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Bounds of the Radio Number of Stacked-Book Graph with Odd Paths

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Abstract: A stacked-book graph $G_{m,n}$ is obtained from the Cartesian product of a star graph S_m and a path P_n, where m and s are the orders of the star graph and the path respectively. Obtaining the radio number of a graph is a rigorous process, which is dependent on diameter of G and positive difference of non-negative integer labels $f(u)$ and $f(v)$ assigned to any two u, v in the vertex set $V(G)$ of G. This paper obtains tight upper and lower bounds of the radio number of $G_{m,n}$ where the path P_n has an odd order. The case where P_n has an even order has been investigated.

Key Words: Radio labeling, Smarandachely radio labeling, direct product of graphs, cross product of graphs, star, path.

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§1. Introduction

All graphs mentioned in this work are simple and undirected. The vertex and edge sets of a graph G are designated as $V(G)$ and $E(G)$ respectively and $e = uv \in E(G)$ connects $u, v \in V(G)$ while $d(u, v)$ denotes the shortest distance between $u, v \in V(G)$. We represent the diameter of G as diam(G).

The radio labeling, which often aims to solve signal interference problems in a wireless network, was first suggested in 1980 by Hale [7] and it is described as follows: Suppose that f is a non negative integer function on $V(G)$ and that f satisfies the radio labeling condition, $|f(u) - f(v)| \geq \mathrm{diam(G)} + 1 - \mathrm{d(u,v)}$ for every pair $u, v \in V(G)$. The span$f$ of f is $f_{max}(G) - f_{min}(G)$, where f_{max} and f_{min} are largest and lowest labels, respectively, assigned on $V(G)$ and the lowest value of span$f$ is the radio number, $rn(G)$, of G. Generally, let $V_1 \subset V(G)$ be a subset of vertices in G with property $\mathscr{P}$. If a labeling f satisfying the radio labeling condition for vertices in $V(G)\backslash V_1$ but $|f(u) - f(v)| < \mathrm{diam(G)} + 1 - \mathrm{d(u,v)}$ for every pair $u, v \in V_1$, then f is called a Smarandachely radio labeling of G and span$f$ of f is denoted by span$^S f$. Clearly,

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$\text{span}^S f$=span$f$ if $V_1 = \emptyset$, i.e., the case of radio labeling on G. It is established that to obtain the radio numbers of graphs is hard. However, for certain graphs, the radio numbers have been obtained. Recent results on radio number include those on middle graph of path [3], trees, [4] and edge-joint graphs [12]. Liu and Zhu [11] showed that for path, P_n, $n \geq 3$,

$$rn(P_n) = \begin{cases} 2k(k-1)+1 & \text{if } n = 2k; \\ 2k^2+2 & \text{if } n = 2k+1. \end{cases}$$

Liu and Zhu's results compliment those obtained by Chatrand et. al. in [5] and [6] about the same graph. Liu and Xie worked on square graphs. In [9], they obtained $rn(P_n^2)$ of square of path as follows:

$$rn(P_n^2) = \begin{cases} k^2+2 & \text{if } n \equiv 1(\text{mod } 4), \text{n} \geq 9; \\ k^2+1 & \text{if otherwise.} \end{cases}$$

Other results on squares of graphs include those obtained for C_n^2 in [10], where C_n is a cycle of order n. On Cartesian products graphs, Jiang [8] solved the radio number problem for $(P_m \Box P_n)$, where for $m, n > 2$, and obtains $rn(P_m \Box P_n) = \frac{mn^2+nm^2-n}{2} - mn - m + 2$, for m odd and n even. Saha and Panigrahi [13] worked on Cartesian products of Cycles while Ajayi and Adefokun in [1] and [2] probe on the radio number of the Cartesian product of path and star graph called the stacked-book graph $G = S_m \Box P_n$. They observed in [1] that $rn(S_m \Box P_n) \leq n^2 m + 1$, a result the authors noted, citing a existing result in [8], is not a tight bound.) In [1], they obtained improve the results for path P_n, where n is even.

In this paper, we investigate further on the radio number of stacked-book graphs, $S_m \Box P_n$, in the case where n is odd and combined with [2], we improve the weak bounds obtained in [1].

