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Application of
Abstract Algebra to Musical Notes and Indian Music

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Abstract: This paper critically analysis the behavior and the relationship that exist between musical notes and abstract algebra. The musical notes form additive Abelian group modulo 12. Finally, the work come up with some propositions due to the musical notes behavior and their proofs, one of which was name Dido's Theorem.

Key Words: Abstract algebra, Abelian group modulo 12, Abelian group modulo 7, musical notes, Indian sargam, Dido's Theorem, transposition, inversion, Smarandache multigroup.

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§1. Introduction

In mathematics and group theory, abstract algebra, studies the algebraic structures known as groups. The concept of a group is central to abstract algebra. Other well-known algebraic structures, such as rings, fields, and vectors space can all be seen as groups endowed with additional operations and axioms. Various physical systems, such as crystals and the hydrogen atom, can be modeled by symmetry groups. Thus, abstract algebra has many important applications in physics, chemistry, and materials science. Abstract algebra is also central to public key cryptography. The modern concept of abstract group developed out of several fields of mathematics (Wussing, 2007). The idea of group theory although developed from the concept of abstract algebra, yet can be applied in many other areas of mathematical areas and other field in sciences and as well as in music. Music theory is a big field within mathematics and lots of different people have taken it in different directions. Music is that one of the fine arts which is concerned with the combination of sounds with a view to beauty of form and the expression of thought or feeling. Music itself is not complete without musical notes. And the musical notes and Indian music are: $\mathbf{C}, \mathbf{C\#}, \mathbf{D}, \mathbf{D\#}, \mathbf{E}, \mathbf{F}, \mathbf{F\#}, \mathbf{G}, \mathbf{G\#}, \mathbf{A}, \mathbf{A\#}, \mathbf{B}$. Now in India, Indian seven sargam are: Sa. Re, Ga, Ma, Pa, Dha, Ni. For years, many people find it difficult to comprehend some concept in group theory satisfactorily, but with the behavior of these musical notes, group theory can be studied. "All is number" (only musical notes) is the motto of the Pythagorean School. This school was founded by the Greek mathematician and philosopher Pythagoras (ca. 580-500 B.C.). The members of the school pursued the study of mathematics, philosophy, astronomy and music.

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§2. Group Theory

Group theory is the branch of pure mathematics which is emanated from abstract algebra. Due to its abstract nature, it was seeming to be an arts subject rather than a science subject. In fact, was considered pure abstract and not practical. Even students of group theory after being introduced to the course seems not to believe as to whether the subject has any practical application in real life, because of its abstract nature The problem prompts the researchers to study the different ways in which group can be express concretely both from theoretical and practical point of view, with intention of bringing its real- life application in musical notes. This paper aim at taking some concepts of group theory to study and understand musical notes in relation to the group's axioms. The main objective is to see these musical notes interpretation algebraically as regard to their behavior. This work focus on the behavior of musical notes which largely depend on groups axioms, theorems such as two left cosets, cyclic groups, Langrange's and Sylow's first theorem.

§3. Definition of Terms

We present here some few definitions that will help us to be familiar with concepts in music and abstract algebra.

3.1 Musical Notes

Musical notes are the following notes:

$\mathrm{C, C\#, D, D\#, E, F, F\#, G, G\#,\ A,\ A\#,\ B.}$

When logically combined, give out pleasant sound to the ear. The first note which is C, is called the root note; C# is called the 2^{nd} note; D is called the 3^{rd} note; $D\#$ is called the 4^{th} note; E is called the 5^{th} note; F is called the 5^{th} note; F# is called the 6^{th} note; G is called the 7^{th} note; G is called the 8^{th} note; A is called the 10^{th} note; A# is called the 11^{th} note and B is called the 12^{th} note.

