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$*$-Ricci Tensor on Generalized Sasakian-Space-Form

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Abstract: The aim of the paper is to study $*$-Ricci tensor in generalized Sasakian space form. We study the generalized Sasakian space form admitting $*$-conformal η- Ricci soliton and analyse the behaviour of the soliton. Also, we prove $*$-Ricci semi-symmetric and Pseudo $*$-Ricci semisymmetric generalized Sasakian space forms are $*$-Ricci flat.

Key Words: Einstein manifold, η-Einstein manifold, $*\eta$-Einstein manifold, Ricci soliton, $*$-Ricci tensor.

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§1. Introduction

The generalized Sasakian space forms have been investigated by numerous researchers like Alegre and Carriazo [6-8]. Thereafter generalized Sasakian spaceform [GSSF] have been studied by many authors [10, 16, 19, 36, 37].

An almost contact metric manifold M is a GSSF if there exist three functions f_1, f_2, f_3 on M such that curvature tensor R is given by

$$\begin{aligned} R(X_1, X_2)X_3 =& f_1\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\} + f_2\{g(X_1, \phi X_3)\phi X_2 \\ & - g(X_2, \phi X_3)\phi X_1 + 2g(X_1, \phi X_2)\phi X_3\} \\ & + f_3\{\eta(X_1)\eta(X_3)X_2 - \eta(X_2)\eta(X_3)X_1 \\ & + g(X_1, X_3)\eta(X_2)\xi - g(X_2, X_3)\eta(X_1)\xi\} \end{aligned} \tag{1.1}$$

for any vector fields X_1, X_2, X_3 on M. In such a case we represent the manifold as $M(f_1, f_2, f_3)$.

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If $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, then a GSSF with Sasakian structure develops Sasakian-space-forms.

A self-similizing elucidation to the Ricci flow [14,15] is said to be a Ricci soliton [42] in case it moves by only a one parameter family of diffeomorphism and scaling. Ricci soliton has been studied by many authors (See [1, 3, 9, 11, 12, 18,31C 34, 38, 39]) and is defined as:

$$L_V g + 2Ric + 2\lambda g = 0. \tag{1.2}$$

The η-Ricci soliton [12] and the Conformal η-Ricci soliton [26] are defined respectively as

$$L_V g + 2Ric = 2\lambda g + 2\mu\eta \otimes \eta, \tag{1.3}$$

$$L_V g + 2Ric + \left[2\lambda - \left(P + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0, \tag{1.4}$$

where L_V is the Lie derivative in the direction of V, Ric is the Ricci tensor, g is the Riemannian metric, V is a vector field, and λ and μ are parameters. The Ricci soliton is said to be shrinking, steady and expanding if λ is negative, zero and positive, respectively. Some related developments can be found in [1, 2, 4, 13, 20C24, 28C35, 38C41].

§2. **Preliminaries**

A $(2n+1)$-dim Riemannian manifold (M, g) is called an almost compact manifold if the following results hold [6]

$$-X_1 + \eta(X_1)\xi = \phi^2(X_1), \tag{2.1}$$

$$1 = \eta(\xi), \tag{2.2}$$

$$g(X_1, \xi) = \eta(X_1), \eta(\phi\xi) = 0, \tag{2.3}$$

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2), \tag{2.4}$$

$$g(X_1, \phi X_2) = -g(\phi X_1, X_2), \tag{2.5}$$

$$g(\phi X_1, X_1) = 0, \tag{2.6}$$

$$(\nabla_{X_1}\eta)(X_2) = g(\nabla_{X_1}\xi, X_2), \tag{2.7}$$

$$\nabla_{X_1}\xi = -\beta\phi X_1, \tag{2.8}$$

for all $X_1 \in TM$ and a function β such that $\xi\beta = 0$.