§2. Preliminaries

Let S_m be a star of order $m \geq 3$ and let v_1 be the center vertex of S_m and $v_2, v_3, \cdots, v_m$ are adjacent to v_1 and let P_n be a path containing n vertices starting from u_1 to u_n. Furthermore, $P = u \xrightarrow{a} v \xrightarrow{b} w$ represents a path of length $a+b$, for which $d(u,v) = a$ and $d(u,w) = b$, where a and b are positive integers. If a stacked-book graph is obtained from the Cartesian product $G_{m,n} = S_m \Box P_n$ of S_m and P_n, then $V(G_{m,n})$ is the Cartesian product of $V(S_m)$ and $V(P_n)$, for which if $u_i v_j \in V(G_{m,n})$, then $u_i \in V(S_m)$, $v_j \in E(P_n)$, while, if $u_i v_j u_k v_i$ forms an edge in $E(G_{m,n})$, then $u_i = u_k$ and $v_j v_l \in E(P_n)$ or $v_j = v_l$ and $u_i u_k \in E(S_m)$.

Some of the following are adopted from [2].

Definition 2.1 *Where it is convenient, we denote $u_i v_j$ as u_{ij} and edge $u_i v_j u_k v_l$ as $u_{ij} u_{kl}$.*

Remark 2.1 Stacked-book graph $G_{m,n}$ contains n number of S_m stars, which can be expressed as the set $\left\{S_{m(i)} : 0 \leq i \leq n\right\}$.

Definition 2.2 *For $G_{m,n} = S_m \Box P_n$, $V_{(i)} \subset V(G_{m,n})$ is the set of vertices on $S_{m(i)}$ stated as $V_{(i)} = u_1 v_i, u_2 v_i, \cdots, u_m v_i$.*

Remark 2.2 We must mention that u_1v_i in the set in the last definition is the center vertex of $S_m(i)$.

Definition 2.3 *Let $G_{m,n} = S_m \square P_n$, n odd, the pair $S_{m(i)}, S_{m(i+\frac{n-1}{2})}$ is a subgraph $G''(i) \subseteq G_{m,n}$, which is induced by V_i and $V_{i+\frac{n-1}{2}}$, where $i \notin \{1, \frac{n+1}{2}, n\}$.*

Remark 2.3 It can be seen that with n odd, every $G_{m,n}$ contains $\frac{n-2}{3}$ number of $G''(i)$ subgraphs and the diameter $diam(G''(i))$ of $G''(i)$ is $\frac{n+3}{3}$.

Next, we introduce the following definitions:

Definition 2.4 *Let $G_{m,n} = S_m \square P_n$. Then, $\bar{G}_{m,n} \subseteq G_{m,n}$ is a subgraph of $G_{m,n}$ induced by the stars $S_{m(1)}, S_{m(\frac{n+1}{2})}, S_n$.*

We now define a class of paths $P'(i)$.

Definition 2.5 *Let $\{P'(t)\}_{t=1}^m$ be a class of paths in $G_{m,n}$, where $P'(t) := v_{j(1)} \xrightarrow{\alpha} v_{k(\frac{n+1}{2})} \xrightarrow{\beta} v_{l(n)}$, such that $j \neq k \neq l$, $v_{j(1)} \in V_{(i)}, v_{k(\frac{n+1}{2})} \in V_{(\frac{n+1}{2})}$ and $v_{l(n)} \in V_{(n)}$ and $1 \leq j, k, l \leq m$.*

It can be verified that $\{P'(t)\}_{t=1}^m$ contains two other sub-classes defined without loss of generality as follows:

$P'_1(t) = \{v_{1(1)} \xrightarrow{\frac{n+1}{2}} v_{3(\frac{n+1}{2})} \xrightarrow{\frac{n+3}{2}} v_{2(n)}, v_{2(1)} \xrightarrow{\frac{n+1}{2}} v_{1(\frac{n+1}{2})} \xrightarrow{\frac{n+1}{2}} v_{3(n)}, v_{3(1)} \xrightarrow{\frac{n+3}{2}} v_{2(\frac{n+1}{2})} \xrightarrow{\frac{n+1}{2}} v_{1(n)}\}$

$P'_2(t) = v_{a(1)} \xrightarrow{\frac{n+3}{2}} v_{b(\frac{n+1}{2})} \xrightarrow{\frac{n+3}{2}} v_{c(n)}$, $a \neq b \neq c$, $4 \leq a, b, c \leq m$. Clearly, $|P'_1(t)| = 3$ and $|P'_2(t)| = m - 2$.

