

Information Fusion on Belief Networks

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Abstract

This paper will focus on the process of “fusing” several observations or models of uncertainty into a single resultant model. Many existing approaches to fusion use subjective quantities such as “strengths of belief” and process these quantities with heuristic algorithms. This paper argues in favor of quantities that can be objectively measured, as opposed to the subjective “strength of belief” values. This paper will focus on probability distributions, and more importantly, structures that denote sets of probability distributions known as “credal sets”. The novel aspect of this paper will be a taxonomy of models of fusion that use specific types of credal sets, namely probability interval distributions and Dempster-Shafer models. An objective requirement for information fusion algorithms is provided, and is satisfied by all models of fusion presented in this paper. Dempster’s rule of combination is shown to not satisfy this requirement. This paper will also assess the computational challenges involved for the proposed fusion approaches.

I. INTRODUCTION

The problem of “fusion” stems from the need to combine information from various sources. Each of these source is assumed to provide either an observation, or a “model of uncertainty”. Several approaches to fusing uncertainty models have been investigated in [Delmotte and Smets(2004)] (the transferable belief model), [Yager(2004)] (uninorm aggregation), [Yager(1987)] (Dempster-Shafer fusion), and [Dezert and Smarandache(2003)], [Van Norden et al.(2008)Van Norden, Bolderheij and Jonker], [Tchamova et al.(2003)Tchamova, Semerdjiev and Dezert] (Dezert-Smarandache fusion). A survey of contemporary fusion approaches is given in [Khaleghi et al.(2013)Khaleghi, Khamis, Karray and Razavi]. Most of these approaches however rely on quantities such as “strengths of belief” that are highly subjective. In addition, the heuristics used to handle strengths of belief are algorithms designed so that the outputs “make sense” as opposed to obeying an objective criteria.

This paper will utilize convex sets of probability distributions, known as credal sets, to form the model of uncertainty in probability distributions. However, unlike most approaches to credal sets which maintain a list of the extreme points, this paper will focus on “subtypes” of credal sets, namely probability interval distributions and Dempster-Shafer models (Dempster-Shafer models can also describe credal sets in addition to heuristic belief functions, as will be discussed in section VI). The fusion of credal sets as models of uncertainty is described in [Karlsson(2010)], [Karlsson et al.(2011)Karlsson, Johansson and Andler], [Karlsson and Steinhauer(2013)], and it is already established that fusion can be performed in an efficient and exact manner when credal sets are denoted by listing their extreme points. It is known however, that credal set subtypes such as probability interval distributions and Dempster-Shafer models can describe certain credal sets using dramatically less data than the number of extreme points (see [Tessem(1992)], [Zaffalon(2002)] for a discussion on the number of extreme points of probability interval distributions). When a credal set is the set of probability distributions denoted by a specific probability interval distribution or Dempster-Shafer model, the probability interval distribution or Dempster-Shafer model is the more efficient representation in terms of space (and subsequently computational complexity).

This is the prime motivation for using the credal set subtypes of probability interval distributions and Dempster-Shafer models, and gives practical purpose to the catalog of fusion approaches presented in this paper.

The use of intervals to describe probabilities is formally developed in [De Campos et al.(1994)De Campos, Huete and Moral] and [Klir(2005), chapter 5] and is described in detail in section V.

Dempster-Shafer (DS) theory is described in [Klir(2005), chapter 5], [Yager(1987)] and section VI. In many publications such as [Yager and Filev(1995)], Dempster-Shafer models are interpreted as follows: the belief and plausibility respectively form lower and upper bounds for the true probability. This is the interpretation of Dempster-Shafer theory that will be used in this paper. With this interpretation, Dempster-Shafer models effectively denote a credal set. Dempster's rule of combination, described in [Klir(2005), chapter 5] and [Yager(1987)], is a popular approach to fusing Dempster-Shafer models. There is however, a major inconsistency with Dempster's rule of combination when Dempster-Shafer models are interpreted as credal sets: the fused Dempster-Shafer model does not describe a credal set that contains all possible probability distributions that result from fusing probability distributions chosen from the credal set of each input Dempster-Shafer model (see definition 4). This inconsistency is described in section VI-D.

Many approaches to fusion using Dempster-Shafer models focus on "redistributing conflict" such as from [Dezert et al.(2006)Dezert, Tchamova, Smarandache and Konstantinova], [Smarandache and Dezert(2005a)], [Smarandache and Dezert(2005b)]. "Conflict redistribution" focuses on minimizing or eliminating the renormalization that occurs when Dempster-Shafer models are combined/fused. This paper, due to its focus on credal sets, will not focus on approaches such as conflict redistribution, since these approaches do not treat Dempster-Shafer models as credal sets.

An important aspect of this paper is a look at the various algorithms that fuse credal sets. Existing work on this topic include the known Bayesian fusion of credal sets from [Karlsson et al.(2011)Karlsson, Johansson and Andler], the calculation of posterior probability intervals from [De Campos et al.(1994)De Campos, Huete and Moral], [Walley(1996)], [Antonucci et al.(2013a)Antonucci, De Campos, Huber and Zaffalon], and the creation of software packages that calculate posterior credal sets such as "CREDO" [Antonucci et al.(2013b)Antonucci, Huber, Zaffalon, Luginbuhl, Chapman and Ladouceur]. This paper will propose a catalog of fusion approaches that utilize probability interval distributions and Dempster-Shafer models. This taxonomy will include existing work and algorithms generated specifically for this paper.

The structure of this paper is as follows: Section III will review two different modes of fusion using point probability distributions. Section IV will review the requirements for fusion involving sets of probability distributions. Section V will cover the use of probability intervals. Section VI will cover the use of Dempster-Shafer models, and propose an alternative to Dempster's rule of combination.

II. CONTRIBUTIONS

The contributions of this paper are:

- The most important contribution of this paper is a taxonomy and catalog of fusion approaches and algorithms that utilize "subtypes" of credal sets, in this case "probability interval distributions" and "Dempster-Shafer models". All fusion approaches will satisfy an important objective criteria, referred to in this paper as the "containment property". Special attention is paid to the computational challenges involved. Various approaches are given, which exhibit trade-offs between accuracy and computational complexity. Some of the fusion approaches are already known to the literature (such as context specific fusion with probability intervals described in [Walley(1996)]), and others were created specifically for this paper.

- A proposed objective criteria for information fusion referred to as the “containment property” (see section IV for the definitions) is given. Dempster’s rule of combination is shown to violate the containment property.
- A distinction is made between two types of information fusion, referred to as “context specific” and “general fusion”. Each type of fusion has different information requirements, and the algorithms are different. Context specific fusion requires more prior information, but is less computationally intensive than general fusion. The important distinction between context specific fusion and general fusion is that context specific fusion only requires raw observations as input, while general fusion requires complete credal sets. Context specific fusion follows the hypothesis-observation models used in publications such as [Delmotte and Smets(2004)], [Zaffalon(2002)] and [Walley(1996), section 4, calculus], and the algorithms are generally polynomial time with respect to the size of the input. General fusion is similar to the direct Bayesian fusion of credal sets. While the direct Bayesian fusion of credal sets can be performed exactly in polynomial time [Karlsson et al.(2011)Karlsson, Johansson and Andler, Theorem 2], when credal sets are restricted to specific subtypes, general fusion becomes much more difficult.

III. BACKGROUND

In the literature, convex sets of probability distributions are referred to as “credal sets”.

Definition 1: A **credal set**, is a convex set of probability distributions. All probability distributions in a credal set cover the same variables and have the same domain.

Specific “subtypes” of credal sets that will be the focus of investigation include “probability interval distributions” and “Dempster-Shafer models”. In [Karlsson et al.(2011)Karlsson, Johansson and Andler], credal sets are denoted by listing their “extreme points”. The extreme points are points that belong to the credal set, but are not a convex combination of other points in the credal set [Karlsson et al.(2011)Karlsson, Johansson and Andler]. Each subtype of credal set however has a more compact style of representation that comes with the restriction that there are some credal sets that cannot be represented by the current subtype.

In this paper, a credal set subtype is considered to be “non-trivial” if and only if it denotes a set of probability distributions as opposed to a single probability distribution.

The following notation will be used with respect to credal sets:

- Given a single variable x , the set $\{x\}$ will be denoted by simply using x .
- Given a set of variables X , $\text{Val}(X)$ is the set of all possible complete assignments to the variables in X .
- Given a set of variables X , the set of sets $2^{\text{Val}(X)} \setminus \{\emptyset\} = \{A \subseteq \text{Val}(X) : A \neq \emptyset\}$ is denoted by $\text{Set}(X)$.
- Given a credal set S ,
 - $\text{Var}(S)$ is the set of variables covered by each probability distribution from S .
 - $\text{Val}(S)$ denotes $\text{Val}(\text{Var}(S))$.
 - $\text{Set}(S)$ denotes $\text{Set}(\text{Var}(S))$.
- Given an arbitrary condition C ,
 - $\text{Pr}_L(C) = \min(\text{Pr}(C))$ denotes the smallest probability that C is satisfied.
 - $\text{Pr}_U(C) = \max(\text{Pr}(C))$ denotes the largest probability that C is satisfied.

In addition, pseudo code will be used to describe various algorithms. Comments in the pseudo code are denoted using a double forward slash: `//`, or are enclosed by: `/* . . . */`.

A. Two Approaches to Fusion

In this paper, fusion will occur within the context of trying to identify an object using information gathered by remote sensors.

There are two fusion problems that will be considered by this paper:

Problem 1: Context Specific Fusion: Consider a variable of interest, hypothesis variable H . Given several ($N \geq 1$) observations O_1, O_2, \dots, O_N , we wish to generate a “posterior” credal set $S_\bullet = \mathbf{F}(O_1, O_2, \dots, O_N)$ that covers the hypothesis variable H . S_\bullet should consolidate all of the observations O_1, O_2, \dots, O_N . The hypothesis variable, H , will be assumed to have $M \geq 2$ possible values, denoted by: $1, 2, \dots, M$. The subtype of the posterior credal set will be the same as the subtype of the credal set that is the “prior” for H .

General Fusion: Consider a variable of interest, hypothesis variable H . Given several ($N \geq 2$) credal sets S_1, S_2, \dots, S_N that cover the hypothesis variable H , we wish to generate a “posterior” credal set $S_\bullet = \mathbf{F}(S_1, S_2, \dots, S_N)$ that covers H and consolidates all of the information from S_1, S_2, \dots, S_N . The hypothesis variable, H , will be assumed to have $M \geq 2$ possible values, denoted by: $1, 2, \dots, M$. The subtype of the posterior credal set will be the same as the subtype of the input credal sets.

