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A Study of
Kenmotsu Manifolds with Semi-Symmetric Metric Connection

B. Laha

(Department of Mathematics, Shri Shikshayatan College, Kolkata, India)

A. Mallick

(Department of Mathematics, Heramba Chandra College, Kolkata, India)

E-mail: barnali.laha87@gmail.com, arupkm14@gmail.com

Abstract: The present paper aims to study semi-symmetric metric connection on Kenmotsu Manifolds. First section introduces us with the development of Kenmotsu manifolds. Next section gives us some preliminary ideas about the manifold. Here we have studied the necessary condition under which a vector field will be a strict-contact vector field. In the next section we have extended our study to generalized ϕ-recurrent $n = 2m+1$-dimensional Kenmotsu manifold with respect to semi-symmetric metric connection. Further we have studied this manifold satisfying the condition $L.S = 0$ w.r.t semi-symmetric connection. Lastly we have cited an example of Kenmotsu manifold with semi-symmetric metric connection.

Key Words: Kenmotsu manifolds, semi-symmetric metric connection, conharmonically curvature tensor, extended generalized $\phi-$recurrent Kenmotsu manifolds.

AMS(2010): 53C15, 53C25, 53C40.

§1. Introduction

In [24], S.Tanno classified the connected almost contact metric manifold whose automorphism group has maximum dimension, which are three classes following:

a) the homogeneous normal contact Riemannian manifolds with constant $\phi-$ holomorphic sectional curvature if the sectional curvature of the plain section containing ξ, say $C(X, \xi) > 0$.

b) the global Riemanian product of a line or a circle and a Käehlerian manifold with constant holomorphic sectional curvature, $C(X, \xi) = 0$.

c) a warped product space $RX_{\lambda}C^n$, if $C(X, \xi) < 0$.

The manifold of class (a) are characterized by some tensor equations, it has a Sasakian structure and manifolds of class (b) are characterized by a tensorial relation admitting a cosymplectic structure. In 1972 Kenmotsu has introduced a new class of almost contact Riemannian manifolds which are nowadays called Kenmotsu manifolds [11]. He obtained some tensorial equations to characterize manifolds of class (c).

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Let (M, ϕ, ξ, η, g) be a $n = 2m + 1$ dimensional almost contact metric manifold. Then the product $\bar{M} = MXR$ has a natural almost complex structure J with the product metric G being Hermitian manifold $(\bar{M}, J, G)$. The notion of trans-sasakian manifolds was introduced by Oubina [15] in 1985. In general, a Trans-sasakian manifold $(M, \phi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type (α, β). Trans-Sasakian manifold of type $(0, \beta)$ is called β-Kenmotsu manifolds. In 1932, Hayden has given the notion of metric connection with torsion on Riemannian manifold [10]. The semi-symmetric connection on Riemannian manifold was studied by K.Yano [25] in 1970. The SemiC symmetric connections on Riemannian manifold was also studied by K.S. Amur [2], S.S. Pujar, C.S. Bagewadi [4] et.al in 1976.

The notion of local symmetry of a Riemannian manifolds has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi [22] introduced the notion of locally $\phi-$symmetry on a Sasakian manifolds. Generalizing the notion of $\phi-$symmetry one of the authors in [3] introduced the notion of $\phi-$recurrent Kenmotsu manifolds.

The notion of generalized recurrent manifolds has been introduced by Dubey [9] and studied by De and Guha [7]. Again, the notion of generalized Ricci-recurrent manifolds has been introduced and studied by De et al. [8]. A Riemannian manifold $(M^n, g), n > 2$, is called generalized recurrent [9], [7] if its curvature tensor R satisfies the condition

$$\nabla R = A \otimes R + B \otimes G, \tag{1.1}$$

where A and B are non-vanishing 1-forms defined by $A(\delta) = g(\delta, \rho 1), B(\delta) = g(\delta, \rho 2)$ and the tensor G is defined by

