4–Remainder Cordial of Some Tree Related Graphs

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Abstract: Let \( G \) be a \((p, q)\) graph. Let \( f \) be a map from \( V(G) \) to the set \( \{1, 2, \cdots, k\} \) where \( k \) is an integer \( 2 < k \leq |V(G)| \). For each edge uv assign the label \( r \) where \( r \) is the remainder when \( f(u) \) is divided by \( f(v) \) (or) \( f(v) \) is divided by \( f(u) \) according as \( f(u) \geq f(v) \) or \( f(v) \geq f(u) \). The function \( f \) is called a \( k \)-remainder cordial labeling of \( G \) if \( |v_f(i) - v_f(j)| \leq 1 \), \( i, j \in \{1, \cdots, k\} \) where \( v_f(x) \) denote the number of vertices labelled with \( x \) and \( |\eta_e - \eta_o| \leq 1 \) where \( \eta_e \) and \( \eta_o \) respectively denote the number of edges labeled with even integers and number of edges labelled with odd integers. A graph with a \( k \)-remainder cordial labeling is called a \( k \)-remainder cordial graph. In this paper we investigate the 4–remainder cordial labeling behavior of banana tree, coconut tree, bamboo tree.

Key Words: Tree, banana tree, coconut tree, double coconut tree, bamboo tree, \( k \)-remainder cordial labeling, Smarandachely \( k \)-remainder cordial labeling.

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§1. Introduction

All graphs in this paper are finite, undirected and simple. The vertex set and edge set of a graph are denoted by \( V(G) \) and \( E(G) \) respectively. Ponraj et al. defined the \( k \)-remainder cordial labeling of a graph in [3]. \( k \)-Remainder cordial labeling behavior of path, cycle, star, complete graph, wheel, comb etc have been investigated in [3]. Here we investigate the 4-Remainder cordial labeling behavior of Banana tree,Coconut tree,Double coconut tree,Bamboo tree,caterpillar tree.

§2. 4- Remainder Cordial Labeling

Definition 2.1 Let \( G \) be a \((p, q)\) graph. Let \( f \) be a map from \( V(G) \) to the set \( \{1, 2, \cdots, k\} \) where \( k \) is an integer \( 2 < k \leq |V(G)| \). For each edge uv assign the label \( r \) where \( r \) is the remainder when \( f(u) \) is divided by \( f(v) \) (or) \( f(v) \) is divided by \( f(u) \) according as \( f(u) \geq f(v) \) or \( f(v) \geq f(u) \). The function \( f \) is called a \( k \)-remainder cordial labeling of \( G \) if \( |v_f(i) - v_f(j)| \leq 1 \),

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where $v_f(x)$ denote the number of vertices labelled with $x$ and $|\eta_e - \eta_o| \leq 1$
where $\eta_e$ and $\eta_o$ respectively denote the number of edges labeled with even integers and number
of edges labelled with odd integers. A graph with a $k$-remainder cordial labeling is called a $k$-remainder
cordial graph.

Conversely, the function $f$ is called a Smarandachely $k$-remainder cordial labeling of $G$ if
there an integer pair $\{i, j\} \subset \{1, 2, \cdots, k\}$ with $|v_f(i) - v_f(j)| > 1$ and $|\eta_e - \eta_o| \leq 1$. Such a
graph with a Smarandachely $k$-remainder cordial labeling is called a Smarandachely $k$-remainder
cordial graph.

**Definition 2.2** The banana tree $B(m, n)$ is a graph obtained by connecting one leaf of each of
$m-$ copies of the star $K_{1, n}$ with a single root vertex that is distinct from all the stars.

**Definition 2.3** The coconut tree $CT(m, n)$ is a graph obtained from the path $P_n$ by appending
$m$ new pendant edges at an end vertex of $P_n$.

**Definition 2.4** The double Coconut tree $DCT(m, n, r)$ is a tree obtained by attaching $m > 1$
pendent vertices to one end of the path $P_n$ and $r > 1$ pendant vertices to the other end of $P_n$.

**Definition 2.5** The fire cracker $F_{n,k}$ is obtained by the concatenation of $n-$ copies of $k-$
stars by linking one leaf from each.

**Definition 2.6** The bamboo tree $BT(n, m, k)$ is a tree obtained from $k$ copies of paths $P_n$ of
length $n - 1$ and $K_{1, m}$ stars.Identify one of the two pendant vertices of the $j^{th}$ path with centre
of the $j^{th}$ star.Identify the other pendant vertex of $a$ each path with a single vertex $u_0$.

**Definition 2.7** The caterpillar $S_n(m_1, m_2, \ldots, m_n)$ is a tree obtained from the path $u_1u_2\cdots u_n$
by identifying the centre of the star $K_{1, m_i} (1 \leq i \leq n)$ with $u_i$. And let the vertex set and edge
set of star $K_{1, m_i}$ be
\[ V(K_{1, m_i}) = \{v_i, u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}, \quad E(K_{1, m_i}) = \{v_iu_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}, \]
identifying $u_i$ with $v_i (1 \leq i \leq n)$.

§3. Main Results

**Theorem 3.1** The banana tree $B(m, n)$ is 4-remainder cordial for $m \equiv 0 \pmod{4}$ and $n$ is
any positive integer.

**Proof** Let the vertex set and edge set of $B(m, n)$ be
\[ V(B(m, n)) = \{a_1, a_{i,1}, a_{i,2}, \cdots, a_{i,m} : 1 \leq i \leq n\}, \]
\[ E(B(m, n)) = \{a_1a_{i,1}, a_{i,2}, a_{i,3}, \cdots, a_{i,m} : 1 \leq i \leq m\} \cup \{a_{2,j}a_{i,j} : 3 \leq i \leq n, 1 \leq j \leq m\}. \]

First, assign the label 3 to the vertex $a_1$ which has degree $n$. Next assign the labels 1, 2, 3, 4
respectively to the vertices $a_{1,i} (1 \leq i \leq 4)$ which has degree 2 and assign the labels 1, 2, 3, 4
respectively to the vertices $a_{1,i}$ $(5 \leq i \leq 8)$ which has degree 2. Proceeding like this until we reach the vertex $a_{1,m}$.

Next, move to the vertices which has degree $n - 1$. Define

$$f(a_2,i) = \begin{cases} 2 & \text{if } f(a_1,i) = 1, \\ 3 & \text{if } f(a_1,i) = 2, \\ 4 & \text{if } f(a_1,i) = 3, \\ 1 & \text{if } f(a_1,i) = 4. \end{cases}$$

Finally, move to the pendant vertices. Assign the labels to the vertices $f(a_{i,j})$, ($(1 \leq j \leq m)$ and $(3 \leq i \leq n)$) by

$$f(a_{i,j}) = \begin{cases} 1 & \text{if } f(a_1,i) = 1, \\ 2 & \text{if } f(a_1,i) = 3, \\ 3 & \text{if } f(a_1,i) = 2, \\ 4 & \text{if } f(a_1,i) = 4. \end{cases}$$

Thus, this vertex labeling shows that $f$ is the 4-remainder cordial labeling of $B(m,n)$ for $m \equiv 0 \pmod{4}$. Since $v_1 = v_f(1) = v_f(2) = v_f(4) = \frac{mn}{4}$, $v_f(3) = \frac{mn+4}{4}$ and $\eta_v = \eta_o = \frac{mn}{2}$. This completes the proof. □

A 4-remainder cordial labeling of $B(4,5)$ is given in Figure 1.

![Figure 1](image)

**Theorem 3.2** The coconut tree $CT(n,n)$ is 4-remainder cordial for all values of $n$.

**Proof** Let $P_n$ be the path $v_1v_2\cdots v_n$ and let $V(K_1,n) = \{w_i : 1 \leq i \leq n\}$ are the vertex set of $CT(n,n)$ and edge set $E(CT(n,n)) = E(P_n) \cup E(K_1,n)$ identifying the $w$ with $v_n$. The proof of this theorem this proved in the following four cases.