The process of context specific fusion is shown in figure 1(a). Observations O_1, O_2, \dots, O_N are acquired from various sources: in this case, the sources are sensors. Alongside existing data in the form of a prior credal set for H and probability ranges for each observation given each possible value of H , the observations are fused to produce a “posterior” credal set that describes H . This is the approach to fusion used in [Delmotte and Smets(2004)], [Zaffalon(2002)] and [Walley(1996), section 4, calculus].

The process of general fusion is shown in figure 1(b). “Prior” credal sets S_1, S_2, \dots, S_N are acquired from various sources: in this case, the sources are sensors. The credal sets are fused to produce a “posterior” credal set that describes H . Unlike context specific fusion, general fusion does not require existing data. Despite this however, each sensor must be equipped with the capacity to return a credal set about H , as opposed to raw data and observations. This is the approach to fusion used in many publications such as [Guo and Tanaka(2010)], [Karlsson et al.(2011)]Karlsson, Johansson and Andler], [Yager(2004)], [Yager and Petry(2016)].

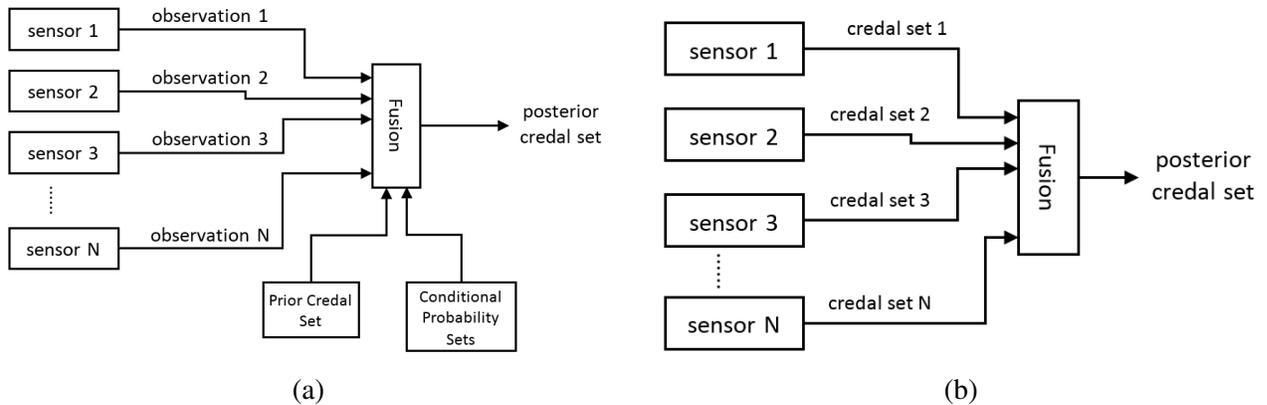


Fig. 1. (a) The process of Context Specific Fusion. (b) The process of General Fusion.

To describe in detail how each fusion process works, the concept of causal networks needs to be established:

Definition 2: A **Causal Network** is a directed acyclic graph wherein random variables are depicted as nodes. A directed edge from node x to y indicates that variable x has a direct/causal influence on

y . Without any conditioning of any random variables, the model of uncertainty (probability distribution, credal set, or other model) that describes y is a function of the set of “parents” of y . A parent of y is a variable like x which is the source of an edge that terminates on y . Unlike Bayesian networks, the precise nature of a child node’s dependency of the values of the parent node does not need to be specified in a causal network.

When conditional probability tables are used to describe how each node depends on its parents, the causal network becomes the well-known “Bayesian network”.

When the probability distributions in the conditional probability tables in a Bayesian network are replaced with credal sets, the result is a “credal network”. Theory related to credal networks can be found in [Cozman(2000)]. An important concept related to credal networks is the concept of the “strong extension”, which is the tightest convex hull that contains all joint probability distributions allowed by the credal network.

The fact that in “causal networks”, the manner of a child node’s dependency on its parent does not need to be specified, means that causal networks can include Bayesian networks and credal networks.

A causal network provides a concrete high level model of the scenario of interest. In the case of fusion, there will be a distinct causal network for each fusion approach.

Figure 2(a) displays the causal network that describes the scenario used for context specific fusion. In this scenario, the hypothesis variable H influences each of the observations O_1, O_2, \dots, O_N . For each possible hypothesis, the observations all occur independently (there are no causal links between observations).

Figure 2(b) displays the causal network that describes the scenario used for general fusion. Unlike the causal network for context specific fusion, there are instead N hypothesis variables H_1, H_2, \dots, H_N that correspond to each of the N credal sets S_1, S_2, \dots, S_N . The hypothesis variables are all independent (there are no causal links between the H_i ’s). There is then a binary variable E which attains 1 if and only if all of the hypothesis variables are equal and 0 if otherwise.

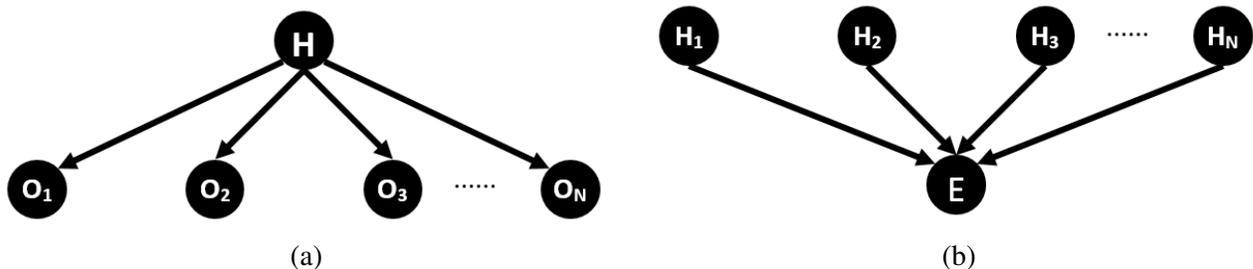


Fig. 2. (a) The causal network that describes the scenario envisioned for context specific fusion. (b) The causal network that describes the scenario envisioned for general fusion.

In this paper, the term “prior” refers to probabilities before the values of any of the variables are known; and the term “posterior” refers to probabilities after the values of certain variables have become known.

During context specific fusion, the observation variables O_1, O_2, \dots, O_N are fixed to their observed values, and the posterior credal set for H is computed. During general fusion, the binary variable E is fixed to 1 which forces all of the hypothesis variables to have the same value. The posterior credal set that describes this common value is the resultant posterior credal set for H .

B. Context Specific Fusion using point probabilities

This section describes the process of context specific fusion when the credal sets consist of single probability distributions. A credal set that contains a single probability distribution is referred to as a “point probability distribution”. Before any fusion can occur, a “prior” probability distribution for H is required. Let the prior probability of $H = j$ for each $j = 1, 2, \dots, M$ be denoted by p_j . In addition, for each O_i ($i = 1, 2, \dots, N$), an observation o_i is received. For each $j = 1, 2, \dots, M$, $p_{i,j}$ will denote the probability of $O_i = o_i$ provided that $H = j$: $p_{i,j} = \Pr(O_i = o_i | H = j)$. Bayes’ rule gives the following posterior probability distribution for H :

$$\forall j' \in \{1, 2, \dots, M\} : \Pr(H = j' | \forall i \in \{1, 2, \dots, N\} : O_i = o_i) = \frac{p_{j'} \prod_{i=1}^N p_{i,j'}}{\sum_{j=1}^M p_j \prod_{i=1}^N p_{i,j}}$$

As an example of context specific fusion, consider a machine that can be in one of 2 states: “functional”; or “non-functional”. The machine’s state is the hypothesis variable H , and the $M = 2$ states are respectively enumerated by 1 and 2. Imagine there are $N = 3$ sensors in place to determine the state of the machine: sensor 1 (O_1) can return either “low temperature” or “high temperature”; sensor 2 (O_2) can return either “low load” or “high load”; and sensor 3 (O_3) can return either “low current” or “high current”. Through careful experimentation, it is known that:

$$\begin{aligned} \Pr(H = 1) &= 0.9 & \Pr(H = 2) &= 0.1 \\ \Pr(O_1 = \text{“low temperature”} | H = 1) &= 0.9 & \Pr(O_1 = \text{“low temperature”} | H = 2) &= 0.4 \\ \Pr(O_2 = \text{“low load”} | H = 1) &= 0.3 & \Pr(O_2 = \text{“low load”} | H = 2) &= 0.6 \\ \Pr(O_3 = \text{“low current”} | H = 1) &= 0.7 & \Pr(O_3 = \text{“low current”} | H = 2) &= 0.2 \end{aligned}$$

If sensor 1 returns $O_1 = \text{“high temperature”}$; sensor 2 returns $O_2 = \text{“low load”}$; and sensor 3 returns $O_3 = \text{“low current”}$; then the posterior probability distribution for H is:

$$\begin{aligned} \Pr(H = 1 | O_1, O_2, O_3) &= \frac{(0.9)(0.1)(0.3)(0.7)}{(0.9)(0.1)(0.3)(0.7) + (0.1)(0.6)(0.6)(0.2)} \approx 0.7241 \\ \Pr(H = 2 | O_1, O_2, O_3) &= \frac{(0.1)(0.6)(0.6)(0.2)}{(0.9)(0.1)(0.3)(0.7) + (0.1)(0.6)(0.6)(0.2)} \approx 0.2759 \end{aligned}$$

It should also be noted that observations can be fused in a sequential fashion. For example, the observations O_1, O_2, \dots, O_N can be fused simultaneously with one large fusion step, but it is also possible to fuse the observations in a sequential fashion. This sequential fusion proceeds as follows. Let \Pr_0 be the prior probability distribution of H . Now fuse the single observation O_1 to get the posterior probability distribution \Pr_1 . To fuse on observation O_2 , the prior distribution \Pr_0 for H should be replaced with \Pr_1 , and then the single observation O_2 should be fused using the new prior. This process continues until all of O_1, O_2, \dots, O_N have been fused. Fusing observations in a sequential manner also provides a means of performing context specific fusion with a computational complexity of $O(N)$. The computational complexity’s dependence on M , the domain size of the hypothesis variable, depends on the subtype of credal set used. In the case of point probabilities however, the computational complexity with respect to M is $O(M)$. The overall computational complexity for fusing point probability distributions in a context specific manner is $O(NM)$.

