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A Note on Smarandachely Consistent Symmetric n-Marked Graphs

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Abstract: A *Smarandachely k-marked graph* is an ordered pair $S = (G, \mu)$ where $G = (V, E)$ is a graph called *underlying graph of* S and $\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ is a function, where each $\overline{e}_i \in \{+, -\}$. An *$n$-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A *Smarandachely symmetric n-marked graph* is an ordered pair $S_n = (G, \mu)$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\mu : V \to H_n$ is a function. In this note, we obtain two different characterizations of Smarandachely consistent symmetric n-marked graphs. Also, we obtain some results by introducing special types of complementations.

Key Words: Smarandachely symmetric n-marked graphs, consistency, balance, complementation.

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§1. Introduction

For graph theory terminology and notation in this paper we follow the book [2]. All graphs considered here are finite and simple.

A *Smarandachely k-marked graph* is an ordered pair $S = (G, \mu)$ where $G = (V, E)$ is a graph called *underlying graph of* S and $\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ is a function, where each $\overline{e}_i \in \{+, -\}$.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$. A *Smarandachely symmetric n-marked graph* is an ordered pair $S_n = (G, \mu)$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\mu : V \to H_n$ is a function.

In this paper, by an *n-tuple/n-marked graph* we always mean a symmetric n-tuple /

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Smarandachely symmetric n-marked graph.

An n-tuple $(a_1, a_2, ..., a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-marked graph $S_n = (G, \mu)$ a vertex labelled with the identity n-tuple is called an *identity vertex*, otherwise it is a *non-identity vertex*. Further, in an n-marked graph $S_n = (G, \mu)$, for any $A \subseteq V(G)$ the *n-tuple* $\mu(A)$ is the product of the n-tuples on the vertices of A.

In [3], the authors defined different notions of balance in an n-marked graph $S_n = (G, \mu)$ as follows:

(i) S_n is *μi-balanced*, if product of n-tuples on each component of S_n is identity n-tuple.

(ii) S_n is *consistent* (*inconsistent*), if product of n-tuples on each cycle of S_n is identity n-tuple (non-identity n-tuple).

(iii) S_n is *balanced*, if every cycle (component) contains an even number of non-identity edges.

Note: (1)A μi-balanced (consistent) n-marked graph need not be balanced and conversely.
(2)A consistent n-marked graph need not be μi-balanced and conversely.

Proposition 1(Characterization of consistent n-marked graphs) *An n-marked graph $S_n = (G, \mu)$ is consistent if, and only if, for each k, $1 \leq k \leq n$, the number of n-tuples in any cycle whose k^{th} co-ordinate is $-$ is even.*

Proof Suppose S_n is consistent and let C be a cycle in S_n with number of n-tuples in any cycle whose k^{th} co-ordinate is $-$ is odd, for some k, $1 \leq k \leq n$. Then, the k^{th} co-ordinate in cycle of n-tuples on the vertices of the cycle C is $-$ and C is inconsistent cycle in S_n. Hence S_n is inconsistent a contradiction.

Converse part follows from the definition of consistent n-marked graphs. □

In [1], Acharya defined trunk on graphs as follows: Given a $u - v$ path $P = (u = u_0, u_1, u_2, ..., u_{m-1}, u_m = v)$ of length $m \geq 2$ in a graph G, the subpath $P' = (u_1, u_2, ..., u_{m-1})$ of P is called a *$u - v$ trunk* or the trunk of P. The following result will give the another characterization of consistent n-marked graph.

Proposition 2 *An n-marked graph $S_n = (G, \mu)$ is consistent if, and only if, for any edge $e = uv$, the n-tuple of the trunk of every $u - v$ path of length ≥ 2 is $\mu(u)\mu(v)$.*

Proof **Necessity:** Suppose $S_n = (G, \mu)$ is consistent. Let $e = uv$ be any edge of S_n and $P = (u = u_0, u_1, u_2, ..., u_{m-1}, u_m = v)$ be any $u - v$ path of length $m \geq 2$ in S_n. Then $C = P \cup \{e\}$ is a cycle in S_n which must have the number of n-tuples whose k^{th} co-ordinate is $-$ is even. Therefore,

$$\mu(P')\mu(u)\mu(v) = \mu(P) = \mu(C) = identity \;\; n - tuple \tag{1}$$

where P' is the trunk of P. Clearly (1), implies that $\mu(P')$ and $\mu(u)\mu(v)$ are equal. Since P was an arbitrarily chosen $u - v$ path of length ≥ 2 and also since the edge e was arbitrary by choice the necessary condition follows.