In view of (2.8), we get

$$(\nabla_{X_1}\eta)(X_2) = g(\nabla_{X_1}\xi, X_2) = -\beta g(\phi X_1, X_2). \tag{2.9}$$

For a $(GSSF)_{2n+1}$, we have

$$R(X_1, X_2)\xi = (f_1 - f_3)[\eta(X_2)X_1 - \eta(X_1)X_2], \tag{2.10}$$

$$R(\xi, X_1)X_2 = (f_1 - f_3)[g(X_1, X_2)\xi - \eta(X_2)X_1], \tag{2.11}$$

$$g(R(\xi, X_1)X_2, \xi) = (f_1 - f_3)g(\phi X_1, \phi X2), \tag{2.12}$$

$$R(\xi, X_1)\xi = (f_1 - f_3)\phi^2 X_1, \tag{2.13}$$

$$S(X_1, X_2) = (2nf_1 + 3f_2 - f_3)g(X_1, X_2) - (3f_2 + (2n-1)f_3)\eta(X_1)\eta(X_2), \tag{2.14}$$

where S is the Ricci tensor and r is the scalar curvature tensor of the space-forms.

§3. ∗-Ricci Tensor in GSSF

Let M be an GSSF with Ricci tensor S. The ∗-Ricci tensor and ∗-scalar curvature of M are defined by

$$S^*(X_1, X_2) = \sum_{i=1}^{2n+1} R(X_1, e_i, \phi e_i, \phi X_2), \qquad r^* = \sum_{i=1}^{2n+1} S^*(e_i, e_i) \tag{3.1}$$

for all $X_1, X_2 \in TM$, where $e_1, \cdots, e_{2n+1}$ is an orthonormal basis of the tangent space TM. By using the first Bianchi identity and (3.1) we get

$$S^*(X_1, X_2) = \frac{1}{2}\sum_{i=1}^{2n+1} g(\phi R(X_1, \phi X_2)e_i, e_i). \tag{3.2}$$

Let M is a GSSF, replace $X_3 = \phi X_3$ in (1.1) and taking inner product with ϕW, and then using (2.1) and (2.2) the resultant equation becomes

$$\begin{aligned} R(X_1, X_2, \phi X_3, \phi W) = {} & f_1\{g(X2, \phi X_3)g(X_1, \phi W) - g(X_1, \phi X_3)g(X_2, \phi W)\} \\ & + f_2\{-g(X_1, X_3)g(\phi X_2, \phi W) + \eta(X_1)\eta(X_3)g(\phi X_2, \phi W) \\ & + g(X_2, X_3)g(\phi X_1, \phi W) - \eta(X_2)\eta(X_3)g(\phi X_1, \phi W) \\ & - 2g(X_1, \phi X_2)g(X_3, \phi W)\}. \end{aligned} \tag{3.3}$$

Let $[e_i]_{i=1}^{2n+1}$ be an orthonormal basis of the tangent space at each point of the manifold. Then setting $X_2 = X_3 = e_i$ in (3.3) and proceeding summation over $1 \le i \le 2n+1$ and also by using (2.1) and (2.3), we get

$$R(X_1, e_i, \phi e_i, \phi W) = f_1 g(\phi X_1, \phi W) + f_2(2n+1)g(\phi X_1, \phi W). \tag{3.4}$$

Hence, we have the following result.

Theorem 3.1 *In a GSSF $M(f_1, f_2, f_3)$, the $*$-Ricci tensor is obtained by*

$$S^*(X_1, W) = [f_1 + (2n+1)f_2]g(X_1, W) - [f_1 + (2n+1)f_2]\eta(X_1)\eta(W). \tag{3.5}$$

The following corollary is immediate.

Corollary 3.1 *A GSSF $M(f_1, f_2, f_3)$ is an $*$-η-Einstein manifold.*

§4. $*$-Ricci Semisymmetric GSSF

A GSSF $M(f_1, f_2, f_3)$ is called Ricci semisymmetric if $R(X_1, X_2) \cdot S = 0$ for all $X_1, X_2 \in TM$. Similarly, we define $*$-Ricci semisymmetric by $R(X_1, X_2) \cdot S^* = 0$.