C. General Fusion using point probabilities

This section describes the general fusion process when the credal sets consist of single probability distributions. For now, assume that each S_i is a single probability distribution with respective probabilities $p_{i,1}, p_{i,2}, \dots, p_{i,M}$, where $p_{i,j}$ is the prior probability that $H_i = j$. With probability distributions, it is required that $\sum_{j=1}^M p_{i,j} = 1$. Since it is required that $H_1 = H_2 = \dots = H_N (= H)$, we know that $E = 1$. Bayes' rule gives the following posterior probability distribution for H :

$$\forall j' \in \{1, 2, \dots, M\} : \Pr(H_1 = j' | H_1 = H_2 = \dots = H_N) = \frac{\prod_{i=1}^N p_{i,j'}}{\sum_{j=1}^M \prod_{i=1}^N p_{i,j}}$$

As an example of general fusion, consider the same machine from the context specific fusion example: a machine that can be in one of 2 states: “functional”; or “non-functional”. The machine’s state is the hypothesis variable H , and the $M = 2$ states are respectively enumerated by 1 and 2. Again there are $N = 3$ sensors in place to determine the state of the machine, but instead of these sensors simply returning a direct observation, they instead return their own guess at the probability distribution for H .

Assume that sensor 1 returns a 45% probability of $H_1 = 1$ (the machine is functional); sensor 2 returns a 60% probability of $H_2 = 1$; and sensor 3 returns a 10% probability of $H_3 = 1$. Since it is known that $H_1 = H_2 = H_3$ (which is equivalent to requiring that $E = 1$), the posterior probability distribution for H , the common value, is:

$$\Pr(H = 1) = \frac{\Pr(H_1 = H_2 = H_3 = 1)}{\Pr(H_1 = H_2 = H_3)} = \frac{(0.45)(0.6)(0.1)}{(0.45)(0.6)(0.1) + (0.55)(0.4)(0.9)} = 0.12$$

$$\Pr(H = 2) = \frac{\Pr(H_1 = H_2 = H_3 = 2)}{\Pr(H_1 = H_2 = H_3)} = \frac{(0.55)(0.4)(0.9)}{(0.45)(0.6)(0.1) + (0.55)(0.4)(0.9)} = 0.88$$

There is additional complexity to the sensors since they now have to return probability distributions as opposed to raw data. Unlike context specific fusion however, prior probability values and conditional probabilities do not have to be accumulated ahead of time (except possibly for the purpose of “calibrating” each sensor to return probabilities).

Similar to context specific fusion, credal sets can also be fused in a sequential fusion. Let credal sets S_1, S_2, \dots, S_N cover the hypothesis variable H . S_1, S_2, \dots, S_N can be fused in a single large fusion step, but it is also possible to fuse these credal sets in a sequential fashion as follows: S_1 and S_2 are fused to form S'_2 ; then S'_2 and S_3 are fused to form S'_3 ; and so on. Again, sequential fusion allows general fusion to proceed with a computational complexity of $O(N)$ with respect to N . The computational complexity with respect to M depends on the subtype of credal set used, but for point probabilities the computational complexity is again $O(M)$ with respect to M . The overall computational complexity for fusing point probability distributions in a general manner is $O(NM)$.

The following sections will now focus on fusion where the credal set subtype is a set of probability distributions, as opposed to a single probability distribution.

IV. FUSION USING NONTRIVIAL CREDAL SETS

This section will describe both context specific and general fusion using credal sets that denote sets of probability distributions as opposed to single probability distributions. These credal sets are referred to as being “nontrivial”.

The basic idea of generalizing fusion to nontrivial credal sets, is that the output credal set should contain every possible probability distribution that results from fusion using probability distributions chosen from each input credal set. Ideally, the output credal set should denote as small a set as possible. Details related to context specific and general fusion are given in the next sections.

A. Context Specific Fusion using Nontrivial Credal Sets

A high level description of the process of context specific fusion using credal sets can be found in [Zaffalon(2002)], [Karlsson et al.(2011)Karlsson, Johansson and Andler].

For context specific fusion, a “prior” credal set of the chosen subtype S_0 is needed that describes H . For each observation O_i ($i = 1, 2, \dots, N$), let o_i denote the value assigned to O_i . For each $j = 1, 2, \dots, M$, let $P_{i,j}$ denote the set of all possible values of $\Pr(O_i = o_i | H = j)$. The resultant credal set S_\bullet should now satisfy the following **containment property**:

Definition 3: The Containment Property for Context Specific Fusion:

First, choose an arbitrary probability distribution p_1, p_2, \dots, p_M from S_0 . For each $i = 1, 2, \dots, N$, and $j = 1, 2, \dots, M$ consider an arbitrary choice of probability $p_{i,j}$ where $p_{i,j} \in P_{i,j}$. The probability distribution given by:

$$\forall j' = 1, 2, \dots, M : p_{\bullet,j'} = \frac{p_{j'} \prod_{i=1}^N p_{i,j'}}{\sum_{j=1}^M p_j \prod_{i=1}^N p_{i,j}}$$

should now be contained by the posterior credal set S_\bullet , no matter the choice of p_j 's and $p_{i,j}$'s. S_\bullet is considered “tight” if no other probability distributions are contained. S_\bullet is considered “maximally tight” if there is no other credal set of the *same subtype* S'_\bullet that satisfies the containment property and is a proper subset of S_\bullet .

Like with point probabilities, context specific fusion using nontrivial credal sets can proceed in a sequential manner. However, the resultant credal set may not be as tight as the credal set that results from simultaneous fusion.

It is also important to note that **the cost of acquiring each $P_{i,j}$ will not be counted as part of our analysis of the computational complexity** of various algorithms for context specific fusion.

B. General Fusion using Nontrivial Credal Sets

General fusion using credal sets is described in [Karlsson et al.(2011)Karlsson, Johansson and Andler].

For general fusion, the resultant credal set S_\bullet should satisfy the following **containment property**:

Definition 4: The Containment Property for General Fusion:

Choose arbitrary probability distributions $\Pr_1, \Pr_2, \dots, \Pr_N$ from S_1, S_2, \dots, S_N respectively. The resultant probability distribution from fusing $\Pr_1, \Pr_2, \dots, \Pr_N$ should be contained by S_\bullet , no matter the choice of \Pr_i 's. S_\bullet is considered “tight” if no other probability distributions are contained. S_\bullet is considered “maximally tight” if there is no other credal set of the *same subtype* S'_\bullet that satisfies the containment property and is a proper subset of S_\bullet .

Like with point probabilities, general fusion using nontrivial credal sets can proceed in a sequential manner. However, the resultant credal set may not be as tight as the structure that results from simultaneous fusion.

When the causal network from Figure 2(b) is treated as a credal network, the strong extension [Cozman(2000)] bears a similarity to the containment property. It is important to note however, that the “strong extension” of credal networks requires tightness, something that is not a requirement of the containment property. Also in the context of this paper, the maximally tight posterior credal set of the correct subtype may not be as tight as the convex hull that constitutes the strong extension.

C. Lower and Upper Probability Bounds

As noted in [De Campos et al.(1994)De Campos, Huete and Moral], [Zaffalon(1999)], [Zaffalon(2002)], determining the lower and upper bounds for posterior probabilities requires the simultaneous minimization and maximization of probabilities. Let A denote a condition of interest, and let B denote a condition that is forced to be true (B denotes $\forall i = 1, 2, \dots, N : O_i = o_i$ in the case of context specific fusion, and B denotes $E = 1$ in the case of general fusion). If Pr_L and Pr_U denote lower and upper probability bounds respectively, then ([De Campos et al.(1994)De Campos, Huete and Moral]):

$$\text{Pr}_L(A|B) = \frac{\text{Pr}_L(A \wedge B)}{\text{Pr}_L(A \wedge B) + \text{Pr}_U(\neg A \wedge B)} \quad \text{and} \quad \text{Pr}_U(A|B) = \frac{\text{Pr}_U(A \wedge B)}{\text{Pr}_U(A \wedge B) + \text{Pr}_L(\neg A \wedge B)}$$

For the credal set subtypes considered by this paper, the minimization (maximization) of $\text{Pr}(A \wedge B)$ does not interfere or interact with the maximization (minimization) of $\text{Pr}(\neg A \wedge B)$.

D. Approximate approaches

In many cases, credal sets are denoted by listing their “extreme points”. When credal sets are denoted by listing their extreme points, they are not confined to any subtype such as probability interval distributions or Dempster-Shafer models. The paper [Karlsson et al.(2011)Karlsson, Johansson and Andler] states and proves a theorem (referred to in [Karlsson et al.(2011)Karlsson, Johansson and Andler] as Theorem 2) that implies that context specific and general fusion using credal sets can be done exactly, meaning that the containment property is satisfied and that the resultant credal set is “tight”. This is not necessarily the case if the credal sets are restricted to a specific subtype.

In the context of this paper, the output credal set has the same subtype as the credal sets used for the prior data in the case of context specific fusion, and the same subtype as the input credal sets in the case of general fusion. However, due to the limitations on the expressive power of each subtype of credal set, it is rarely possible to return a credal set of the desired subtype that is “tight”. Moreover, in many cases, finding the tightest possible output credal set of the desired subtype may be computationally intractable, as will be seen in the subsequent sections. Both of these limitations imply that most fusion approaches discussed here will not return a tight credal set of the desired subtype. However, all fusion approaches will satisfy the containment property, something that Dempster’s rule of combination (section VI-D) fails to satisfy.

In sections V-A and VI-A, probability interval distributions and Dempster-Shafer models are shown to be a more memory efficient alternative to listing the extreme points of credal sets. This increase in memory efficiency is argued to compensate for the decrease in accuracy caused when non-tight credal sets of the desired subtype are returned by fusion.

V. PROBABILITY INTERVAL FUSION

The use of probability intervals as opposed to point probabilities is discussed in [De Campos et al.(1994)De Campos, Huete and Moral], [Guo and Tanaka(2010)] and [Walley(1996), section 4].

Definition 5: A **probability interval distribution** S over the values $1, 2, \dots, M$ is a set of closed intervals $[l_1, u_1], [l_2, u_2], \dots, [l_M, u_M]$. A probability distribution p_1, p_2, \dots, p_M is contained by S if and

B. Context Specific Fusion with Probability Intervals

A high level description of the process of context specific fusion using probability intervals can be found in [Walley(1996), section 4, calculus], and [Zaffalon(2002)].

Let S_0 and $[l_1, u_1], [l_2, u_2], \dots, [l_M, u_M]$ denote the prior probability interval distribution for H .

After the observations $O_i = o_i$ have been received for each $i = 1, 2, \dots, N$, for each $j = 1, 2, \dots, M$, the set of possible values of $\Pr(O_i = o_i | H = j)$ is an interval $P_{i,j} = [l_{i,j}, u_{i,j}]$. Note that for each $i = 1, 2, \dots, N$, that the intervals $[l_{i,1}, u_{i,1}], [l_{i,2}, u_{i,2}], \dots, [l_{i,M}, u_{i,M}]$ do not collectively form a probability interval distribution.