**Case 1.** $n \equiv 0 \pmod{4}$.

First, assign the label to the vertices of the path $P_n$. Assign the labels 1, 2, 3, 4 respectively to the vertices $v_1, v_2, v_3, v_4$ and assign the labels 1, 2, 3, 4 respectively to the vertices $v_5, v_6, v_7, v_8$. Next assign the labels 1, 2, 3, 4 respectively to the vertices $v_9, v_{10}, v_{11}, v_{12}$. Proceeding
like this until we reach the vertex $v_{n-4}$ and assign the labels $1, 1, 2, 3$ respectively to the vertices $v_{n-3}, v_{n-2}, v_{n-1}, v_n$.

Next, move to the vertices of the star $K_{1,n}$. Assign the label 1 to the vertices $w_1, w_5, w_9, \ldots, w_{n-7}$ and assign the label 2 to the vertices $w_2, w_6, w_{10}, \ldots, w_{n-6}$. Secondly assign the label 3 to the vertices $w_3, w_7, w_{11}, w_{n-5}$ and assign the label 4 to the vertices $w_4, w_8, w_{12}, \ldots, w_{n-4}$. Finally assign the labels $2, 3, 4, 4$ respectively to the vertices $w_{n-3}, w_{n-2}, w_{n-1}, w_n$.

**Case 2.** $n \equiv 1 \pmod{4}$

First, assign the label to the vertices of the path $P_n$. Assign the label 1 to the vertices $v_1, v_5, v_9, \ldots, v_{n-4}$ and assign the label 2 to the vertices $v_2, v_6, v_{10}, \ldots, v_{n-3}$. Secondly assign the label 3 to the vertices $v_3, v_7, v_{11}, \ldots, v_{n-2}$ and assign the label 4 to the vertices $v_4, v_8, v_{12}, \ldots, v_{n-1}$. Finally assign the labels 3 to the vertex $v_n$.

Next, move to the vertices of the star $K_{1,n}$. Assign the label 1 to the vertices $w_1, w_8, w_{12}, \ldots, w_{n-5}$ and assign the label 2 to the vertices $w_2, w_9, w_{13}, \ldots, w_{n-4}$. Secondly assign the label 3 to the vertices $w_3, w_{10}, w_{14}, \ldots, w_{n-3}$ and assign the label 4 to the vertices $w_4, w_{11}, w_{15}, \ldots, w_{n-2}$. Finally assign the label 1 to the vertices $w_1$ and $w_{n-1}$ and assign the label 3 to the vertex $w_2$ then assign the label 4 to the vertex $w_3$ and assign the label 2 to the vertex $w_n$.

**Case 3.** $n \equiv 2 \pmod{4}$

First, assign the label to the vertices of the path $P_n$. Assign the label 1 to the vertices $v_1, v_5, v_9, \ldots, v_{n-5}$ and assign the label 2 to the vertices $v_2, v_6, v_{10}, \ldots, v_{n-4}$. Secondly assign the label 3 to the vertices $v_3, v_7, v_{11}, \ldots, v_{n-3}$ and assign the label 4 to the vertices $v_4, v_8, v_{12}, \ldots, v_{n-2}$. Finally assign the labels 1, 3 respectively to the vertices $v_{n-1}, v_n$.

Next, assign the labels to the vertices of the star $K_{1,n}$. Assign the label 1 to the vertices $w_4, w_8, w_{12}, \ldots, w_{n-6}$ and assign the label 2 to the vertices $w_5, w_9, w_{13}, \ldots, w_{n-5}$. Secondly assign the label 3 to the vertices $w_6, w_{10}, w_{14}, \ldots, w_{n-4}$ and assign the label 4 to the vertices $w_7, w_{11}, w_{15}, \ldots, w_{n-3}$. Finally, assign the label 1 to the vertex $w_1$ and assign the label 2 to the vertices $w_2$ and $w_{n-2}$, then assign the label 3 to the vertex $w_{n-1}$ and assign the labels 4 to the vertices $w_3$ and $w_n$.

**Case 4.** $n \equiv 3 \pmod{4}$

First, assign the labels to the vertices of the path $P_n$. Assign the label 1 to the vertices $v_1, v_5, v_9, \ldots, v_{n-2}$ and assign the label 2 to the vertices $v_2, v_6, v_{10}, \ldots, v_{n-1}$. Secondly assign the label 3 to the vertices $v_3, v_7, v_{11}, \ldots, v_n$ and assign the label 4 to the vertices $v_4, v_8, v_{12}, \ldots, v_{n-3}$.

Next, assign the labels to the vertices of the star $K_{1,n}$. Assign the label 1 to the vertices $w_2, w_6, w_{10}, \ldots, w_{n-1}$ and assign the label 2 to the vertices $w_3, w_7, w_{11}, \ldots, w_n$. Secondly assign the label 3 to the vertices $w_4, w_8, w_{12}, \ldots, w_{n-3}$ and assign the label 4 to the vertices $w_5, w_9, w_{13}, \ldots, w_{n-2}$. Finally, assign the labels 4 to the vertex $w_1$ and assign the labels 1, 2 to the vertices $w_{n-1}, w_n$.

Thus, the Table 1 given below shows that coconut tree graph admits the $4-$remainder cordial labeling.
Table 1.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$v_f(1)$</th>
<th>$v_f(2)$</th>
<th>$v_f(3)$</th>
<th>$v_f(4)$</th>
<th>$\eta_e$</th>
<th>$\eta_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 \pmod{4}$</td>
<td>$\frac{n}{2}$</td>
<td>$\frac{n}{2}$</td>
<td>$\frac{n}{2}$</td>
<td>$\frac{n}{2}$</td>
<td>$n - 1$</td>
<td>$n$</td>
</tr>
<tr>
<td>$n \equiv 1 \pmod{4}$</td>
<td>$\frac{n+1}{2}$</td>
<td>$\frac{n-1}{2}$</td>
<td>$\frac{n+1}{2}$</td>
<td>$\frac{n-1}{2}$</td>
<td>$n - 1$</td>
<td>$n$</td>
</tr>
<tr>
<td>$n \equiv 2 \pmod{4}$</td>
<td>$\frac{n}{2}$</td>
<td>$\frac{n}{2}$</td>
<td>$\frac{n}{2}$</td>
<td>$\frac{n}{2}$</td>
<td>$n - 1$</td>
<td>$n$</td>
</tr>
<tr>
<td>$n \equiv 3 \pmod{4}$</td>
<td>$\frac{n+1}{2}$</td>
<td>$\frac{n-1}{2}$</td>
<td>$\frac{n-1}{2}$</td>
<td>$\frac{n-1}{2}$</td>
<td>$n - 1$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

This completes the proof. \( \square \)

A 4-remainder cordial labeling of $CT(6,6)$ is given in Figure 2.

**Figure 2**

**Theorem 3.3** A double coconut tree $DCT(n,n,n)$ is 4- remainder cordial for all values of $n$.

**Proof** Let $P_n$ be the path $w_1w_2 \cdots w_n$ and $V(DCT(n,n,n)) = V(P_n) \cup \{u_i, v_i : 1 \leq i \leq n\}$ and $E(DCT(n,n,n)) = E(P_n) \cup \{v_iw_1 : 1 \leq i \leq n\} \cup \{u_iw_n : 1 \leq i \leq n\}$. Clearly $DCT(n,n,n)$ has $3n$ vertices and $3n - 1$ edges.

Now we describe the vertex labeling as follows. There are four cases arises.

