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Backlund Transformations of Non-Null Curve Flows with Respect to Frenet Frame

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Abstract: In this paper we present three classes of curve evolution connected with the generalized nonlinear Schrödinger equation and the generalized nonlinear heat system using visco-Da Rios equation in Minkowski 3-space. Later we obtain Backlund transformations of nonnul curve flows associated with visco-Da Rios equation in Minkowski 3-space.

Key Words: The visco Da Rios equation, Schrödinger equation, Backlund transformation

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§1. Introduction

The classic Backlund transformations are used to construct constant negative Gaussian curvature surfaces. It is used to obtain new pseudospherical surfaces using solution of an integrable partial differential equation.

In the last years Backlund transformations have been studied by many authors [1-9]. Nemeth studied Backlund transformations of constant torsion curves in 3-dimensional constant curvature spaces in Euclidean 3-space [5]. Palmer studied Backlund Transformations for surfaces in Minkowski Space [6]. Gürbüz extended Backlund transformations of constant torsion curves n dimensional Lorenzian space [7]. Abdel-Baky showed that the Minkowski versions of the Backlund's theorem and its application by using the method of moving frames [8]. Bracken studied Backlund transformations for several cases of a generalized KdV equation in Euclidean space [9]. Grbovic and Nesovic studied Backlund transformation and vortex filament equation for pseudo null curves in Minkowski 3-space [10]

The connection between moving curves and integrable systems has been studied by numerous authors [11-23]. If the position vector of a vortex filament is $\beta(\sigma, u)$, the following equation

$$\beta_u = \beta_\sigma \times \beta_{\sigma\sigma}$$

is called the Da Rios equation [11]. Hasimoto discovered the connection between the motion of vortex filament in an incompressible, inviscid three-dimensional fluid and solutions of the Nonlinear Schrödinger NLS equation in Euclidean 3-space [12]. Lakshamanan *et al.* presented motion of curves and surfaces and nonlinear evolution equations in (2+1) dimensions [13].

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Barros et al. found out explicit solutions of the Betchov-Da Rios soliton equation in three-dimensional Lorentzian space forms [14].

Murugesh and Balakrishnan presented three classes of curve evolution associated with numerous soliton equations with respect to Frenet frame in Euclidean 3-space [24]. Gürbüz obtained three classes of curve evolution connected with the nonlinear Schrödinger equation according to Frenet frame in Minkowski 3-space [25].

The case when viscosity effects on a fluid dynamic, the following equation is

$$\beta_u = \beta_\sigma \times \beta_\sigma + \varsigma\beta_\sigma \tag{1}$$

is called the visco-Da Rios equation [26]. Here ς denotes viscosity and non-negatif constant. Gürbüz and Yoon presented the visco modified Heisenberg magnet model and physical applications with respect to the Frenet frame in Minkowski 3-space [27].

Qu and Kang studied Bäcklund transformations for integrable geometric curve flows in Euclidean 3-space [28]. In this paper, one give three classes of curve evolution connected with the visco Da Rios equation with respect to the Frenet frame in Minkowski 3-space are presented. Later Backlund transformations of three classes of the curve evolution connected with the generalized nonlinear heat flow and the repulsive type generalized nonlinear Schrödinger flow with respect to Frenet frame in Minkowski 3-space are studied.

§2. Backlund Transformations of the $GNLS^-$ and $GNLH$ Flows in $\mathbf{R}_1^3$

Let the β be a non-null curve with the arc length σ in 3 dimensional Minkowski space $\mathbf{R}_1^3$. Formulae of derivative of the Frenet frame $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ have the following form [29]

$$\begin{aligned}\mathbf{T}_\sigma &= \varepsilon_2\kappa\mathbf{N},\\ \mathbf{N}_\sigma &= -\varepsilon_1\kappa\mathbf{T}+\varepsilon_3\tau\mathbf{B},\\ \mathbf{B}_\sigma &= -\varepsilon_2\tau\mathbf{N}\end{aligned}$$

satisfying

$$\begin{aligned}\langle\mathbf{T},\mathbf{T}\rangle_L &= \varepsilon_1, \langle\mathbf{N},\mathbf{N}\rangle_L = \varepsilon_2,\ \langle\mathbf{B},\mathbf{B}\rangle_L = \varepsilon_3,\\ \mathbf{T}\times_L\mathbf{N} &= \varepsilon_3,\ \ \mathbf{N}\times_L\mathbf{B} = \varepsilon_1\mathbf{T},\ \mathbf{B}\times_L\mathbf{T} = \varepsilon_2\mathbf{N},\ \varepsilon_i = \pm1, i = 1,2,3\end{aligned}$$

where $\times$ is cross product. The $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are tangent, normal and binormal vectors. The κ,τ are curvatures and torsion of the β.

In this section one present Backlund transformations of three classes of curve evolution connected with the generalized nonlinear heat GNLH flow and the repulsive type generalized nonlinear Schrödinger $GNLS^-$ flow in Minkowski 3-space $\mathbf{R}_1^3$.

Class I. Let β_1 be spacelike curve with timelike binormal according to the Frenet frame in $\mathbf{R}_1^3$. Assume that

$$\beta_{1\sigma} = \mathbf{T}. \tag{2}$$

The visco-Da Rios equation associated with the generalized nonlinear heat system is given by

$$\beta_{1u} = \beta_{1\sigma} \times \beta_{1\sigma\sigma} + \varsigma\beta_{1\sigma} \tag{3}$$

where ς is the viscosity and non-negatif constant.