The posterior probability interval distribution for H is determined by computing the smallest and largest possible posterior probabilities for each value of H . Let this posterior distribution be denoted by $[l_{\bullet,1}, u_{\bullet,1}], [l_{\bullet,2}, u_{\bullet,2}], \dots, [l_{\bullet,M}, u_{\bullet,M}]$.

To find these extremes, let p_1, p_2, \dots, p_M denote an arbitrary prior probability distribution for H that is contained by S_0 , and let $p_{i,j}$ for each $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, M$ denote an arbitrary probability from the interval $[l_{i,j}, u_{i,j}]$.

For an arbitrary $j' = 1, 2, \dots, M$, in order to compute $l_{\bullet,j'}$, the probability of $H = j' \wedge \forall i \in \{1, 2, \dots, N\} : O_i = o_i$ should be minimized, while the probability of $H \neq j' \wedge \forall i \in \{1, 2, \dots, N\} : O_i = o_i$ should be maximized. This can be done by setting $p_{j'} = l_{j'}$; $p_{i,j'} = l_{i,j'}$ for each $i = 1, 2, \dots, N$; and $p_{i,j} = u_{i,j}$ for each $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, M$ where $j \neq j'$. To decide upon each p_j where $j \neq j'$, a greedy maximization approach is used. Each p_j is set to l_j by default, and the following process is repeated: Find $j \in \{1, 2, \dots, M\} \setminus \{j'\}$ that maximizes $c_j = \prod_{i=1}^N p_{i,j}$. Next, p_j should be set to the highest allowed probability (the probability is limited by both u_j and the fact that $\sum_{j=1}^M p_j = 1$). j should then be removed from the set $j \in \{1, 2, \dots, M\} \setminus \{j'\}$, and a new j should be chosen. This process repeats until $\sum_{j=1}^M p_j = 1$. A similar process is used to compute each $u_{\bullet,j'}$.

The following algorithm depicts the process of context specific fusion using probability intervals. To save space, the steps involved in computing the upper bounds $u_{\bullet,j'}$ will be shown in parentheses beside the steps for computing the lower bounds $l_{\bullet,j'}$.

```

for  $j = 1$  to  $M$  do
   $l_{\Pi,j} \leftarrow \prod_{i=1}^N l_{i,j}$ 
   $u_{\Pi,j} \leftarrow \prod_{i=1}^N u_{i,j}$ 
end for
for  $j' = 1$  to  $M$  do
  //  $l_{\bullet,j'}$  ( $u_{\bullet,j'}$ ) will be computed.
  for  $j = 1$  to  $M$  do
     $p_j \leftarrow l_j$  ( $p_j \leftarrow u_j$ )
    if  $j = j'$  then
      /* The prior probability of  $\Pr(H = j' \wedge \forall i = 1, 2, \dots, N : O_i = o_i)$  should
      be minimized (maximized). */
       $c_j \leftarrow l_{\Pi,j}$  ( $c_j \leftarrow u_{\Pi,j}$ )
       $b_j \leftarrow 0$ 
    else
      /* The prior probability of  $\Pr(H \neq j' \wedge \forall i = 1, 2, \dots, N : O_i = o_i)$  should
      be maximized (minimized). */
       $c_j \leftarrow u_{\Pi,j}$  ( $c_j \leftarrow l_{\Pi,j}$ )
       $b_j \leftarrow 1$ 
    end if
  
```

```

end for
 $\sigma \leftarrow \sum_{j=1}^M l_j$  ( $\sigma \leftarrow \sum_{j=1}^M u_j$ )
while  $\sigma < 1$  ( $\sigma > 1$ ) do
    Find the  $j$  where  $b_j = 1$  that maximizes  $c_j$ .
     $p_j \leftarrow p_j + \min(u_j - l_j, 1 - \sigma)$  ( $p_j \leftarrow p_j - \min(u_j - l_j, \sigma - 1)$ )
     $\sigma \leftarrow \sigma + \min(u_j - l_j, 1 - \sigma)$  ( $\sigma \leftarrow \sigma - \min(u_j - l_j, \sigma - 1)$ )
     $b_j \leftarrow 0$ 
end while
 $l_{\bullet, j'} \leftarrow \frac{p_{j'} \cdot c_{j'}}{\sum_{j=1}^M p_j \cdot c_j}$  ( $u_{\bullet, j'} \leftarrow \frac{p_{j'} \cdot c_{j'}}{\sum_{j=1}^M p_j \cdot c_j}$ )
end for

```

The overall time complexity for context specific fusion using probability intervals is $O(NM + M^2)$.

As an example of context specific fusion using probability intervals, the same example used for point probabilities in section III-B will be used. This time, however a ± 0.05 margin will be included on each probability:

$$\begin{aligned}
 \Pr(H = 1) &= [0.85, 0.95] & \Pr(H = 2) &= [0.05, 0.15] \\
 \Pr(O_1 = \text{"low temperature"}|H = 1) &= [0.85, 0.95] & \Pr(O_1 = \text{"low temperature"}|H = 2) &= [0.35, 0.45] \\
 \Pr(O_2 = \text{"low load"}|H = 1) &= [0.25, 0.35] & \Pr(O_2 = \text{"low load"}|H = 2) &= [0.55, 0.65] \\
 \Pr(O_3 = \text{"low current"}|H = 1) &= [0.65, 0.75] & \Pr(O_3 = \text{"low current"}|H = 2) &= [0.15, 0.25]
 \end{aligned}$$

If sensor 1 returns $O_1 = \text{"high temperature"}$; sensor 2 returns $O_2 = \text{"low load"}$; and sensor 3 returns $O_3 = \text{"low current"}$; then the posterior probability interval distribution for H is:

$$\Pr(H = 1|O_1, O_2, O_3) \approx [0.3036, 0.9428] \quad \Pr(H = 2|O_1, O_2, O_3) \approx [0.0572, 0.6964]$$

By comparison with the example from III-B, it can be seen that the containment property is holding.

C. General Fusion with Probability Intervals

Each credal set S_i is a probability interval distribution, and the prior probability of $H_i = j$ is a closed interval $[l_{i,j}, u_{i,j}]$ instead of the point probability $p_{i,j}$. S_i now describes a set of probability distributions as opposed to a single probability distribution.

Here it should be noted that exact fusion is not possible as probability intervals do not have the necessary expressive power to denote the exact set of possible fused probability distributions. Two approaches to approximate fusion will be covered in the next two subsections:

Finding the maximally tight posterior probability interval distribution for general fusion requires an algorithm for solving the following NP-hard problem:

Problem 2: Optimum sum of products

Input:

Two positive integers n and m .

Two $n \times m$ arrays of non-negative real numbers: $a_{i,j}$ and $b_{i,j}$ for each $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. It must be the case that:

$$\forall i \in \{1, 2, \dots, n\} : \forall j \in \{1, 2, \dots, m\} : 0 \leq a_{i,j} \leq b_{i,j}$$

One n length vector of non-negative real numbers: c_i for each $i = 1, 2, \dots, n$. It must be the case that:

$$\forall i \in \{1, 2, \dots, n\} : \sum_{j=1}^m a_{i,j} \leq c_i \leq \sum_{j=1}^m b_{i,j}$$

In addition, a choice between maximization and minimization must be made.

Internal Variables to be optimized:

One $n \times m$ array of non-negative real numbers: $x_{i,j}$ for each $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. The following restrictions hold:

$$\begin{aligned} \forall i \in \{1, 2, \dots, n\} : \forall j \in \{1, 2, \dots, m\} : a_{i,j} \leq x_{i,j} \leq b_{i,j} \\ \forall i \in \{1, 2, \dots, n\} : \sum_{j=1}^m x_{i,j} = c_i \end{aligned}$$

Output:

The maximum, or minimum depending on choice, possible value of the expression

$$\sum_{j=1}^m \prod_{i=1}^n x_{i,j}$$

Problem 2 in essence takes n unnormalized probability interval distributions over a domain of m values: $[a_{i,1}, b_{i,1}], [a_{i,2}, b_{i,2}], \dots, [a_{i,m}, b_{i,m}]$, and extracts from each an unnormalized probability distribution $x_{i,1}, x_{i,2}, \dots, x_{i,m}$ that sums to c_i . The probability distributions are chosen to either maximize or minimize the probability of agreement between all chosen probability distributions. A proof of the NP-hardness of problem 2 is given in the Appendix.

It is not hard to show that the $x_{i,j}$'s that optimize $\sum_{j=1}^m \prod_{i=1}^n x_{i,j}$ attain a ‘‘corner state’’. That is, for each $i = 1, 2, \dots, N$, $x_{i,j} = a_{i,j}$ or $x_{i,j} = b_{i,j}$ for all but one $j = 1, 2, \dots, M$. There are a finite number of corner states, so as noted in [Zaffalon(1999)], [Zaffalon(2002)], problem 2 can be solved via an exhaustive search of the corner states.

Since problem 2 is NP-hard, approximate solutions are necessary for tractable calculations. None of the approximations made in this paper will violate the containment property. In [Antonucci et al.(2013a)Antonucci, De Campos, Huber and Zaffalon], an optimization problem that encompasses problem 2 is solved in an approximate manner using hill climbing iterations.

1) *Approach 1:* If computational intractability is not an issue, problem 2 can be solved to find the tightest possible lower and upper bounds for the posterior probability distribution. The following algorithm depicts the process of general fusion using probability intervals. To save space, the steps involved in computing the upper bounds $u_{\bullet,j'}$ will be shown in parentheses beside the steps for computing the lower bounds $l_{\bullet,j'}$.

for $j' = 1$ to M **do**

// $l_{\bullet,j'}$ ($u_{\bullet,j'}$) will be computed.

$q \leftarrow \prod_{i=1}^N l_{i,j'}$ ($q \leftarrow \prod_{i=1}^N u_{i,j'}$)

// q is the minimized (maximized) prior probability of $H_1 = H_2 = \dots = H_N = j'$.