**Case 1.** $n \equiv 0 \pmod{4}$.

First, assign the label to the vertices of the path $P_n$. Assign the label 3, 2 to the vertices $w_1, w_2$ and assign the label 2, 3 to the vertices $w_{n-1}, w_n$. Next assign the label 1, 2, 3, 4 to the vertices $w_3, w_4, w_5, w_6$ and assign the label 1, 2, 3, 4 to the vertices $w_7, w_8, w_9, w_{10}$. Proceeding like this until we reach $w_{n-2}$.

Next, assign the labels for the pendent vertices. First, assign the label 3 to the vertex $u_{n-3}$ and then assign the labels 4 to the vertices $u_{n-2}, u_{n-1}, u_n$. Secondly assign the label 2 to the vertex $v_{n-3}$ and then assign the labels 1 to the vertices $v_{n-2}, v_{n-1}, v_n$. Next assign the labels for the remaining vertices. Assign the labels 1, 2, 3, 4 to the vertices $u_i$ and $v_i (3 \leq i \leq 6)$. Then assign the labels 1, 2, 3, 4 to the vertices $u_i$ and $v_i (7 \leq i \leq 10)$. Proceeding like this until we reach $u_{n-4}$ and $v_{n-4}$.

**Case 2.** $n \equiv 1 \pmod{4}$.

First, assign the label to the vertices of the path $P_n$. Assign the label 3, 2 to the vertices
Next, assign the labels for the pendant vertices. First assign the label 3 to the vertex $u_n$, and then assign the label 4 to the vertices $w_1, w_2$. Next assign the labels 1 to the vertices $w_3, w_4, w_5, w_6$ and assign the label 2 to the vertices $w_7, w_8, w_9, w_{10}$. Proceeding like this until we reach $w_{n-3}$.

Next, assign the labels for the pendant vertices. First assign the label 3 to the vertex $u_{n-4}, u_{n-3}$ and then assign the label 4 to the vertices $w_{n-2}, w_{n-1}, u_n$. Secondly assign the label 2 to the vertex $v_{n-4}, v_{n-3}$ and then assign the labels 1 to the vertices $v_{n-2}, v_{n-1}, v_n$. Next assign the labels for the remaining vertices. Assign the labels 1, 2, 3, 4 to the vertices $u_i$ and $v_i$ ($3 \leq i \leq 6$). Then assign the labels 1, 2, 3, 4 to the vertices $u_i$ and $v_i$ ($7 \leq i \leq 10$). Proceeding like this until we reach $u_{n-5}$ and $v_{n-5}$.

**Case 3.** $n \equiv 2 \pmod{4}$.

First, assign the label to the vertices of the path $P_n$. Assign the label 3 to the vertices $w_1, w_2$. Next assign the label 1, 2, 3, 4 to the vertices $w_3, w_4, w_5, w_6$ and assign the label 1, 2, 3, 4 to the vertices $w_7, w_8, w_9, w_{10}$. Proceeding like this until we reach $w_n$.

Next, assign the labels for the pendant vertices. First assign the label 3 to the vertices $u_i, (i = 1, 2, \ldots, \frac{n-2}{2})$ which is adjacent with the vertex $w_n$ and assign the label 2 to the vertices $u_i, (i = \frac{n+2}{2}, \frac{n+4}{2}, \ldots, n)$ which is adjacent with vertex $w_n$. Next assign the labels 4 to the vertices $v_i, (i = 1, 2, \ldots, \frac{n}{2})$ which is adjacent with the vertex $w_1$ and then assign the label 1 to the vertices $v_i, (i = \frac{n+2}{2}, \frac{n+4}{2}, \ldots, n)$ which is adjacent with vertex $w_1$.

**Case 4.** $n \equiv 3 \pmod{4}$.

First, assign the label to the vertices of the path $P_n$. Assign the label 3 to the vertices $w_1, w_2$. Next assign the label 1, 2, 3, 4 to the vertices $w_3, w_4, w_5, w_6$ and assign the label 1, 2, 3, 4 to the vertices $w_7, w_8, w_9, w_{10}$. Proceeding like this until we reach $w_{n-1}$ Next assign the label 4 to the vertex $w_n$.

Next, assign the labels for the pendant vertices. First assign the label 3 to the vertices $u_i, (i = 1, 2, \ldots, \frac{n+1}{2})$ which is adjacent with the vertex $w_n$ and assign the label 2 to the vertices $u_i, (i = \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n)$ which is adjacent with vertex $w_n$. Next assign the labels 1 to the vertices $v_i, (i = 1, 2, \ldots, \frac{n+1}{2})$ which is adjacent with the vertex $w_1$ and then assign the label 4 to the vertices $v_i, (i = \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n)$ which is adjacent with vertex $w_1$.

Thus, the Table 2 given below shows that the double coconut tree graph admits the 4–remainder cordial.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$v_f(1)$</th>
<th>$v_f(2)$</th>
<th>$v_f(3)$</th>
<th>$v_f(4)$</th>
<th>$\eta_c$</th>
<th>$\eta_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 \pmod{4}$</td>
<td>$\frac{3n}{4}$</td>
<td>$\frac{3n}{4}$</td>
<td>$\frac{3n}{4}$</td>
<td>$\frac{3n}{4}$</td>
<td>$\frac{3n-2}{2}$</td>
<td>$\frac{3n}{2}$</td>
</tr>
<tr>
<td>$n \equiv 1 \pmod{4}$</td>
<td>$\frac{3n-3}{4}$</td>
<td>$\frac{3n+1}{4}$</td>
<td>$\frac{3n+1}{4}$</td>
<td>$\frac{3n+1}{4}$</td>
<td>$\frac{3n-1}{2}$</td>
<td>$\frac{3n-1}{2}$</td>
</tr>
<tr>
<td>$n \equiv 2 \pmod{4}$</td>
<td>$\frac{3n-2}{4}$</td>
<td>$\frac{3n+2}{4}$</td>
<td>$\frac{3n+2}{4}$</td>
<td>$\frac{3n-2}{4}$</td>
<td>$\frac{3n-2}{2}$</td>
<td>$\frac{3n}{2}$</td>
</tr>
<tr>
<td>$n \equiv 3 \pmod{4}$</td>
<td>$\frac{3n-1}{4}$</td>
<td>$\frac{3n-1}{4}$</td>
<td>$\frac{3n+3}{4}$</td>
<td>$\frac{3n-1}{4}$</td>
<td>$\frac{3n-1}{2}$</td>
<td>$\frac{3n-1}{2}$</td>
</tr>
</tbody>
</table>

**Table 2**

This competes the proof. □

**Theorem 3.4** The fire cracker $F_{n,k}$ is 4–remainder cordial for all values of $n$. 
Proof Let \( V(F_{n,k}) = \{v_o^i, v_1^i, v_2^i : 1 \leq i \leq n \} \) where \( v_o^i \) be the apex vertices of the star and \( E(F_{n,k}) = \{v_o^i v_o^j : 1 \leq i, j \leq n, 1 \leq j \leq 2 \} \cup \{v_o^i v_o^{i+1} : 1 \leq i \leq n-1 \} \). Clearly \( G \) has \( 3n \) vertices and \( 3n - 1 \) edges.

Now we describe the vertex labeling. There are four cases arises.

**Case 1.** \( n \equiv 0 \) (mod 4).

First, assign the labels for the vertices \( v_o^i \). Assign the labels 1, 2, 3, 4 respectively to the vertices \( v_o^1, v_o^2, v_o^3, v_o^4 \) and assign the labels 1, 2, 3, 4 respectively to the vertices \( v_o^5, v_o^6, v_o^7, v_o^8 \). Next assign the labels 1, 2, 3, 4 respectively to the vertices \( v_o^9, v_o^{10}, v_o^{11}, v_o^{12} \). Proceeding like this until we reach \( v_o^n \). In similar way assign the labels for \( v_2^i, 1 \leq i \leq n \). Next assign the labels for \( v_1^i \), assign the labels 4, 3, 2, 1 respectively to the vertices \( v_1^1, v_1^2, v_1^3, v_1^4 \) and assign the labels 4, 3, 2, 1 respectively to the vertices \( v_1^9, v_1^{10}, v_1^{11}, v_1^{12} \). Proceeding like this until we reach \( v_1^n \).

