_n = y_n$ if $x_n = y_n$.

We get $\max(x, y) = z = \ldots z_n \ldots z_1 z_0$, where, for each $n = 0, 1, \ldots$, we set

1. $z_n = y_n$ if $x_n \leq y_n$,
2. $z_n = x_n$ if $x_n \geq y_n$,
3. $z_n = x_n = y_n$ if $x_n = y_n$.

It is important to remark that there exists the maximal number $N_{max} \in \mathbf{Z}_p$ under condition $\mathcal{O}_{\mathbf{Z}_p}$. It is easy to see:

$$N_{max} = -1 = (p-1) + (p-1) \cdot p + \ldots + (p-1) \cdot p^k + \ldots = \sum_{k=0}^{\infty} (p-1) \cdot p^k$$

Therefore

**5** $\min(x, N_{max}) = x$ and $\max(x, N_{max}) = N_{max}$ for any $x \in \mathbf{Z}_p$.

Now consider *$p$-adic valued matrix logic* $\mathfrak{M}_{\mathbf{Z}_p}$.

**Definition 7.** *The ordered system* $<V_{\mathbf{Z}_p}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{N_{max}\}>$ *is called $p$-adic valued matrix logic* $\mathfrak{M}_{\mathbf{Z}_p}$, *where*

1. $V_{\mathbf{Z}_p} = \{0, \ldots, N_{max}\} = \mathbf{Z}_p$,
2. *for all* $x \in V_{\mathbf{Z}_p}$, $\neg x = N_{max} - x$,
3. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \supset y = (N_{max} - \max(x, y) + y)$,
4. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \vee y = (x \supset y) \supset y = \max(x, y)$,

5. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \wedge y = \neg(\neg x \vee \neg y) = \min(x, y)$,

6. *for a subset* $M \subseteq V_{\mathbf{Z}_p}$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* $M$,

7. *for a subset* $M \subseteq V_{\mathbf{Z}_p}$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* $M$,

8. $\{N_{max}\}$ *is the set of designated truth values.*

The truth value $0 \in \mathbf{Z}_p$ is false, the truth value $N_{max} \in \mathbf{Z}_p$ is true, and other truth values $x \in \mathbf{Z}_p$ are neutral.

**Proposition 4.** *The logic* $\mathfrak{M}_{\mathbf{Z}_2} = <V_{\mathbf{Z}_2}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{N_{max}\}>$ *is a Boolean algebra.*

*Proof.* Indeed, the operation $\neg$ in $\mathfrak{M}_{\mathbf{Z}_2}$ is the Boolean complement:

1. $\max(x, \neg x) = N_{max}$,

2. $\min(x, \neg x) = 0$. □

### 7.5.2 $p$-Adic probability theory

#### 7.5.2.1 Frequency theory of $p$-adic probability

Let us remember that the frequency theory of probability was created by Richard von Mises in [10]. This theory is based on the notion of a collective: "We will say that a collective is a mass phenomenon or a repetitive event, or simply a long sequence of observations for which there are sufficient reasons to believe that the relative frequency of the observed attribute would tend to a fixed limit if the observations were infinitely continued. This limit will be called the probability of the attribute considered within the given collective" [10].

As an example, consider a random experiment $\mathcal{S}$ and by $L = \{s_1, \ldots, s_m\}$ denote the set of all possible results of this experiment. The set $\mathcal{S}$ is called the label set, or the set of attributes. Suppose there are $N$ realizations of $\mathcal{S}$ and write a result $x_j$ after each realization. Then we obtain the finite sample: $x = (x_1, \ldots, x_N), x_j \in L$. A collective is an infinite idealization of this finite sample: $x = (x_1, \ldots, x_N, \ldots), x_j \in L$. Let us compute frequencies $\nu_N(\alpha; x) = n_N(\alpha; x)/N$, where $n_N(\alpha; x)$ is the number of realizations of the attribute $\alpha$ in the first $N$ tests. There exists the statistical stabilization of relative frequencies: the frequency $\nu_N(\alpha; x)$ approaches a limit as $N$ approaches infinity for every label $\alpha \in L$. This limit $\mathbf{P}(\alpha) = \lim \nu_N(\alpha; x)$ is said to be the probability of the label $\alpha$ in the frequency theory of probability. Sometimes this probability is denoted by $\mathbf{P}_x(\alpha)$ to show a dependence on the collective $x$. Notice that the limits of relative frequencies have to be stable with respect to a place selection (a choice of a subsequence) in the collective. A. Yu. Khrennikov developed von Mises' idea and proposed the frequency theory of $p$-adic probability in [6, 7]. We consider here Khrennikov's theory.