3.2 Musical Flat b

Musical flats can be defined as the movement of sound from one pitch to the one lower, and it is donated to b. For example, movement from F to any other not

3.3 Musical Sharp

This can be considered as the movement of sound from a pitch (note) to another pitch higher, and it bis denoted by #. For example, movement from F to any other note to the right on the musical notes. Tone This simply meant any movement from a musical note to the next note two steps forward or backward on the musical notes. For example, movement from F to G or to D#. Semitone This can be defined as any movement from a musical note to the next note a step forward or backward on the musical notes (Scales). For example, movement from F to F# or F to E. Now in India Indian seven sargam are:

Sa (For Agni Devta);

Re means Rishabh (For Brahamma Devta);

Ga means Gandhar (For Goddess Saraswati);

Ma means Madhyam (For God Mahadev or Shiv);

Pa means Pancham (For Goddess Laxmi);

Dha means Dhaivata (For Lord Ganesha) and

Ni means Nishad (For Sun God).

3.4 Chord

A chord is produced when two, three or more notes are sounded together.

3.5 Transposition

Transposition involves playing or writing a given melody at a different pitch higher or lower other than the original.

3.6 Abstract Group

A group is a non-empty set $(G, *)$ together with an operation $(*)$ on it which satisfies the following axioms:

C_1 : $\forall \mathrm{a}, \mathrm{b} \in \mathrm{G}, \quad \mathrm{a} * \mathrm{b} \in G$ (Closure);

C_2 : $\forall \mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{G}, \quad (\mathrm{a} * \mathrm{b}) * \mathrm{c} = \mathrm{a} * (\mathrm{b} * \mathrm{c})$ (Associative);

C_3 : $\forall \mathrm{a}, \mathrm{b} \in G, \exists e \in G$, then $\quad \mathrm{a} * e = e * \mathrm{a} = \mathrm{a}$ (Identity);

C_4 : $\forall \mathrm{a} \in G, \exists \mathrm{a}^{-1} \in G$, then $\mathrm{a} * \mathrm{a}^{-1} = \mathrm{a}^{-1} * \mathrm{a} = e$ (Inverse);

C_5 : $\forall \mathrm{a}, \mathrm{b} \in G, \mathrm{a} * \mathrm{b} = \mathrm{b} * \mathrm{a} \in G$ (commutative).

A generalization of group is the multigroup ([7]). Usually, a *Smarandache multigroup*

$$\widetilde{G} = \left(\bigcup_{i=1}^{m} G_i ; \bigcup_{i=1}^{m} \{\cdot_i\} \right)$$

is the union of m groups, i.e., $(G_i; \cdot_i)$ is a group for integers $1 \leq i \leq m$ constraint with conditions on their intersection, for instance their intersection $\bigcap\limits_{i=1}^{m} G_i = \{\mathrm{e}\}$, the identity of all G_i, i.e., $e_i = e$ for integers $1 \leq i \leq m$, which can be also applied to characterize the musical notes.

3.7 Integers Modulo m

This is a finite group that is called the additive group of the residue class of integers modulo m. it is denoted by Z_m.

3.8 p-Group

Let p be an arbitrary but fixed prime number. A finite group G is said to be a p-group if its order is power of p.

If $T \leq G$ and $|T| = p^r$ for an integer $r \geq 0$ then T is called p-subgroup of G.

§4. Theoretical Underpinning

Notice that the musical notes is obviously a multigroup consists of 12 trivial groups. However, several authors worked on the application of group theory to many fields in sciences, games and many more other fields but only few have ventured the field of music. Pythagoras $(428 - 347$ B.C.), who is considered as founder of the first school of mathematics as a purely deductive science is also the founder of a theoretical music. He used to say that "all is number "and musical notes are not exceptional, that is C, C#, D, D#, E, F, F#, G, G#, A, A#, B.

But why "*all is number*"? The Pythagoreans associated certain meanings and characters to numbers. They considered odd numbers as males and even numbers as females. To the Pythagoreans, one is the number of reason, two is the number of opinion, three is the number of harmony, four is the number of justice, five is the number of marriage, six is the number of creation, seven is the number of awe, and ten is the number of the universe. A couple of possible reasons were given. The first one is the Eastern influence.