Let us consider a GSSF $M(f_1, f_2, f_3)$ that satisfies

$$R(X_1, X_2) \cdot S^* = 0. \tag{4.1}$$

From (4.1), we have

$$S^*(R(X_1, X2)U_1, V_1) + S^*(U_1, R(X_1, X_2)V_1) = 0. \tag{4.2}$$

Substituting $X_1 = U_1 = \xi$ we get

$$S^*(R(\xi, X_2)\xi, V_1) + S^*(\xi, R(\xi, X_2)V_1) = 0. \tag{4.3}$$

Using $S^* = 0$ and $S^*(X_1, \xi) = 0$ and (2.10) in (4.3) we obtain

$$(f_1 - f_3)S^*(X_2, V_1) = 0, \tag{4.4}$$

which gives either $(f_1 - f_3) \neq 0$ or $S^*(X_2, V_1) = 0$. Hence, we have the following theorem.

Theorem 4.1 *Let M be a GSSF is $*$-Ricci semisymmetric. Then either $f_1 \neq f_3$ or $M(f_1, f_2, f_3)$ is $*$-Ricci flat.*

§5. ϕ-Pseudo $*$-Ricci Symmetric GSSF

Definition 5.1 *A GSSF M is called ϕ-pseudo Ricci symmetric if the $*$-Ricci operator Q^* satisfies*

$$\phi^2((\nabla_{X_1} Q^*)(X_2)) = 2K(X_1)Q^*(X_2) + K(X_2)Q^* X_1 + S^*(X_2, X_1)\rho, \tag{5.1}$$

for any vector field X_1, X_2 where K is a non-zero 1-form.

If, in particular, $K = 0$ then manifold is called ϕ-$*$-Ricci symmetric [27]. Let us take a GSSF M, which is ϕ-pseudo $*$-Ricci symmetric. Then by virtue of (2.1), it follows from (5.1) that

$$-(\nabla_{X_1}Q^*)(X_2) + \eta(\nabla_{X_1}Q^*)(X_2)\xi = 2K(X_1)Q^*(X_2) + K(X_2)Q^*X_1 + S^*(X_2, X_1)\rho, \quad (5.2)$$

from which it follows that

$$\begin{aligned} -g((\nabla_{X_1}Q^*)(X_2), X_3) \quad &+ \quad S^*(Q\nabla_{X_1}X_2, X_3) + \eta((\nabla_{X_1}Q^*)(X_2))\eta(X_3) \\ &= \quad 2K(X_1)S^*(X_2, X_3) + K(X_2)S^*(X_1, X_3). \end{aligned} \quad (5.3)$$

Take $X_2 = \xi$ in (5.3) and use (2.8), (3.5) to get

$$-\beta S^*(\phi X_1, X_3) + \eta((\nabla_{X_1}Q^*)(\xi))\eta(X_3) = K(\xi)S^*(X_1, X_3). \quad (5.4)$$

Put $X_3 = \phi X_3$ in (5.4) and use (3.5)

$$-\beta Fg(\phi X_1, \phi X_3) = K(\xi)Fg(X_1, X_3). \quad (5.5)$$

By using (2.4) and (3.5), then contracting (5.5) on top of X_1 and X_3, we obtain

$$r^* = \frac{K(\xi)F(2n+1)}{\beta}. \quad (5.6)$$

Hence, we have the following theorem.

Theorem 5.1 *If GSSF M is a ϕ-pseudo $*$ Ricci symmetric, then*

$$r^* = \frac{K(\xi)F(2n+1)}{\beta}.$$

In particular, if $K = 0$, In view of (3.5) and (5.5), we obtain

$$\beta S^*(X_1, X_3) = 0. \quad (5.7)$$

Hence, we have the following corollary.

Corollary 5.1 *A ϕ-$*$-Ricci symmetric GSSF is $*$-Ricci flat provided $\beta \neq 0$.*

§6. ∗-Conformal η-Ricci Soliton in GSSF

Definition 6.1 *The $*$-Conformal η-Ricci soliton is defined as*

$$L_V g + 2Ric^* + \left[2\lambda - \left(P + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0, \quad (6.1)$$

where L_V is the Lie derivative along the vector field V, λ and μ are constants, Ric^* is the $*$-Ricci tensor, P is a scalar non- dynamical field and n is the dimension of manifold.