We shall study some ensembles $S = S_N$, which have a $p$-dic volume $N$, where $N$ is the $p$-adic integer. If $N$ is finite, then $S$ is the ordinary finite ensemble. If $N$ is infinite, then $S$ has essentially $p$-adic structure. Consider a sequence of ensembles $M_j$ having volumes $l_j \cdot p^j$, $j = 0, 1, \ldots$ Get $S = \cup_{j=0}^{\infty} M_j$. Then the cardinality $|S| = N$. We may imagine an ensemble $S$ as being the population of a tower $T = T_S$, which has an infinite number of floors with the following distribution of population through floors: population of $j$-th floor is $M_j$. Set $T_k = \cup_{j=0}^{k} M_j$.

This is population of the first $k+1$ floors. Let $A \subset S$ and let there exists: $n(A) = \lim_{k\to\infty} n_k(A)$, where $n_k(A) = |A \cap T_k|$. The quantity $n(A)$ is said to be a $p$-adic volume of the set $A$.

We define the probability of $A$ by the standard proportional relation:

$$\mathbf{P}(A) \triangleq \mathbf{P}_S(A) = \frac{n(A)}{N}, \tag{7.6}$$

where $|S| = N$, $n(A) = |A \cap S|$.

We denote the family of all $A \subset S$, for which $\mathbf{P}(A)$ exists, by $\mathcal{G}_S$. The sets $A \in \mathcal{G}_S$ are said to be events. The ordered system $(S, \mathcal{G}_S, \mathbf{P}_S)$ is called a *$p$-adic ensemble probability space for the ensemble $S$.*

**Proposition 5.** *Let $F$ be the set algebra which consists of all finite subsets and their complements. Then $F \subset \mathcal{G}_S$.*

*Proof.* Let $A$ be a finite set. Then $n(A) = |A|$ and the probability of $A$ has the form:

$$\mathbf{P}(A) = \frac{|A|}{|S|}$$

Now let $B = \neg A$. Then $|B \cap T_k| = |T_k| - |A \cap T_k|$. Hence there exists $\lim_{k\to\infty} |B \cap T_k| = N - |A|$. This equality implies the standard formula:

$$\mathbf{P}(\neg A) = 1 - \mathbf{P}(A)$$

In particular, we have: $\mathbf{P}(S) = 1$. □

The next propositions are proved in [6]:

**Proposition 6.** *Let $A_1, A_2 \in \mathcal{G}_S$ and $A_1 \cap A_2 = \emptyset$. Then $A_1 \cup A_2 \in \mathcal{G}_S$ and*

$$\mathbf{P}(A_1 \cup A_2) = \mathbf{P}(A_1) + \mathbf{P}(A_2).$$

□

**Proposition 7.** *Let $A_1, A_2 \in \mathcal{G}_S$. The following conditions are equivalent:*

1. $A_1 \cup A_2 \in \mathcal{G}_S$,
2. $A_1 \cap A_2 \in \mathcal{G}_S$,
3. $A_1 \backslash A_2 \in \mathcal{G}_S$,
4. $A_2 \backslash A_1 \in \mathcal{G}_S$. □

But it is possible to find sets $A_1, A_2 \in \mathcal{G}_S$ such that, for example, $A_1 \cup A_2 \notin \mathcal{G}_S$. Thus, the family $\mathcal{G}_S$ is not an algebra, but a semi-algebra (it is closed only with respect to a finite unions of sets, which have empty intersections). $\mathcal{G}_S$ is not closed with respect to countable unions of such sets.