$$G(X, Y)Z = g(Y, Z)X - g(X, Z)Y \tag{1.2}$$

for all $X, Y, Z \in \chi(M)$; $\chi(M)$ being the Lie algebra of smooth vector fields on M and ∇ denotes the operator of covariant differentiation with respect to the metric g. The 1-forms A and B are called the associated 1-forms of the manifold. A Riemannian manifold $(M^n, g), n > 2$, is called generalized Ricci-recurrent [6] if its Ricci tensor S of type $(0, 2)$ satisfies the condition $\nabla S = A \otimes S + B \otimes g$, where A and B are non-vanishing 1-forms. In 2007, Özgür [16] studied generalized recurrent Kenmotsu manifolds. Recently Basari and Murathan [3] introduced the notion of generalized ϕ-recurrent Kenmotsu manifolds generalized the notion of Özgür. For extending the notion of Basari and Murathan in [3], A. Shaikh [20] introduce the notion of extended generalized ϕ-recurrent β-Kenmotsu manifolds. We in this paper have further studied and established few results on generalized ϕ-recurrent Kenmotsu manifolds.

§2. **Preliminaries**

Let $M^n(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold where ϕ is a tensor field of type $(1, 1)$, ξ is a vector field and η is a 1-form and g is the induced Riemannian metric on $\tilde{M}$ satisfying

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \tag{2.2}$$

$$g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, \phi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y) \tag{2.4}$$

for all vector fields X, Y on M. Now if

$$(\nabla_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi, \tag{2.5}$$

where ∇ is the Riemannian connection of g, then(M, ϕ, ξ, η, g) is called a Kenmotsu manifold. On Kenmotsu manifold M, we also have

$$\nabla_X \xi = X - \eta(X)\xi \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$.

Also we have the following relations on Kenmotsu manifolds

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.8}$$

$$S(X, \xi) = -(n-1)\eta(X). \tag{2.9}$$

Since $S(X, Y) = g(QX, Y)$ we can get

$$S(\phi X, \phi Y) = g(Q\phi X, \phi Y). \tag{2.10}$$

We have by using (2.1), (2.9), $Q\phi = \phi Q$ and $g(X, \phi Y) = -g(\phi X, Y)$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y). \tag{2.11}$$

Also, we have

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y) \tag{2.12}$$

and

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X). \tag{2.13}$$

Now we shall mention few definitions which are required to establish the theorems.

Definition 2.1 *A Kenmotsu manifold is said to be $\eta-$Einstein if its Ricci tensor S satisfies the form*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.14}$$

where a, b are smooth functions.

Definition 2.2 *A vector field X on a Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ is said to be Contact vector field if*

$$(\mathcal{L}_X \eta)(Y) = \sigma\eta(y), \tag{2.15}$$

where σ is a scalar function on M and $\mathcal{L}_X$ denote the lie derivative along X. X is called strict Contact vector field if $\sigma = 0$.

A relation between the curvature tensor R and $\bar{R}$ of type $(1,3)$ of the connections ∇ and $\bar{\nabla}$ respectively is given by [25]

$$\bar{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(Z,X)Y. \tag{2.16}$$

Also Ricci tensor satisfies

$$\bar{S}(Y,Z) = S(Y,Z) - 2g(Y,Z) + 2\eta(Z)\eta(Y) + g(\phi Y, Z), \tag{2.17}$$

where $\bar{S}$ and S are Ricci tensor of M with respect to semi-symmetric metric connections $\bar{\nabla}$ and the Levi-Civita connection ∇, respectively. Also we have

$$\bar{S}(Y,Z) = S(Y,Z) + 2mg(Y,Z). \tag{2.18}$$

§3. Geometric Vector Fields on Kenmotsu Manifold with Respect to Semi-Symmetric Metric Connections

In this section we shall give the following proof on vector field.