/* The maximum (minimum) prior probability of $H_1 = H_2 = \dots = H_N \neq j'$ will now be computed using problem 2: */

```

n ← N
m ← M − 1
for i = 1 to N do
  ci ← 1 − li,j' (ci ← 1 − ui,j')
  for j = 1 to M − 1 do
    if j < j' then
      ai,j ← li,j and bi,j ← ui,j
    else
      ai,j ← li,j+1 and bi,j ← ui,j+1
    end if
  end for
end for
Solve problem 2 with the values of n, m, ai,j, bi,j, ci, and use maximization (minimization). Assign
the result to r.
l•,j' ←  $\frac{q}{q+r}$  (u•,j' ←  $\frac{q}{q+r}$ )
end for

```

The overall computational complexity for approach #1 is $O(NM + M \cdot f(N, M - 1))$ where $f(n, m)$ is the computational complexity of problem 2. While the arrays are being freshly generated for each application of problem 2 in the pseudo code above, the computational complexity assumes a single array can be pre-calculated at a cost of $O(NM)$ and used for all applications of problem 2 with different columns ignored.

Problem 2 is NP-hard, but when $n = 2$, the problem is greatly simplified. Problem 2 when $n = 2$ becomes a bilinear programming problem. The resultant bilinear programming programming problem can be more easily solved, and provides a means through which the probability interval distributions can be fused in a sequential manner. It should be noted however, that the resultant probability interval distribution will not be as tight as the probability interval distribution formed through simultaneous fusion.

When probability interval distributions are fused in a pairwise manner, the computational complexity of approach #1 reduces to $O(NM + N \cdot M \cdot f(2, M - 1))$.

2) *Approach 2:* Theory from [De Campos et al.(1994)De Campos, Huete and Moral] can be used for the general fusion of probability interval distributions. It should be noted however, that the approach presented here will fail to be maximally tight for the following reasons: In the context of general fusion, the hypothesis variables H_1, H_2, \dots, H_N are all independent when E is ignored. When a joint probability interval distribution is formed that covers the hypothesis variables, the independence between H_1, H_2, \dots, H_N can no longer be enforced. This makes possible joint probability distributions over H_1, H_2, \dots, H_N that do not satisfy the independence between the hypothesis variables.

When $E = 1$, H will denote the common value of H_1, H_2, \dots, H_N .

Ignoring the variable E , the joint probability interval for $\Pr(\forall i \in \{1, 2, \dots, N\} : H_i = j_i)$ is $\left[\prod_{i=1}^N l_{i,j_i}, \prod_{i=1}^N u_{i,j_i} \right]$

For each $j' \in \{1, 2, \dots, M\}$, the smallest and largest posterior probability $\Pr(H = j' | E = 1)$ is

$$l_{\bullet,j'} = \frac{\Pr_L(H = j' \wedge E = 1)}{\Pr_L(H = j' \wedge E = 1) + \Pr_U(H \neq j' \wedge E = 1)}$$

and

$$u_{\bullet,j'} = \frac{\Pr_U(H = j' \wedge E = 1)}{\Pr_U(H = j' \wedge E = 1) + \Pr_L(H \neq j' \wedge E = 1)}$$

respectively.

For each $j' \in \{1, 2, \dots, M\}$, $\prod_{i=1}^N l_{i,j'}$ is the smallest possible prior probability $\Pr_L(H = j' \wedge E = 1)$. The maximum prior probability $\Pr_U(H \neq j' \wedge E = 1)$ seems to be $\sum_{j:j \neq j'} \prod_{i=1}^N u_{i,j}$. However, while each upper bound is attainable, upper bounds may not be *simultaneously attainable*. Another upper bound on the prior probability $\Pr(H \neq j' \wedge E = 1)$ arises from a lower bound on the prior probability $\Pr(H = j' \vee E = 0)$. A lower bound on the prior probability $\Pr(H = j' \vee E = 0)$ that arises directly from the probability intervals is $\prod_{i=1}^N l_{i,j'} + L$ where:

$$L = \prod_{i=1}^N \sum_{j=1}^M l_{i,j} - \sum_{j=1}^M \prod_{i=1}^N l_{i,j}$$

The maximum prior probability $\Pr_U(H \neq j' \wedge E = 1)$ is: $\min\left(\sum_{j:j \neq j'} \prod_{i=1}^N u_{i,j}, 1 - \prod_{i=1}^N l_{i,j'} - L\right)$

The smallest possible posterior probability $\Pr_L(H = j'|E = 1)$ is:

$$l_{\bullet,j'} = \frac{\prod_{i=1}^N l_{i,j'}}{\min\left(\prod_{i=1}^N l_{i,j'} + \sum_{j:j \neq j'} \prod_{i=1}^N u_{i,j}, 1 - L\right)}$$

Using a similar argument, the largest possible posterior probability $\Pr_U(H = j'|E = 1)$ is:

$$u_{\bullet,j'} = \frac{\prod_{i=1}^N u_{i,j'}}{\max\left(\prod_{i=1}^N u_{i,j'} + \sum_{j:j \neq j'} \prod_{i=1}^N l_{i,j}, 1 - U\right)}$$

where

$$U = \prod_{i=1}^N \sum_{j=1}^M u_{i,j} - \sum_{j=1}^M \prod_{i=1}^N u_{i,j}$$

The above gives the complete approach to computing the posterior probability interval distribution S_{\bullet} for H : $[l_{\bullet,1}, u_{\bullet,1}]; [l_{\bullet,2}, u_{\bullet,2}]; \dots; [l_{\bullet,M}, u_{\bullet,M}]$.

The overall computational complexity for approach #2 is $O(N \cdot M)$. To achieve this efficiency, the following expressions should be computed in the following order:

$$\begin{aligned} \forall i \in \{1, 2, \dots, N\} : l_{i,\Sigma} &= \sum_{j=1}^M l_{i,j} & u_{i,\Sigma} &= \sum_{j=1}^M u_{i,j} & \forall j \in \{1, 2, \dots, M\} : l_{\Pi,j} &= \prod_{i=1}^N l_{i,j} & u_{\Pi,j} &= \prod_{i=1}^N u_{i,j} \\ L_{\Pi\Sigma} &= \prod_{i=1}^N l_{i,\Sigma} & U_{\Pi\Sigma} &= \prod_{i=1}^N u_{i,\Sigma} & L_{\Sigma\Pi} &= \sum_{j=1}^M l_{\Pi,j} & U_{\Sigma\Pi} &= \sum_{j=1}^M u_{\Pi,j} \end{aligned}$$

$$\begin{aligned} \forall j \in \{1, 2, \dots, M\} : l_{\bullet,j} &= \frac{l_{\Pi,j}}{\min(l_{\Pi,j} + (U_{\Sigma\Pi} - u_{\Pi,j}), 1 - (L_{\Pi\Sigma} - L_{\Sigma\Pi}))} \\ \forall j \in \{1, 2, \dots, M\} : u_{\bullet,j} &= \frac{u_{\Pi,j}}{\max(u_{\Pi,j} + (L_{\Sigma\Pi} - l_{\Pi,j}), 1 - (U_{\Pi\Sigma} - U_{\Sigma\Pi}))} \end{aligned}$$

As an example of general fusion using approach #2 for probability intervals, the same example used for point probabilities in section III-C will be used. This time, however a ± 0.05 margin will be included on each probability:

$$\begin{aligned} \Pr(H_1 = 1) &= [0.40, 0.50] & \Pr(H_1 = 2) &= [0.50, 0.60] \\ \Pr(H_2 = 1) &= [0.55, 0.65] & \Pr(H_2 = 2) &= [0.35, 0.45] \\ \Pr(H_3 = 1) &= [0.05, 0.15] & \Pr(H_3 = 2) &= [0.85, 0.95] \end{aligned}$$

The posterior probability distribution for H , the common value, is:

$$\Pr(H = 1) \approx [0.0411, 0.2468] \quad \Pr(H = 2) \approx [0.7532, 0.9589]$$

By comparison with the example from III-C, it can be seen that the containment property is holding.

VI. DEMPSTER-SHAFFER FUSION

A description of Dempster-Shafer theory can be found in [Klir(2005), chapter 5] and [Yager(1987)].

Definition 6: A **Dempster-Shafer model** S over the values $\text{Val}(S) = \{1, 2, \dots, M\}$ is described by a “mass function” $m : \text{Set}(S) \rightarrow [0, 1]$ where $\text{Set}(S) = 2^{\text{Val}(S)} \setminus \{\emptyset\}$. It must be the case that:

$$\sum_{J \in \text{Set}(S)} m(J) = 1$$

A probability distribution p_1, p_2, \dots, p_M is contained by S if and only if

$$\forall J' \subseteq \{1, 2, \dots, M\} : \sum_{J \subseteq J' \wedge J \neq \emptyset} m(J) \leq \sum_{j \in J'} p_j \leq \sum_{J \cap J' \neq \emptyset \wedge J \neq \emptyset} m(J)$$

In other words, the probability of the outcome j being a member of J' is bounded from below by the “belief”:

$$\text{Bel}(J') = \sum_{J \subseteq J' \wedge J \neq \emptyset} m(J)$$

and from above by the “plausibility”:

$$\text{Pl}(J') = \sum_{J \cap J' \neq \emptyset \wedge J \neq \emptyset} m(J)$$

Any probability distribution contained by S can be generated in the following manner: For each $J \in \text{Set}(S)$, the weight contained by $m(J)$ is partitioned between the elements of J . Every and only the probability distributions contained by S can be formed from this process.

Dempster-Shafer models have a greater expressive power than probability intervals. Every probability interval distribution has an equivalent Dempster-Shafer model, but only a small fraction of Dempster-Shafer models have an equivalent probability interval distribution.