**Proposition 8.** *Let $A \in \mathcal{G}_S$, $\mathbf{P}(A) \neq 0$ and $B \in \mathcal{G}_A$. Then $B \in \mathcal{G}_S$ and the following Bayes formula holds:*

$$\mathbf{P}_A(B) = \frac{\mathbf{P}_S(B)}{\mathbf{P}_S(A)} \tag{7.7}$$

*Proof.* The tower $T_A$ of the $A$ has the following population structure: there are $M_{A_j}$ elements on the $j$-th floor. In particular, $T_{A_k} = T_k \cap A$. Thus

$$n_{A_k}(B) = |B \cap T_{A_k}| = |B \cap T_k| = n_k(B)$$

for each $B \subset A$. Hence the existence of $n_A(B) = \lim_{k\to\infty} n_{A_k}(B)$ implies the existence of $n_S(B)$ with $n_S(B) = \lim_{k\to\infty} n_k(B)$. Moreover, $n_S(B) = n_A(B)$. Therefore,

$$\mathbf{P}_A(B) = \frac{n_A(B)}{n_S(A)} = \frac{n_A(B)/|S|}{n_S(A)/|S|}.$$

□

**Proposition 9.** *Let $N \in \mathbf{Z}_p$, $N \neq 0$ and let the ensemble $S_{-1}$ have the p-adic volume $-1 = N_{max}$ (it is the largest ensemble).*

1. *Then $S_N \in \mathcal{G}_{S_{-1}}$ and*

$$\mathbf{P}_{S_{-1}}(S_N) = \frac{|S_N|}{|S_{-1}|} = -N$$

2. *Then $\mathcal{G}_{S_N} \subset \mathcal{G}_{S_{-1}}$ and probabilities $\mathbf{P}_{S_N}(A)$ are calculated as conditional probabilities with respect to the subensemble $S_N$ of ensemble $S_{-1}$:*

$$\mathbf{P}_{S_N}(A) = \mathbf{P}_{S_{-1}}(\frac{A}{S_N}) = \frac{\mathbf{P}_{S_{-1}}(A)}{\mathbf{P}_{S_{-1}}(S_N)}, A \in \mathcal{G}_{S_N}$$

□

#### 7.5.2.2 Logical theory of $p$-adic probability

Transform the matrix logic $\mathfrak{M}_{\mathbf{Z}_p}$ into a $p$-adic probability theory. Let us remember that a formula $\varphi$ has the truth value $0 \in \mathbf{Z}_p$ in $\mathfrak{M}_{\mathbf{Z}_p}$ if $\varphi$ is false, a formula $\varphi$ has the truth value $N_{max} \in \mathbf{Z}_p$ in $\mathfrak{M}_{\mathbf{Z}_p}$ if $\varphi$ is true, and a formula $\varphi$ has other truth values $\alpha \in \mathbf{Z}_p$ in $\mathfrak{M}_{\mathbf{Z}_p}$ if $\varphi$ is neutral.

**Definition 8.** *A function $\mathbf{P}(\varphi)$ is said to be a probability measure of a formula $\varphi$ in $\mathfrak{M}_{\mathbf{Z}_p}$ if $\mathbf{P}(\varphi)$ ranges over numbers of $\mathbf{Q}_p$ and satisfies the following axioms:*

1. $\mathbf{P}(\varphi) = \frac{\alpha}{N_{max}}$*, where $\alpha$ is a truth value of $\varphi$;*
2. *if a conjunction $\varphi \wedge \psi$ has the truth value $0$, then $\mathbf{P}(\varphi \vee \psi) = \mathbf{P}(\varphi) + \mathbf{P}(\psi)$,*
3. $\mathbf{P}(\varphi \wedge \psi) = \min(\mathbf{P}(\varphi), \mathbf{P}(\psi))$.

Notice that:

1. taking into account condition 1 of our definition, if $\varphi$ has the truth value $N_{max}$ for any its interpretations, i. e., $\varphi$ is a tautology, then $\mathbf{P}(\varphi) = 1$ in all possible worlds, and if $\varphi$ has the truth value 0 for any its interpretations, i. e., $\varphi$ is a contradiction, then $\mathbf{P}(\varphi) = 0$ in all possible worlds;

2. under condition 2, we obtain $\mathbf{P}(\neg\varphi) = 1 - \mathbf{P}(\varphi)$.