The general time evolution according to the Frenet frame of the spacelike curve β_1 with timelike binormal for first class in $\mathbf{R}_1^3$ is given by

$$\begin{aligned} \mathbf{T}_u &= \eta_1\mathbf{N} + \eta_2\mathbf{B} & (4)\\ \mathbf{N}_u &= -\eta_1\mathbf{T}+\gamma_1\mathbf{B} & (5)\\ \mathbf{B}_u &= \eta_2\mathbf{T}+\gamma_1\mathbf{N} & (6) \end{aligned}$$

where η_1, η_2 and γ_1 are smooth functions. Using Eqs.(2) and (3) one obtain

$$\mathbf{T}_u = \mathbf{T} \times \mathbf{T}_\sigma + \varsigma\mathbf{T} = \varsigma\mathbf{T} - \kappa\mathbf{B} \tag{7}$$

The compatibility conditions $\beta_{1\sigma u} = \beta_{1u\sigma}$ and $\mathbf{T}_{u\sigma} = \mathbf{T}_{\sigma u}$ the followings are obtained

$$\begin{aligned} \eta_1 &= \kappa(\tau + \varsigma), \ \eta_2 = -\kappa_\sigma & (8)\\ \gamma_1 &= -(\frac{\kappa_{\sigma\sigma}}{\kappa} + \tau(\tau + \varsigma)) & (9)\\ \kappa_u &= (\kappa(\tau + \varsigma))_\sigma + \kappa_\sigma\tau & (10) \end{aligned}$$

respectively, the second frame $\{\phi_1, \phi_2, \phi_3\}$ and Hasimoto-like transformation μ_1 using visco-Da Rios equation with respect to Frenet frame for the first class in Minkowski 3-space are given by

$$\begin{aligned} \phi_1 &= \mathbf{T}, & (11)\\ \phi_2 &= \frac{\mathbf{N}+\mathbf{B}}{\sqrt{2}}e^{\int\tau}, & (12)\\ \phi_3 &= \frac{\mathbf{N}-\mathbf{B}}{\sqrt{2}}e^{-\int\tau}, & (13)\\ \mu_1 &= \frac{\kappa}{\sqrt{2}}e^{\int\tau}. & (14) \end{aligned}$$

From Eqs.(11), (12) (13)and (14), the spatial variation of the frame $\{\phi_1, \phi_2, \phi_3\}$ associated with the generalized nonlinear heat system for first class is given by

$$\begin{aligned} \phi_{1\sigma} &= \mu_2\phi_2 + \mu_1\phi_3\\ \phi_{2\sigma} &= -\mu_1\phi_1\\ \phi_{3\sigma} &= -\mu_2\phi_1, \end{aligned}$$

where $\mu_2 = \frac{\kappa}{\sqrt{2}}e^{-\int\tau}$. The temporal variation of the frame $\{\phi_1, \phi_2, \phi_3\}$ associated with the generalized nonlinear heat system for first class is given by

$$\phi_{1u} = (\mu_{1\sigma} + \varsigma\mu_1)\phi_3 - (\mu_{2\sigma} - \varsigma\mu_2)\phi_2,$$

$$\begin{aligned}\phi_{2u} &= -(\mu_{1\sigma}+\varsigma\mu_1)\phi_1+R_1\phi_2,\\ \phi_{3u} &= (\mu_{2\sigma}-\varsigma\mu_2)\phi_1-R_1\phi_3.\end{aligned}$$

From $\phi_{2u\sigma}=\phi_{2\sigma u}$ one obtain the generalized nonlinear heat GNLH system for spacelike curve with timelike binormal with respect to Frenet frame for first class in Minkowski 3-space:

$$\begin{aligned}\mu_{1u} &= \mu_{1\sigma\sigma}+\varsigma\mu_{1\sigma}+R_1\mu_1\\ \mu_{2u} &= -\mu_{2\sigma\sigma}+\varsigma\mu_{2\sigma}-R_1\mu_2,\ R_1=\mu_1\mu_2.\end{aligned}$$

3.1 **Backlund Transformation of the Generalized Nonlinear Heat Flow with Respect to Frenet Frame for First Class in $\mathbf{R}_1^3$**

Backlund transformation of the generalized nonlinear heat flow with respect to Frenet frame for first class is constructed: Let $\widetilde{\beta}_1$ be another curve associated with the spacelike curve β_1.

$$\widetilde{\beta}_1(\sigma,u)=\beta_1(\sigma,u)+f_1(\sigma,u)\mathbf{T}+f_2(\sigma,u)\mathbf{N}-f_3(\sigma,u)\mathbf{B}, \tag{15}$$

where f_1, f_2 and f_3 are functions of u and σ. Respectively, the derivative of according σ and u of Eq.(15) are obtained by

$$\widetilde{\beta}_{1\sigma}=(1+f_{1\sigma}-f_2\kappa)\mathbf{T}+(f_{2\sigma}+f_1\kappa+f_3\tau)\mathbf{N}-(f_{3\sigma}+f_2\tau)\mathbf{B}, \tag{16}$$

$$\begin{aligned}\widetilde{\beta}_{1u} &= (\varsigma+f_{1u}-f_2\kappa(\tau+\varsigma)+f_3\kappa_\sigma)\mathbf{T}\\ &\quad +(f_{2u}+f_1\kappa(\tau+\varsigma)-f_3(\frac{\kappa_{\sigma\sigma}}{\kappa}+\tau(\tau+\varsigma))\mathbf{N}\\ &\quad +(f_1\kappa_\sigma-f_2(\frac{\kappa_{\sigma\sigma}}{\kappa}+\tau(\tau+\varsigma))+f_{3u}-\kappa)\mathbf{B}\end{aligned} \tag{17}$$