In a manner similar to the use of probability intervals, a Dempster-Shafer model can be completely characterized by the lower bound “belief function” $\text{Bel} : \text{Set}(S) \rightarrow [0, 1]$. The belief function must satisfy the following properties:

(The belief/lower bound must be contained by $[0, 1]$)

$$\forall J \in \text{Set}(S) : 0 \leq \text{Bel}(J) \leq 1$$

(The lower bounds must respect the union of disjoint sets)

$$\forall J_1, J_2 \in \text{Set}(S) : J_1 \cap J_2 = \emptyset \implies \text{Bel}(J_1 \cup J_2) \geq \text{Bel}(J_1) + \text{Bel}(J_2)$$

(The lower bound must be 1 for the entire domain)

$$\text{Bel}(\text{Val}(S)) = 1$$

A Dempster-Shafer model can also be completely characterized by the upper bound “plausibility function” $\text{Pl} : \text{Set}(S) \rightarrow [0, 1]$. The belief function must satisfy the following properties:

(The plausibility/upper bound must be contained by $[0, 1]$)

$$\forall J \in \text{Set}(S) : 0 \leq \text{Pl}(J) \leq 1$$

(The upper bounds must respect the union of disjoint sets)

$$\forall J_1, J_2 \in \text{Set}(S) : J_1 \cap J_2 = \emptyset \implies \text{Pl}(J_1 \cup J_2) \leq \text{Pl}(J_1) + \text{Pl}(J_2)$$

(The upper bound must be 1 for the entire domain)

$$\text{Pl}(\text{Val}(S)) = 1$$

Given a valid belief/lower bound function or a valid plausibility/upper bound function, the mass function can be computed via the inclusion/exclusion principle [Yager and Liu(2008), pg. 4]:

$$\forall J' \in \text{Set}(S) : m(J') = \sum_{J \subseteq J' \wedge J \neq \emptyset} (-1)^{|J'|+|J|} \text{Bel}(J)$$

$$\forall J' \in \text{Set}(S) : m(J') = \sum_{J \supseteq (\text{Val}(S) \setminus J') \wedge J \neq \emptyset} (-1)^{1+|\text{Val}(S)|+|J'|+|J|} \text{Pl}(J)$$

A. Dempster-Shafer models and credal sets

Like with probability intervals, a Dempster-Shafer model S over the domain $1, 2, \dots, M$ for which the number of extreme points greatly exceeds the size of S will be constructed to prove the utility of Dempster-Shafer models in comparison with listing the extreme points. The size of the Dempster-Shafer model is $O(2^M)$. In this case, the number of extreme points will be $\Omega(M!)$. Let $M \gg 1$ be arbitrary: For each $J \in \text{Set}(S)$, let $m(J) = \frac{1}{2^M - 1}$. An extreme probability distribution is formed by choosing a permutation of $1, 2, \dots, M$. Let $\rho : \text{Val}(S) \rightarrow \text{Val}(S)$ denote this permutation. All probability mass gravitates to $\rho(1)$; followed by $\rho(2)$; and so on. The probability assigned to $\rho(j)$ for each $j = 1, 2, \dots, M$ is: $p_j = \frac{2^{\binom{M-j}{2}}}{(2^M - 1)}$. The number of permutations ρ is $M!$, so the number of extreme points is $\Omega(M!)$.

Again, it is clear that representing a Dempster-Shafer model by its extreme points is not computationally efficient. For instance, if $M = 20$, a Dempster-Shafer model requires $2^{20} - 1 = 1048575$ values, while the number of extreme points is $20! \approx 2.4329 \times 10^{18}$.

B. Context Specific Fusion with Dempster-Shafer models

An approach to context specific fusion using Dempster-Shafer models is given in [Delmotte and Smets(2004)]. The approach presented here however, will differ from the approach given in [Delmotte and Smets(2004)], since the presented approach will aim to satisfy the containment property. The approach presented here will bear similarities to the approach from [Zaffalon(2002)].

Context specific fusion using Dempster-Shafer models proceeds in a very similar manner to context specific fusion using probability intervals.

Let $S_0; m_0 : \text{Set}(H) \rightarrow [0, 1]$; and $\text{Bel}_0 : \text{Set}(H) \rightarrow [0, 1]$ all denote the prior Dempster-Shafer model for H .

After the observations $O_i = o_i$ have been received for each $i = 1, 2, \dots, N$, for each $j = 1, 2, \dots, M$, the set of possible values of $\text{Pr}(O_i = o_i | H = j)$ is an interval $P_{i,j} = [l_{i,j}, u_{i,j}]$. Each $P_{i,j}$ is simply an interval, as a full Dempster-Shafer model that covers O_i is not required. Only the scenario of $O_i = o_i$ is under consideration.

The posterior Dempster-Shafer model for H is determined by computing the smallest (or largest) possible posterior probabilities for each nonempty subset of $\text{Val}(H)$. Let the lower (or upper) posterior probability of $J \in \text{Set}(H)$ be denoted by $\text{Bel}_\bullet(J)$ (or $\text{Pl}_\bullet(J)$).

The following algorithm depicts the process of context specific fusion using probability intervals. To save space, the steps involved in computing the plausibilities Pl_\bullet will be shown in parentheses beside the steps for computing the beliefs Bel_\bullet . (Note however, that only computing the beliefs are necessary for the posterior Dempster-Shafer model.)

```

for  $j = 1$  to  $M$  do
   $l_{\Pi,j} \leftarrow \prod_{i=1}^N l_{i,j}$ 
   $u_{\Pi,j} \leftarrow \prod_{i=1}^N u_{i,j}$ 
end for
for all  $J' \in \text{Set}(H)$  do
  //  $\text{Bel}_\bullet(J')$  ( $\text{Pl}_\bullet(J')$ ) will be computed.
  for  $j = 1$  to  $M$  do
     $p_j \leftarrow 0$ 
    if  $j \in J'$  then
      /* The prior probability of  $\text{Pr}(H \in J' \wedge \forall i = 1, 2, \dots, N : O_i = o_i)$  should
      be minimized (maximized). */
       $c_j \leftarrow l_{\Pi,j}$  ( $c_j \leftarrow u_{\Pi,j}$ )
    else
      /* The prior probability of  $\text{Pr}(H \notin J' \wedge \forall i = 1, 2, \dots, N : O_i = o_i)$  should
      be maximized (minimized). */
       $c_j \leftarrow u_{\Pi,j}$  ( $c_j \leftarrow l_{\Pi,j}$ )
    end if
  end for
  for all  $J \in \text{Set}(H)$  do
    if  $J \subseteq J'$  then
      Find the  $j \in J$  that minimizes (maximizes)  $c_j$ .
    else if  $J \cap J' = \emptyset$  then
      Find the  $j \in J$  that maximizes (minimizes)  $c_j$ .
    else
      Find the  $j \in J \setminus J'$  ( $j \in J \cap J'$ ) that maximizes  $c_j$ .
  end for

```

```

end if
 $p_j \leftarrow p_j + m_0(J)$ 
end for
 $\text{Bel}_\bullet(J') \leftarrow \frac{\sum_{j' \in J'} p_{j'} \cdot c_{j'}}{\sum_{j=1}^M p_j \cdot c_j}$  ( $\text{Pl}_\bullet(J') \leftarrow \frac{\sum_{j' \in J'} p_{j'} \cdot c_{j'}}{\sum_{j=1}^M p_j \cdot c_j}$ )
end for

```

The overall time complexity for context specific fusion (including the cost of determining the final masses using the inclusion/exclusion principle) using Dempster-Shafer models is $O(NM + 2^{2M})$. Since the size D of the prior Dempster-Shafer model for H is approximately 2^M , the time complexity is in fact $O(NM + D^2)$.

As an example of context specific fusion using Dempster-Shafer models, the same example used for point probabilities in section III-B will be used. A ± 0.05 margin will be included to form the prior Dempster-Shafer model for H and each probability interval for the observed evidence:

$$\begin{array}{lll}
m_0(\{1\}) = 0.85 & m_0(\{2\}) = 0.05 & m_0(\{1, 2\}) = 0.1 \\
\Pr(o_1|H = 1) = [0.05, 0.15] & \Pr(o_1|H = 2) = [0.55, 0.65] & \\
\Pr(o_2|H = 1) = [0.25, 0.35] & \Pr(o_2|H = 2) = [0.55, 0.65] & \\
\Pr(o_3|H = 1) = [0.65, 0.75] & \Pr(o_3|H = 2) = [0.15, 0.25] &
\end{array}$$

The posterior Dempster-Shafer model for H is:

$$m_\bullet(\{1\}) \approx 0.3036 \quad m_\bullet(\{2\}) \approx 0.0572 \quad m_\bullet(\{1, 2\}) \approx 0.6392$$

By comparison with the example from III-B, it can be seen that the containment property is holding.

C. General Fusion with Dempster-Shafer models

Dempster's rule of combination (described in [Yager(1987)]) performs general fusion of Dempster-Shafer models. However, Dempster's rule of combination fails to satisfy the containment property.

Each structure S_i is a Dempster-Shafer model. S_i is denoted by either the mass function $m_i : \text{Set}(H_i) \rightarrow [0, 1]$; or the belief function $\text{Bel}_i : \text{Set}(H_i) \rightarrow [0, 1]$. Also, the resultant Dempster-Shafer model S_\bullet is denoted by either $m_\bullet : \text{Set}(H) \rightarrow [0, 1]$; or $\text{Bel}_\bullet : \text{Set}(H) \rightarrow [0, 1]$.

Finding the tightest possible Dempster-Shafer model for general fusion requires an algorithm for solving the following problem:

Problem 3: Optimum sum of products, Dempster-Shafer variant

Input:

Two positive integers n and m .

A set A with m distinct quantities.

An n length array of functions: $f_i : 2^A \setminus \{\emptyset\} \rightarrow [0, +\infty)$ for each $i = 1, 2, \dots, n$.

In addition, a choice between maximization and minimization must be made.

Internal Variables to be optimized:

An n length array of functions: $g_i : 2^A \setminus \{\emptyset\} \rightarrow A$ for each $i = 1, 2, \dots, n$. The following restriction must hold:

$$\forall i \in \{1, 2, \dots, n\} : \forall J \in 2^A \setminus \{\emptyset\} : g_i(J) \in J$$

An $n \times m$ array of non-negative real numbers: $x_{i,j}$ for each $i = 1, 2, \dots, n$ and $j \in A$ where:

$$\forall i \in \{1, 2, \dots, n\} : \forall j \in A : x_{i,j} = \sum_{J \in 2^A \setminus \{\emptyset\} \wedge g_i(J)=j} f_i(J)$$

Output:

The maximum, or minimum depending on choice, possible value of the expression

$$\sum_{j \in A} \prod_{i=1}^n x_{i,j}$$

Problem 3, which is directly analogous to problem 2, in essence takes n unnormalized Dempster-Shafer models over a domain of m values: $A = \{1, 2, \dots, m\}$. From each Dempster-Shafer model $i = 1, 2, \dots, n$, each probability mass is focused onto a single element, forming an unnormalized probability distribution $x_{i,1}, x_{i,2}, \dots, x_{i,m}$. The probability distributions are chosen to either maximize or minimize the probability of agreement between all chosen probability distributions.