§3. Results

In the next results, we establish a lower bound of the radio number $rn(G_{m,n})$ of a stacked-book graph $G_{m,n}$.

Lemma 3.1 *Let f be the radio labeling function on $G_{m,n}$, n odd, and let*

$$V_{(\frac{n+1}{2})} = \left\{v_{1(\frac{n+1}{2})}, v_{2(\frac{n+1}{2})}, v_{3(\frac{n+1}{2})}, \left\{v_{d(\frac{n+1}{2}):4\leq d\leq m}\right\}\right\}$$

be the vertex set of the mid vertices in $P(t) \subseteq \{P'(t)\}_{t=1}^m$. Now, let $v \in V_{\frac{n+1}{2}}$ be some vertex in $V_{\frac{n+1}{2}}$. If $f(v)$ is f_{max} on $V(P(t))$, then

$$f(v) = \begin{cases} \frac{n+5}{2} & \text{if } v \in \left\{v_{1(\frac{n+1}{2})}, v_{2(\frac{n+1}{2})}, v_{3(\frac{n+1}{2})}\right\}; \\ \frac{n+3}{2} & \text{otherwise.} \end{cases}$$

Proof Since $P(t) \subset G_{m,n}$, then, radio labeling of any vertex on $V(P(t))$ is based on $diam(G_{m,n})$ and for $u, v \in V(P(t))$, $d(u, v) = k$, where k is the distance between u and v in

$G_{m,n}$. We consider the three paths in $P_1'(t)$ next.

Case 1(a) For $P_1'(1) := v_{1(1)} \xrightarrow{\frac{n+1}{2}} v_{3(\frac{n+1}{2})} \xrightarrow{\frac{n+3}{2}} v_{2(n)}$, let $f(v_{1(1)}) = 0$. Now $d(v_{1(1)}, v_{2(n)}) = n$. Therefore $f(v_{2(n)}) \geq f(v_{1(1)}) + diam(G_{m,n}) + 1 - n$=2. Also, $d(v_{2(n)}, v_{3(\frac{n+1}{2})}) = \frac{n+3}{2}$ and thus, $f(v_{3(\frac{n+1}{2})}) \geq f(v_{2(n)}) + dim(G_{m,n}) + 1 - \frac{n+3}{2} \geq \frac{n+5}{2}$. (It should be note that if we set $f(v_{2(n)}) = 0$, then, $f(v_{3(\frac{n+1}{2})}) \geq \frac{n+7}{2}$.)

Case 1(b) For $P_1'(2) := v_{2(1)} \xrightarrow{\frac{n+1}{2}} v_{1(\frac{n+1}{2})} \xrightarrow{\frac{n+1}{2}} v_{3(n)}$, let $f(v_{2(1)}) = 0$, then $d(v_{2(1)}, v_{3(n)}) = n+1$ and thus, $f(v_3(n)) \geq n + 2 - (n+1) = 1$. Likewise, $d(v_{3(n)}, v_{1(\frac{n+1}{2})}) = \frac{n+2}{2}$ and therefore, $f(v_{3(n)}) \geq n + 3 - (\frac{n+1}{2}) = \frac{n+5}{2}$.

