

Extension of Inagaki General Weighted Operators
and
A New Fusion Rule Class of Proportional Redistribution of Intersection
Masses

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Abstract.

In this paper we extend Inagaki Weighted Operators fusion rule (WO) [see 1, 2] in information fusion by doing redistribution of not only the conflicting mass, but also of masses of non-empty intersections, that we call Double Weighted Operators (DWO).

Then we propose a new fusion rule Class of Proportional Redistribution of Intersection Masses (CPRIM), which generates many interesting particular fusion rules in information fusion.

Both formulas are presented for 2 and for $n \geq 3$ sources.

An application and comparison with other fusion rules are given in the last section.

Keywords: Inagaki Weighted Operator Rule, fusion rules, proportional redistribution rules, DSm classic rule, DSm cardinal, Smarandache codification, conflicting mass

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1. Introduction.

Let $\theta = \{\theta_1, \theta_2, \dots, \theta_n\}$, for $n \geq 2$, be the frame of discernment, and $S^\theta = (\theta, \cup, \cap, \tau)$ its super-power set, where $\tau(x)$ means complement of x with respect to the total ignorance.

Let $I_t =$ total ignorance $= \theta_1 \cup \theta_2 \cup \dots \cup \theta_n$, and Φ be the empty set.

$S^\theta = 2^{\wedge \theta_{\text{refined}}} = 2^{\wedge (2^\wedge \theta)} = D^{\theta \cup \theta^c}$, when refinement is possible, where $\theta^c = \{\tau(\theta_1), \tau(\theta_2), \dots, \tau(\theta_n)\}$.

We consider the general case when the domain is S^θ , but S^θ can be replaced by $D^\theta = (\theta, \cup, \cap)$ or by $2^\theta = (\theta, \cup)$ in all formulas from below.

Let $m_1(\cdot)$ and $m_2(\cdot)$ be two normalized masses defined from S^θ to $[0, 1]$.

We use the conjunction rule to first combine $m_1(\cdot)$ with $m_2(\cdot)$ and then we redistribute the mass of $m(X \cap Y) \neq 0$, when $X \cap Y = \Phi$.

Let's denote $m_{2\cap}(A) = (m_1 \oplus m_2)(A) = \sum_{\substack{X, Y \in S^\theta \\ (X \cap Y) = A}} m_1(X)m_2(Y)$ using the conjunction rule.

Let's note the set of intersections by:

$$S_\cap = \left\{ \begin{array}{l} X \in S^\theta \mid X = y \cap z, \text{ where } y, z \in S^\theta \setminus \{\Phi\}, \\ X \text{ is in a canonical form, and} \\ X \text{ contains at least an } \cap \text{ symbol in its formula} \end{array} \right\}. \quad (1)$$

In conclusion, S_\cap is a set of formulas formed with singletons (elements from the frame of discernment), such that each formula contains at least an intersection symbol \cap , and each formula is in a canonical form (easiest form).

For example: $A \cap A \notin S_\cap$ since $A \cap A$ is not a canonical form, and $A \cap A = A$. Also, $(A \cap B) \cap B$ is not in a canonical form but $(A \cap B) \cap B = A \cap B \in S_\cap$.

Let

$$S_\cap^\Phi = \text{the set of all empty intersections from } S_\cap,$$

and

$S_{\cap,r}^{non\Phi} = \{\text{the set of all non-empty intersections from } S_\cap^{non\Phi} \text{ whose masses are redistributed to other sets, which actually depends on the sub-model of each application}\}.$

2. Extension of Inagaki General Weighted Operators (WO).

Inagaki general weighted operator (WO) is defined for two sources as:

$$\forall A \in 2^\theta \setminus \{\Phi\}, m_{(WO)}(A) = \sum_{\substack{X, Y \in 2^\theta \\ (X \cap Y) = A}} m_1(X)m_2(Y) + W_m(A) \cdot m_{2\cap}(\Phi), \quad (2)$$

where

$$\sum_{X \in 2^\theta} W_m(X) = 1 \text{ and all } W_m(\cdot) \in [0, 1]. \quad (3)$$

So, the conflicting mass is redistributed to non-empty sets according to these weights $W_m(\cdot)$.