Theorem 3.1 *Every contact vector field on a Kenmotsu manifold leaving the Ricci tensor with respect to semi-symmetric connection invariant is a strict contact vector field.*

Proof Let a Contact vector field X on a Kenmotsu manifolds leaves the Ricci tensor with respect to semi-symmetric metric connections invariant i.e

$$(\mathcal{L}_X \bar{S})(Y,Z) = 0. \tag{3.1}$$

From (3.1) we have $(\mathcal{L}_X \bar{S})(Y,Z) = \bar{S}(\mathcal{L}_X Y, Z) + \bar{S}(Y, \mathcal{L}_X Z)$. Putting $Z = \xi$ we obtain

$$(\mathcal{L}_X \bar{S})(Y,\xi) = \bar{S}(\mathcal{L}_X Y, \xi) + \bar{S}(Y, \mathcal{L}_X \xi). \tag{3.2}$$

Putting $Z = \xi$ in (2.16) we can get

$$\bar{S}(Y,\xi) = -(n-1)\eta(Y). \tag{3.3}$$

Taking Lie derivative on both the sides of the above equation and using definition (2.2) we can obtain

$$-(n-1)\sigma\eta(Y) = \bar{S}(Y, \mathcal{L}_X \xi). \tag{3.4}$$

Taking $Y = \xi$ in (3.4) we find

$$\eta(\mathcal{L}_X \xi) = \sigma. \tag{3.5}$$

Again from (2.14) snd using the definition for Lie derivative we can infer

$$-\eta(\mathcal{L}_X\xi) = \sigma. \tag{3.6}$$

Hence, combining (3.5) and (3.6) we conclude that $\sigma = 0$ and therefore the proof. □

§4. On Extended Generalized ϕ-Recurrent Kenmotsu Manifold with Respect to Semi-Symmetric Metric Connections

For this section we first define the following terms.

Definition 4.1 *A Kenmotsu manifolds with respect to semi-symmetric connection is said to be a ϕ-recurrent manifold if there exists a non-zero $1-$form B such that*

$$\phi^2((\bar{\nabla}_W R)(X,Y)Z) = B(W)R(X,Y)Z$$

for arbitrary vector fields X, Y, Z, W.

Again we define $\phi-$generalized recurrent Kenmotsu manifold.

Definition 4.2 *A Riemannian manifold (M^n, g) is called $\phi-$generalized recurrent [7], if its curvature tensor R satisfies the condition*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]$$

where A and B are two $1-$forms, B is non zero and these are defined by

$$g(W,\rho_1) = A(W), g(W,\rho_2) = B(W)$$

for all $W \in \chi(M)$. Here ρ_1 and ρ_2 being the vector fields associated to the $1-$form A and B respectively.

Lastly we define an extended generalized $\phi-$recurrent Kenmotsu manifolds.

Definition 4.3 *A Kenmotsu manifold is said to be an extended generalized $\phi-$recurrent Kenmotsu manifolds if its Curvature tensor R satisfies the relation*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)\phi^2(R(X,Y)Z) + B(W)\phi^2[g(Y,Z)X - g(X,Z)Y]$$

for all $X, Y, Z, W \in \chi(M)$ where A, B are two non-vanishing 1-forms such that $g(W,\rho_1) = A(W)$ and $g(W,\rho_2) = B(W)$ for all $W \in \chi(M)$ with ρ_1 and ρ_2 being the vector fields associated 1-form A and B, respectively [16].

In this connection, we mention the works of Prakasha [17] on Sasakian manifolds. Then we can state the following theorem.

Theorem 4.4 *Suppose M^n is an $\eta-$Einstein Kenmotsu manifolds. If b and a are constant functions then either M^n is an Einstein manifold or M^n is an α Kenmotsu manifolds.*

Proof We first suppose that M^n is an $\eta-$Einstein Kenmotsu manifolds. Then the Ricci tensor satisfies the following relation

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{4.1}$$

where a, b are smooth functions on M^n. Putting $X = Y = \xi$ in (4.1) we get

$$S(\xi, \xi) = a + b.$$

Therefore from above we can calculate that

$$a + b = -(n+1). \tag{4.2}$$

In local coordinate (4.1) can be written as

$$R_{ij} = ag_{ij} + b\eta_i\eta_j. \tag{4.3}$$

On contraction of (4.3) with g^{ij} we get

$$r = 3a + b. \tag{4.4}$$

Taking Covariant derivative with respect to k from the equation (4.3) we obtain

$$R_{ij.k} = a_{.k}g_{ij} + b_{.k}\eta_i\eta_j + b\eta_{i.k}\eta_j + b\eta_i\eta_{j.k}. \tag{4.5}$$

