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A Note on Laplacian Coefficients of the Characteristic Polynomial of L-Matrix of a Marked Graph

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Abstract: With this article in mind, we have found the characteristic polynomial of a Laplacian L-matrix of a graph with signs. Using the trace of the Laplacian L-matrix and the number of vertices of the marked graph, the coefficients of the characteristic polynomial have been found. Also we have shown that the same characteristic polynomial coefficients can be obtained using Laplacian eigenvalues of L-matrix. Further, we have obtained an upper bound for the largest eigenvalue of a signed graph.

Key Words: Signed graphs, neutrosophic singed graph, marked graphs, balance, switching.

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§1. Introduction

For all phrases and notes of graph theory, readers should refer to [6]. Here we consider simple and non-loop graphs including the restricted number of vertices and edges.

A *signed graph* $\Gamma = (G(V, E), \triangle)$ is a graph that describes each edge as an edge with positive or negative signs, where G is the graph without sign and $\triangle : E \longrightarrow \{+, -\}$ is a function. Generally, a *neutrosophic singed graph* G^N is such a graph with a bijective $\triangle^N : E \longrightarrow \{+, -\}$ for $e \in E$ neutrosophically, i.e., there is a partition on E with T of positive labels, I of negative labels and F of fail set such that $T \cup I \cup F = E$, where the labels on edges in F are undetermined, which may be all positive, negative or random completely. Particularly, if $F = \emptyset$, a neutrosophic singed graph is just a signed graph.

Usually, people and their relations are represented by a signed graph. One of the main applications of a signed graph is to study the relationships among people as explained in [3].

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Based on the nature of the relationship we can term them as positive or negative. If two people are friendly then their relationship is positive whereas if two people detest each other, there is a negative association between them. These concepts are discussed in [13] and [5].

The applications of signed graphs have been proposed and presented in [7], [8] and [19]. Harary [9] invented the phrase *balanced signed graph* to describe how a balanced signed graph has an even number of negative edges in each cycle.

The positive cycle in a signed graph is the cycle in which the product of signs of the edges is positive. This is referred to as balanced signed graph. Unbalanced signed graph is one which is not balanced (see Harary [9]). A simple *balanced signed graph* can be found using the graph algorithm created by Harary et.al.[11].

A *marking* of a graph Γ is a function $\mu : V(G) \to \{+,-\}$. A *marking* of Γ is a function $\mu : V(G) \to \{+,-\}$; A signed graph Γ together with a marking μ is denoted by Γ_μ. Given a signed graph Γ one can easily define a marking μ of Γ as for any vertex $v \in V(\Gamma)$,

$$\mu(v) = \prod_{uv \in E(\Gamma)} \triangle(uv)$$

and the marking μ of Γ is called *canonical marking* of Γ.

R. Abelson and Rosenberg in [16] were the first to present the *switching signed graph* as a tool to study social behavior. In [19] T.Zaslavasky has clearly explained the relevance of *switching signed graphs* mathematically. *Switching signed graph*, denoted as $\Gamma_\mu(\Gamma)$ and referred to as Γ_μ is a function that employs the marking $\flat$ to alter the mark of each edge of Γ.

Signed graphs $\Gamma_1 = (G_1, \triangle)$ and $\Gamma_2 = (G_2, \triangle')$ are isomorphic if their underline graphs G_1 and G_2 are also *isomorphic*. Here G_1 and G_2 are not signed graphs. Therefore, Γ_1 and Γ_2 are *switching equivalent* which is represented as $\Gamma_1 \sim \Gamma_2$. Specifically $\Gamma_\mu(\Gamma_1) \sim \Gamma_2$ for any marking μ and G_1 and G_2 remain unaltered. Furthermore, two signed graphs Γ_1, Γ_2 are said to cycle *isomorphic* if the cycles of two signed graphs have the same sign.

The following proposition is the characterization of switching signed graph, given by T.Zaslavasky [18].

Proposition 1.1 ([18]) *Two signed graphs Γ_1 and Γ_2 with the same underline graphs are switching equivalent if and only if they are cycle isomorphic.*

In a signed graph, the degree of each vertex can be calculated by $d = d^+ + d^-$ so that the degree of vertex in a signed graph Γ and their underline graphs is the same and in the adjacent matrix of a signed graph, if two vertices are adjacent then the entry a_{ij} is 1 along with the sign of the edge, otherwise the entry is zero. Furthermore, in a Laplacian matrix, if two vertices v_i and v_j are adjacent then the entries a_{ij} are 1 with the opposite sign of the corresponding adjacent edge v_iv_j, otherwise a_{ij} is zero and the diagonal entries a_{ii} being the degree of the vertex.

