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A Note on Product Irregularity Strength of Graphs

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Abstract: For a graph G without isolated vertices and without isolated edges, a *product irregular labeling* $w : E(G) \to \{1, 2, \cdots .m\}$ is a labeling of the edges of G such that for any two distinct vertices u and v of G the product of labels of the edges incident with u is different from the product of labels of the edges incident with v. The minimal m for which there exist a product irregular labeling is called the *product irregularity strength* of G and is denoted by $ps(G)$. In this note, we find the product irregularity strength of block graph of cycle-star graph and sunlet graph.

Key Words: Smarandachely H product-irregular labeling, product-irregular labeling, product irregularity strength, block graph, cycle-star graph, sunlet graph.

AMS(2010): 05C05, 05C15, 05C78.

§1. Introduction

Throughout this paper let G be a simple graph, i.e., a graph without loops or multiple edges, without isolated vertices and without isolated edges. Let the vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. Let $w : E(G) \to \{1, 2, \cdots .m\}$ be an integer labeling of the edges of G. Then the *product degree* $pd_G(v)$ of a vertex $v \in V(G)$ in the graph G with respect to the labeling w is defined by

$$pd_G(v) = \prod_{v \in e} w(e).$$

A labeling w is said to be *product-irregular* if for every pair of vertices $u, v \in V(G)$, $u \neq v$,

$$pd_G(u) \neq pd_G(v).$$

Generally, for a typical subgraph $H \prec G$, a labeling w is said to be *Smarandachely H product-irregular* if for every pair of vertices $u, v \in V(G)$, $u \neq v$, there are $pd_G(u) \neq pd_G(v)$ for $u, v \in V(G) \backslash V(H)$ but $pd_G(u) = pd_G(v)$ for $u, v \in V(H)$. Clearly, if $H = \emptyset$, such a Smarandachely H product-irregular property is nothing else but the product-irregular property.

The *product irregularity strength* $ps(G)$ of G is the smallest value of m for which there exists a product-irregular labeling $w : E(G) \to \{1, 2, \cdots .m\}$.

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This concept was first introduced by Anholcer in [1] as a multiplicative version of the well studied concept of irregularity strength of graphs introduced by Chartrand et al. in [3].

The corona product of two graphs G and H, denoted by $G \bigodot H$, is a graph obtained by taking one copy of G (which has n vertices) and n copies $H_1, H_2, \cdots, H_n$ of H, and then joining the ith vertex of G to every vertex in H_i. The corona product $C_n \bigodot K_1$ is called the *sunlet graph.*

A graph G is *connected* if between any two distinct vertices there is a path. A maximal connected subgraph of G is called a *component* of G. A *cut-vertex* of a graph is one whose removal increases the number of components. A *non-separable graph* is connected, nontrivial, and has no cut-vertices. A *block* of a graph is a maximal non-separable subgraph. If two distinct blocks B_1 and B_2 are incident with a common cut-vertex, then they are called *adjacent blocks.*

There are many graph operators (or graph valued functions) with which one can construct a new graph from a given graph, such as the line graphs, the block graphs, and their generalizations.

The *block graph* of a graph G, written $B(G)$, is the graph whose vertices are the blocks of G and in which two vertices are adjacent whenever the corresponding blocks have a cut-vertex in common.

Jelena Sedlar [5] introduced the concept of cycle-star graph as follows: The *cycle-star graph*, written $CS_{k,n-k}$, is a graph with n vertices consisting of the cycle graph of length k and $n-k$ leafs appended to the same vertex of the cycle.

