ERDOS-SMARANDACHE MOMENTS NUMBERS

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The starting point of this article is represented by a recent work of Finch [2000]. Based on two asymptotic results concerning the Erdos function, he proposed some interesting equation concerning the moments of the Smarandache function. The aim of this note is give a bit modified proof and to show some computation results for one of the Finch equation. We will call the numbers obtained from computation '*Erdos-Smarandache Moments Number*'. The Erdos-Smarandache moment number of order 1 is obtained to be the Golomb-Dickman constant.

1. INTRODUCTION

We briefly present the results used in this article. These concern the relationship between the Smarandache and the Erdos functions and some asymptotic equations concerning them. These are important functions in Number Theory defined as follows:

- The Smarandache function [Smarandache, 1980] is $S: N^* \to N$,

$$S(n) = \min\{k \in N | k!\, \mathrm{X} n\} \ (\forall n \in N^*) . \qquad (1)$$

- The Erdos function is $P : N^* \to N$,

$$P(n) = \min\{ p \in N \mid n \mathrm{X} p \wedge p \text{ is prim}\} (\forall n \in N^* \setminus \{1\}),\ P(1) = 0 . \qquad (2)$$

Their main properties are:

$$(\forall a, b \in N^*)\ (a,b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\},\ P(a \cdot b) = \max\{P(a), P(b)\} . \qquad (3)$$

$$(\forall a \in N^*)\ P(a) \le S(a) \le a \text{ and the equalities occur } \textit{iif} \text{ a is prim.} \qquad (4)$$

An important equation between these functions was found by Erdos [1991]

$$\lim_{n\to\infty} \frac{\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right|}{n} = 0 , \qquad (5)$$

which was extended by Ford [1999] to

$$\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right| = n \cdot e^{-\left(\sqrt{2} + a_n\right) \cdot \sqrt{\ln n \cdot \ln \ln n}} , \text{ where } \lim_{n\to\infty} a_n = 0 . \qquad (6)$$

Equations (5-6) are very important because create a similarity between these functions especially for asymptotic properties. Moreover, these equations allow us to translate convergence properties on the Smarandache function to convergence properties on the Erdos function and vice versa. The main important equations that have been obtained by this translation are presented in the following.

THE AVERAGE VALUES

$\frac{1}{n}\sum_{i=2}^{n} S(i) = O(\frac{n}{\log n})$ [Luca, 1999] $\frac{1}{n}\sum_{i=2}^{n} P(i) = O(\frac{n}{\log n})$ [Tabirca, 1999] and their generalizations

$$\frac{1}{n}\sum_{i=2}^{n} P^{a}(i) = \frac{z\,(a+1)}{a+1}\cdot\frac{n^{a}}{\ln(n)} + O\left(\frac{n^{a}}{\ln^{2}(n)}\right)$$ [Knuth and Pardo 1976]

$$\frac{1}{n}\sum_{i=2}^{n} S^{a}(i) = \frac{z\,(a+1)}{a+1}\cdot\frac{n^{a}}{\ln(n)} + O\left(\frac{n^{a}}{\ln^{2}(n)}\right)$$ [Finch, 2000]

THE HARMONIC SERIES

$\lim_{n\to\infty}\sum_{i=2}^{n}\frac{1}{S^{a}(i)} = \infty$ [Luca, 1999] $\lim_{n\to\infty}\sum_{i=2}^{n}\frac{1}{P^{a}(i)} = \infty$ [Tabirca, 1999]

THE LOG-AVERAGE VALUES

$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\frac{\ln P(i)}{\ln i} = l$ [Kastanas, 1994] $\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\frac{\ln S(i)}{\ln i} = l$ [Finch, 1999] and their generalizations

$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln P(i)}{\ln i}\right)^{a} = l_{a}$ [Shepp, 1964] $\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln S(i)}{\ln i}\right)^{a} = l_{a}$ [Finch, 2000].

2. THE ERDOS-SMARANDACHE MOMENT NUMBERS

From a combinatorial study of random permutation Sheep and Lloyd [1964] found the following integral equation

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln P(i)}{\ln i}\right)^{a} = \int_{0}^{\infty}\frac{x^{a-1}}{a!}\cdot\exp\left(-x-\int_{-x}^{\infty}\frac{\exp(-y)}{y}\,dy\right)dx := l_{a}\,. \tag{7}$$

Finch [2000] started from Equation (7) and translated it from the Samrandache function.

Theorem [Finch, 2000] *If a is a positive number then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln S(i)}{\ln i}\right)^{a} = l_{a}\,. \tag{8}$$

Proof

Many terms of the difference $\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln S(i)}{\ln i}\right)^{a}-\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln P(i)}{\ln i}\right)^{a}$ are equal, therefore there will be reduced. This difference is transformed as follows:

$$\left|\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln S(i)}{\ln i}\right)^{a}-\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln P(i)}{\ln i}\right)^{a}\right|=\frac{1}{n}\cdot\left|\sum_{i=2}^{n}\left[\left(\frac{\ln S(i)}{\ln i}\right)^{a}-\left(\frac{\ln P(i)}{\ln i}\right)^{a}\right]\right|=$$

$$=\frac{1}{n}\cdot\left|\sum_{i:S(i)>P(i)}\left[\left(\frac{\ln S(i)}{\ln i}\right)^{a}-\left(\frac{\ln P(i)}{\ln i}\right)^{a}\right]\right|\leq\frac{1}{n}\cdot\sum_{i:S(i)>P(i)}\frac{\left|\ln^{a} S(i)-\ln^{a} P(i)\right|}{\ln^{a} i}\leq$$

$$\leq\frac{1}{n}\cdot\sum_{i:S(i)>P(i)}\frac{\left|\ln^{a} S(i)-\ln^{a} P(i)\right|}{\ln^{a} i}\leq\frac{1}{n}\cdot$$

n the following we will present a proof for the result The Erdos harmonic series can be defined Sgby $\sum_{n\geq 2}\frac{1}{P^{a}(n)}$. This is one of the important series with the Erdos function and its convergence is studied starting from the convergence of the Smarandache harmonic series $\sum_{n\geq 2}\frac{1}{S^{a}(n)}$. Some results concerning series with the function *S* are reviewed briefly in the following:

- If $(x_n)_{n>0}$ is an increasing sequence such that $\lim_{n\to\infty} x_n=\infty$, then the series $\sum_{n>1}\frac{x_{n+1}-x_n}{S(x_n)}$ is divergent. [Cojocaru, 1997]. (7)
- The series $\sum_{n\geq 2}\frac{1}{S^{2}(n)}$ is divergent. [Tabirca, 1998] (8)
- The series $\sum_{n\geq 2}\frac{1}{S^{a}(n)}$ is divergent for all *a*>0. [Luca, 1999] (9)

These above results are translated to the similar properties on the Erdos function.

Theorem 1. If $(x_n)_{n>0}$ is an increasing sequence such that $\lim_{n\to\infty} x_n=\infty$, then the series $\sum_{n>1}\frac{x_{n+1}-x_n}{P(x_n)}$ is divergent.

Proof The proof is obvious based on the equation $P(x_n) \le S(x_n)$. Therefore, the equation $\frac{x_{n+1}-x_n}{P(x_n)} \ge \frac{x_{n+1}-x_n}{S(x_n)}$ and the divergence of the series $\sum_{n>1} \frac{x_{n+1}-x_n}{S(x_n)}$ give that the series $\sum_{n>1} \frac{x_{n+1}-x_n}{P(x_n)}$ is divergent. ♦

A direct consequence of Theorem 1 is the divergence of the series $\sum_{n>1} \frac{1}{P(a \cdot n+b)}$, where *a*,*b*>0 are positive numbers. This gives that $\sum_{n\ge 2} \frac{1}{P(n)}$ is divergent and moreover that $\sum_{n\ge 2} \frac{1}{P^a(n)}$ is divergent for all *a*<1.

Theorem 2. The series $\sum_{n\ge 2} \frac{1}{P^a(n)}$ is divergent for all *a*>1.

Proof The proof studies two cases.

Case 1. $a \ge \frac{1}{2}$.

In this case, the proof is made by using the divergence of $\sum_{n\ge 2} \frac{1}{S^a(n)}$.

Denote $A = \{i = \overline{2,n} \mid S(i) = P(i)\}$ and $B = \{i = \overline{2,n} \mid S(i) > P(i)\}$ a partition of the set $\{i = \overline{1,n}\}$. We start from the following simple transformation

$$\sum_{i=2}^{n} \frac{1}{P^a(i)} = \sum_{i=2}^{n} \frac{1}{S^a(i)} + \sum_{i\in B} \left[\frac{1}{P^a(i)} - \frac{1}{S^a(i)}\right] = \sum_{i=2}^{n} \frac{1}{S^a(i)} + \sum_{i\in B} \frac{S^a(i)-P^a(i)}{P^a(i)\cdot S^a(i)}. \tag{10}$$

An $i \in B$ satisfies $S^a(i) - P^a(i) \ge 1$ and $P(i) < S(i) \le n$ thus, (10) becomes

$$\sum_{i=2}^{n} \frac{1}{P^a(i)} \ge \sum_{i=2}^{n} \frac{1}{S^a(i)} + \sum_{i\in B} \frac{1}{n^{2\cdot a}} = \sum_{i=2}^{n} \frac{1}{S^a(i)} + \frac{1}{n^{2\cdot a}} \cdot |B|. \tag{11}$$

The series $\sum_{n\ge 2} \frac{1}{P^a(n)}$ is divergent because the series $\sum_{n\ge 2} \frac{1}{S^a(n)}$ is divergent and

$$\lim_{n\to\infty} \frac{|B|}{n^{2\cdot a}} = \lim_{n\to\infty} \frac{n \cdot e^{-(\sqrt{2}+a_n)\cdot\sqrt{\ln n \cdot \ln\ln n}}}{n^{2\cdot a}} = \lim_{n\to\infty} \frac{1}{n^{2\cdot a-1} \cdot e^{(\sqrt{2}+a_n)\cdot\sqrt{\ln n \cdot \ln\ln n}}} = 0 .$$

Case 2. $\frac{1}{2} > a > 1$.

The first case gives that the series $\sum_{n\geq 2}\frac{1}{P^{\frac{1}{2}}(n)}$ is divergent.

Based on $P^{\frac{1}{2}}(n) > P^{a}(n)$, the inequality $\sum_{i=2}^{n}\frac{1}{P^{a}(i)} > \sum_{i=2}^{n}\frac{1}{P^{\frac{1}{2}}(i)}$ is found. Thus, the series $\sum_{n\geq 2}\frac{1}{S^{a}(n)}$ is divergent. ♦

The technique that has been applied to the proof of Theorem 2 can be used in the both ways. Theorem 2 started from a property of the Smarandache function and found a property of the Erdos function. Opposite, Finch [1999] found the property $\lim_{n\to\infty}\frac{\sum_{i=2}^{n}\frac{\ln S(i)}{\ln i}}{n} = l$ based on the similar property $\lim_{n\to\infty}\frac{\sum_{i=2}^{n}\frac{\ln P(i)}{\ln i}}{n} = l$, where λ=0.6243299 is the Golomb-Dickman constant. Obviously, many other properties can be proved using this technique. Moreover, Equations (5-6) gives a very interesting fact - **"the Smarandache and Erdos functions may have the same behavior especially on the convergence problems."**

References

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