Let M be a GSSF admiting $*$-conformal η-Ricci soliton (g, v, λ, μ). When $V = \xi$ in (6.1),

$$L_\xi g(X_1, X_2) + 2S^*(X_1, X_2) + \left[2\lambda - \left(P + \frac{2}{n}\right)\right] g(X_1, X_2) + 2\mu\eta(X_1)\eta(X_2) = 0. \tag{6.2}$$

This can be written as

$$\begin{aligned} g(\nabla_{X_1}\xi, X_2) &+ g(X_1, \nabla_{X_2}\xi) + 2S^*(X_1, X_2) \\ &+ \left[2\lambda - \left(P + \frac{2}{n}\right)\right] g(X_1, X_2) + 2\mu\eta(X_1)\eta(X_2) = 0. \end{aligned} \tag{6.3}$$

Using (2.8) in (6.3), we get

$$\begin{aligned} g(-\beta\phi X_1, X_2) &+ g(X_1, -\beta\phi X_2) + 2S^*(X_{1,2}) \\ &+ \left[2\lambda - \left(P + \frac{2}{n}\right)\right] g(X_1, X_2) + 2\mu\eta(X_1)\eta(X_2) = 0. \end{aligned} \tag{6.4}$$

Since M is a GSSF, making use of (2.5) in (6.4), we know

$$S^*(X_1, X_2) = -\left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right] g(X_1, X_2) - \mu\eta(X_1)\eta(X_2). \tag{6.5}$$

From (6.5), we have the $*$-scalar curvature

$$r^* = -\left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right) + \mu\right](2n + 1),$$

which is a constant. In view of (3.5) and (6.5), we have

$$\begin{aligned} &\left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right] g(X_1, X_2) + \mu\eta(X_1)\eta(X_2) \\ &+[f_1 + (2n + 1)f_2]g(X_1, X_2) - [f_1 + (2n + 1)f_2]\eta(X)\eta(X_2) = 0. \end{aligned} \tag{6.6}$$

Using X_1 by ϕX_1 in (6.6), we have

$$\left\{(f_1 + (2n + 1)f_2) + \left(\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right)\right\} g(\phi X_1, X_2) = 0. \tag{6.7}$$

Interchanging X_1 and X_2 we obtain

$$\left\{(f_1 + (2n + 1)f_2) + \left(\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right)\right\} g(\phi X_2, X_1) = 0. \tag{6.8}$$

Solving (6.7) and (6.8), we get

$$\lambda = \frac{1}{2}\left(P + \frac{2}{n}\right) - F, \tag{6.9}$$

where $F = (f_1 + (2n + 1)f_2)$. Thus, we have the following theorem.

Theorem 6.1 *Let M be a GSSF admitting $*$-conformal η-Ricci soliton. Then, the nature of soliton is*

(1) *steady when* $P = 2F - \frac{2}{n}$;

(2) *expanding when* $P > 2F - \frac{2}{n}$;

(3) *shrinking when* $P < 2F - \frac{2}{n}$.

Making use of (6.9) in (6.5), the following corollary is immediate.

Corollary 6.1 *If a GSSF $M(f_1, f_2, f_3)$ admitting a $*$-conformal η-Ricci soliton, then M is Sasaki-$*$-η-Einstein.*

Setting $X_1 = \xi$ in (6.6), we get

$$\left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right) + \mu\right] \eta(X_2) = 0. \tag{6.10}$$

Take $X_2 = \xi$ in (6.10) we obtain

$$\mu = -\left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]. \tag{6.11}$$

Making use of (6.9) and (6.11) in (6.5) we have

$$S^*(X_1, X_2) = Fg(X_1, X_2) - \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right] \eta(X_1)\eta(X_2). \tag{6.12}$$

In view of (6.12) and (5.7), by putting $X_1 = X_2 = \xi$, we have the following corollary.

Corollary 6.2 *If a $*$-conformal η-Ricci soliton on a ϕ-psuedo $*$-Ricci symmetric GSSF, then the nature of soliton is*

(1) *steady when* $P = 2F - \frac{2}{n}$;

(2) *expanding when* $P > 2F - \frac{2}{n}$;

(3) *shrinking when* $P < 2F - \frac{2}{n}$.