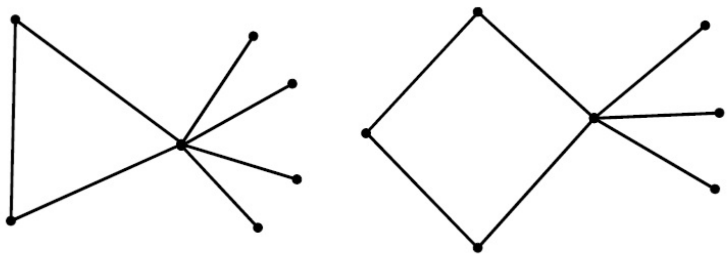

Figure 1 The cycle-star graphs $CS_{3,4}$ and $CS_{4,3}$

§2. Preliminary Results

Let n_d denote the number of vertices of degree d, where $\delta(G) \leq d \leq \Delta(G)$. Anholcer in [1] showed that

$$ps(G) \geq max_{\delta(G)\leq d\leq \Delta(G)} \left\{ \left\lceil \frac{d}{e} n_d^{\frac{1}{d}} - d + 1 \right\rceil \right\}. \tag{1}$$

If the graph G is $r-$regular, then the expression (1) reduces to

$$ps(G) \geq \left\lceil \frac{r}{e} n^{\frac{1}{r}} - r + 1 \right\rceil. \tag{2}$$

Also, for a cycle C_n on $n \geq 3$ vertices, the bounds on $ps(C_n)$ is given in [1]. That is, for $n \geq 3$,

$$ps(C_n) \geq \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil$$

for $n > 17$,

$$ps(C_n) \geq \left\lceil \left(\frac{n}{1 - log_e 2} \right)^{\frac{1}{2}} \right\rceil$$

and that for every $\epsilon > 0$ there exists n_0 such that for every $n \geq n_0$,

$$ps(C_n) \leq \left\lceil (1+\epsilon)\sqrt{2n}\ log_e n \right\rceil .$$

Anholcer in [2] considered the product irregularity strength of complete bipartite graphs $K_{m,n}$ and proved that for two integers m and n such that $2 \geq m \geq n$, $ps(K_{m,n}) = 3$ if and only if $n \geq \binom{m+2}{2}$.

However, the studies on the product irregularity strength of the intersection graph on the vertex set of a graph was not attempted. In this paper we have made an attempt to fill this gap and study the the product irregularity strength of the block graph of cycle-star graph and sunlet graph.

§3. Product Irregularity Strength of Block Graph of Cycle-Star Graph $CS_{k,n-k}$

The following result in [4] determines the exact value of product irregularity strength of a complete graph K_n on $n \geq 3$ vertices.

Theorem 3.1 *For every complete graph K_n on $n \geq 3$ vertices, $ps(K_n) = 3$.*

We now use Theorem 3.1 to find the exact value of product irregularity strength of block graph of cycle-star graph $CS_{k,n-k}$ for $k \geq 3$ and $n - k \geq 2$.

Theorem 3.2 *Let $G = CS_{k,n-k}$ be a cycle-star graph, where $k \geq 3$ and $n - k \geq 2$. Then $ps(B(G)) = 3$.*

Proof Since the block graph a cycle-star graph $CS_{k,n-k}$ with $k \geq 3$ and $n - k \geq 2$ leafs is a complete graph K_n on $n \geq 3$ vertices, from Theorem 3.1, it follows that $ps(B(G)) = 3$. This completes the proof. □

§4. Product Irregularity Strength of Block Graph of Sunlet Graph $C_n \bigodot K_1$

In this section we find the exact value of product irregularity strength of block graph of sunlet graph $C_n \bigodot K_1, n \geq 3$.

Theorem 4.1 *Let $G = C_n \bigodot K_1$, $n \geq 3$, be a sunlet graph. Then $ps(B(G)) = n$.*

Proof Let $G = C_n \bigodot K_1$, $n \geq 3$, be a sunlet graph. By definition, the block graph of sunlet graph is a star graph $K_{1,n}$ on $n \geq 3$ vertices. Let $v_1, v_2, \cdots, v_n$ be pendant vertices and v_0 be the central vertex of $K_{1,n}$. At first, let us weight all the edges consecutively starting from 1 to n. Then the product degree of vertices $v \in B(G)$ is $pd_{B(G)}(v_i) = i$ for $1 \leq i \leq n$ and $pd_{B(G)}(v_i) = n!$. Clearly, product degrees of all vertices are distinct. Hence $ps(B(G)) = n$. This completes the proof. □

§5. Conclusion

In this note, we have found the exact values of product irregularity strength of block graph of cycle-star graph and sunlet graph. However, to find the exact values of product irregularity strength of different graph operators still remain open.

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