Let $\widetilde{\sigma}$ be the arclength parameter of the curve $\widetilde{\beta}_1$. Via Eq.(16), one derive

$$d\widetilde{\sigma}=\sqrt{(1+f_{1\sigma}-f_2\kappa)^2+(f_{2\sigma}+f_1\kappa+f_3\tau)^2-(f_{3\sigma}+f_2\tau)^2}d\sigma=\Theta d\sigma. \tag{18}$$

The tangent vector of $\widetilde{\beta}_1$ is derived by

$$\widetilde{\mathbf{T}}=\frac{d\widetilde{\beta}_1}{d\sigma}\frac{d\sigma}{d\widetilde{\sigma}}=\delta_1\mathbf{T}+\delta_2\mathbf{N}+\delta_3\mathbf{B}, \tag{19}$$

where

$$\delta_1 = \Theta^{-1}(1+f_{1\sigma}-f_2\kappa),\ \delta_2=\Theta^{-1}(f_{2\sigma}+f_1\kappa+f_3\tau),\ \delta_3=-\Theta^{-1}(f_{3\sigma}+f_2\tau)$$

From Eq.(19),

$$\frac{d^2\widetilde{\beta}_1}{d\widetilde{\sigma}^2}=\frac{\delta_{1\sigma}-\delta_2\kappa}{\Theta}\mathbf{T}+\frac{\delta_{2\sigma}+\delta_1\kappa-\delta_3\tau}{\Theta}\mathbf{N}+\frac{\delta_{3\sigma}-\delta_2\tau}{\Theta}\mathbf{B}. \tag{20}$$

From Eq.(20), the curvature $\widetilde{\kappa}$ of the curve $\widetilde{\beta}_1$ is derived by

$$\widetilde{\kappa}=\frac{\sqrt{(\delta_{1\sigma}-\delta_2\kappa)^2+(\delta_{2\sigma}+\delta_1\kappa-\delta_3\tau)^2-(\delta_{3\sigma}-\delta_2\tau)^2}}{\Theta}=\frac{\Lambda}{\Theta}. \tag{21}$$

The principal normal vector $\widetilde{\mathbf{N}}$ of the curve $\widetilde{\beta}_1$ is obtained by

$$\widetilde{\mathbf{N}}=\frac{\delta_{1\sigma}-\delta_2\kappa}{\Lambda}\mathbf{T}+\frac{\delta_{2\sigma}+\delta_1\kappa-\delta_3\tau}{\Lambda}\mathbf{N}+\frac{\delta_{3\sigma}-\delta_2\tau}{\Lambda}\mathbf{B}. \tag{22}$$

The binormal vector of $\widetilde{\mathbf{B}}$ of the curve $\widetilde{\beta}_1$ is found in the following

$$\widetilde{\mathbf{B}}=\omega_1\mathbf{T}+\omega_2\mathbf{N}+\omega_3\mathbf{B}, \tag{23}$$

where

$$\begin{aligned}
\omega_1 &= -\frac{\delta_3(\delta_{2\sigma}+\delta_1\kappa-\delta_3\tau)-\delta_2(\delta_{3\sigma}-\delta_2\tau)}{\Lambda}\\
\omega_2 &= \frac{\delta_3(\delta_{1\sigma}-\delta_2\kappa)-\delta_1(\delta_{3\sigma}-\delta_2\tau)}{\Lambda}\\
\omega_3 &= -\frac{\delta_1(\delta_1\kappa+\delta_{2\sigma}-\delta_3\tau)-\delta_2(\delta_{1\sigma}-\delta_2\kappa)}{\Lambda}
\end{aligned}$$

Since $\widetilde{\beta}_1$ and β_1 have with same integrable systems, the curve $\widetilde{\beta}_1$ satisfies the generalized nonlinear heat flow $\widetilde{\beta}_{1u}$ of the visco Da Rios equation for the first class in Minkowski 3-space following.

$$\widetilde{\beta}_{1u}=-\widetilde{\kappa}\widetilde{\mathbf{B}}+\varsigma\widetilde{\mathbf{T}}. \tag{24}$$