We know that $L(\Gamma,+)$ and $L(\Gamma,-)$ are the Laplacian matrices of the signed graphs $(\Gamma,+)$ and $(\Gamma,-)$ whose edges are all positive and negative respectively. Also $L(\Gamma,+)$ is the signless Laplacian matrix of Γ which is the sum of the diagonal matrix and the adjacent matrix.

Prof. E. Sampathkumar and M. A Sriraj in [21] have introduced a new matrix $A_{\mathtt{L}}(G)$ called $\mathtt{L}$-matrix of a vertex labelled graph G=(V,E), whose elements are defined as follows:

$$a_{ij} = \begin{cases} 2, \text{ if } v_i \text{ and } v_j \text{ are adjacent with } \mu(v_i) = \mu(v_j) \\ 1, \text{ if } v_i \text{ and } v_j \text{ are adjacent with } \mu(v_i) \neq \mu(v_j) \\ -1, \text{ if } v_i \text{ and } v_j \text{ are non adjacent with } \mu(v_i) = \mu(v_j) \\ 0, \text{ otherwise.} \end{cases}$$

With the motivation of $A_{\mathtt{L}}(G)$ (or $A_{\mathtt{L}}(\Gamma)$) we have Laplacian $\mathtt{L}$-matrix of a signed graph $\mathtt{L}(\Gamma)$ which is defined as follows:

$$b_{ij} = \begin{cases} -2, \text{ if } v_i \text{ and } v_j \text{ are adjacent with } \mu(v_i) = \mu(v_j) \\ -1, \text{ if } v_i \text{ and } v_j \text{ are adjacent with } \mu(v_i) \neq \mu(v_j) \\ 1, \text{ if } v_i \text{ and } v_j \text{ are non adjacent with } \mu(v_i) = \mu(v_j) \\ d(v_i), \text{ if i = j} \\ 0, \text{ otherwise.} \end{cases}$$

Also, $\mathtt{L}(\Gamma) = D(\Gamma) - A_{\mathtt{L}}(\Gamma)$, where $D(\Gamma)$ is the diagonal matrix and $A_{\mathtt{L}}(\Gamma)$ is the Adjacent $\mathtt{L}$-matrix of a signed graph Γ.

Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_n$ be the eigenvalues of the Laplacian $\mathtt{L}$-matrix of signed graph $\Gamma = (G, \triangle)$, having n vertices. In 2003 Yaoping Hou. et. al.[20] have established new bounds for eigenvalues of a signed graph as stated in the following theorem.

Theorem 1.2 ([20]) *Let $\Gamma = (G, \triangle)$ be a signed graph with n vertices. Then*

$$\lambda_1 \leq 2(n-1)$$

with equality holds if and only if, Γ is switching equivalent to a complete graph with all edges being negative.

§2. Laplacian Coefficients of the Characteristic Polynomial of $\mathtt{L}$-Matrix of a Marked Graph

The coefficients of a characteristic polynomial are useful in the study of chemical properties of molecules. In [14], Ivailo M. Mladenov et. al. have given very elegant algorithm to find coefficients of a characteristic polynomial using trace of the Laplacian matrix. In [2], Carla Silva Oliveria et. al. have found second and third Laplacian coefficients of a characteristic polynomial. Francesco Belardo et. al. [12] have found Laplacian coefficients of a marked signed graph as stated in the following theorem.

Theorem 2.1 ([12]) *The Laplacian characteristic polynomial of any signed graph* Γ *is*

$$\psi(\Gamma, x) = x^n + q_1 x^{n-1} + \cdots + q_{n-1}x + q_n,$$

where

$$q_p = (-1)^p \sum_{H \in H_p} w(H)$$

and the set of signed TU-sub graphs of Γ *with* p *edges is* H_p.

Using the above theorem and by the motivation of Faddeev LeVerrier algorithm and [14], [1] we present a new algorithm to find signed Laplacian coefficients of a characteristic polynomial using the trace of a Laplacian $\mathtt{L}$-matrix of Γ, whose underline graph is complete and $\mu(v_i) = \mu(v_j)$.