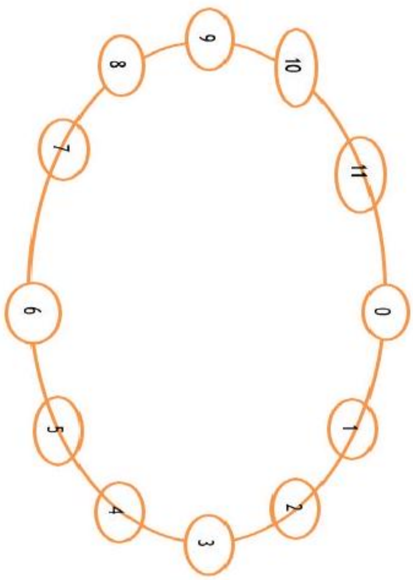

Figure 1 Musical clock

Having traveled to Egypt and Babylon, Pythagoras might have been influenced by numerology, which deals with numbers and mystical relations among them, that was common in these two regions. A second possible reason is to give an alternative view to the contemporary belief in Greek concerning the principles of things. At the time, it was believed that earth, air, fire and water are the four basics principles of things. This did not convince Pythagoras in explaining the principles of immaterial things. A third possibility comes from astronomy, a subject that was studied by Pythagoras. In studying stars, one observes that each constellation can be characterized by the number of stars composing it and the geometrical figure that they form. The fourth possible reason comes from music. The members of the school practiced music. Pythagoras observed that musical notes produced from a vibrating string of some length could be characterized by (ratios of) numbers. Dividing a vibrating string by some movable object into two different lengths produced different types of musical notes. These notes are then described by the ratios of the lengths of the parts of the vibrating string. Explaining musical notes and describing stars by numbers may have then led the Pythagoreans to think that numbers can also be used to explain other phenomena (Heath [6] and Thomas M. Flore [9]). He referred to C, C#, D, D#, E, F, F#, G, G#, A, A#, B. As the Z_{12} Model of pitch class. He constructed a musical clock shown in Figure 1.

He also said that there is a bijection between the set of pitch classes and Z_{12}. He defined transposition as: $T_n : Z_{12} \to Z_{12}$ then their exist $T_n(X) : X + n$ and inversion was also defined as $I_n : Z_{12} \to Z_{12}$ then their exist $I_n(X) : -x + n$, where n is in mod 12.

Ada Zhang [1] considered possibly musical notes with corresponding integers as

C	$C\#$	D	$D\#$	E	F	$F\#$	G	$G\#$	A	$A\#$
0	1	2	3	4	5	6	7	8	9	10

He defined transposition T_n as that moves a pitch-class or pitch-class set up by n $(\bmod 12)$ [1].

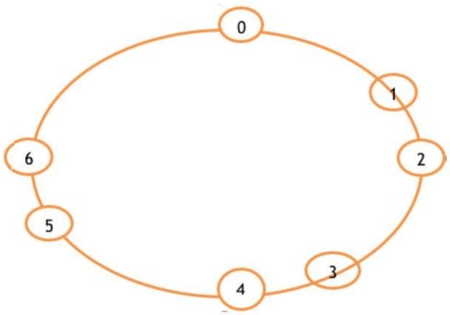

Figure 2 Indian musical clock

In my ideas considered possibly indian musical sargam with corresponding integers as

Sa	Re	Ga	Ma	Pa	Dha	Ni
0	1	2	3	4	5	6

We defined transposition, T_n as that moves a sargam class by $n(\text{mod } 7)$ And inversion was also defined here as T_nI as the pitch (A) about $C(0)$ and then transposes it by n. that is, $T_nI(a) = -a + n(\text{mod}12)$. Then further, laid out all the pitches in a circular pattern on a 12 -sided polygon. That is, consider the transposition T_{11}. It sends C to B, C to C, Alissa [3] assert that the musical actions of the dihedral groups. This paper considers two ways in which the dihedral groups act on the set of major and minor triads.