Since $\mathbf{P}(N_{max}) = 1$, we have

$$\mathbf{P}(\max\{x \in V_{\mathbf{Z}_p}\}) = \sum_{x \in V_{\mathbf{Z}_p}} \mathbf{P}(x) = 1$$

All events have a conditional plausibility in the logical theory of $p$-adic probability:

$$\mathbf{P}(\varphi) \equiv \mathbf{P}(\varphi / N_{max}), \tag{7.8}$$

i. e., for any $\varphi$, we consider the conditional plausibility that there is an event of $\varphi$, given an event $N_{max}$,

$$\mathbf{P}(\varphi/\psi) = \frac{\mathbf{P}(\varphi \wedge \psi)}{\mathbf{P}(\psi)}. \tag{7.9}$$

### 7.5.3 $p$-Adic fuzzy logic

The probability interpretation of the logic $\mathfrak{M}_{\mathbf{Z}_p}$ shows that this logic is a special system of fuzzy logic. Indeed, we can consider the membership function $\mu_A$ as a $p$-adic valued predicate.

**Definition 9.** *Suppose $X$ is a non-empty set. Then a $p$-adic-valued fuzzy set $A$ in $X$ is a set defined by means of the membership function $\mu_A$: $X \mapsto \mathbf{Z}_p$, where $\mathbf{Z}_p$ is the set of all $p$-adic integers.*

It is obvious that the set $A$ is completely determined by the set of tuples $\{<u, \mu_A(u)>: u \in X\}$. We define a norm $|\cdot|_p \colon \mathbf{Q}_p \mapsto \mathbf{R}$ on $\mathbf{Q}_p$ as follows:

$$|n = \sum_{k=-N}^{+\infty} \alpha_k \cdot p^k|_p \triangleq p^{-L},$$

where $L = \max\{k \colon n \equiv 0 \mod p^k\} \geq 0$, i. e. $L$ is an index of the first number distinct from zero in $p$-adic expansion of $n$. Note that $|0|_p \triangleq 0$. The function $|\cdot|_p$ has values 0 and $\{p^\gamma\}_{\gamma \in \mathbf{Z}}$ on $\mathbf{Q}_p$. Finally, $|x|_p \geq 0$ and $|x|_p = 0 \equiv x = 0$. A set $A \subset X$ is called *crisp* if $|\mu_A(u)|_p = 1$ or $|\mu_A(u)|_p = 0$ for any $u \in X$. Notice that $|\mu_A(u) = 1|_p = 1$ and $|\mu_A(u) = 0|_p = 0$. Therefore our membership function is an extension of the classical characteristic function. Thus, $A = B$ causes $\mu_A(u) = \mu_B(u)$ for all $u \in X$ and $A \subseteq B$ causes $|\mu_A(u)|_p \leqslant |\mu_B(u)|_p$ for all $u \in X$.

In $p$-adic fuzzy logic, there always exists a non-empty intersection of two crisp sets. In fact, suppose the sets $A$, $B$ have empty intersection and $A$, $B$ are crisp. Consider two cases under condition $\mu_A(u) \neq \mu_B(u)$ for any $u$. First, $|\mu_A(u)|_p = 0$ or $|\mu_A(u)|_p = 1$ for all $u$ and secondly $|\mu_B(u)|_p = 0$ or $|\mu_B(u)|_p = 1$ for all $u$. Assume we have $\mu_A(u_0) = N_{max}$ for some $u_0$, i. e., $|\mu_A(u_0)|_p = 1$. Then $\mu_B(u_0) \neq N_{max}$, but this doesn't mean that $\mu_B(u_0) = 0$. It is possible that $|\mu_A(u_0)|_p = 1$ and $|\mu_B(u_0)|_p = 1$ for $u_0$.

Now we set logical operations on $p$-adic fuzzy sets:

1. $\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$;
2. $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$;
3. $\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \min(\mu_A(x), \mu_B(x))$;
4. $\mu_{\neg A}(x) = \neg\mu_A(x) = N_{max} - \mu_A(x) = -1 - \mu_A(x)$.

## 7.6 Conclusion

In this chapter, one has constructed on the basis of infinite DSm models three logical many-valued systems: $\mathfrak{M}_{\mathbf{Z}_p}$, $\mathfrak{M}_{^*\mathbf{Q}}$, and $\mathfrak{M}_{^*\mathbf{R}}$. These systems are principal versions of the non-Archimedean logic and they can be used in probabilistic and fuzzy reasoning. Thus, the DSm models assumes many theoretical and practical applications.

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