Proposition 2.2 *If* $\mathtt{L}$ *is Laplacian* $\mathtt{L}$*-matrix of a complete signed graph* Γ *and* $\mu(v_i) = \mu(v_j)$ *then,*

$$tr(\mathtt{L}^\ell) = (n-1)(n+1)^\ell + (-1)^\ell (n-1)^\ell.$$

Proof Here, the proof is by induction.

$$\begin{aligned} tr(\mathtt{L}) &= \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n \\ &= (n+1) + (n+1) + \ldots + (-1)(n-1) \\ &= (n-1)(n+1) + (-1)(n-1). \end{aligned}$$

We get

$$tr(\mathtt{L}^2) = (n-1)(n+1)^2 + (-1)^2(n-1)^2,$$

$$tr(\mathtt{L}^3) = (n-1)(n+1)^3 + (-1)^3(n-1)^3.$$

Similarly, for an integer k,

$$tr(\mathtt{L}^k) = (n-1)(n+1)^k + (-1)^k(n-1)^k,$$

$$\begin{aligned} tr(\mathtt{L}^{k+1}) &= \lambda_1^{k+1} + \lambda_2^{k+1} + \lambda_3^{k+1} + \cdots + \lambda_n^{k+1} \\ &= (n+1)^{k+1} + (n+1)^{k+1} + ... + (n+1)^{k+1} + (-1)^{k+1}(n-1)^{k+1} \\ &= (n-1)(n+1)^{k+1} + (-1)^{k+1}(n-1)^{k+1}. \end{aligned}$$

Hence, by induction,

$$tr(\mathtt{L}^\ell) = (n-1)(n+1)^\ell + (-1)^\ell(n-1)^\ell.$$

This completes the proof. □

For an example, we put $\ell = 1,2,3,4$ in the above expression, then

$$\begin{aligned}
tr(\mathtt{L}) &= n(n-1),\\
tr(\mathtt{L}^2) &= n^3+2n^2-3n,\\
tr(\mathtt{L}^3) &= n^4+n^3+3n^2-5n,\\
tr(\mathtt{L}^4) &= n^5+4n^4-2n^3+4n^2-7n.
\end{aligned}$$

The coefficients q_1, q_2, q_3 and q_4 of characteristic equation of a Laplacian $\mathtt{L}$-matrix of a signed graph Γ are calculated as follows:

$$\begin{aligned}
q_1 &= -tr(\mathtt{L}) = -n(n-1),\\
q_2 &= -\frac{1}{2}tr\left\{Z_1\mathtt{L}\right\} \quad \text{where } \mathrm{Z_1 = \mathtt{L} + q_1 I)}\\
&= -\frac{1}{2}tr\left\{(\mathtt{L}^2)+(-n^2+n)\mathtt{L}\right\}\\
&= -\frac{1}{2}\left\{tr(\mathtt{L}^2)-n^2tr(\mathtt{L})+ntr(\mathtt{L})\right\}\\
&= -\frac{1}{2}\left\{(n^3+2n^2-3n)-n^2(n^2-n)+n(n^2-n)\right\}\\
&= -\frac{1}{2}\left\{-n^4+3n^3+n^2-3n\right\},\\
q_3 &= -\frac{1}{3}tr\left\{Z_2\mathtt{L}\right\} \quad \text{where } \mathrm{Z_2 = Z_1\mathtt{L} + q_2 I)}\\
&= -\frac{1}{3}tr\left\{(\mathtt{L}+q_1I)\mathtt{L}^2+q_2L\right\}\\
&= -\frac{1}{3}tr\left\{\mathtt{L}^3+q_1\mathtt{L}^2+q_2\mathtt{L}\right\}\\
&= -\frac{1}{3}tr\left\{\mathtt{L}^3+(-n^2+n)\mathtt{L}^2+(\frac{-1}{2}(-n^4+3n^3+n^2-3n))\mathtt{L}\right\}\\
&= -\frac{1}{3}\left\{tr(\mathtt{L}^3)+(-n^2+n)tr(\mathtt{L}^2)-(\frac{1}{2}(-n^4+3n^3+n^2-3n)tr(\mathtt{L})\right\}\\
&= -\frac{1}{3}\big\{(n^4+n^3+3n^2-5n)+(-n^2+n)(n^3+2n^2-3n)\\
&\quad -\frac{1}{2}(-n^4+3n^3+n^2-3n)(n(n-1)\Big\}\\
&= -\frac{1}{6}(n^6-6n^5+2n^4+16n^3-3n^2-10n),\\
q_4 &= -\frac{1}{4}tr(Z_3\mathtt{L}) \quad (\text{where } \mathrm{Z_3 = Z_2\mathtt{L} + q_3 I)}\\
&= -\frac{1}{4}tr\left((Z_1\mathtt{L}+q_2)\mathtt{L}^2+q_3\mathtt{L}\right)\\
&= -\frac{1}{4}tr\left((\mathtt{L}+q_1)\mathtt{L}^3+q_2\mathtt{L}^2+q_3\mathtt{L}\right)\\
&= -\frac{1}{4}tr(\mathtt{L}^4+q_1\mathtt{L}^3+q_2\mathtt{L}^2+q_3\mathtt{L})\\
&= -\frac{1}{4}\left(tr(\mathtt{L}^4)+q_1tr(\mathtt{L}^3)+q_2tr(\mathtt{L}^2)+q_3tr(\mathtt{L})\right)
\end{aligned}$$