1) *Approach 1:* Like with probability intervals, if computational intractability is not an issue, problem 3 can be solved to find the tightest Dempster-Shafer model for the posterior probability distribution. The following algorithm depicts the process of general fusion using Dempster-Shafer models. To save space, the steps involved in computing the plausibilities Pl_\bullet will be shown in parentheses beside the steps for computing the beliefs Bel_\bullet . (Note however, that only computing the beliefs are necessary for the posterior Dempster-Shafer model.)

```

for all  $J' \in \text{Set}(H)$  do
  //  $\text{Bel}_\bullet(J')$  ( $\text{Pl}_\bullet(J')$ ) will be computed.
  /* The minimum prior probability of  $E = 1 \wedge H \in J'$  ( $E = 1 \wedge H \notin J'$ ) will
  now be computed using problem 3: */
   $n \leftarrow N$ 
   $m \leftarrow |J'|$  ( $m \leftarrow M - |J'|$ )
   $A \leftarrow J'$  ( $A \leftarrow \text{Val}(H) \setminus J'$ )
  for  $i = 1$  to  $n$  do
    for all  $J \in 2^A \setminus \{\emptyset\}$  do
       $f_i(J) \leftarrow m_i(J)$ 
    end for
  end for
  Solve problem 3 with the values of  $n, m, A, f_i$ , and use minimization. Assign the result to  $q$ .
  /* The maximum prior probability of  $E = 1 \wedge H \notin J'$  ( $E = 1 \wedge H \in J'$ ) will
  now be computed using problem 3: */
   $m \leftarrow M - |J'|$  ( $m \leftarrow |J'|$ )
   $A \leftarrow \text{Val}(H) \setminus J'$  ( $A \leftarrow J'$ )
  for  $i = 1$  to  $n$  do
    for all  $J \in 2^A \setminus \{\emptyset\}$  do
      /* Since the prior probability of  $E = 1 \wedge H \in A$  is being maximized,
      probability mass gravitates into  $A$ : */
       $f_i(J) \leftarrow \sum_{J'' \in \text{Set}(H) \wedge J'' \cap A = J} m_i(J'')$ 
    end for
  end for
  Solve problem 3 with the values of  $n, m, A, f_i$ , and use maximization. Assign the result to  $r$ .

```

$\text{Bel}_\bullet(J') \leftarrow \frac{q}{q+r} (\text{Pl}_\bullet(J') \leftarrow \frac{r}{q+r})$
end for

The overall computational complexity for approach #1 is $O(N \cdot 2^{2M} + 2^M \cdot g(N, M))$ where $g(n, m)$ is the computational complexity of problem 3. While the input functions are being freshly generated for each application of 3 in the pseudo code above, the computational complexity assumes that all input functions can be pre-calculated at a cost of $O(N \cdot 2^{2M})$ and used for all applications of problem 3 with different entries ignored.

When Dempster-Shafer models are fused in a pairwise manner, the computational complexity of approach #1 reduces to $O(N \cdot 2^{2M} + N \cdot 2^M \cdot g(2, M))$.

2) *Approach 2*: The second approach to general fusion using Dempster-Shafer models also uses theory from [De Campos et al.(1994)De Campos, Huete and Moral] and is similar to general fusion approach #2 for probability intervals. Like with probability intervals, a joint Dempster-Shafer model is created that covers the variables H_1, H_2, \dots, H_N . Again, like with probability intervals, the joint Dempster-Shafer model will fail to enforce the independence between variables H_1, H_2, \dots, H_N . For this fusion approach the assumption that $N \geq 2$ is important.

The joint Dempster-Shafer model for the variables H_1, H_2, \dots, H_N , denoted by S_\times , is created as follows: consider $J_\times = J_1 \times J_2 \times \dots \times J_N$ for arbitrary $J_1, J_2, \dots, J_N \in \text{Set}(H)$. Let the mass assigned to J_\times be: $m_\times(J_\times) = m_1(J_1)m_2(J_2) \dots m_n(J_N)$. For any $J_\times \in \text{Set}(\{H_1, H_2, \dots, H_N\})$, if there does not exist any $J_1, J_2, \dots, J_N \in \text{Set}(H)$ such that $J_\times = J_1 \times J_2 \times \dots \times J_N$, then the mass assigned to J_\times is 0: $m_\times(J_\times) = 0$.

The calculation of $\text{Bel}_\bullet(J')$ (and $\text{Pl}_\bullet(J')$) for an arbitrary $J' \in \text{Set}(H)$ will now be the focus. When point probabilities are used, the posterior probability for $H \in J'$ is: $\Pr(H \in J') = \frac{\Pr(H \in J' \wedge E=1)}{\Pr(H \in J' \wedge E=1) + \Pr(H \notin J' \wedge E=1)}$. The condition that $E = 1$ requires that $H_1 = H_2 = \dots = H_N$, and H denotes the common value. As noted in section IV,

$$\Pr_L(H \in J' | E = 1) = \frac{\Pr_L(H \in J' \wedge E = 1)}{\Pr_L(H \in J' \wedge E = 1) + \Pr_U(H \notin J' \wedge E = 1)}$$

and

$$\Pr_U(H \in J' | E = 1) = \frac{\Pr_U(H \in J' \wedge E = 1)}{\Pr_U(H \in J' \wedge E = 1) + \Pr_L(H \notin J' \wedge E = 1)}$$

Therefore:

$$\forall J' \in \text{Set}(H) : \text{Bel}_\bullet(J') = \frac{\text{Bel}_\times(H \in J' \wedge E = 1)}{\text{Bel}_\times(H \in J' \wedge E = 1) + \text{Pl}_\times(H \in (\text{Val}(H) \setminus J') \wedge E = 1)}$$

$$\forall J' \in \text{Set}(H) : \text{Pl}_\bullet(J') = \frac{\text{Pl}_\times(H \in J' \wedge E = 1)}{\text{Pl}_\times(H \in J' \wedge E = 1) + \text{Bel}_\times(H \in (\text{Val}(H) \setminus J') \wedge E = 1)}$$

where

$$\begin{aligned} \forall J' \in \text{Set}(H) : \quad \text{Bel}_\times(H \in J' \wedge E = 1) &= \sum_{j \in J'} \prod_{i=1}^N m_i(\{j\}) \\ \forall J' \in \text{Set}(H) : \quad q_\times(H \in J' \wedge E = 1) &= \prod_{i=1}^N \sum_{J \supseteq J'} m_i(J) \\ \forall J' \in \text{Set}(H) : \quad \text{Pl}_\times(H \in J' \wedge E = 1) &= \sum_{J \subseteq J' \wedge J \neq \emptyset} (-1)^{1+|J|} q_\times(H \in J \wedge E = 1) \end{aligned}$$

Note that the quantities $\text{Bel}_\times(H \in \emptyset \wedge E = 1)$ and $\text{Pl}_\times(H \in \emptyset \wedge E = 1)$ default to 0.

The expression $\sum_{j \in J'} \prod_{i=1}^N m_i(\{j\})$ is non-zero if and only if there exists some $j \in J'$ for which $m_i(\{j\}) > 0$ for all $i = 1, 2, \dots, N$. For this approach to general fusion using Dempster-Shafer models, masses assigned to singleton elements of $\text{Set}(H)$ are important for the creation of non-trivial Dempster-Shafer models.

The computational complexity of approach #2 is $O(N \cdot 2^{2M})$.

As an example of general fusion using approach #2 for Dempster-Shafer models, the same example used for point probabilities in section III-C will be used. This time however, a ± 0.05 margin will be included to form each Dempster-Shafer model:

$$\begin{array}{lll} m_1(\{1\}) = 0.40 & m_1(\{2\}) = 0.50 & m_1(\{1, 2\}) = 0.10 \\ m_2(\{1\}) = 0.55 & m_2(\{2\}) = 0.35 & m_2(\{1, 2\}) = 0.10 \\ m_3(\{1\}) = 0.05 & m_3(\{2\}) = 0.85 & m_3(\{1, 2\}) = 0.10 \end{array}$$

The posterior probability distribution for H , the common value, is:

$$m_\bullet(\{1\}) \approx 0.0411 \quad m_\bullet(\{2\}) \approx 0.7532 \quad m_\bullet(\{1, 2\}) \approx 0.2057$$

By comparison with the example from III-C, it can be seen that the containment property is holding.

D. Dempster's Rule of Combination

Dempster's rule of combination performs general fusion of Dempster Shafer models. Dempster's rule of combination, described in [Yager(1987)], proceeds as follows:

$$\forall J' \in \text{Set}(H) : m_\bullet(J') = \frac{1}{K} \sum_{\substack{J_1, J_2, \dots, J_N \in \text{Set}(H) \\ J_1 \cap J_2 \cap \dots \cap J_N = J'}} m_1(J_1) m_2(J_2) \dots m_N(J_N)$$

where K is a normalization constant that ensures that $\sum_{J' \in \text{Set}(H)} m_\bullet(J') = 1$.

As will be shown in the following example, Dempster's rule of combination fails to satisfy the containment property. The example is from [Eastwood and Yanushkevich(2016)].

As an example of Dempster's rule of combination failing to satisfy the containment property, let $N = 2$ and $M = 2$. Let Dempster-Shafer model S_1 be defined by:

$$m_1(\{1\}) = 0.1 \quad m_1(\{2\}) = 0.1 \quad m_1(\{1, 2\}) = 0.8$$

Let Dempster-Shafer model S_2 be the same: $S_2 = S_1$.

Dempster's rule of combination gives the following resultant Dempster-Shafer model S_\bullet :

$$m_\bullet(\{1\}) \approx 0.1735 \quad m_\bullet(\{2\}) \approx 0.1735 \quad m_\bullet(\{1, 2\}) \approx 0.6531$$

Now consider probability distribution $\text{Pr}_1 \in S_1$:

$$\text{Pr}_1(1) = 0.1 \quad \text{Pr}_1(2) = 0.9$$

Let probability distribution $\text{Pr}_2 \in S_2$ be the same: $\text{Pr}_2 = \text{Pr}_1$.

Fusing Pr_1 and Pr_2 gives Pr_\bullet :

$$\text{Pr}_\bullet(1) \approx 0.0122 \quad \text{Pr}_\bullet(2) \approx 0.9878$$

It is readily apparent that $\text{Pr}_\bullet \notin S_\bullet$, which violates the containment property for general fusion. This example demonstrates that Dempster's rule of combination violates the containment property for general fusion.

Due to the fact that Dempster's rule of combination fails to satisfy the containment property, general fusion approach #2 is proposed as an alternative to Dempster's rule of combination.

VII. CONCLUSION

This paper has given a taxonomy of approaches to both context specific and general fusion using both probability interval distributions and Dempster-Shafer models. Fusion approaches that were covered include:

- Point probability distributions:
 - Context specific fusion (section III-B): The computational complexity is $O(NM)$.
 - General fusion (section III-C): The computational complexity is $O(NM)$.
- Probability Interval Distributions:
 - Context specific fusion (section V-B): The computational complexity is $O(NM + M^2)$, and the posterior is maximally tight.
 - General fusion approach #1 (section V-C1): The computational complexity is $O(NM + Mf(N, M - 1))$, and the posterior is maximally tight ($f(n, m)$ is the complexity of problem 2).
 - General fusion approach #2 (section V-C2): The computational complexity is $O(NM)$, and the posterior is not maximally tight.
- Dempster-Shafer models:
 - Context specific fusion (section VI-B): The computational complexity is $O(NM + 2^{2M})$, and the posterior is maximally tight.
 - General fusion approach #1 (section VI-C1): The computational complexity is $O(N \cdot 2^{2M} + 2^M g(N, M))$, and the posterior is maximally tight ($g(n, m)$ is the complexity of problem 3).
 - General fusion approach #2 (section VI-C2): The computational complexity is $O(N \cdot 2^{2M})$, and the posterior is not maximally tight.