**Case 2.** \( n \equiv 1 \) (mod 4).

As in Case 1, assign the labels for the vertices \( v_o^i, v_1^i, v_2^i, 1 \leq i \leq n-1 \). Finally assign the label 1 to the vertex \( v_o^n \) and assign the label 2 to the vertex \( v_1^n \), then assign the label 3 to the vertex \( v_2^n \).

**Case 3.** \( n \equiv 2 \) (mod 4).

As in Case 1, assign the labels for the vertices \( v_o^i, v_1^i, v_2^i, 1 \leq i \leq n-2 \). Finally assign the label 1 to the vertex \( v_o^{n-1} \) and assign the label 2 to the vertex \( v_o^n \), then assign the label 2 to the vertex \( v_1^{n-1} \) and assign the label 3 to the vertex \( v_1^n \). Next assign the label 3 to the vertex \( v_2^{n-1} \) and assign the label 4 to the vertex \( v_2^n \).

**Case 4.** \( n \equiv 3 \) (mod 4).

As in Case 1, assign the labels for the vertices \( v_o^i, v_1^i, v_2^i, 1 \leq i \leq n \) and \( v_2^i, 1 \leq i \leq n-1 \). Note that in this process the vertex \( v_o^{n-2} \) and \( v_2^{n-2} \) gets the label 1, and the vertex \( v_o^{n-1} \) and \( v_2^{n-1} \) gets the label 2, the vertex \( v_o^n \) gets the label 3 and the vertex \( v_1^{n-2} \) get the label 4. Next the vertex \( v_1^{n-1} \) get the label 3 and to the vertex \( v_1^n \) get the label 2. Finally assign the label 4 to the vertex \( v_2^2 \).

Thus, the Table 3 below shows that this vertex labeling gives the 4—remainder cordial labeling of fire cracker.

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( v_f(1) )</th>
<th>( v_f(2) )</th>
<th>( v_f(3) )</th>
<th>( v_f(4) )</th>
<th>( \eta_o )</th>
<th>( \eta_o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0 ) (mod 4)</td>
<td>( \frac{3n}{4} )</td>
<td>( \frac{3n}{4} )</td>
<td>( \frac{3n}{4} )</td>
<td>( \frac{3n}{4} )</td>
<td>( \frac{3n-2}{2} )</td>
<td>( \frac{3n}{2} )</td>
</tr>
<tr>
<td>( n \equiv 1 ) (mod 4)</td>
<td>( \frac{3n+1}{4} )</td>
<td>( \frac{3n+1}{4} )</td>
<td>( \frac{3n+1}{4} )</td>
<td>( \frac{3n+1}{4} )</td>
<td>( \frac{3n-1}{2} )</td>
<td>( \frac{3n-1}{2} )</td>
</tr>
<tr>
<td>( n \equiv 2 ) (mod 4)</td>
<td>( \frac{3n-2}{4} )</td>
<td>( \frac{3n+2}{4} )</td>
<td>( \frac{3n+2}{4} )</td>
<td>( \frac{3n+2}{4} )</td>
<td>( \frac{3n-2}{2} )</td>
<td>( \frac{3n}{2} )</td>
</tr>
<tr>
<td>( n \equiv 3 ) (mod 4)</td>
<td>( \frac{3n-1}{4} )</td>
<td>( \frac{3n+3}{4} )</td>
<td>( \frac{3n-1}{4} )</td>
<td>( \frac{3n-1}{4} )</td>
<td>( \frac{3n-1}{2} )</td>
<td>( \frac{3n-1}{2} )</td>
</tr>
</tbody>
</table>

Table 3

This completes the proof. \( \square \)
Theorem 3.5 The bamboo tree $BT(n, n, n)$ is $4-$ remainder cordial for $n \equiv 0, 2 \pmod{4}$.

Proof Let

$$
V(BT(n, n, n)) = \{u, u_{i,j}, w_{i,i} : 1 \leq i \leq n, 1 \leq j \leq n - 1\},
$$

$$
E(BT(n, n, n)) = \{u, u_{i,j}, w_{i,i} : 1 \leq i \leq n\} \cup \{u_{i,i+1} : 1 \leq i \leq n - 2\}
$$

We prove this theorem in two cases.

Case 1. $n \equiv 0 \pmod{4}$.

First, assign the label 3 to the vertex $u$. Let $f : V \to \{1, 2, 3, 4\}$ defined by

$$
f(u_{i,i}) = \begin{cases} 
1 & \text{for all } i = 1, 5, 9, \ldots, i + 4, \ldots, n - 3, \\
2 & \text{for all } i = 2, 6, 10, \ldots, i + 4, \ldots, n - 2, \\
3 & \text{for all } i = 3, 7, 11, \ldots, i + 4, \ldots, n - 1, \\
4 & \text{for all } i = 4, 8, 12, \ldots, i + 4, \ldots, n.
\end{cases}
$$

Next, assign the labels for the vertices of the paths. Consider the path $P_1, P_2, \cdot \cdot \cdot , P_{n-3}$. Assign the labels 4, 1, 4, 1, $\cdot \cdot \cdot$, 4, 1 consecutively to the vertices of this paths. Next consider the paths $P_2, P_6, \cdot \cdot \cdot , P_{n-2}$. Assign the labels 3, 2, 3, $\cdot \cdot \cdot$, 3, 2 consecutively to the vertices of these paths. Next consider the paths $P_3, P_7, \cdot \cdot \cdot , P_{n-1}$. Assign the labels 2, 3, 2, $\cdot \cdot \cdot$, 2, 3 consecutively to the vertices of these paths, lastly consider the paths $P_4, P_8, \cdot \cdot \cdot , P_n$. Assign the labels 1, 4, 1, $\cdot \cdot \cdot$, 1, 4 consecutively to the vertices of these paths.

And then, move to the vertices of the star. Assign the label 1 to the vertices $w_{i,i}, (1 \leq i \leq n)$ if $u_{i,n-1}, (1 \leq i \leq n)$ gets label 1. Secondly assign the label 3 to the vertices $w_{i,i}, (1 \leq i \leq n)$ if $u_{i,n-1}, (1 \leq i \leq n)$ gets label 2. Then assign the label 2 to the vertices $w_{i,i}, (1 \leq i \leq n)$ if $u_{i,n-1}, (1 \leq i \leq n)$ gets label 3. Finally assign the label 4 to the vertices $w_{i,i}, (1 \leq i \leq n)$ if $u_{i,n-1}, (1 \leq i \leq n)$ gets label 4.

Case 2. $n \equiv 2 \pmod{4}$.