In the extension of this WO, which we call the Double Weighted Operator (DWO), we redistribute not only the conflicting mass $m_{2\cap}(\Phi)$ but also the mass of some (or all) non-empty intersections, i.e. those from the set $S_{\cap,r}^{non\Phi}$, to non-empty sets from S^θ according to some weights $W_m(\cdot)$ for the conflicting mass (as in WO), and respectively according to the weights $V_m(\cdot)$ for the non-conflicting mass of the elements from the set $S_{\cap,r}^{non\Phi}$:

$$\forall A \in (S^\theta \setminus S_{\cap,r}^{non\Phi}) \setminus \{\Phi\}, m_{DWO}(A) = \sum_{\substack{X, Y \in S^\theta \\ (X \cap Y) = A}} m_1(X)m_2(Y) + W_m(A) \cdot m_{2\cap}(\Phi) + V_m(A) \cdot \sum_{z \in S_{\cap,r}^{non\Phi}} m_{2\cap}(z),$$

(4)

where

$$\sum_{X \in S^\theta} W_m(X) = 1 \text{ and all } W_m(\cdot) \in [0,1], \text{ as in (3)}$$

and

$$\sum_{z \in S_{\cap,r}^{non\Phi}} V_m(z) = 1 \text{ and all } V_m(\cdot) \in [0,1]. \quad (5)$$

In the free and hybrid modes, if no non-empty intersection is redistributed, i.e. $S_{\cap,r}^{non\Phi}$ contains no elements, DWO coincides with WO .

In the Shafer's model, always DWO coincides with WO .

For $s \geq 2$ sources, we have a similar formula:

$$\forall A \in (S^\theta \setminus S_{\cap,r}^{non\Phi}) \setminus \{\Phi\}, m_{DWO}(A) = \sum_{\substack{X_1, X_2, \dots, X_s \in S^\theta \\ \bigcap_{i=1}^s X_i = A}} \prod_{i=1}^s m_i(X_i) + W_m(A) \cdot m_{s\cap}(\Phi) + V_m(A) \cdot \sum_{z \in S_{\cap,r}^{non\Phi}} m_{s\cap}(z) \quad (6)$$

with the same restrictions on $W_m(\cdot)$ and $V_m(\cdot)$.

3. A Fusion Rule Class of Proportional Redistribution of Intersection Masses

For $A \in (S^\theta \setminus S_{\cap,r}^{non\Phi}) \setminus \{\Phi, I_i\}$ for two sources we have:

$$m_{CPRIM}(A) = m_{2\cap}(A) + f(A) \cdot \frac{\sum_{\substack{X, Y \in S^\theta \\ \{\Phi = X \cap Y \text{ and } A \subseteq M\} \\ \text{or } \{\Phi \neq X \cap Y \in S_{\cap,r}^{non\Phi} \text{ and } A \subseteq N\}}} m_1(X)m_2(Y)}{\sum_{z \subseteq M} f(z)}, \quad (7)$$

where $f(X)$ is a function directly proportional to X , $f : S^\theta \rightarrow [0, \infty]$. (8)

For example, $f(X) = m_{2\cap}(X)$, or (9)

$$f(X) = \text{card}(X), \text{ or}$$

$$f(X) = \frac{\text{card}(X)}{\text{card}(M)} \text{ (ratio of cardinals), or}$$

$$f(X) = m_{2\cap}(X) + \text{card}(X), \text{ etc.};$$

and M is a subset of S^θ , for example: (10)

$$M = \tau(X \cup Y), \text{ or}$$

$$M = (X \cup Y), \text{ or}$$

$$M \text{ is a subset of } X \cup Y, \text{ etc.},$$

where N is a subset of S^θ , for example: (11)

$$N = X \cup Y, \text{ or}$$

$$N \text{ is a subset of } X \cup Y, \text{ etc.}$$

And

$$m_{CPRIM}(I_t) = m_{2\cap}(I_t) + \sum_{X,Y \in S^\theta} m_1(X)m_2(Y) \cdot \left\{ X \cap Y = \Phi \text{ and } (M = \Phi \text{ or } \sum_{z \subseteq M} f(z) = 0) \right\} \quad (12)$$