Sufficiency: Suppose that S_n satisfies the condition stated in the Proposition. We need to show that S_n is consistent. Let $C = (v_1, v_2, ..., v_h, v_1)$ be any cycle in S_n. Consider any edge $e = v_iv_{i+1}$ of C where indices are reduced modulo h. Then by the condition, we have

$$\mu(v_i)\mu(v_{i+1}) = \prod_{j\in \mathbf{h}-\{i,i+1\}} \mu(v_j), \qquad \mathbf{h} = \{1, 2, ..., h\} \tag{2}$$

because the section of P of C, not containing the edge v_iv_{i+1}, which is a $v_i - v_{i+1}$ path of length ≥ 2 in S_n satisfies the condition. Equation (2) shows that the number of same non-identity vertices in $\{v_i, v_{i+1}\}$ must be of the even or odd as the number of same non-identity vertices in $V(C) - \{v_i, v_{i+1}\}$. Clearly, this is possible if, and only if, the number of n-tuples cycle C whose k^{th} co-ordinate is $-$ is even if, and only if, C is consistent. Since C was an arbitrarily chosen cycle in S_n, it follows that S_n must be consistent. □

If we take $n = 1$ in the above Proposition, then the following result regarding 1-marked graph (i.e, marked graph).

Corollary 3(B. D. Acharya [1]) *A marked graph $S = (G, \mu)$ is consistent if, and only if, for any edge $e = uv$, the sign of the trunk of every $u - v$ path of length ≥ 2 is $\mu(u)\mu(v)$.*

§2. Complementation

In this section, we investigate the notion of complementation of graphs with multiple signs on their vertices. For any $t \in H_n$, the *t-complement* of $a = (a_1, a_2, ..., a_n)$ is: $a^t = at$. The *reversal* of $a = (a_1, a_2, ..., a_n)$ is: $a^r = (a_n, a_{n-1}, ..., a_1)$. For any $T \subseteq H_n$, and $t \in H_n$, the *t-complement* of T is $T^t = \{a^t : a \in T\}$.

Let $S_n = (G, \mu)$ and $S'_n = (G', \mu')$ be two n-marked graphs. Then S_n is said to be *isomorphic* to S'_n and we write $S_n \cong S'_n$, if there exists a bijection $\phi : V \to V'$ such that if $e = uv$ is an edge in S_n, u and v is labeled by $a = (a_1, a_2, ..., a_n)$ and $a' = (a'_1, a'_2, ..., a'_n)$ respectively, then $\phi(u)\phi(v)$ is an edge in S'_n and $\phi(u)$ and $\phi(v)$ which is labeled by a and a' respectively, and conversely.

For each $t \in H_n$, an n-marked graph $S_n = (G, \mu)$ is *t-self complementary*, if $S_n \cong S^t_n$.

Proposition 4 *For all $t \in H_n$, an n-marked graph $S_n = (G, \mu)$ is t-self complementary if, and only if, S^a_n is t-self complementary, for any $a \in H_n$.*

Proof Suppose S_n is t-self complementary. Then, $S_n \cong S^t_n$. This implies $S^a_n \cong S^{at}_n$.

Conversely, suppose that S^a_n is t-self complementary. Then, $S^a_n \cong (S^a_n)^t$. Since $(S^a_n)^a = S_n$. Hence $S_n \cong (S^{at}_n)^a = S^t_n$. □

Proposition 5 *Let $S_n = (G, \mu)$ be an n-marked graph. Suppose the underlying graph of S_n is bipartite. Then, for any $t \in H_n$, S_n is consistent if, and only if, its t-complement S^t_n is consistent.*

Proof Since S_n is consistent, by Proposition 1, for each $k, 1 \leq k \leq n$, the number of n-tuples on any cycle C in G whose k^{th} co-ordinate is $-$ is even. Also, since G is bipartite, for each $k, 1 \leq k \leq n$, number of n-tuples on C whose k^{th} co-ordinate is $+$ is also even. This implies that the same thing is true in any t-complement of S_n, where t can be any element of H_n. Hence S_n^t is i-balanced. Similarly, the converse follows, since for each $t \in H_n$, the underlying graph of S_n^t is also bipartite. $\square$

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