Contracting (4.5) with g^{ik} we get

$$R^k_{j.k} = a_{.j} + b_{.k}\xi^k\eta_j + b\eta_{i.k}g^{ik}\eta_j + b\eta_i\eta_{j.k}g^{ik}. \tag{4.6}$$

We also know that

$$R^a_i = g^{aj}R_{ij}.$$

From Bianchi's Identity we have

$$R^a_{ijk.a} + R^a_{ika.j} + R^a_{iaj.k} = 0.$$

We can write from above equation

$$R^a_{ijk.a} + R_{ik,j} - R_{ij.k} = 0.$$

Multiplying above equation by g^{ij} we can obtain

$$g^{ij}R^a_{ijk.a} + g^{ij}R_{ik.j} - g^{ij}R_{ij.k} = 0.$$

Simplifying the above equation we have

$$2R^k_{j.k} = r_{.j}. \tag{4.7}$$

From (4.6) and (4.7) we can obtain

$$r_{.j} = 2R^k_{j.k} = 2[a_{.j} + b_{.k}\xi^k\eta_j + b\eta_{i.k}g^{ik}\eta_j + b\eta_i\eta_{j.k}g^{ik}]. \tag{4.8}$$

Solving above we get

$$\eta_{i.k}g^{ik} = n - \eta^k\eta_k = n - 1. \tag{4.9}$$

Therefore

$$r_{.j} = 2[a_{.j} + b_{.k}\xi^k\eta_j + (n-1)b\eta_j]. \tag{4.10}$$

Again taking Covariant derivative w.r.t k from equation (4.2)

$$a_{.k} + b_{.k} = 0. \tag{4.11}$$

Also taking Covariant derivative w.r.t j from equation (4.4) we can calculate

$$r_{.j} = 3a.j + b_{.j} = 2a.j + a.j + b_{.j} = 2a.j. \tag{4.12}$$

From equation (4.10) and (4.12) we have

$$(n-1)b = 0.$$

If $n > 1$ then above equation yields $b = 0$. Hence, we get the theorem. □

Theorem 4.5 *An extended generalized $\phi-$recurrent Kenmotsu manifold (M^n, g) with respect to semi-symmetric metric connection is an Einstein manifold and the $1-$forms A and B are related as $(n-1)A(W) - 2B(W) = 0$.*

Proof Consider an extended generalized $\phi-$recurrent Kenmotsu manifold $(M^n, \phi, \eta, \xi, g)$ with respect to semi-symmetric metric connection. The we have from definition (4.3)

$$\phi^2((\bar\nabla_W\bar R)(X,Y)Z) = A(W)\phi^2(\bar R(X,Y)Z) + B(W)\phi^2[g(Y,Z)X - g(X,Z)Y]. \tag{4.13}$$

Using (1.2), (2.1) and (4.13) we can obtain

$$\begin{aligned} &-(\bar\nabla_W\bar R)(X,Y)Z + \eta((\bar\nabla_W\bar R)(X,Y)Z)\xi \\ &= A(W)[-\bar R(X,Y)Z + \eta(\bar R(X,Y)Z)\xi] + B(W)[-g(Y,Z)X + g(X,Z)Y \\ &\quad +\eta(X)g(Y,Z)\xi - g(X,Z)\eta(Y)\xi]. \end{aligned} \tag{4.14}$$

Taking inner product of (4.14) with U and using (2.3) we can calculate

$$-g((\bar\nabla_W\bar R)(X,Y)Z, U) + \eta((\bar\nabla_W\bar R)(X,Y)Z)g(\xi, U)$$

$$= A(W)[-g(\bar{R}(X,Y)Z,U) + \eta(\bar{R}(X,Y)Z)g(\xi,U)] + B(W)[-g(Y,Z)g(X,U)$$
$$+g(X,Z)g(Y,U) + \eta(X)g(Y,Z)g(\xi,U) - g(X,Z)\eta(Y)g(\xi,U)] \tag{4.15}$$