First assign the label 3 to the vertex $u$. Define

$$
f(u_{i,i}) = \begin{cases} 
1 & \text{for all } i = 1, 5, 9, \ldots, i + 4, \ldots, n - 1, \\
2 & \text{for all } i = 2, 6, 10, \ldots, i + 4, \ldots, n, \\
3 & \text{for all } i = 3, 7, 11, \ldots, i + 4, \ldots, n - 3, \\
4 & \text{for all } i = 4, 8, 12, \ldots, i + 4, \ldots, n - 2.
\end{cases}
$$

As in Case 1 assign the labels to the vertices of paths $u_{i,i+1}, (1 \leq i \leq n)$ and the vertices of the stars $w_{i,i}, (1 \leq i \leq n - 2)$. Finally assign the label 1 to the vertices $w_{i,n-1}, (1 \leq i \leq \frac{n}{2})$ if $f(u_{i,n-1}) = 2$ and assign the label 2 to the vertices $w_{n-1,i}, \frac{n+2}{2} \leq i \leq n$ if $f(u_{i,n-1}) = 1$, then assign the label 4 to the vertices $w_{n,i}, 1 \leq i \leq \frac{n}{2}$ if $f(u_{i,n-1}) = 2$ and assign the label 3 to the vertices $w_{n,i}, \frac{n+2}{2} \leq i \leq n$ if $f(u_{i,n-1}) = 2$.

Thus, the Table 4 below shows that this vertex labeling gives the 4–remainder cordial
labeling of bamboo tree.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$v_f(1)$</th>
<th>$v_f(2)$</th>
<th>$v_f(3)$</th>
<th>$v_f(4)$</th>
<th>$\eta_c$</th>
<th>$\eta_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 \pmod{4}$</td>
<td>$\frac{2n^2-n}{4}$</td>
<td>$\frac{2n^2-n}{4}$</td>
<td>$\frac{2n^2-n+2}{4}$</td>
<td>$\frac{2n^2-n}{4}$</td>
<td>$\frac{2n^2-n}{2}$</td>
<td>$\frac{2n^2-n}{2}$</td>
</tr>
<tr>
<td>$n \equiv 2 \pmod{4}$</td>
<td>$\frac{2n^2-n+2}{4}$</td>
<td>$\frac{2n^2-n+2}{4}$</td>
<td>$\frac{2n^2-n+2}{4}$</td>
<td>$\frac{2n^2-n-2}{4}$</td>
<td>$\frac{2n^2-n}{2}$</td>
<td>$\frac{2n^2-n}{2}$</td>
</tr>
</tbody>
</table>

Table 4

This completes the proof. \(\square\)

A 4-remainder cordial labeling of bamboo tree is given in Figure 3.

![Figure 3](image)

**Theorem 3.6** The caterpillar $S_n(m,m,\ldots,m)$ is 4-remainder cordial for all values of $n$.

**Proof** Taking the vertex set and edge set is of $S_n(m,m,\ldots,m)$ as in Definition 2.8. We prove this theorem in two cases.

**Case 1.** $m = n$.

**Subcase 1.1** $n \equiv 0 \pmod{4}$.

First, assign the labels 1, 2, 3, 4 respectively to the vertices $u_1, u_2, u_3, u_4$ and assign the labels 1, 2, 3, 4 respectively to the vertices $u_5, u_6, u_7, u_8$.

Next assign the labels 1, 2, 3, 4 respectively to the vertices $u_9, u_{10}, u_{11}, u_{12}$.

Proceeding like this until we reach the vertices $u_{n-3}, u_{n-2}, u_{n-1}, u_n$.

Next, assign the label to the pendant vertices $u_{i,j}$ ($1 \leq i, j \leq n$) by

$$f(u_{i,j}) = \begin{cases} 
1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \ldots, n-3, \\
2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \ldots, n-2, \\
3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \ldots, n-1, \\
4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \ldots, n.
\end{cases}$$

**Subcase 1.1** $n \equiv 1 \pmod{4}$.
First, assign the labels 1, 2, 3, 4 respectively to the vertices $u_1, u_2, u_3, u_4$ and assign the labels 1, 2, 3, 4 respectively to the vertices $u_5, u_6, u_7, u_8$. Next assign the labels 1, 2, 3, 4 respectively to the vertices $u_9, u_{10}, u_{11}, u_{12}$. Proceeding like this until we assign the labels 1, 2, 3, 4 to the vertices $u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}$ and then assign the label 3 to the vertex $u_n$.

Next, assign the label to the pendent vertices $u_{i,j}$ $(1 \leq i, j \leq n)$ by

$$f(u_{i,j}) = \begin{cases} 
1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \ldots, n - 4, \\
2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \ldots, n - 3, \\
3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \ldots, n - 2, \\
4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \ldots, n - 1.
\end{cases}$$

Finally, assign the label 1, 2, 3, 4 respectively to the vertices $u_{n,1}, u_{n,2}, u_{n,3}, u_{n,4}$ and assign the label 1, 2, 3, 4 respectively to the vertices $u_{n,5}, u_{n,6}, u_{n,7}, u_{n,8}$. Proceeding like this until we reach the vertex $u_{n,(n-1)}$ then assign the label 1 to the vertex $u_{n,n}$.

**Subcase 1.3** \( n \equiv 2 \pmod{4} \).

First, assign the labels 1, 2, 3, 4 respectively to the vertices $u_1, u_2, u_3, u_4$ and assign the labels 1, 2, 3, 4 respectively to the vertices $u_5, u_6, u_7, u_8$. Next assign the labels 1, 2, 3, 4 respectively to the vertices $u_9, u_{10}, u_{11}, u_{12}$. Proceeding like this until we assign the labels 1, 2, 3, 4 to the vertices $u_{n-5}, u_{n-4}, u_{n-3}, u_{n-2}$ and then assign the label 2, 3 to the vertices $u_{n-1}, u_n$.

Next, assign the label to the pendent vertices $u_{i,j}$ $(1 \leq i, j \leq n)$ by

$$f(u_{i,j}) = \begin{cases} 
1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \ldots, n - 5, \\
2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \ldots, n - 4, \\
3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \ldots, n - 3, \\
4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \ldots, n - 2.
\end{cases}$$

Finally, assign the label 1 to the vertices $u_{(n-1),i} (i = 1, 2, \ldots, \frac{n}{2})$ and assign the label 2 to the vertices $u_{(n-2),i} (i = \frac{n+2}{2}, \frac{n+4}{2}, \ldots, n)$. Then assign the label 3 to the vertices $u_{n,1} (i = 1, 2, \ldots, \frac{n}{2})$ and then assign the label 4 to the vertices $u_{n,i} (i = \frac{n+2}{2}, \frac{n+4}{2}, \ldots, n)$.

**Subcase 1.4** \( n \equiv 3 \pmod{4} \).

First, assign the labels 1, 2, 3, 4 respectively to the vertices $u_1, u_2, u_3, u_4$ and assign the labels 1, 2, 3, 4 respectively to the vertices $u_5, u_6, u_7, u_8$. Next assign the labels 1, 2, 3, 4 respectively to the vertices $u_9, u_{10}, u_{11}, u_{12}$. Proceeding like this until we reach $u_{n-3}$, then assign the label 1, 2, 3 to the vertices $u_{n-2}, u_{n-1}, u_n$.

Next, assign the label to the pendent vertices $u_{i,j}$ $(1 \leq i, j \leq n)$ by

$$f(u_{i,j}) = \begin{cases} 
1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \ldots, n - 6, \\
2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \ldots, n - 5, \\
3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \ldots, n - 4, \\
4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \ldots, n - 3.
\end{cases}$$
Finally, assign the label 1 to the vertices $u_{(n-2),i}, (i = 1, 2, \cdots, \frac{3n-1}{4})$ and assign the label 2 to the vertices $u_{(n-2),i}, (i = \frac{3n+3}{4}, \frac{3n+7}{4}, \cdots, n)$ and $u_{(n-1),i}, (i = 1, 2, \cdots, \frac{n-1}{2}).$ Then assign the label 3 to the vertices $u_{(n-1),i}, (i = \frac{n+1}{4}, \frac{n+5}{4}, \cdots, n)$ and $u_{n,i}, (i = 1, 2, \cdots, \frac{n-3}{4})$ and then assign the label 4 to the vertices $u_{n,i}, (i = \frac{n+1}{4}, \frac{n+5}{4}, \cdots, n).