$$\begin{aligned} &= -\frac{1}{4}\left\{(n^5+4n^4-2n^3+4n^2-7n)+q_1(n^4+n^3+3n^2-5n)\right.\\ &\quad \left.+\,q_2(n^3+2n^2-3n)+q_3(-n(n-1))\right\}\\ &= -\frac{1}{24}\{-n^8+10n^7-17n^6-38n^5+61n^4+70n^3-43n^2-42n\}. \end{aligned}$$

Theorem 2.3 *If the Laplacian characteristic equation of* $\mathtt{L}$*-matrix of* Γ *is*

$$\psi(\Gamma,t)=y^n+t_1y^{n-1}+\cdots+t_{n-1}y+t_n$$

with $\mu(v_i)=\mu(v_j)$ *and* $t_0=1$*, then,*

$$t_\hbar=\frac{-1}{\hbar}\sum_{j=0}^{\hbar-1}t_jtr(L^{\hbar-j}),$$

where, $t_\hbar$ *are the coefficients of the characteristic polynomial and* $\hbar\neq 0$.

Proof Here,

$$\begin{aligned} t_1 &= -tr(\mathtt{L}),\\ t_2 &= -\frac{1}{2}tr(t_1\mathtt{L})=-\frac{1}{2}tr((\mathtt{L}+t_1)\mathtt{L})\\ &= -\frac{1}{2}\{tr(\mathtt{L}^2)+t_1tr(\mathtt{L})\}=-\frac{1}{2}\{t_0tr(\mathtt{L}^2)+t_1tr(\mathtt{L})\}. \end{aligned}$$

Similarly, for an integer k,

$$t_k=-\frac{1}{k}\sum_{j=0}^{k-1}t_j\ tr(\mathtt{L}^{k-j})$$

and we have,

$$\begin{aligned} t_{k+1} &= -\frac{1}{k+1}tr(t_k\mathtt{L})=-\frac{1}{k+1}tr((t_{k-1}L+t_k)\mathtt{L})\\ &= -\frac{1}{k+1}tr((\mathtt{L}t_{k-2}+t_{k-1})\mathtt{L}^2+t_k\mathtt{L})\\ &= -\frac{1}{k+1}(tr(\mathtt{L}^3t_{k-2})+t_{k-1}tr(\mathtt{L}^2)+t_ktr(\mathtt{L}))\\ &= -\frac{1}{k+1}(tr(\mathtt{L}^4t_{k-3})+tr(\mathtt{L}^3)t_{k-1}+tr(\mathtt{L}^2)t_k+tr(\mathtt{L})),\\ \cdots &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,\\ t_{k+1} &= \frac{-1}{k+1}\{t_0tr(\mathtt{L}^{k+1})+t_1tr(\mathtt{L}^k)+t_2tr(\mathtt{L}^{k-1})+\cdots+t_ktr(\mathtt{L})\}, \end{aligned}$$

i.e.,

$$t_{k+1}=-\frac{1}{k+1}\sum_{j=0}^{k}t_j\ tr(\mathtt{L}^{k+1-j}).$$

Hence by induction,

$$t_\hbar = -\frac{1}{\hbar}\sum_{j=0}^{\hbar-1} t_j tr(\mathtt{L}^{\hbar-j}),$$

where $\hbar \neq 0$. □

Corollary 2.4 *For any signed graph Γ, let $\mu(v_i) = \mu(v_j)$ and $t_0 = 1$. Then, the Laplacian characteristic polynomial of $\mathtt{L}-$ matrix of Γ is*

$$\psi(\Gamma, y) = y^n + t_1 y^{n-1} + \cdots + t_{n-1} y + t_n$$

with $t_0 = 1$ and

$$t_\imath = -\frac{1}{\imath}\sum_{j=0}^{\imath-1} t_j((n-1)(n+1)^{\imath-j} + (-1)^{\imath-j}(n-1)^{\imath-j})$$

where $\imath \neq 0$.