Theorem 3.1 *The generalized nonlinear heat flow Eq.(24) is invariant according to Backlund transformation Eq.(15) for second class in Minkowski 3-space if f_1, f_2 and f_3 satisfy the following system*

$$\begin{aligned}
\varsigma+f_{1u}-f_2\kappa(\tau+\varsigma)+f_3\kappa_\sigma &= -\frac{\Lambda}{\Theta}\omega_1+\varsigma\delta_1 \qquad (25)\\
f_{2u}+f_1\kappa(\tau+\varsigma)+f_3\left(\frac{\kappa_{\sigma\sigma}}{\kappa}+\tau(\tau+\varsigma)\right) &= -\frac{\Lambda}{\Theta}\omega_2+\varsigma\delta_2 \qquad (26)\\
-f_1\kappa_\sigma-f_2\left(\frac{\kappa_{\sigma\sigma}}{\kappa}+\tau(\tau+\varsigma)\right)-f_{3u}-\kappa &= -\frac{\Lambda}{\Theta}\omega_3+\varsigma\delta_3 \qquad (27)
\end{aligned}$$

Proof Via Eqs.(17), (19), (21), (23), (24) one obtain Eqs.(25), (26) and (27). □

Class II. Let β_2 be a spacelike curve with timelike binormal according to Frenet frame in $\mathbf{R}_1^3$. Assume that

$$\beta_{2\sigma}=\mathbf{B}. \tag{28}$$

The modified visco-Da Rios equation associated with the repulsive type generalized nonlinear Schrödinger $GNLS^-$ equation is given by

$$\beta_{2u} = \beta_{2\sigma} \times \beta_{2\sigma\sigma} + \varsigma\beta_{2\sigma}, \tag{29}$$

where ς is the viscosity and non-negative constant.

The general time evolution of the Frenet frame for the second class in $\mathbf{R}_1^3$ is given by

$$\begin{aligned}
\mathbf{T}_u &= \gamma_2\mathbf{N}+\zeta_1\mathbf{B}, &(30)\\
\mathbf{N}_u &= -\gamma_2\mathbf{T}+\zeta_2\mathbf{B}, &(31)\\
\mathbf{B}_u &= \zeta_1\mathbf{T}+\zeta_2\mathbf{N}. &(32)
\end{aligned}$$

Via Eqs.(28) and (29) it is derived by

$$\mathbf{B}_{2u} = \mathbf{B}_{2\sigma} \times \mathbf{B}_{2\sigma\sigma} + \varsigma\mathbf{B}_{2\sigma} = \tau\mathbf{T}+\varsigma\mathbf{B} \tag{33}$$

From $\beta_{2\sigma u} = \beta_{2u\sigma}$ and $\mathbf{B}_{u\sigma} = \mathbf{B}_{\sigma u}$ via Eq.(29), Eq.(32) and Eq.(33) one can derive

$$\begin{aligned}
\zeta_1 &= \tau_\sigma,\ \zeta_2 = \tau(\kappa-\varsigma), &(34)\\
\gamma_2 &= \frac{\tau_{\sigma\sigma}}{\tau} - \kappa(\kappa-\varsigma), &(35)\\
\tau_u &= -\kappa\tau_\sigma - (\tau(\kappa-\varsigma))_\sigma &(36)
\end{aligned}$$

The second frame $\{\xi_1, \xi_2, \xi_2^*\}$ connected with the repulsive type $GNLS^-$ equation Eq.(28) and Hasimoto-like transformation ψ_2 are given by

$$\begin{aligned}
\xi_1 &= \mathbf{B}, &(37)\\
\xi_2 &= \frac{\mathbf{T}+i\mathbf{N}}{\sqrt{2}}e^{i\int\kappa}, &(38)\\
\xi_2^* &= \frac{\mathbf{T}-i\mathbf{N}}{\sqrt{2}}e^{-i\int\kappa} &(39)
\end{aligned}$$

and

$$\psi_2 = \frac{\tau}{\sqrt{2}}e^{i\int\kappa}, \tag{40}$$

where ξ_2^* is the complex conjugate of ξ_2. Using Eqs.(37), (38), (39) and (40), the spatial and time evolutions of the frame $\{\xi_1, \xi_2, \xi_2^*\}$ connected with the $GNLS^-$ equation for second class, respectively are given by

$$\begin{aligned}
\xi_{1\sigma} &= -i\psi_2\xi_2^* + i\psi_2^*\xi_2,\\
\xi_{2\sigma} &= -i\psi_2\xi_1,\\
\xi_{2\sigma}^* &= i\psi_2^*\xi_1;\\
\xi_{1u} &= (\psi_{2\sigma}^* + i\varsigma\psi_2^*)\xi_2 + (\psi_{2\sigma} - i\varsigma\psi_2)\xi_2^*,\\
\xi_{2u} &= (\psi_{2\sigma} - i\varsigma\psi_2)\xi_1 + iR_2\xi_2,
\end{aligned}$$

$$\xi^*_{2u} = (\psi^*_{2\sigma} + i\varsigma\psi^*_2)\xi_1 - iR_2\xi^*_2.$$

From $\xi_{2u\sigma} = \xi_{2\sigma u}$ one can obtain the repulsive type $GNLS^-$ equation for second class in in $\mathbf{R}^3_1$

$$\psi_{2u} = i\psi_{2\sigma\sigma} + iR_1\psi_2 + \varsigma\psi_{2\sigma},\ R_2 = -\psi_2\psi^*_2$$

3.2 Backlund Transformation of the $GNLS^-$ Flow with Respect to Frenet Frame for Second Class in $\mathbf{R}^3_1$

Backlund transformation of the $GNLS^-$ flow of spacelike curve β_2 with timelike binormal is constructed as the following.