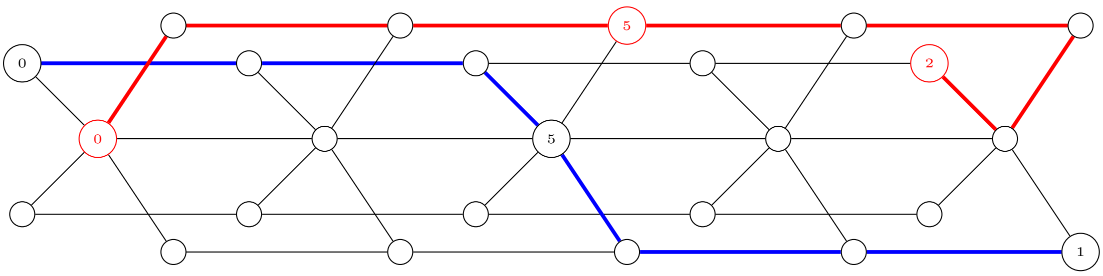

Figure 1 Illustration of Case 1(a) and (b) in a $G_{5,5}$ stacked-book graph

Case 1(c) Now for $P_1'(3) := v_{3(1)} \xrightarrow{\frac{n+3}{2}} v_{2(\frac{n+1}{2})} \xrightarrow{\frac{n+1}{2}} v_{1(n)}$, we assume $f(v_{1(n)}) = 0$. Also, $d(v_{3(1)}, v_1(n)) = n + 1$ in $G_{m,n}$. Thus, $f(v_{3(1)}) \geq 2$ and since $d(v_{3(1)}, v_{2\frac{n+1}{2}}) \geq \frac{n+3}{2}$, then, $f(v_{2(\frac{n+1}{2})}) \geq 2 + n + 2 - (\frac{n+3}{2}) \geq \frac{n+5}{2}$.

Next we consider the paths in $P_2'(t)$.

Case 2. Every path in $P_2'(t)$ are geometrically similar and are of the form $P_2'(4) = v_{a(1)} \xrightarrow{\frac{n+3}{2}} v_{b(\frac{n+1}{2})} \xrightarrow{\frac{n+3}{2}} v_{n(n)}$, such that $d(v_{a(1)}, v_{c(n)}) = n + 1$ and $d(v_{c(n)}, v_b(\frac{n+1}{2})) = \frac{n+3}{2}$, in $G_{m,n}$ and for all $a \neq b \neq c \neq m$, without loss of generality. Thus, suppose that $f(v_{a(1)}) = 0$, then $f(v_c(n)) \geq 1$ and $f(v_{b(\frac{n+1}{2})}) \geq \frac{n+3}{2}$. $\square$

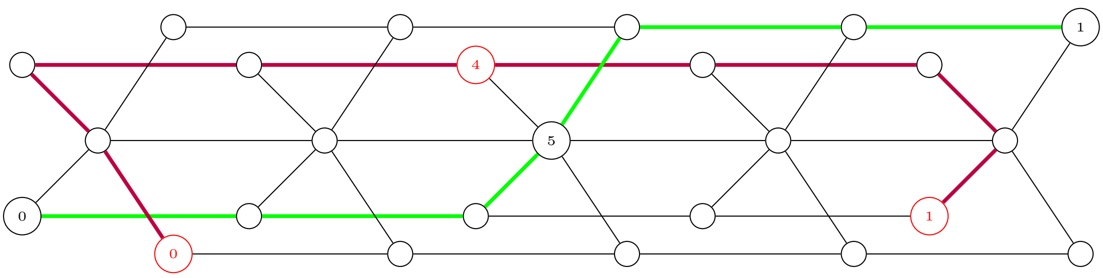

Figure 2 Illustration of Case 1(c) and Case 2 in a $G_{5,5}$ stacked-book graph

Remark 3.1 In (a) and (c) of Case 1, if the respective center vertices $v_{1(1)}$ and $v_{1(n)}$ of stars $S_{(1)}$ and $S_{(n)}$ are labeled $f(v_{1(1)}) = f(v_{1(n)}) = 0$, the radio labels on the mid vertices of their paths would be at least $\frac{n+7}{2}$.

Remark 3.2 For the m paths in $\{P't\}_{t=1}^{m}$, the sum of all the radio labels on the center vertices ($span(f)$ of f on $P'(t)$ is $3(\frac{n+5}{2}) + (m-3)(\frac{n+3}{2}) = \frac{1}{2}(mn + 3m + 6)$.

Next, we obtained a lower bound for $\{P(t)\}_{t=1}^{m}$.