Thus, the Table 5 below shows that this vertex labeling gives the 4—remainder cordial labeling of caterpillar.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$v_f(1)$</th>
<th>$v_f(2)$</th>
<th>$v_f(3)$</th>
<th>$v_f(4)$</th>
<th>$\eta_c$</th>
<th>$\eta_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 \pmod{4}$</td>
<td>$\frac{n^2+n}{4}$</td>
<td>$\frac{n^2+n}{4}$</td>
<td>$\frac{n^2+n}{4}$</td>
<td>$\frac{n^2+n}{4}$</td>
<td>$\frac{n^2+n}{2}$</td>
<td>$\frac{n^2+n-2}{2}$</td>
</tr>
<tr>
<td>$n \equiv 1 \pmod{4}$</td>
<td>$\frac{n^2+n+2}{4}$</td>
<td>$\frac{n^2+n-2}{4}$</td>
<td>$\frac{n^2+n+2}{4}$</td>
<td>$\frac{n^2+n-2}{4}$</td>
<td>$\frac{n^2+n-2}{2}$</td>
<td>$\frac{n^2+n}{2}$</td>
</tr>
<tr>
<td>$n \equiv 2 \pmod{4}$</td>
<td>$\frac{n^2+n-2}{4}$</td>
<td>$\frac{n^2+n+2}{4}$</td>
<td>$\frac{n^2+n+2}{4}$</td>
<td>$\frac{n^2+n-2}{4}$</td>
<td>$\frac{n^2+n-2}{2}$</td>
<td>$\frac{n^2+n}{2}$</td>
</tr>
<tr>
<td>$n \equiv 3 \pmod{4}$</td>
<td>$\frac{n^2+n}{4}$</td>
<td>$\frac{n^2+n}{4}$</td>
<td>$\frac{n^2+n}{4}$</td>
<td>$\frac{n^2+n}{4}$</td>
<td>$\frac{n^2+n}{2}$</td>
<td>$\frac{n^2+n}{2}$</td>
</tr>
</tbody>
</table>

Table 5

**Case 2.** $m \neq n.$

**Subcase 2.1** $n \equiv 0 \pmod{4}, m \equiv 0 \pmod{4}.$

As in Case 1 assign the label to the vertices $u_i$ and $u_{i,j}.$

**Subcase 2.2** $n \equiv 1 \pmod{4}, m \equiv 0 \pmod{4}.$

As in Case 1 assign the label to the vertices $u_i, 1 \leq i \leq n - 5$ and assign the label 1, 3, 2, 4, 3 respectively to the vertices $u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n.$ Next assign the labels to the vertices $u_{i,j}(1 \leq i \leq n - 1, 1 \leq j \leq m)$ by

$$f(u_{i,j}) = \begin{cases} 
1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \cdots, n-4, \\
2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \cdots, n-3, \\
3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \cdots, n-2, \\
4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \cdots, n-1.
\end{cases}$$

Finally, assign the label 1, 2, 3, 4 respectively to the vertices $u_{n,1}, u_{n,2}, u_{n,3}, u_{n,4}$ and assign the label 1, 2, 3, 4 respectively to the vertices $u_{n,5}, u_{n,6}, u_{n,7}, u_{n,8}.$ Proceeding like this until we reach the vertex $u_{n,m}.$

**Subcase 2.3** $n \equiv 2 \pmod{4}, m \equiv 0 \pmod{4}.$

First, assign the labels 1, 2, 3, 4 respectively to the vertices $u_1, u_2, u_3, u_4$ and assign the labels 1, 2, 3, 4 respectively to the vertices $u_5, u_6, u_7, u_8.$ Next assign the labels 1, 2, 3, 4 respectively to the vertices $u_9, u_{10}, u_{11}, u_{12}.$ Proceeding like this until we assign the labels 1, 2, 3, 4 to the vertices $u_{n-5}, u_{n-4}, u_{n-3}, u_{n-2}$ and then assign the label 1 to the vertices $u_{n-1}, u_n.$
Next assign the label to the pendent vertices \( u_{i,j} \) \((1 \leq i \leq n - 2, 1 \leq j \leq m)\) by

\[
f(u_{i,j}) = \begin{cases} 
1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \cdots, n - 6, \\
2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \cdots, n - 4, \\
3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \cdots, n - 3, \\
4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \cdots, n - 2.
\end{cases}
\]

Finally, assign the label 1 to the vertices \( u_{(n-1),i}, (i = 1, 2, \cdots, \frac{n}{2}) \) and assign the label 3 to the vertices \( u_{(n-1),i}, (i = \frac{m+2}{2}, \frac{m+4}{2}, \cdots, m) \). Then assign the label 2 to the vertices \( u_{n,i}, (i = 1, 2, \cdots, \frac{n}{2}) \) and then assign the label 4 to the vertices \( u_{n,i}, (i = \frac{m+2}{2}, \frac{m+4}{2}, \cdots, m) \).

**Subcase 2.4** \( n \equiv 3 \pmod{4}, m \equiv 0 \pmod{4} \).

As in Case 1 assign the label to the vertices \( u_i, 1 \leq i \leq n \) and then assign the label to the pendent vertices \( u_{i,j} \) \((1 \leq i \leq n - 3, 1 \leq j \leq m)\) by

\[
f(u_{i,j}) = \begin{cases} 
1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \cdots, n - 6, \\
2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \cdots, n - 5, \\
3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \cdots, n - 4, \\
4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \cdots, n - 3.
\end{cases}
\]

Finally, assign the label 1 to the vertices \( u_{(n-2),i}, (i = 1, 2, \cdots, \frac{3n}{4}) \) and assign the label 3 to the vertices \( u_{(n-2),i}, (i = \frac{3m+4}{4}, \frac{3m+8}{4}, \cdots, m) \) and \( u_{(n-1),i}, (i = 1, 2, \cdots, \frac{m}{2}) \). Then assign the label 2 to the vertices \( u_{(n-1),i}, (i = \frac{m+2}{2}, \frac{m+4}{2}, \cdots, m) \) and \( u_{n,i}, (i = 1, 2, \cdots, \frac{m}{2}) \) and then assign the label 4 to the vertices \( u_{n,i}, (i = \frac{m+4}{2}, \frac{m+8}{2}, \cdots, m) \).

Thus, the Table 6 below shows that this vertex labeling gives the 4–remainder cordial labeling of caterpillar.

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( v_f(1) )</th>
<th>( v_f(2) )</th>
<th>( v_f(3) )</th>
<th>( v_f(4) )</th>
<th>( \eta_c )</th>
<th>( \eta_o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0 ) ( \pmod{4} )</td>
<td>( \frac{nm+n}{4} )</td>
<td>( \frac{nm+n}{4} )</td>
<td>( \frac{nm+n}{4} )</td>
<td>( \frac{nm+n}{4} )</td>
<td>( \frac{nm+n}{2} )</td>
<td>( \frac{nm+n}{2} )</td>
</tr>
<tr>
<td>( n \equiv 1 ) ( \pmod{4} )</td>
<td>( \frac{nm+n-1}{4} )</td>
<td>( \frac{nm+n+3}{4} )</td>
<td>( \frac{nm+n-1}{4} )</td>
<td>( \frac{nm+n-1}{4} )</td>
<td>( \frac{nm+n-2}{2} )</td>
<td>( \frac{nm+n}{2} )</td>
</tr>
<tr>
<td>( n \equiv 2 ) ( \pmod{4} )</td>
<td>( \frac{nm+n+2}{4} )</td>
<td>( \frac{nm+n-2}{4} )</td>
<td>( \frac{nm+n+2}{4} )</td>
<td>( \frac{nm+n-2}{4} )</td>
<td>( \frac{nm+n-1}{2} )</td>
<td>( \frac{nm+n-1}{2} )</td>
</tr>
<tr>
<td>( n \equiv 3 ) ( \pmod{4} )</td>
<td>( \frac{nm+n+1}{4} )</td>
<td>( \frac{nm+n+1}{4} )</td>
<td>( \frac{nm+n+1}{4} )</td>
<td>( \frac{nm+n+1}{4} )</td>
<td>( \frac{nm+n-3}{2} )</td>
<td>( \frac{nm+n-1}{2} )</td>
</tr>
</tbody>
</table>

**Table 6**

**Subcase 2.5** \( n \equiv 0 \pmod{4}, m \equiv 1 \pmod{4} \).

As in Case 1 assign the label to the vertices \( u_i \) and \( u_{i,j} \).