The containment property, which requires that the fusion of any choice of point probability distributions be contained in the resultant credal set, is presented as an objective requirement that all fusion approaches should satisfy. Dempster’s rule of combination is shown to not satisfy the containment property (see section VI-D).

Credal sets are convex sets of probability distributions, and a typical approach to denoting credal sets is to list their extreme points. It has been shown in [Karlsson et al.(2011)Karlsson, Johansson and Andler] that context specific fusion and general fusion can be exactly and computationally efficiently performed by listing the extreme points of credal sets. Exact fusion requires that the containment property holds and that the resultant model is tight. This at first seems to imply that listing the extreme points of credal sets are the optimal approach to describing convex sets of probability distributions. This paper shows however, that representing probability interval distributions and Dempster-Shafer models using lists of their extreme points can lead to excessive memory requirements and poor computational efficiency. Therefore, this paper proposes probability intervals and Dempster-Shafer models as a computationally tractable alternative to the listing of extreme points.

Unlike listing extreme points, context specific and general fusion using probability interval distributions and Dempster-Shafer models can rarely be performed exactly. Moreover, probability intervals and Dempster-Shafer models lack the expressive power to denote a tight posterior credal set. All approaches to fusion proposed here satisfy the containment property, and the approaches presented have varying levels of speed and accuracy.

There are many directions for future work. Problems 2 and 3 can be further investigated for more accurate and computationally efficient algorithms despite problem 2 being NP-hard. The algorithms for context specific fusion and general fusion can be generalized to “credal networks” (existing work on credal networks can be found in [Antonucci et al.(2013a)Antonucci, De Campos, Huber and Zaffalon], [Antonucci et al.(2013b)Antonucci, Huber, Zaffalon, Luginbuhl, Chapman and Ladouceur], [Cano et al.(2007)Cano, Gomez, Moral and Abellan], [Cozman(2005)]). In addition, the presented approaches can be investigated for specific applications of sensor fusion.

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APPENDIX

The satisfiability problem (SAT) from propositional logic is known to be NP-complete by the Cook-Levin theorem [Sipser(2006), pg. 276].

The formulation of the SAT problem given in [Sipser(2006), pg. 271] is:

Problem 4: Satisfiability (SAT) formulation 1

Input A propositional formula ϕ of length m , with at most n binary propositional variables.

Output A binary yes/no that indicates if there exists an assignment to the n propositional variables such that ϕ evaluates to “true”.

Here however, an alternate formulation of the SAT problem is used that is equivalent to the first formulation:

Problem 5: Satisfiability (SAT) formulation 2

Input A set of n binary propositional variables $x_1, x_2, \dots, x_n \in \{0, 1\}$.

A set of m clauses $\phi_1, \phi_2, \dots, \phi_m$. Each clause is a disjunction: $\phi_j = l_{j,1} \vee l_{j,2} \vee \dots \vee l_{j,n}$ where $l_{j,i}$ is either F (false); x_i ; or $\neg x_i$.

Output A binary yes/no that indicates if there exists an assignment to the n propositional variables such that every ϕ_j evaluates to “true”.

Both formulations of SAT are polynomial time reducible to each other. Formulation 2 can be envisioned as a specific instance of formulation 1 with $\phi = \phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_m$, and so formulation 2 is readily polynomial time reducible to formulation 1. Formulation 1 can be reduced to formulation 2 in polynomial time via the following process: given the expression tree for ϕ , an extra propositional variable can be created for each interior node. A node’s dependence on its children can be encoded via a small set of disjunctive clauses. Hence, the condition that ϕ return true can be encoded by a set of disjunctive clauses that can be generated in polynomial time. This set of disjunctive clauses constitutes the polynomial time reduction of formulation 1 to formulation 2.

To establish that problem 2 is NP-hard, it is sufficient to show that SAT (formulation 2) is polynomial time reducible to problem 2. Polynomial time reducible means that SAT can be solved in polynomial time provided that a polynomial time algorithm exists for problem 2. SAT can be solved by problem 2 in the following manner:

Start with the input to SAT: A set of n binary propositional variables $x_1, x_2, \dots, x_n \in \{0, 1\}$, and a set of m clauses $\phi_1, \phi_2, \dots, \phi_m$.

SAT is solved via problem 2 by the following algorithm:

```

 $n' \leftarrow n + m$ 
 $m' \leftarrow 2n$ 
for  $i' = 1$  to  $n$  do
  for  $i = 1$  to  $n$  do
    if  $i = i'$  then
       $a_{i',2i-1} \leftarrow 0$  and  $a_{i',2i} \leftarrow 0$ 
    else
       $a_{i',2i-1} \leftarrow 1$  and  $a_{i',2i} \leftarrow 1$ 
    end if
     $b_{i',2i-1} \leftarrow 1$  and  $b_{i',2i} \leftarrow 1$ 
  end for
 $c_{i'} \leftarrow 2n - 1$ 
end for
for  $j = 1$  to  $m$  do
  for  $i = 1$  to  $n$  do
    if  $l_{j,i} \equiv x_i$  then
       $a_{n+j,2i-1} \leftarrow 0$  and  $a_{n+j,2i} \leftarrow 1$ 
    else if  $l_{j,i} \equiv \neg x_i$  then
       $a_{n+j,2i-1} \leftarrow 1$  and  $a_{n+j,2i} \leftarrow 0$ 
    else
       $a_{n+j,2i-1} \leftarrow 1$  and  $a_{n+j,2i} \leftarrow 1$ 
    end if

```

```

     $b_{n+j,2i-1} \leftarrow 1$  and  $b_{n+j,2i} \leftarrow 1$ 
  end for
   $c_{n+j} \leftarrow 2n - 1$ 
end for
Solve problem 2 with the values of  $n = n'$ ,  $m = m'$ ,  $a_{i,j}$ ,  $b_{i,j}$ ,  $c_i$ , and use maximization. Assign the
result to  $r$ .
if  $r \geq n$  then
  return: yes ( $\phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_m$  is satisfiable)
else
  return: no ( $\phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_m$  is unsatisfiable)
end if

```

Figure 3 depicts the use of problem 2 to solve the SAT problem involving the clauses: $\phi_1 = x_1 \vee x_2 \vee x_3$; $\phi_2 = F \vee x_2 \vee \neg x_3$; $\phi_3 = \neg x_1 \vee \neg x_2 \vee F$; and $\phi_4 = \neg x_1 \vee \neg x_2 \vee \neg x_3$. Note that problem 2 is optimized by a “corner state”, wherein the parameters do not take on intermediate values. For each of the top n rows, one of x_i or $\neg x_i$ is chosen to be true by forcing the corresponding row entry to 0. The product of the corresponding column is forced to 0. The products of at least n columns are 0, so the sum of products is at most n . For each of the bottom m rows, a supporting literal for clause ϕ_j is chosen by again forcing the corresponding row entry to 0. If the chosen supporting literal does not match the choice of x_i 's in the top n rows, then another column has a 0 product and the sum of products falls below n . If the clauses are all simultaneously satisfiable, then there exists a choice of assignments to each x_i and a choice of supporting literal for each ϕ_j so that the product of n columns is 1, and the sum of products attains a maximum of n .

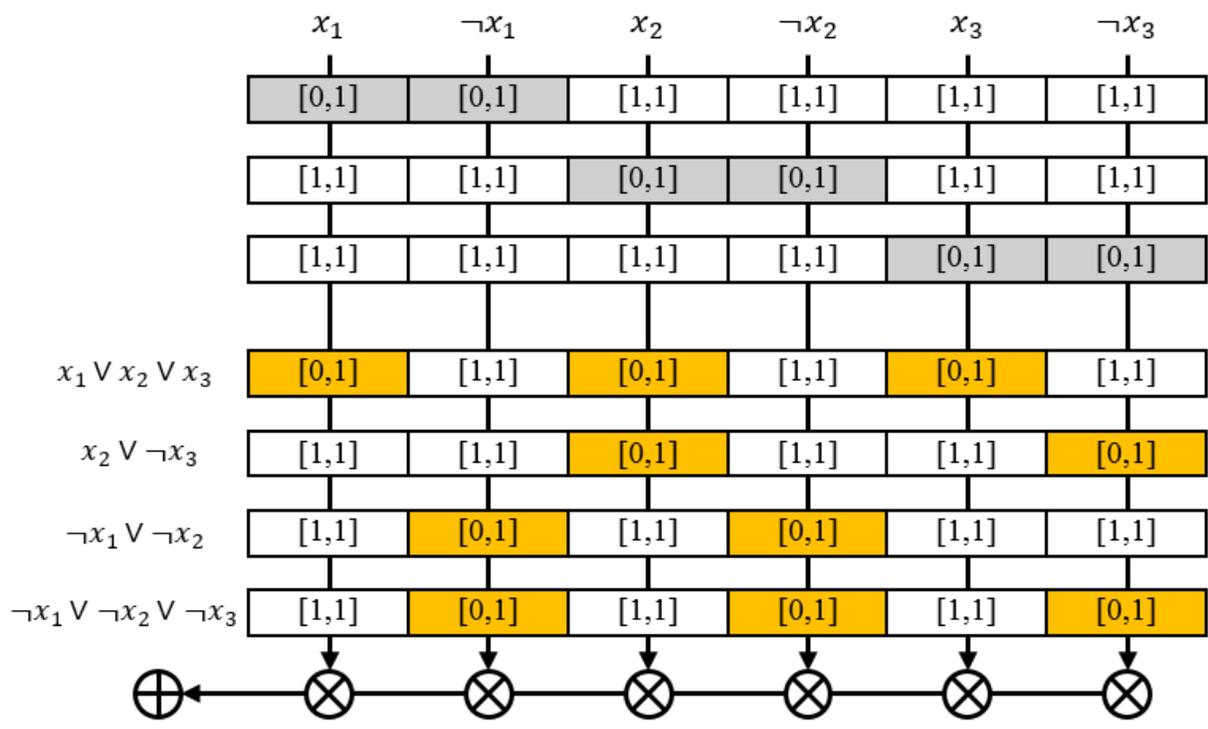


Fig. 3. A visual depiction of setting up problem 2 to solve the SAT problem.