Let $\widetilde{\beta}_2$ be another curve associated with the spacelike curve β_2 with timelike binormal $\mathbf{B}$ for second class in $\mathbf{R}^3_1$, it can be expressed

$$\widetilde{\beta}_2(\sigma,u) = \beta_2(\sigma,u) + g_1(\sigma,u)\mathbf{T}+g_2(\sigma,u)\mathbf{N} - g_3(\sigma,u)\mathbf{B}, \tag{41}$$

where g_1, g_2 and g_3 are the smooth functions of u and σ. By differentiating of according σ and u of Eq.(41), the following are obtained by

$$\frac{\partial\widetilde{\beta}_2}{\partial\sigma} = \widetilde{\beta}_{2\sigma} = (g_{1\sigma} - g_2\kappa)\mathbf{T}+(g_1\kappa + g_{2\sigma} + g_3\tau)\mathbf{N}+(1 - g_2\tau - g_{3\sigma})\mathbf{B}, \tag{42}$$

$$\begin{aligned}\widetilde{\beta}_{2u} &= (\tau + g_{1u} - g_2(\frac{\tau_{\sigma\sigma}}{\tau} - \kappa(\kappa-\varsigma)) - g_3\tau_\sigma)\mathbf{T} \\ &+(g_1(\frac{\tau_{\sigma\sigma}}{\tau} - \kappa(\kappa-\varsigma)) + g_{2u} - g_3\tau(\kappa-\varsigma))\mathbf{N} \\ &+(\varsigma + g_1\tau_\sigma + g_2\tau(\kappa-\varsigma) - g_{3u})\mathbf{B}.\end{aligned} \tag{43}$$

For the arclength parameter $\widetilde{\sigma}$ is of the curve $\widetilde{\beta}_2$ in $\mathbf{R}^3_1$ one can obtains

$$\begin{aligned} d\widetilde{\sigma} &= \sqrt{(g_{1\sigma} - g_2\kappa)^2 + (g_{2\sigma} + g_1\kappa + g_3\tau)^2 - (1 - g_2\tau - g_{3\sigma})^2}d\sigma \\ &= \Gamma d\sigma \end{aligned} \tag{44}$$

The tangent vector $\widetilde{\mathbf{T}}$ of the curve $\widetilde{\beta}_2$ for the second class in $\mathbf{R}^3_1$ is obtained by

$$\widetilde{\mathbf{T}} = \frac{d\widetilde{\beta}_2}{d\sigma}\frac{d\sigma}{d\widetilde{\sigma}} = \varphi_1\mathbf{T}+\varphi_2\mathbf{N}+\varphi_3\mathbf{B}, \tag{45}$$

where

$$\begin{aligned}\varphi_1 &= \Gamma^{-1}(g_{1\sigma} - g_2\kappa)\mathbf{T},\ \ \varphi_2 = \Gamma^{-1}(g_{2\sigma} + g_1\kappa + g_3\tau) \\ \varphi_3 &= -\Gamma^{-1}(1 - g_2\tau - g_{3\sigma}).\end{aligned}$$

Using Eq.(45), one can obtains

$$\frac{d^2\widetilde{\beta}}{d\widetilde{\sigma}^2} = \frac{\varphi_{1\sigma} - \varphi_2\kappa}{\Gamma}\mathbf{T}+\frac{\varphi_{2\sigma} + \varphi_1\kappa - \varphi_3\tau}{\Gamma}\mathbf{N} + \frac{\varphi_{3\sigma} - \varphi_2\tau}{\Gamma}\mathbf{B}. \tag{46}$$

Using Eqs.(46) the curvature $\widetilde{\kappa}$ of the curve $\widetilde{\beta}_2$ is obtained by

$$\widetilde{\kappa} = \frac{\sqrt{(\varphi_{1\sigma} - \varphi_2\kappa)^2 + (\varphi_{2\sigma} + \varphi_1\kappa - \varphi_3\tau)^2 - (\varphi_{3\sigma} - \varphi_2\tau)^2}}{\Gamma} = \frac{\Phi}{\Gamma}. \tag{47}$$

With aid of Eqs.(46) and (47), the principal normal vector $\widetilde{\mathbf{N}}$ of the curve $\widetilde{\beta}_2$ is obtained by

$$\widetilde{\mathbf{N}} = \frac{\varphi_{1\sigma} - \varphi_2\kappa}{\Phi}\mathbf{T} + \frac{\varphi_{2\sigma} + \varphi_1\kappa - \varphi_3\tau}{\Phi}\mathbf{N} + \frac{\varphi_{3\sigma} - \varphi_2\tau}{\Phi}\mathbf{B}. \tag{48}$$

From Eqs.(46), (47) and (48), the binormal vector of $\widetilde{\mathbf{B}}$ of the curve $\widetilde{\beta}_2$ can be obtained by

$$\widetilde{\mathbf{B}} = \rho_1\mathbf{T} + \rho_2\mathbf{N} + \rho_3\mathbf{B}, \tag{49}$$

where

$$\begin{aligned}
\rho_1 &= \frac{\varphi_2(\varphi_{3\sigma} - \varphi_2\tau) - \varphi_3(\varphi_{2\sigma} + \varphi_1\kappa - \varphi_3\tau)}{\Phi} \\
\rho_2 &= \frac{-\varphi_1(\varphi_{3\sigma} - \varphi_2\tau) + \varphi_3(\varphi_{1\sigma} - \varphi_2\kappa)}{\Phi} \\
\rho_3 &= \frac{-\varphi_1(\varphi_1\kappa + \varphi_{2\sigma} - \varphi_3\tau) + \varphi_2(\varphi_{1\sigma} - \varphi_2\kappa)}{\Phi}.
\end{aligned}$$