Again applying (4.15) we have

$$-g((\bar{\nabla}_W\bar{R})(X,Y)Z,U) + \eta((\bar{\nabla}_W\bar{R})(X,Y)Z)\eta(U)$$
$$= A(W)[-g(\bar{R}(X,Y)Z,U) + \eta(\bar{R}(X,Y)Z)\eta(U)] + B(W)[-g(Y,Z)g(X,U)$$
$$+g(X,Z)g(Y,U) + \eta(X)g(Y,Z)\eta(U) - g(X,Z)\eta(Y)\eta(U)]. \tag{4.16}$$

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis for the tangent space of M^n at a point $p \in M^n$. Putting $X = U = e_i$ in (4.16) and taking summation over i from 1 to n, we have

$$-(\bar{\nabla}_W\bar{S})(Y,Z) + \sum_{i=1}^{n}\eta((\bar{\nabla}_W\bar{R})(e_i,Y)Z)\eta(e_i)$$
$$= A(W)[-\bar{S}(Y,Z) + \eta(\bar{R}(\xi,Y)Z)] + B(W)[-g(Y,Z) - \eta(Y)\eta(Z)] \tag{4.17}$$

Putting $Z = \xi$ in (4.17) we get

$$-(\bar{\nabla}_W\bar{S})(Y,\xi) + \sum_{i=1}^{n}\eta((\bar{\nabla}_W\bar{R})(e_i,Y)\xi)\eta(e_i)$$
$$= A(W)[-\bar{S}(Y,\xi) + \eta(\bar{R}(\xi,Y)\xi)] + B(W)[-g(Y,\xi) - \eta(Y)\eta(\xi)] \tag{4.18}$$

On simplifying above equation we have

$$-(\bar{\nabla}_W\bar{S})(Y,\xi) + \sum_{i=1}^{n}\eta((\bar{\nabla}_W\bar{R})(e_i,Y)\xi)\eta(e_i) = -A(W)\bar{S}(Y,\xi) - 2B(W)\eta(Y). \tag{4.19}$$

Taking the second term of (4.19) we can calculate

$$\begin{aligned}\eta((\bar{\nabla}_W\bar{R})(e_i,Y)\xi)\eta(e_i) &= g(\bar{\nabla}_W\bar{R}(e_i,Y)\xi,\xi) - g(\bar{R}(\bar{\nabla}_We_i,Y)\xi,\xi)\\ &\quad -g(\bar{R}(e_i,\bar{\nabla}_WY)\xi,\xi) - g(\bar{R}(e_i,Y)\bar{\nabla}_W\xi,\xi).\end{aligned} \tag{4.20}$$

Let $p \in M^n$, since e_i is an orthonormal basis, so $\bar{\nabla}_We_i = 0$ at p. Also

$$g(\bar{R}(e_i,Y)\xi,\xi) = -g(\bar{R}(\xi,\xi)Y,e_i) = 0. \tag{4.21}$$

Since $\nabla_Wg = 0$, we have

$$g(\bar{\nabla}_W\bar{R}(e_i,Y)\xi,\xi) + g(\bar{R}(e_i,Y)\xi,\bar{\nabla}_W\xi) = 0. \tag{4.22}$$

From (4.20) and (4.22) we can obtain

$$g((\bar{\nabla}_W\bar{R})(e_i,Y)\xi,\xi) = -g(\bar{R}(e_i,Y)\xi,\bar{\nabla}_W\xi) - g(\bar{R}(\bar{\nabla}_We_i,Y)\xi,\xi)$$

$$-g(\bar{R}(e_i, \bar{\nabla}_W Y)\xi, \xi) - g(\bar{R}(e_i, Y)\bar{\nabla}_W \xi, \xi). \tag{4.23}$$

We also know

$$g(\bar{R}(e_i, \bar{\nabla}_W Y)\xi, \xi) = 0 = g(\bar{R}(\bar{\nabla}_W e_i, Y)\xi, \xi). \tag{4.24}$$

