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A Note on Congruences for the Sum of Squares and Triangular Numbers

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Abstract: We derive some congruences for $r_k(n)$ and $t_k(n)$ using their generating functions, where $r_k(n)$ and $t_k(n)$ denote the number of representations of n as a sum of k squares and number of representations of n as a sum of k triangular numbers, respectively.

Key Words: Divisor sum, triangular numbers, sum of squares.

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§1. Introduction

The *sum of squares function*, denoted by $r_k(n)$, gives the number of representations of n as a sum of k squares, where zeros and distinguishing signs and order are allowed. For example, 5 can be written as a sum of two squares in the following ways

$$\begin{aligned} 5 &= (-2)^2 + (-1)^2 = (-2)^2 + (1)^2 \\ &= (2)^2 + (-1)^2 = (2)^2 + (1)^2 \\ &= (-1)^2 + (-2)^2 = (-1)^2 + (2)^2 \\ &= (1)^2 + (-2)^2 = (1)^2 + (2)^2. \end{aligned}$$

So, $r_2(5) = 8$.

The *generating function* for $r_k(n)$ is given by

$$\theta(q)^k = \sum_{n=0}^{\infty} (-1)^n \, r_k(n) \, q^n, \tag{1.1}$$

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where

$$\theta(q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \qquad (|q| < 1).$$

By Gauss's formula [1, formula 7.324], we know that

$$\theta(q) = \prod_{j=1}^{\infty} \frac{1-q^j}{1+q^j} = \prod_{n\geq 1} (1-q^{2n})(1-q^{2n-1})^2 \qquad (|q| < 1). \tag{1.2}$$

For any positive integer n, the numbers $n(n+1)/2$ are the *triangular numbers*. The *sum of triangular numbers function*, denoted by $t_k(n)$, gives the number of representations of n as a sum of r triangular numbers where representations with different orders are counted as unique. For instance, $t_2(7) = 2$ since $7 = 1 + 6 = 6 + 1$.

The generating function for $t_k(n)$ is given by

$$\Psi^k(q) = \sum_{n=0}^{\infty} t_k(n)\, q^n, \tag{1.3}$$

where

$$\Psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = 1 + q + q^3 + q^6 + \cdots \qquad (|q| < 1).$$

By Gauss's formula [1, Eq.7. 321 on p.6], we have

$$\Psi(q) = \prod_{j=1}^{\infty} \frac{(1-q^{2j})^2}{(1-q^j)} = \prod_{j=1}^{\infty} (1+q^j)^2\, (1-q^j) \qquad |q| < 1. \tag{1.4}$$

2. Some Congruences for $r_k(n)$ and $t_k(n)$

Lemma 2.1 *Let $S_1(n) = \sum_{odd\ d|n} \frac{2}{d}$, where $\sum_{odd\ d|n}$ denotes the sum over all odd divisors d of n. Then*

$$r_k(n) = \frac{-k}{n} \sum_{j=1}^{n} (-1)^j\, j\, S_1(j)\, r_k(n-j) \qquad (k, n \geq 1). \tag{2.1}$$

Proof Taking logarithm on both sides of equation (1.2), we have

$$\begin{aligned}
\log\theta(q) &= \sum_{j=1}^{\infty} \log(1-q^j) - \sum_{j=1}^{\infty} \log(1+q^j) \\
&= -\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} \frac{q^{lj}}{l} + \sum_{j'=1}^{\infty}\sum_{l'=1}^{\infty} \frac{q^{l'j'}(-1)^{l'}}{l'} \\
&= -\sum_{n=1}^{\infty} q^n \left(\sum_{d|n} \frac{1-(-1)^d}{d} \right).
\end{aligned}$$

From the equation (1.1), we get

$$\log\left\{\sum_{n=0}^{\infty}(-1)^n\, r_k(n)\, q^n\right\} = -k\sum_{n=1}^{\infty} S_1(n)\, q^n.$$

Differentiating the preceding equation with respect to q gives

$$\sum_{n=1}^{\infty}(-1)^n\, r_k(n)\, n\, q^{n-1} = -k\sum_{n=1}^{\infty} S_1(n)\, n\, q^{n-1}\sum_{n=0}^{\infty}(-1)^n\, r_k(n)\, q^n.$$

Comparing coefficients of q^n on both sides of the above equation we get equation (2.1). □

Lemma 2.2 *Let* $S_2(n) = \sum_{d|n}\frac{1+2\,(-1)^d}{d}$, *where* $\sum_{d|n}$ *denotes sum over all divisors* d *of* n. *Then*

$$t_k(n) = \frac{-k}{n}\sum_{j=1}^{n} j\, S_2(j)\, t_k(n-j) \qquad (k, n \geq 1). \tag{2.2}$$

Proof Taking logarithm on both sides of equation (1.4), we have

$$\begin{aligned}\log(\Psi(q)) &= \sum_{j=1}^{\infty} 2\,\log(1+q^j) + \sum_{j=1}^{\infty}\log(1-q^j)\\ &= -\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} 2\,\frac{(-1)^l\, q^{lj}}{l} - \sum_{j'=1}^{\infty}\sum_{l'=1}^{\infty}\frac{q^{l'j'}}{l'}\\ &= -\sum_{n=1}^{\infty} q^n \sum_{d|n}\frac{1+2\,(-1)^d}{d}.\end{aligned}$$

Then, we proceed as in the proof of the preceding lemma to arrive at equation (2.2). □

From equations (2.1) and (2.2), we deduce the following theorem.

Theorem 2.3 *Let* n *and* k *be integers such that* $(n, k) = 1$. *Then*

$$r_k(n) \equiv 0 \pmod{k}, \tag{2.3}$$

and

$$t_k(n) \equiv 0 \pmod{k}. \tag{2.4}$$

From equations (1.2) and (1.4), we deduce the following theorem.

Theorem 2.4 *For all primes* p, *we have*

$$r_{kp}(np) \equiv r_k(n) \pmod{p}, \tag{2.5}$$

and

$$t_{kp}(np) \equiv t_k(n) \pmod{p}. \tag{2.6}$$

Theorem 2.5 *If p is a prime number, then*

$$r_{p+1}(n) \equiv \sum_j r_1(t) \pmod{p}, \tag{2.7}$$

where j is an integer and $t = (n - j^2)/p$ is integer.

Proof Using

$$\theta(q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \qquad (|q| < 1),$$

we have

$$\sum_{n=0}^{\infty}(-1)^n \, r_{p+1}(n) \, q^n = \sum_{n=0}^{\infty}(-1)^n \, r_p(n) \, q^n \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}$$

Comparing coefficients of q^n on both sides in the above equation, we have

$$r_{p+1}(n) = \sum_j r_p(n - j^2).$$

Now from equation (2.3), we know that $r_p(n-j^2) \equiv 0 \pmod{p}$ if p and $n-j^2$ are co-prime. Also, from equation (2.5), when $(n - j^2)$ is divisible by p, we have

$$r_p(n - j^2) \equiv r_1(t) \pmod{p},$$

where $t = (n - j^2)/p$. □

Proceeding as in the proof of above theorem, we deduce the following theorem.

Theorem 2.6 *If p is a prime number, then*

$$t_{p+1}(n) \equiv \sum_j t_1(t) \pmod{p}, \tag{2.8}$$

where j is positive integer and $t = \left(n - \frac{j(j+1)}{2}\right)/p$ is integer.

References

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