The torsion vector $\widetilde{\tau}$ of the curve $\widetilde{\beta}_2$ using Eqs.(48) and (49) in Minkowski 3-space is derived by

$$\widetilde{\tau} = -\frac{\Psi}{\Gamma\Phi}, \tag{50}$$

where

$$\begin{aligned}
\Psi = {} & (\rho_{1\sigma} - \rho_2\kappa)(\varphi_{1\sigma} - \varphi_2\kappa) + (\rho_{2\sigma} + \rho_1\kappa - \rho_3\tau)(\varphi_{2\sigma} + \varphi_1\kappa - \varphi_3\tau) \\
& -(\varphi_{3\sigma} - \varphi_2\tau)(\rho_{3\sigma} - \rho_2\tau).
\end{aligned}$$

Since the curve $\widetilde{\beta}_2$ and the curve β_2 are expressed with same integrable systems, the curve $\widetilde{\beta}_2$ satisfies the following generalized nonlinear Schrödinger flow of the visco Da Rios equation connected with the second class in Minkowski 3-space:

$$\widetilde{\beta}_{2u} = \widetilde{\tau}\widetilde{\mathbf{T}} + \varsigma\widetilde{\mathbf{B}}. \tag{51}$$

Theorem 3.2 *The generalized Schrödinger flow Eq.(51) is invariant according to Backlund transformation Eq.(41) for second class in Minkowski 3-space if g_1, g_2 and g_3 satisfies the following system*

$$\begin{aligned}
\tau + g_{1u} - g_2\left(\frac{\tau_{\sigma\sigma}}{\tau} - \kappa(\kappa - \varsigma)\right) - g_3\tau_\sigma &= -\frac{\Psi}{\Gamma\Phi}\varphi_1 + \varsigma\rho_1 && (52) \\
g_1\left(\frac{\tau_{\sigma\sigma}}{\tau} - \kappa(\kappa - \varsigma)\right) + g_{2u} - g_3\tau(\kappa - \varsigma) &= -\frac{\Psi}{\Gamma\Phi}\varphi_2 + \varsigma\rho_2 && (53) \\
\varsigma + g_1\tau_\sigma + g_2\tau(\kappa - \varsigma) - g_{3u} &= -\frac{\Psi}{\Gamma\Phi}\varphi_3 + \varsigma\rho_3. && (54)
\end{aligned}$$

Proof Via Eqs.(43), (48), (49), (50), (51) one can obtain Eqs.(52), (53) and (54). □

Class 3 Let β_3 be spacelike curve with timelike binormal according to the Frenet frame in $\mathbf{R}_1^3$. Assume that

$$\beta_{3\sigma} = \mathbf{N}. \tag{55}$$

The modified visco-Da Rios equation associated with the repulsive type generalized nonlinear Schrödinger $GNLS^-$ equation is given by

$$\beta_{3u} = \beta_{3\sigma} \times \beta_{3\sigma\sigma} + \varsigma\beta_{3\sigma}, \tag{56}$$

where ς is the viscosity and non-negative constant. The general time evolution of the Frenet frame

$$\begin{aligned} \mathbf{T}_u &= -h_1\mathbf{N} + \gamma_3\mathbf{B} && (57)\\ \mathbf{N}_u &= h_1\mathbf{T}+h_2\mathbf{B} && (58)\\ \mathbf{B}_u &= \gamma_3\mathbf{T} - h_2\mathbf{N}. && (59) \end{aligned}$$

Via Eqs.(55) and (56) it is found

$$\beta_{3u} = -\tau\mathbf{T} + \varsigma\mathbf{N} - \kappa\mathbf{B}. \tag{60}$$

From $\beta_{3u\sigma} = \beta_{3\sigma u}$ and $\mathbf{N}_{3u\sigma} = \mathbf{N}_{3\sigma u}$ the following are obtained by

$$h_1 = -\tau_\sigma - \varsigma\kappa,\ h_2 = -(\kappa_\sigma + \varsigma\tau),\ \gamma_3 = \frac{1}{2}(\tau^2 - \kappa^2)\ , \tag{61}$$

$$\begin{aligned} \kappa_u &= \tau_{\sigma\sigma} + \varsigma\kappa_\sigma + \frac{1}{2}\tau(\kappa^2 - \tau^2),\\ \tau_u &= \kappa_{\sigma\sigma} + \varsigma\tau_\sigma + \frac{1}{2}\kappa(\kappa^2 - \tau^2). \end{aligned}$$