Now using (4.24) in (4.23) and using the fact that R is skew-symmetric we get

$$g((\bar{\nabla}_W \bar{R})(e_i, Y)\xi, \xi) = 0. \tag{4.25}$$

Therefore second term of (4.19) is zero, i.e.

$$\sum_{i=1}^{n} \eta((\bar{\nabla}_W \bar{R})(e_i, Y)\xi)\eta(e_i) = 0. \tag{4.26}$$

On using (4.26) in (4.19) we have

$$-(\bar{\nabla}_W \bar{S})(Y, \xi) = -A(W)\bar{S}(Y, \xi) - 2B(W)\eta(Y). \tag{4.27}$$

Now we know

$$(\bar{\nabla}_W \bar{S})(Y, \xi) = \bar{\nabla}_W \bar{S}(Y, \xi) - \bar{S}(\bar{\nabla}_W Y, \xi) - \bar{S}(Y, \bar{\nabla}_W \xi). \tag{4.28}$$

Using (2.6), (2.9) and (2.12) in (4.28) we can get

$$(\bar{\nabla}_W \bar{S})(Y, \xi) = -(n-1)g(Y, W) - S(Y, W). \tag{4.29}$$

From (4.27) and (4.29) we have

$$-(n-1)g(Y, W) - S(Y, W) = -A(W)\bar{S}(Y, \xi) - 2B(W)\eta(Y). \tag{4.30}$$

Putting $Y = \xi$ in (4.30) we get

$$(n-1)A(W) - 2B(W)\eta(Y) = 0. \tag{4.31}$$

Hence from (4.30) and (4.31) we can infer

$$S(Y, W) = -(n-1)g(Y, W), \tag{4.32}$$

where $a = -(n-1)$ and $b = 0$. Therefore M^n is an Einstein manifold. □

§5. Conharmonic Curvature Tensor on a Kenmotsu Manifolds with Respect to Semi-Symmetric Metric Connection

A conharmonic curvature tensor has been studied by Ozgur [16], M. Tarafdar and Bhattacharyya [23] in 2003. Further studies were carried in 2010 by Siddique and Ahsan [19]. In almost contact manifolds M of dimension $n \geq 3$, the conharmonic curvature tensor $\bar{L}$ with respect to semi-symmetric metric connection $\bar{\nabla}$ is given by

$$\bar{L}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-2}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \quad (5.1)$$

for $X,Y,Z \in \chi(M)$ where $\bar{R},\bar{S},\bar{Q}$ are the Riemannian curvature tensor, Ricci tensor and the Ricci operator with respect to semi-symmetric connection $\bar{\nabla}$, respectively.

A conharmonic curvature tensor $\bar{L}$ with respect to semi-symmetric metric connection $\bar{\nabla}$ is said to be flat if it vanishes identically with respect to the Connection $\bar{\nabla}$. On the basis of above definitions we can state the following theorem.

Theorem 5.1 *If a $n(\geq 3)$ dimensional Kenmotsu manifolds with respect to semi-symmetric metric connection admitting a conharmonic curvature tensor and a non-zero Ricci tensor satisfies $\bar{L}(X,Y)\bar{S}=0$, then the modulus of non-zero eigen values of the endomorphism $\bar{Q}$ of the tangent space corresponding to $\bar{S}$ is 0 where α,β are smooth functions on M^n.*

Proof We consider a $n(n \geq 3)$dimensional Kenmotsu manifolds with respect to semi-symmetric metric connection, satisfying the condition $\bar{L}(X,Y)\bar{S}=0$. Then we have

$$\bar{S}(\bar{L}(X,Y)U,V) + \bar{S}(U,\bar{L}(X,Y)V) = 0 \quad (5.2)$$

for all $X,Y,U,V \in \chi(M)$. Substituting X by ξ in the above equation we can obtain

$$\bar{S}(\bar{L}(\xi,Y)U,V) + \bar{S}(U,\bar{L}(\xi,Y)V) = 0. \quad (5.3)$$

Let $\bar{\lambda}$ be the eigen values of the endomorphism $\bar{Q}$ corresponding to an eigenvector X, then