Remark 3.3 From Remark 3.2, we notice that for optimum labeling of the three vertices on each of the paths in $\{P(t)\}_{t=1}^{m}$, the end vertex, which closest to the mid vertex is most suitable to be labeled first. These are $v_{1(1)} \in P_1'(1), v_1(n) \in P_1'(3)$ and any end vertex in the remaining paths. We refer to each of these ends vertices as initial label vertex.

Lemma 3.2 *Let $G(*)$ be a subgraph of $G_{m,n}$, induced by all the end point vertices and the midpoint vertices of $\{P_1'(t)\}_{t=1}^{m}$ i.e $S_{m(1)}, S_{m(\frac{n+1}{2})}, S_{m(n)}$. Then $rn(G(*)) \geq \frac{1}{2}(2mn+4m-n+5)$ in $G_{m,n}$.*

Proof Let v_1 and v_2 be center vertices on $S_{m(1)}$ and $S_{m(n)}$ respectively. There exist vertices $u_\alpha, u_\beta \in S_{m(\frac{n+1}{2})}$, $\alpha \neq \beta$, u_α, u_β not center vertices of $S_{m(\frac{n+1}{2})}$ such that $d(v_1, u_\alpha) = d(v_2, u_\beta) = \frac{n+1}{2}$. Also, there exists a subset $A = \{\omega_r\}$ in $S_{m(1)}$, (or $S_{m(n)}$) such that $|A| = m - 3$, and $B = \{x_s\}$ in $S_{m(\frac{n+1}{2})}$, $|B| = m-1$, such that for $r \neq s$, $d(\omega_r, x_s) = \frac{n+3}{2}$. Now, the sum of $span(f)$ of f for all the pair (ω_r, x_s) will be $(m-1)(\frac{n+1}{2}) = \frac{1}{2}(mn + m - n - 1)$ and thus,

$$\begin{aligned} rn(G(*)) &\geq \frac{1}{2}(mn + m - n - 1) + \frac{1}{2}mn + 3m + 6] \\ &\geq \frac{1}{2}(2mn + 4m - n + 5). \end{aligned}$$

This completes the proof. □

We extend the result in Lemma 3.2 in other to obtain a lower bound for the radio number of stacked book graph $G_{m,n}$, with off $n \geq 5$.

Definition 3.1 *Let $G_{m,n}$ be a stacked-book graph with odd n, $n \geq 5$, and $m \geq 4$. Also, let $G(*)$ as defined earlier. A subgraph $G(**)$ of $G_{m,n}$ as $G(**) = G_{m,n} \backslash G(*)$.*

Remark 3.4 We can see that $G(**)$ is a subgraph of $G_{m,n}$, induced by $\{S_{m(i)}\}_{i=2}^{n-1}$, $i \neq \frac{n+1}{2}$.

Definition 3.2 *The subgraph of $G(**)$, induced by $S_{m(t)}$ and $S_{m(t+\frac{n-1}{2})}$ is denoted by $G''(t)$.*

Remark 3.5 It should be noted that $G(**) \subset G_{m,n}$ contains exactly $\frac{n-3}{2}$ $G''(t)$ subgraphs.

Remark 3.6 Let $G''(t)$ be induced by $S_{m(t)}$ and $S_{m(t+\frac{n-1}{2})}$ and let $V(S_{m(t)}) = \{u_1, u_2, \cdots, u_m\}$ and $V(S_{m(t+\frac{n-1}{2})}) = \{v_1, v_2, \cdots, v_m\}$ be the vertex sets of $S_{m(t)}$ and $S_{m(t+\frac{n-1}{2})}$ where u_1 and v_1 are the respective center vertices. It can be seen that, $d(u_i, v_j) \in \{\frac{n+1}{2}, \frac{n+3}{2}\}$, where $i \neq j$.

Remark 3.7 For $i \neq j$, $d(u_1, v_j) = d(u_j, v_1) = \frac{n+1}{2}$ and for $i \neq j$, $i, j \neq 1$, $d(u_i, v_j) = \frac{n+3}{2}$.

Now we obtain a lower bound value for the radio number labeling of $G''(t)$ in $G_{m,n}$.