The third frame $\{\pi_1, \pi_2, \pi_3\}$ connected with the generalized nonlinear heat system and Hasimoto-like transformation Ω_1 for the spacelike curve with timelike binormal are given by:

$$\begin{aligned} \pi_1 &= \mathbf{N}, && (62)\\ \pi_2 &= \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{B}), && (63)\\ \pi_3 &= \frac{1}{\sqrt{2}}(\mathbf{B} - \mathbf{T}). && (64) \end{aligned}$$

and

$$\Omega_1 = \frac{1}{\sqrt{2}}(\kappa + \tau). \tag{65}$$

Using Eqs.(62), (63), (64) and (65), the spatial evolution of the frame $\{\pi_1, \pi_2, \pi_3\}$ connected with the generalized nonlinear heat equation of the spacelike curve with timelike binormal for third class, respectively are given by

$$\begin{aligned}\pi_{1\sigma} &= -\Omega_1\pi_2+\Omega_2\pi_3\\ \pi_{2\sigma} &= \Omega_2\pi_1\\ \pi_{3\sigma} &= -\Omega_1\pi_1.\end{aligned}$$

where

$$\Omega_2=\frac{1}{\sqrt{2}}(\kappa-\tau).$$

The time evolution of the frame $\{\pi_1,\pi_2,\pi_3\}$ connected with the generalized nonlinear heat equation of the spacelike curve with timelike binormal for the third class, respectively are given by

$$\begin{aligned}\pi_{1u} &= -(\Omega_{1\sigma}-\varsigma\Omega_1)\pi_2-(\Omega_{2\sigma}+\varsigma\Omega_2)\pi_3\\ \pi_{2u} &= -(\Omega_{2\sigma}+\varsigma\Omega_2)\pi_1+R_3\pi_2\\ \pi_{3u} &= -(\Omega_{1\sigma}-\varsigma\Omega_1)\pi_1-R_3\pi_3\end{aligned}$$

From $\pi_{2u\sigma}=\pi_{2\sigma u}$ one obtains the generalized nonlinear heat system for the spacelike curve with timelike binormal with respect to Frenet frame for third class in Minkowski 3-space:

$$\begin{aligned}\Omega_{2u} &= -\Omega_{2\sigma\sigma}+\varsigma\Omega_{2\sigma}+R_3\Omega_2,\ R_3=\Omega_1\Omega_2\\ \Omega_{1u} &= \Omega_{1\sigma\sigma}+\varsigma\Omega_{1\sigma}-R_3\Omega_1,\ R_3=-\Omega_1\Omega_2\end{aligned}$$

3.3 Backlund Transformation Connected with visco-Da Rios Equation for Third Class in $\mathbf{R}_1^3$

Backlund transformation of the generalized Schrödinger flow for the spacelike curve β_3 with timelike binormal for the third class is constructed as the following:

Consider other nonnul curve $\widetilde{\beta}_3$ connected with β_3,

$$\widetilde{\beta}_3(\sigma,u)=\beta_3(\sigma,u)+h_1(\sigma,u)\mathbf{T}+h_2(\sigma,u)\mathbf{N}-h_3(\sigma,u)\mathbf{B}, \tag{66}$$

where h_1, h_2 and h_3 are functions of u and σ. If the derivatives of according σ and u of Eq.(66) are taken, one can obtain

$$\widetilde{\beta}_{3\sigma}=(h_{1\sigma}-h_2\kappa)\mathbf{T}+(1+h_1\kappa+h_{2\sigma}+h_3\tau)\mathbf{N}-(h_{3\sigma}+h_2\tau)\mathbf{B}, \tag{67}$$

$$\begin{aligned}\widetilde{\beta}_{3u} = {} & (-\tau+h_{1u}-h_2(\tau_\sigma+\varsigma\kappa)-h_3(\frac{\tau^2-\kappa^2}{2}))\mathbf{T}\\ &+(\varsigma+h_1(\tau_\sigma+\varsigma\kappa)+h_{2u}-h_3(\kappa_\sigma+\varsigma\tau)\mathbf{N}\\ &+(h_1(\frac{\tau^2-\kappa^2}{2})-h_2(\kappa_\sigma+\varsigma\tau)-h_{3u}-\kappa)\mathbf{B}\end{aligned} \tag{68}$$

Let $\widetilde{\sigma}$ be the arc length parameter of the curve $\widetilde{\beta}_3$, in this case

$$\begin{aligned} d\widetilde{\sigma} &= \sqrt{(h_{1\sigma}-h_2\kappa)^2+(1+h_1\kappa+h_{2\sigma}+h_3\tau)^2-(h_{3\sigma}+h_2\tau)^2}d\sigma \\ &= \chi d\sigma \end{aligned} \tag{69}$$

The tangent vector of the curve $\widetilde{\beta}_3$ is obtained by

$$\widetilde{\mathbf{T}} = \frac{d\widetilde{\beta}_3}{d\sigma}\frac{d\sigma}{d\widetilde{\sigma}} = m_1\mathbf{T}+m_2\mathbf{N}+m_3\mathbf{B}. \tag{70}$$

From Eq.(70),

$$\frac{d^2\widetilde{\beta}_3}{d\widetilde{\sigma}^2} = \frac{m_{1\sigma}-m_2\kappa}{\chi}\mathbf{T}+\frac{m_{2\sigma}-m_1\kappa+m_3\tau}{\chi}\mathbf{N}+\frac{m_2\tau+m_{3\sigma}}{\chi}\mathbf{B}. \tag{71}$$