Definition 6.2 *A GSSF is said to be $*$-weakly symmetric if there exists 1-form A, B, C, D, E on M such that the condition*

$$\begin{aligned}(\nabla_{X_1} S^*)(X_3, W_1) =& A(X_1)S^*(X_3, W_1) + B(R(X_1, X_3)W_1) \\ &+ C(X_3)S^*(X_1, W_1) + D(W_1)S^*(X_1, X_3) + E(R(X_1, W_1)X_3),\end{aligned} \tag{6.13}$$

where the 1-form E is defined by $E(X_1) = g(X_1, V_1)$, $\forall x \in \chi(M)$.

Definition 6.3([11]) *A GSSF is said to be $*$-weakly Ricci-symmetric if there exists 1-form*

ε, σ, E on M such that the condition

$$(\nabla_{X_1} S^*)(X_2, X_3) = \varepsilon(X_1)S^*(X_2, X_3) + \sigma(X_2)S^*(X_1, X_3) + E(X_3)S^*(X_1, X_2), \tag{6.14}$$

holds for all vector fields $X_1, X_2, X_3, W \in \chi(M)$. If $\varepsilon = \sigma = E$, then M is said to be pseudo Ricci-symmetric.

Let M be a weakly symmetric GSSF. Then substituting $W = \xi$ in (6.13), we have

$$\begin{aligned}(\nabla_{X_1} S^*)(X_3, \xi) \;&=\; A(X_1)S^*(X_3, \xi) + B(R(X_1, X_3)\xi) \\ &\quad + C(X_3)S^*(X_1, \xi) + D(\xi)S^*(X_1, X_3) + E(R(X_1, \xi)X_3). \end{aligned} \tag{6.15}$$

In view of (2.10) and (6.12), equation (6.15) reduces to

$$\begin{aligned}(\nabla_{X_1} S^*)(X_3, \xi) \;&=\; A(X_1)\left\{F\eta(Z) - \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\eta(X_3)\right\} \\ &\quad + B(f_1 - f_3)\{\eta(Z)X - \eta(X_1)X_3\} \\ &\quad + C(X_3)\left\{F\eta(X_1) - \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\eta(X_1)\right\} \\ &\quad + D(\xi)\left\{Fg(X_1, X_3) - \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\eta(X_1)\eta(X_3)\right\} \\ &\quad + E(R(X_1, \xi)X_3). \end{aligned} \tag{6.16}$$

Considering the covariant derivative of the $*$-Ricci tensor S^* along the vector field X_1, we obtain

$$(\nabla_{X_1} S^*)(X_3, \xi) = \nabla_{X_1} S^*(X_3, \xi) - S^*(\nabla_{X_1} X_3, \xi) - S^*(X_3, \nabla_{X_1}\xi). \tag{6.17}$$

By the use of (2.8) and (6.12) above equation takes the form

$$(\nabla_{X_1} S^*)(X_3, \xi) = -B\left\{\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right\} g(\phi X_1, X_3). \tag{6.18}$$

In view of (6.16) and (6.18), we obtain

$$\begin{aligned} &A(X_1)\left\{F\eta(X_3) - \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\eta(X_3)\right\} + B(f_1 - f_3)\{\eta(X_3)X_1 - \eta(X_1)X_3\} \\ &\quad + C(Z)\left\{F\eta(X_1) - \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\eta(X_1)\right\} \\ &\quad + D(\xi)\left\{Fg(X_1, X_3) - \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\eta(X_1)\eta(X_3)\right\} \\ &\quad + E(R(X_1, \xi)X_3) = -B\left\{\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right\} g(\phi X_1, X_3). \end{aligned} \tag{6.19}$$

Setting $X_1 = X_3 = \xi$ in (6.19) and on simplification, it yields

$$\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\{A(\xi) + C(\xi) + D(\xi)\} = 0 \tag{6.20}$$

which implies that the vanishing of the 1-form $A + C + D$ over the vector field ξ is necessary in order that M is a Ricci soliton on weakly symmetric GSSF. As similar to the previous calculation, it can be easily shown that

$$\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\{A(\xi) + C(\xi) + D(\xi)\} = 0$$

holds for arbitrary vector field X_1 on M. This gives the following theorem.