$$\bar{Q}X = \bar{\lambda}X. \quad (5.4)$$

We know $g(\bar{Q}X,Y) = \bar{S}(X,Y) = \bar{\lambda}g(X,Y)$. On using $(2.16),(2.18),(5.1)$ and (5.3) we can calculate

$$\eta(U)\bar{S}(Y,V) - \eta(V)\bar{S}(U,Y) = 0. \quad (5.5)$$

Putting $U=\xi$ in (5.5) we get $\bar{S}(Y,V)=0$. Hence, from (2.18) we know that

$$S(Y,V) = -2mg(Y,V). \quad (5.6)$$

On putting $Y=X=\xi$ in the relation $\bar{S}(X,Y)=\bar{\lambda}g(X,Y)$ we get $\bar{\lambda}=0$. Therefore, we get the theorem. □

§6. Example of a Kenmotsu Manifold with Respect to Semi-Symmetric Metric Connections

Let $M = \{(x,y,z) \in \mathbf{R}^3 | (x,y,z) \neq (0,0,0)\}$ be a three-dimensional manifold [13]. The vector fields $e_1 = z\frac{\partial}{\partial x}, e_2 = z\frac{\partial}{\partial y}, \xi = e_3 = -z\frac{\partial}{\partial z}$ are linearly independent at each point of M. We define the Rimannian metric g by

$$g(e_i, e_i) = 1,\ g(e_i, e_j) = 0, \text{ where } i, j \in \{1, 2, 3\} \text{ and } i \neq j.$$

The (1,1) tensor field ϕ is defined as

$$\phi(e_1) = -e_2,\ \phi(e_2) = e_1,\ \phi(e_3) = 0.$$

If η is 1-form then $\eta(e_3) = g(e_3, e_3) = 1$. We can easily verify by the linearity of ϕ and g that (ϕ, ξ, η, g) is an almost contact metric structure on M.
Let ∇ be the Levi-Civita connection on $\mathbf{R}^3$. Then we have

$$[e_1, e_2]= 0,\ [e_1, e_3] = e_1,\ [e_2, e_3] = e_2.$$

By using Koszul's formula for the Riemannian metric g, we can find

$$\nabla_{e_1} e_1 = -e_3,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = e_1,$$
$$\nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = -e_3,\ \nabla_{e_2} e_3 = e_2,$$
$$\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0.$$

Using these we can verify $\nabla_X \xi = X - \eta(X)\xi$. Hence the manifold is a Kenmotsu manifold. We consider the linear connection $\tilde{\nabla}$ such that

$$\tilde{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + \eta(e_j) e_i - g(e_i, e_j) e_3.$$

From above relation we can calculate the non-zero components

$$\tilde{\nabla}_{e_1} e_1 = -2e_3,\ \tilde{\nabla}_{e_1} e_3 = 2e_1,\ \tilde{\nabla}_{e_1} e_2 = -e_3,\ \tilde{\nabla}_{e_2} e_3 = 2e_2.$$

Let $\bar{T}$ is the torsion tensor of metric connection $\bar{\nabla}$, then we have

$$\bar{T}(X, Y) = \eta(Y)X - \eta(X)Y.$$

On calculation we can see that

$$\bar{T}(X, Y) = 0.$$

We know that

$$(\nabla_X g)(Y, Z) = Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

and

$$(\bar{\nabla}_X g)(Y, Z) = Xg(Y, Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z).$$

Using above formula we can calculate

$$(\bar{\nabla}_{e_1} g)(e_2, e_3) = 0 = (\bar{\nabla}_{e_1} g)(e_3, e_2) = (\bar{\nabla}_{e_2} g)(e_1, e_3) = (\bar{\nabla}_{e_3} g)(e_2, e_1).$$

Therefore we can view that

$$(\bar{\nabla}_X g)(Y, Z) = 0$$

for all X, Y and $Z \in \chi(M)$. Hence $\bar{\nabla}$ is a semi-symmetric metric connection on M.

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