From Eq.(70), the curvature $\widetilde{\kappa}$ of $\widetilde{\beta}_2$ is given by

$$\widetilde{\kappa} = \frac{\sqrt{(m_{1\sigma}-m_2\kappa)^2+(m_{2\sigma}+m_1\kappa-m_3\tau)^2-(m_{3\sigma}-m_2\tau)^2}}{\chi} = \frac{\Upsilon}{\chi}. \tag{72}$$

The normal vector $\widetilde{\mathbf{N}}$ of the curve $\widetilde{\beta}_3$ is given by

$$\widetilde{\mathbf{N}} = \frac{m_{1\sigma}-m_2\kappa}{\Upsilon}\mathbf{T}+\frac{m_{2\sigma}-m_1\kappa+m_3\tau}{\Upsilon}\mathbf{N}+\frac{m_2\tau+m_{3\sigma}}{\Upsilon}\mathbf{B}. \tag{73}$$

The vector of $\widetilde{\mathbf{B}}$ of the curve $\widetilde{\beta}_3$ is given by

$$\widetilde{\mathbf{B}} = n_1\mathbf{T}+n_2\mathbf{N}+n_3\mathbf{B}, \tag{74}$$

where

$$\begin{aligned} n_1 &= \frac{m_2(m_{3\sigma}+m_2\tau)-m_3(m_{2\sigma}-m_1\kappa+m_3\tau)}{\Upsilon} \\ n_2 &= \frac{m_3(m_{1\sigma}-m_2\kappa)-m_1(m_{3\sigma}+m_2\tau)}{\Upsilon} \\ n_3 &= \frac{m_2(m_{1\sigma}-m_2\kappa)-m_1(m_{2\sigma}-m_1\kappa+m_3\tau)}{\Upsilon} \end{aligned}$$

The torsion vector $\widetilde{\tau}$ of the curve $\widetilde{\beta}_3$ is obtained by

$$\widetilde{\tau} = \frac{\alpha}{\Upsilon\chi}, \tag{75}$$

where

$$\begin{aligned} \alpha &= -(n_{1\sigma}-n_2\kappa)(m_{1\sigma}-m_2\kappa)-(n_{2\sigma}+n_1\kappa-n_3\tau)(m_{2\sigma}-m_1\kappa+m_3\tau) \\ &\quad +(n_{3\sigma}-n_2\tau)(m_{3\sigma}+m_2\tau) \end{aligned}$$

Since the curve $\widetilde{\beta}_3$ and the curve β_3 have with same integrable systems, the curve $\widetilde{\beta}_3$ yields the following generalized nonlinear heat $GNLH$ flow of the visco Da Rios equation for the third class in Minkowski 3-space

$$\widetilde{\beta}_{3u} = -\widetilde{\tau}\widetilde{\mathbf{T}} - \widetilde{\kappa}\mathbf{B} + \varsigma\widetilde{\mathbf{N}}. \tag{76}$$

Theorem 3.3 *The generalized nonlinear heat flow Eq.(76) is invariant according to Backlund transformation Eq.(66) for third class in Minkowski 3-space if h_1, h_2 and h_3 satisfies the following system:*

$$\begin{aligned}
-\tau + h_{1u} - h_2(\tau_\sigma + \varsigma\kappa) - h_3\left(\frac{\tau^2-\kappa^2}{2}\right) &= -\frac{\alpha}{\Upsilon\chi}m_1 - \frac{\Upsilon}{\chi}n_1 \\
&\quad + \varsigma\frac{m_{1\sigma} - m_2\kappa}{\Upsilon} && (77)\\
\varsigma + h_1(\tau_\sigma + \varsigma\kappa) + h_{2u} - h_3(\kappa_\sigma + \varsigma\tau) &= -\frac{\alpha}{\Upsilon\chi}m_2 - \frac{\Upsilon}{\chi}n_2 \\
&\quad + \varsigma\frac{m_{2\sigma} - m_1\kappa + m_3\tau}{\Upsilon} && (78)\\
h_1\left(\frac{\tau^2-\kappa^2}{2}\right) - h_2(\kappa_\sigma + \varsigma\tau) - h_{3u} - \kappa &= -\frac{\alpha}{\Upsilon\chi}m_3 - \frac{\Upsilon}{\chi}n_3 \\
&\quad + \varsigma\frac{m_2\tau + m_{3\sigma}}{\Upsilon} && (79)
\end{aligned}$$

Proof Via Eqs.(68), (70), (72), (73), (74), (76) one can obtain Eqs.(77), (78) and (79). □

§4. Conclusions

In this paper one obtained the first class connected with the generalized nonlinear heat $GNLH$ system to the spacelike curve evolution with timelike binormal according to Frenet frame in $\mathbf{R}_1^3$. Later, one presented Backlund transformation of $GNLH$ flow with the Frenet frame in Minkowski 3-space. We gave the second class connected with generalized nonlinear Schrödinger equation $GNLS^-$ of spacelike curve evolution with timelike binormal according to Frenet frame in $\mathbf{R}_1^3$ and obtained Backlund transformation of $GNLS$ flow with Frenet frame in Minkowski 3-space in $\mathbf{R}_1^3$. Finally, one presented the third class connected with the generalized nonlinear Schrödinger equation $GNLH$ of spacelike curve evolution with timelike binormal according to Frenet frame in $\mathbf{R}_1^3$ and obtained Backlund transformation of $GNLH$ flow with Frenet frame in Minkowski 3-space in $\mathbf{R}_1^3$.

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