**Subcase 2.6** \( n \equiv 1 \pmod{4}, m \equiv 1 \pmod{4} \).

As in Case 1 assign the label to the vertices \( u_i \) and \( u_{i,j} \).

**Subcase 2.7** \( n \equiv 2 \pmod{4}, m \equiv 1 \pmod{4} \).

First, assign the labels \( 1, 2, 3, 4 \) respectively to the vertices \( u_1, u_2, u_3, u_4 \) and assign the
labels 1, 2, 3, 4 respectively to the vertices $u_5, u_6, u_7, u_8$. Next assign the labels 1, 2, 3, 4 respectively to the vertices $u_9, u_{10}, u_{11}, u_{12}$. Proceeding like this until we assign the labels 1, 2, 3, 4 to the vertices $u_{n-5}, u_{n-4}, u_{n-3}, u_{n-2}$ and then assign the label 2, 3 to the vertices $u_{n-1}, u_n$.

Next, assign the label to the pendant vertices $u_{i,j} (1 \leq i \leq n-2, 1 \leq j \leq m)$ by

$$f(u_{i,j}) = \begin{cases} 
1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \ldots, n-5, \\
2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \ldots, n-4, \\
3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \ldots, n-3, \\
4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \ldots, n-2.
\end{cases}$$

Finally, assign the label 1 to the vertices $u_{(n-1),i}, (i = 1, 2, \ldots, \frac{m+1}{2})$ and assign the label 3 to the vertices $u_{(n-1),i}, (i = \frac{m+3}{2}, \frac{m+5}{2}, \ldots, m)$. Then assign the label 2 to the vertices $u_{n,i}, (i = 1, 2, \ldots, \frac{m-1}{2})$ and assign the label 4 to the vertices $u_{n,i}, (i = \frac{m+1}{2}, \frac{m+3}{2}, \ldots, m)$.

**Subcase 2.8** $n \equiv 3 \pmod{4}, m \equiv 1 \pmod{4}$

As in Case 1 assign the label to the vertices $u_i, 1 \leq i \leq n$ by

$$f(u_{i,j}) = \begin{cases} 
1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \ldots, n-6, \\
2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \ldots, n-5, \\
3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \ldots, n-4, \\
4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \ldots, n-3.
\end{cases}$$

Finally, assign the label 1 to the vertices $u_{(n-2),i}, (i = 1, 2, \ldots, \frac{m+1}{2})$ and assign the label 2 to the vertices $u_{(n-2),i}, (i = \frac{m+3}{2}, \frac{m+5}{2}, \ldots, m)$ and $u_{(n-1),i}, (i = 1, 2, \ldots, \frac{m-1}{2})$. Then assign the label 3 to the vertices $u_{(n-1),i}, (i = \frac{m+1}{2}, \frac{m+3}{2}, \ldots, m)$ and $u_{n,i}, (i = 1, 2, \ldots, \frac{m-1}{2})$ and then assign the label 4 to the vertices $u_{n,i}, (i = \frac{m+1}{4}, \frac{m+3}{4}, \ldots, m)$.

Thus, the Table 7 below shows that this vertex labeling gives the 4–remainder cordial labeling of caterpillar.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$v_f(1)$</th>
<th>$v_f(2)$</th>
<th>$v_f(3)$</th>
<th>$v_f(4)$</th>
<th>$\eta_c$</th>
<th>$\eta_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 \pmod{4}$</td>
<td>$\frac{nm+n}{4}$</td>
<td>$\frac{nm+n}{4}$</td>
<td>$\frac{nm+n}{4}$</td>
<td>$\frac{nm+n}{4}$</td>
<td>$\frac{nm+n}{2}$</td>
<td>$\frac{nm+n}{2}$</td>
</tr>
<tr>
<td>$n \equiv 1 \pmod{4}$</td>
<td>$\frac{nm+n+2}{4}$</td>
<td>$\frac{nm+n-2}{4}$</td>
<td>$\frac{nm+n+2}{4}$</td>
<td>$\frac{nm+n-2}{4}$</td>
<td>$\frac{nm+n}{2}$</td>
<td>$\frac{nm+n}{2}$</td>
</tr>
<tr>
<td>$n \equiv 2 \pmod{4}$</td>
<td>$\frac{nm+n}{4}$</td>
<td>$\frac{nm+n}{4}$</td>
<td>$\frac{nm+n}{4}$</td>
<td>$\frac{nm+n}{4}$</td>
<td>$\frac{nm+n-1}{2}$</td>
<td>$\frac{nm+n-1}{2}$</td>
</tr>
<tr>
<td>$n \equiv 3 \pmod{4}$</td>
<td>$\frac{nm+n-2}{4}$</td>
<td>$\frac{nm+n+2}{4}$</td>
<td>$\frac{nm+n+2}{4}$</td>
<td>$\frac{nm+n-2}{4}$</td>
<td>$\frac{nm+n+2}{2}$</td>
<td>$\frac{nm+n-2}{2}$</td>
</tr>
</tbody>
</table>

**Table 7**

**Subcase 2.9** $n \equiv 0 \pmod{4}, m \equiv 2 \pmod{4}$

As in Case 1 assign the label to the vertices $u_i$ and $u_{i,j}$.

**Subcase 2.10** $n \equiv 1 \pmod{4}, m \equiv 2 \pmod{4}$

As in Case 1 assign the label to the vertices $u_i, 1 \leq i \leq n-5$ and assign the label 1, 3, 2, 4, 3
respectively to the vertices \( u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n \). Next assign the labels to the vertices \( u_{i,j} \) by

\[
 f(u_{i,j}) = \begin{cases} 
 1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \ldots, n - 4, \\
 2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \ldots, n - 2, \\
 3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \ldots, n - 3, \\
 4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \ldots, n - 1. 
\end{cases}
\]

Finally, assign the label 1, 2, 3, 4 respectively to the vertices \( u_{n,1}, u_{n,2}, u_{n,3}, u_{n,4} \) and assign the label 1, 2, 3, 4 respectively to the vertices \( u_{n,5}, u_{n,6}, u_{n,7}, u_{n,8} \). Proceeding like this until we reach the vertex \( u_{n,(m-2)} \) then finally assign the label 1, 2 to the vertices \( u_{n,(m-1)}, u_{n,m} \).

**Subcase 2.11** \( n \equiv 2 \pmod{4}, m \equiv 2 \pmod{4} \).

As in Case 1 assign the label to the vertices \( u_i \) and \( u_{i,j} \).