According to David Wright [4], referred to the musical notes with their corresponding integers as in Ada Zhang [1] as M_{12}, that is the Mathieu group. He asserts that this can be generated by just two permutations Expressed below in both two-line notation and cycle notation. We denote these generating permutations as P_1 and P_0

Adam ([2] defined transposition and inversion as: Transposition is define as $T_n : Z_{12} \to Z_{12}$ then their exist $(X) : x + n \bmod 12$ and he also define Inversion as $I_n(X) : Z_{12} \to Z_{12}$ then their exist $I_n(X) : -x + n$ where n is in mod12.

§5. Methodology

We need a few of conclusions in group theory following ([7]).

Lemma 5.1 *Let $H \leq G$ be groups and $g \in G$. Then,*

(i) $g \in gH$;

(ii) Two left cosets of H in G are either identical or disjoint;

(iii) The number of elements in gH is $|H|$.

Lemma 5.2(Langrange's Theorem) *The order of a subgroup of a finite group is a factor of the order of the group.*

Lemma 5.3 *Every subgroup of a cyclic group is cyclic.*

Lemma 5.4(First Sylow's Theorem) *Let G be a finite group, p a prime and p^r the highest power of P diving the order of G. Then there is a subgroup of G of order of G. Then there is a subgroup of G of order p^r.*

§6. Results and Discussion

The numbering of the musical notes are listed following:

C	C#	D	D#	E	F	F#	G	G#	A	A#	B
0	1	2	3	4	5	6	7	8	9	10	11

Note that $B\# = C$. It shows that the musical notes form a group of integers of Modulo 12. That is $Z_{12} = \{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B\}$. Let the operation be $* = \# = +$. The behavior of the musical note on groups musical notes related with groups axiom is shown in the following:

(i) Closure $E, F \in Z_{12}$. Hence $E * F = A \in Z_{12}$.

(ii) Associative E, F and $F\# \in Z_{12}$. Hence, $(E * F) * F\# = E * (F * F\#) = A * F\# = E * B = D\# = D\#$.

With the behavior of the musical notes of Indian sargam we have just seen, we personally suggest for the root note of musical scales (notes) to be algebraically named as the identity note. Table 1 lists the musical notes and their inverse.

Sargam	Inverse
Re	Ni
Ga	Dha
Ma	Pa

Table 1

References

[1] Ada Zhang, The framework of music theory as represented with groups, www.researchgate.net/publication/23777304 (2009), 9-11.

[2] Adam Townsend (2011), Mathematics and music theory, A paper presentation of undergraduate Mathematics Colloquium at the University College London (www.ucl.sneffel.com), 5-8.

[3] Alissa S. Crans, Thomas M.Fiore and Ramon Satuendra, *Musical actions of dihedral groups*, *The Mathematical Association of America*, 116 (2009), 479–482.

[4] David Wright, Mathematics and Music, *Mathematical World*, Vol.28, American Mathematical Society, 2009.

[5] H. Wussing, *The Genesis of the Abstract Group Concept: A Contribution to the History of the Origin of Abstract Group Theory*, New York: Dover Publications, 2007.

[6] Heath Thomas, *A History of Greek Mathematics*, Vol. 1, Oxford University Press, 1965.

[7] Linfan Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries* (Second edition), The Education Publisher Inc., USA, 2011.

[8] Samuel H. Tsok and I.B. Mshelia, Application of group theory to a local game called "T-sorry checkerboard" (A case of Klein four- group), *International Organization of Scientific Research Journal of Mathematics*, Vol.7, 3(2013), 04-06 .

[9] Thomas M. Fiore (1993), *Music and Mathematics*, 7-15 (fioret@umich.edu).