Proof By Proposition 2.2, we have

$$tr(\mathtt{L}^{\imath-j}) = (n-1)(n+1)^{\imath-j} + (-1)^{\imath-j}(n-1)^{\imath-j}$$

. By Theorem 2.3, we get

$$t_\imath = -\frac{1}{\imath}\sum_{j=0}^{\imath-1} t_j((n-1)(n+1)^{\imath-j} + (-1)^{\imath-j}(n-1)^{\imath-j})$$

where $\imath \neq 0$. □

Corollary 2.5 *For any signed graph Γ and $\Gamma \sim (K_n, -)$, the Laplacian characteristic polynomial of Γ is $\psi(\Gamma, y) = y^n + t_1 y^{n-1} + \cdots + t_{n-1} y + t_n$ with $t_0 = 1$ and $\lambda_1, \lambda_n, \lambda_{n-1}$ are the Laplacian eigenvalues of $\mathtt{L}$-matrix of signed graph Γ then,*

$$t_\varsigma = -\frac{1}{\varsigma}\sum_{r=0}^{\varsigma-1} t_r \lambda_n(\lambda_1^{\varsigma-r} + (-1)^{\varsigma-r}\lambda_n^{\varsigma-r-1}),$$

where $\varsigma \neq 0$.

Proof Since $\lambda_1 = (n+1)$, $\lambda_n = -(n-1)$and hence by Corollary 2.4,

$$t_\varsigma = -\frac{1}{\varsigma}\sum_{r=0}^{\varsigma-1} t_r \lambda_n(\lambda_1^{\varsigma-r} + (-1)^{\varsigma-r-1}\lambda_n^{\varsigma-r})$$

where $\varsigma \neq 0$. □

With the motivation of Theorem 1.2, we will find the following upper bound.

Proposition 2.6 *For any graph with sign Γ, let λ_1 be the maximum Laplacian eigenvalue of*

$\mathtt{L}$*-matrix of* Γ. *Then,*

$$\lambda_1 \leq \sqrt{n^3 - 2n^2 + 5n}.$$

Proof By the Cauchy-Schwartz inequality,

$$\begin{aligned}(\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n)^2 &\leq n\ (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \cdots + \lambda_n^2), \\ (\lambda_1 + (n-2)(n+1) + 2)^2 &\leq n((n-1)(n+1)^2 + 2^2).\end{aligned}$$

This leads to

$$\lambda_1^2 \leq n^3 - 2n^2 + 5n$$

and the proof is completes. □

§3. Conclusion

In this paper we have found the coefficients of the characteristic polynomial of a Laplacian $\mathtt{L}$-matrix of a signed graph Γ in two ways:

1) By using the number of vertices of Γ as shown in the following table;
2) By using the Laplacian eigenvalues of $\mathtt{L} - matrix$ of signed graph Γ.

Also, we have found an upper bound for the largest eigenvalue of a Laplacian $\mathtt{L}$-matrix.

Number of Vertices	**Coefficients of Characteristic Polynomial**
3	t_1=-6, t_2=0, t_3=32
4	t_1=-12, t_2=30, t_3=100, t_4=-375
5	t_1=-20, t_2=-120, t_3=0, t_4=-2160, t_5=5184
6	t_1=-30, t_2=315, t_3=-980, t_4=-5145, t_5=43218, t_6=-8405,
7	t_1=-42, t_2=672, t_3=-4480, t_4=0, t_5=172032, t_6=-917504, t_7=1572864
8	t_1=-56, t_2=1260, t_3=-13608, t_4=51030, t_5=367416, t_6=-4960116, t_7=21257640, t_8=-33480783
9	t_1=-72, t_2=2160, t_3=-33600, t_4=252000, t_5=0, t_6=-16800000, t_7=144000000, t_8=-540000000, t_9=800000000
10	t_1=-90, t_2=3465, t_3=-72600, t_4=838530, t_5=-3689532, t_6=-33820710, t_7=637761960, t_8=-4384613475, t_9=15005121670, t_{10}=-21221529219

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