**Subcase 2.12** \( n \equiv 3 \pmod{4}, m \equiv 2 \pmod{4} \).

As in Case 1 assign the label to the vertices \( u_i, 1 \leq i \leq n \) by

\[
 f(u_{i,j}) = \begin{cases} 
 1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \ldots, n - 6, \\
 2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \ldots, n - 5, \\
 3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \ldots, n - 4, \\
 4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \ldots, n - 3. 
\end{cases}
\]

Finally, assign the label 1 to the vertices \( u_{(n-2),i}, (i = 1, 2, \ldots, \frac{3m+4}{4}) \) and assign the label 3 to the vertices \( u_{(n-2),i}, (i = \frac{3m+6}{4}, \ldots, m) \) and \( u_{(n-1),i}, (i = 1, 2, \ldots, \frac{m+2}{4}) \). Then assign the label 2 to the vertices \( u_{(n-1),i}, (i = \frac{m+2}{4}, \ldots, m) \) and \( u_{n,i}, (i = 1, 2, \ldots, \frac{n+1}{4}) \) and then assign the label 4 to the vertices \( u_{n,i}, (i = \frac{n+1}{4}, \frac{n+10}{4}, \ldots, m) \).

Thus, the Table 8 below shows that this vertex labeling gives the 4–remainder cordial labeling of caterpillar.

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( e_f(1) )</th>
<th>( e_f(2) )</th>
<th>( e_f(3) )</th>
<th>( e_f(4) )</th>
<th>( \eta_e )</th>
<th>( \eta_o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0 \pmod{4} )</td>
<td>( \frac{nm+n}{4} )</td>
<td>( \frac{nm+n}{4} )</td>
<td>( \frac{nm+n}{4} )</td>
<td>( \frac{nm+n}{4} )</td>
<td>( \frac{nm+n-2}{2} )</td>
<td>( \frac{nm+n}{4} )</td>
</tr>
<tr>
<td>( n \equiv 1 \pmod{4} )</td>
<td>( \frac{nm+n+1}{4} )</td>
<td>( \frac{nm+n+1}{4} )</td>
<td>( \frac{nm+n+1}{4} )</td>
<td>( \frac{nm+n+1}{4} )</td>
<td>( \frac{nm+n-1}{2} )</td>
<td>( \frac{nm+n-1}{2} )</td>
</tr>
<tr>
<td>( n \equiv 2 \pmod{4} )</td>
<td>( \frac{nm+n-2}{4} )</td>
<td>( \frac{nm+n+2}{4} )</td>
<td>( \frac{nm+n+2}{4} )</td>
<td>( \frac{nm+n-2}{4} )</td>
<td>( \frac{nm+n-2}{2} )</td>
<td>( \frac{nm+n}{4} )</td>
</tr>
<tr>
<td>( n \equiv 3 \pmod{4} )</td>
<td>( \frac{nm+n+3}{4} )</td>
<td>( \frac{nm+n-1}{4} )</td>
<td>( \frac{nm+n-1}{4} )</td>
<td>( \frac{nm+n-1}{4} )</td>
<td>( \frac{nm+n}{4} )</td>
<td>( \frac{nm+n}{4} )</td>
</tr>
</tbody>
</table>

**Table 8**

**Subcase 2.13** \( n \equiv 0 \pmod{4}, m \equiv 3 \pmod{4} \).

As in Case 1 assign the label to the vertices \( u_i \) and \( u_{i,j} \).

**Subcase 2.14** \( n \equiv 1 \pmod{4}, m \equiv 3 \pmod{4} \).

As in Case 1 assign the label to the vertices \( u_i, 1 \leq i \leq n - 5 \) and assign the label 1, 3, 2, 4, 3 respectively to the vertices \( u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n \). Next assign the labels to the vertices \( u_{i,j} \).
by

\[
f(u_{i,j}) = \begin{cases} 
1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \ldots, n - 4, \\
2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \ldots, n - 2, \\
3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \ldots, n - 3, \\
4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \ldots, n - 1.
\end{cases}
\]

Finally, assign the label 1, 2, 3, 4 respectively to the vertices \(u_{n,1}, u_{n,2}, u_{n,3}, u_{n,4}\) and assign the label 1, 2, 3, 4 respectively to the vertices \(u_{n,5}, u_{n,6}, u_{n,7}, u_{n,8}\). Proceeding like this until we reach the vertex \(u_{n,(m-3)}\) then finally assign the label 1, 2, 4 to the vertices \(u_{n,(m-2)}, u_{n,(m-1)}, u_{n,m}\).

**Subcase 2.15** \(n \equiv 2 \pmod{4}, m \equiv 3 \pmod{4}\).

As in Case 1 assign the label to the vertices \(u_i, 1 \leq i \leq n - 6\) and assign the label 1, 3, 2, 4, 2, 3 respectively to the vertices \(u_{n-5}, u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\). Next assign the labels to the vertices \(u_{i,j}\) by

\[
f(u_{i,j}) = \begin{cases} 
1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \ldots, n - 5, \\
2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \ldots, n - 3, \\
3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \ldots, n - 4, \\
4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \ldots, n - 2.
\end{cases}
\]

Finally, assign the label 1, 2, 3, 4 respectively to the vertices \(u_{(n-1),1}, u_{(n-1),2}, u_{(n-1),3}, u_{(n-1),4}\) and assign the label 1, 2, 3, 4 respectively to the vertices \(u_{(n-1),5}, u_{(n-1),6}, u_{(n-1),7}, u_{(n-1),8}\). Proceeding like this until we reach the vertex \(u_{(n-1),(m-3)}\) then finally assign the label 1, 2, 4 to the vertices \(u_{(n-1),(m-2)}, u_{(n-1),(m-1)}, u_{n,m}\). Finally assign the label 1, 2, 3, 4 respectively to the vertices \(u_{n,1}, u_{n,2}, u_{n,3}, u_{n,4}\) and assign the label 1, 2, 3, 4 respectively to the vertices \(u_{n,5}, u_{n,6}, u_{n,7}, u_{n,8}\). Proceeding like this until we reach the vertex \(u_{n,(m-3)}\) then finally assign the label 2, 4, 4 to the vertices \(u_{n,(m-2)}, u_{n,(m-1)}, u_{n,m}\).

**Subcase 2.16** \(n \equiv 3 \pmod{4}, m \equiv 3 \pmod{4}\).

As in Case 1 assign the label to the vertices \(u_i\) and \(u_{i,j}\).

Thus, the Table 9 below shows that this vertex labeling gives the 4− remainder cordial labeling of caterpillar.

<table>
<thead>
<tr>
<th>Nature of (n)</th>
<th>(v_f(1))</th>
<th>(v_f(2))</th>
<th>(v_f(3))</th>
<th>(v_f(4))</th>
<th>(\eta_c)</th>
<th>(\eta_o)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n \equiv 0 \pmod{4})</td>
<td>(\frac{nm+n}{4})</td>
<td>(\frac{nm+n}{4})</td>
<td>(\frac{nm+n}{4})</td>
<td>(\frac{nm+n-2}{2})</td>
<td>(\frac{nm+n}{2})</td>
<td></td>
</tr>
<tr>
<td>(n \equiv 1 \pmod{4})</td>
<td>(\frac{nm+n}{4})</td>
<td>(\frac{nm+n}{4})</td>
<td>(\frac{nm+n}{4})</td>
<td>(\frac{nm+n-2}{2})</td>
<td>(\frac{nm+n}{2})</td>
<td></td>
</tr>
<tr>
<td>(n \equiv 2 \pmod{4})</td>
<td>(\frac{nm+n}{4})</td>
<td>(\frac{nm+n}{4})</td>
<td>(\frac{nm+n}{4})</td>
<td>(\frac{nm+n-2}{2})</td>
<td>(\frac{nm+n}{2})</td>
<td></td>
</tr>
<tr>
<td>(n \equiv 3 \pmod{4})</td>
<td>(\frac{nm+n}{4})</td>
<td>(\frac{nm+n}{4})</td>
<td>(\frac{nm+n}{4})</td>
<td>(\frac{nm+n-2}{2})</td>
<td>(\frac{nm+n+2}{2})</td>
<td></td>
</tr>
</tbody>
</table>

**Table 9**

This completes the proof. \(\square\)
References