These formulas are easily extended for any $s \geq 2$ sources $m_1(\cdot), m_2(\cdot), \dots, m_s(\cdot)$.

Let's denote, using the conjunctive rule:

$$m_{s\cap}(A) = (m_1 \oplus m_2 \oplus \dots \oplus m_s)(A) = \sum_{\substack{X_1, X_2, \dots, X_s \in S^\theta \\ \bigcap_{i=1}^s X_i = A}} \prod_{i=1}^s m_i(X_i) \quad (13)$$

$$m_{CPRIM}(A) = m_{s\cap}(A) + f(A) \cdot \sum_{\substack{X_1, X_2, \dots, X_s \in S^\theta \\ \left\{ \begin{array}{l} \Phi = \bigcap_{i=1}^s X_i \text{ and } A \subseteq M \\ \text{or} \\ \Phi \neq \bigcap_{i=1}^s X_i \in S_{\cap, r}^{non\Phi} \text{ and } A \subseteq N \end{array} \right\}}} \frac{\prod_{i=1}^s m_i(X_i)}{\sum_{z \subseteq M} f(z) \neq 0} \quad (14)$$

where $f(\cdot)$, M , and N are similar to the above where instead of $X \cup Y$ (for two sources) we take $X_1 \cup X_2 \cup \dots \cup X_s$ (for s sources), and instead of $m_{2\cap}(X)$ for two sources we take $m_{s\cap}(X)$ for s sources.

4. Application and Comparison with other Fusion Rules.

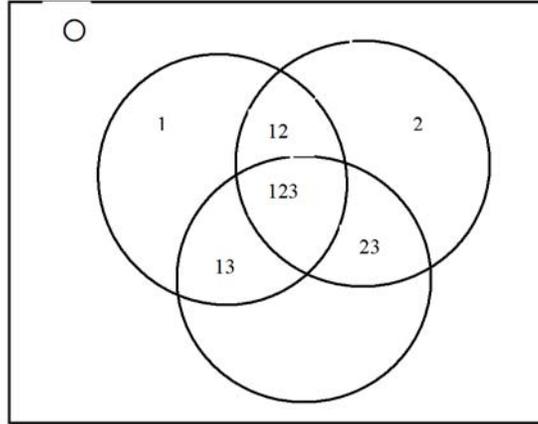
Let's consider the frame of discernment $\Theta = \{A, B, C\}$, and two independent sources $m_1(\cdot)$ and $m_2(\cdot)$ that provide the following masses:

| | A | B | C | $A \cup B \cup C$ |
|--------------|-----|-----|-----|-------------------|
| $m_1(\cdot)$ | 0.3 | 0.4 | 0.2 | 0.1 |
| $m_2(\cdot)$ | 0.5 | 0.2 | 0.1 | 0.2 |

Now, we apply the conjunctive rule and we get:

| | A | B | C | $A \cup B \cup C$ | $A \cap B$ | $A \cap C$ | $B \cap C$ |
|---------------------|------|------|------|-------------------|------------|------------|------------|
| $m_{12\cap}(\cdot)$ | 0.26 | 0.18 | 0.07 | 0.02 | 0.26 | 0.13 | 0.08 |

Suppose that all intersections are non-empty {this case is called: free DSm (Dezert-Smarandache) Model}. See below the Venn Diagram using the Smarandache codification [3]:



Applying DSm Classic rule, which is a generalization of classical conjunctive rule from the fusion space (Θ, \cup) , called *power set*, when all hypotheses are supposed exclusive (i.e. all intersections are empty) to the fusion space (Θ, \cup, \cap) , called *hyper-power set*, where hypotheses are not necessarily exclusive (i.e. there exist non-empty intersections), we just get:

| | A | B | C | $A \cup B \cup C$ | $A \cap B$ | $A \cap C$ | $B \cap C$ |
|-------------------|------|------|------|-------------------|------------|------------|------------|
| $m_{DSmC}(\cdot)$ | 0.26 | 0.18 | 0.07 | 0.02 | 0.26 | 0.13 | 0.08 |

DSmC and the Conjunctive Rule have the same formula, but they work on different fusion spaces.

Inagaki rule was defined on the fusion space (Θ, \cup) . In this case, since all intersections are empty, the total conflicting mass, which is $m_{12\cap}(A \cap B) + m_{12\cap}(A \cap C) + m_{12\cap}(B \cap C) = 0.26 + 0.13 + 0.08 = 0.47$, and this is redistributed to the masses of A, B, C, and $A \cup B \cup C$ according to some weights w_1, w_2, w_3 , and w_4 respectively, depending to each particular rule, where:

$0 \leq w_1, w_2, w_3, w_4 \leq 1$ and $w_1 + w_2 + w_3 + w_4 = 1$. Hence

| | A | B | C | $A \cup B \cup C$ |
|----------------------|------------------|------------------|------------------|-------------------|
| $m_{Inagaki}(\cdot)$ | $0.26 + 0.47w_1$ | $0.18 + 0.47w_2$ | $0.07 + 0.47w_3$ | $0.02 + 0.47w_4$ |

Yet, Inagaki rule can also be straightly extended from the power set to the hyper-power set.

Suppose in DWO the user finds out that the hypothesis $B \cap C$ is not plausible, therefore $m_{12\cap}(B \cap C) = 0.08$ has to be transferred to the other non-empty elements: A, B, C, $A \cup B \cup C$, $A \cap B$, $A \cap C$, according to some weights v_1, v_2, v_3, v_4, v_5 , and v_6 respectively, depending to the particular version of this rule is chosen, where:

$0 \leq v_1, v_2, v_3, v_4, v_5, v_6 \leq 1$ and $v_1 + v_2 + v_3 + v_4 + v_5 + v_6 = 1$. Hence

| | A | B | C | $A \cup B \cup C$ | $A \cap B$ | $A \cap C$ |
|------------------|------------------|------------------|------------------|-------------------|------------------|------------------|
| $m_{DWO}(\cdot)$ | $0.26 + 0.08v_1$ | $0.18 + 0.08v_2$ | $0.07 + 0.08v_3$ | $0.02 + 0.08v_4$ | $0.26 + 0.08v_5$ | $0.13 + 0.08v_6$ |

Now, since CPRIM is a particular case of DWO, but CPRIM is a class of fusion rules, let's consider a sub-particular case for example when the redistribution of $m_{12\cap}(B \cap C) = 0.08$ is done proportionally with respect to the DSm cardinals of B and C which are both equal to 4. DSm

cardinal of a set is equal to the number of disjoint parts included in that set upon the Venn Diagram (see it above).

Therefore 0.08 is split equally between B and C, and we get:

| | A | B | C | $A \cup B \cup C$ | $A \cap B$ | $A \cap C$ |
|-----------------------|------|------------------|------------------|-------------------|------------|------------|
| $m_{CPRIM_{card}}(.)$ | 0.26 | $0.18+0.04=0.22$ | $0.07+0.04=0.11$ | 0.02 | 0.26 | 0.13 |

Applying one or another fusion rule is still debating today, and this depends on the hypotheses, on the sources, and on other information we receive.

5. Conclusion.

A generalization of Inagaki rule has been proposed in this paper, and also a new class of fusion rules, called **Class of Proportional Redistribution of Intersection Masses (CPRIM)**, which generates many interesting particular fusion rules in information fusion.

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