Theorem 6.2 *If (g, ξ, λ, μ) is a $*$-conformal η-Ricci soliton on a weakly symmetric GSSF, then the sum of 1-form is zero everywhere, provided*

$$F + (\lambda - \frac{1}{2}(P + \frac{2}{n})) \neq 0.$$

Suppose that M is a $*$- weakly Ricci symmetric GSSF. Taking $X_3 = \xi$ in (6.14) and by use of (6.12), we have

$$\begin{aligned}(\nabla_{X_1} S^*)(X_2, \xi) &= \varepsilon(X_1)\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\eta(X_2) \\ &\quad + \sigma(X_2)\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\eta(X_1) \\ &\quad + E(\xi)\left\{Fg(X_1, X_2) + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\eta(X_1)\eta(X_2)\right\}. \qquad (6.21)\end{aligned}$$

Again replacing X_3 by X_2 in (6.18), we get

$$(\nabla_{X_1} S^*)(X_2, \xi) = -B\left\{\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right\}g(\phi X_1, X_2). \qquad (6.22)$$

Comparing equations (6.21) and (6.22), we get

$$\begin{aligned}&\varepsilon(X_1)\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\eta(X_2) + \sigma(X_2)\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\eta(X_1) \\ &\quad + E(\xi)\left\{Fg(X_1, X_2) + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\eta(X_1)\eta(X_2)\right\} \\ &\quad = -B\left\{\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right\}g(\phi X_1, X_2). \qquad (6.23)\end{aligned}$$

Setting $X_1 = X_2 = \xi$ in (6.23), we have

$$\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\{\varepsilon(\xi) + \sigma(\xi) + E(\xi)\} = 0. \qquad (6.24)$$

Again, putting $X_1 = \xi$ in (6.23), we have

$$\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\sigma(X_2) = \sigma(\xi)\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}. \qquad (6.25)$$

Replacing X_2 with X_1, it yields

$$\left\{F+\left[\lambda-\frac{1}{2}\left(P+\frac{2}{n}\right)\right]\right\}\sigma(X_1)=\sigma(\xi)\left\{F+\left[\lambda-\frac{1}{2}\left(P+\frac{2}{n}\right)\right]\right\}. \tag{6.26}$$

If we take $X_2=\xi$ in (6.23), we get

$$\left\{F+\left[\lambda-\frac{1}{2}\left(P+\frac{2}{n}\right)\right]\right\}\varepsilon(X_1)=\varepsilon(\xi)\left\{F+\left[\lambda-\frac{1}{2}\left(P+\frac{2}{n}\right)\right]\right\}. \tag{6.27}$$

Similarly, we have

$$\left\{F+\left[\lambda-\frac{1}{2}\left(P+\frac{2}{n}\right)\right]\right\}E(X_1)=E(\xi)\left\{F+\left[\lambda-\frac{1}{2}\left(P+\frac{2}{n}\right)\right]\right\}. \tag{6.28}$$

Adding (6.26), (6.27) and (6.28) and using (6.24), we get

$$\left\{F+\left[\lambda-\frac{1}{2}\left(P+\frac{2}{n}\right)\right]\right\}\{\sigma(X_1)+\varepsilon(X_1)+\rho(X_1)\}=0,$$

for all $X_1\in\chi(M)$. Thus, we have the following result.

Theorem 6.3 *Let M be a $*$-weakly Ricci symmetric GSSF admits $*$-conformal η- Ricci soliton. Then the sum of 1-forms is zero, i.e., $\varepsilon+\sigma+\rho=0$ everywhere, provided $F+(\lambda-\frac{1}{2}(P+\frac{2}{n}))\neq 0$.*

§7. Conclusions

In this paper, the generalized Sasakian space form admitting $*$-conformal η- Ricci soliton has been studied and the behaviour of the soliton is analysed. Also, it is proved that the $*$-Ricci semi-symmetric and the pseudo $*$-Ricci semisymmetric generalized Sasakian space forms are $*$-Ricci flat.

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