Lemma 3.3 *Let $G''(t) \subset G_{m,n}$, with $m \geq 4$ and $n \geq 5$, n odd, be a subgraph of $G_{m,n}$. Then*

$$rn(G''(t)) \geq mn + m - \frac{1}{2}(n-3).$$

Proof Let u_1 and v_1 be center vertices of $S_{m(t)}$ and $S_{m(t+\frac{n-1}{2})}$. By Remark 3.7 above, $d(u_1, v_i) = d(u_i, v_1) = \frac{n+1}{2}$, is the shortest distance between the center vertex of a star in $G''(t)$ and a non-center vertex in the other star in $G''(t)$. It is optimal, therefore to label the center vertices as f_{min} and f_{max}. Now, without loss of generality, set $f_{min} = f(v_1) = 0$. Since $d(v_1, u_i) = \frac{n+1}{2}$, $i \in \{2, 3, \cdots, m\}$. Therefore

$$f(u_i) \geq f(v_1) + diam(G_{m,n}) + 1 - d(u_i, v_1).$$

Let $i = 2$. Thus,

$$\begin{aligned} f(u_2) &\geq 0 + n + 2 - \frac{n+1}{2} \\ &\geq \frac{n+3}{2}. \end{aligned}$$

Now $d(u_2, v_3) = \frac{n+3}{2}$ and therefore,

$$\begin{aligned} f(v_3) &\geq \frac{n+3}{2} + n + 2 - \frac{n+3}{2} \\ &\geq \frac{n+3}{2} + \frac{n+1}{2}. \end{aligned}$$

Also, for $d(v_3, u_4) = \frac{n+3}{2}$, $f(v_4) \geq \frac{n+3}{2} + 2(\frac{n+1}{2})$. By continuing the iteration, we have

$$f(v_m) \geq \frac{n+3}{2} + 2m - 3(\frac{n+1}{2}).$$

Lastly,

$$\begin{aligned} f_{max} = f(u_1) &\geq 2(\frac{n+3}{2}) + (2m-3)(\frac{n+1}{2}) \\ &= mn + m - \frac{1}{2}(n-3). \end{aligned}$$ □

Next we extend the last result to obtain a lower bound for $G(**)$.

Lemma 3.4 *For $G(**) \subset G_{m,n}$, $rn(G(**)) \geq \frac{1}{2}(mn^2 - 2mn - 3m + 2n - 12)$.*

Proof From Lemma 3.3, the *span*(f) of f on $G''(t) = mn + m - \frac{1}{2}(n-3)$. For $G''(t)$, $f_{max} = mn + m - \frac{1}{2}(n-3)$. Let $t = 2$ and let $u_1 = v_{m(2)} \in S_{m(2)}$ and $v_1' = v_{m(2+\frac{n+1}{2})} \in S_{m(2+\frac{n+1}{2})}$, be center vertices of $S_{m(2)}$ and $S_{m(2+\frac{n+1}{2})}$. Now, $d(u_i, v_1') = \frac{n+1}{2}$. Thus,

$$f(v_1') \geq f(u_1) + n + 2 - \frac{n+1}{2} = f(u_1) + \frac{n+3}{2}.$$

This implies that for $G''(3)$, induced by $S_{m(3)}$ and $S_{m(2+\frac{n+1}{2})}$, $f_{min} = f(u_1) + \frac{n+3}{2}$, and $f_{max} = f(v_1'')$, where v'' is the center vertex of $S_{m(3)}$. From the precedure in Lemma 3.3, there are $\frac{n-3}{2}$ $G''(t)$ subgraphs in $G_{m,n}$. Therefore, f_{max} of $G(**)$ is $f(v_1^{(k)}) \in S_{(m(\frac{n-1}{2}))}$, where $f(v_1^{(k)})$ is the center vertex of $S_{m(\frac{n-1}{2})}$. Following the iteration,

$$\begin{aligned} f(v_1^{(k)}) &\geq \frac{n-3}{2}[mn + m - \frac{1}{2}(n-3)] + \frac{n-5}{2}\left(\frac{n+3}{2}\right) \\ &\geq \frac{1}{2}(mn^2 - 2mn - 3m + 2n - 12). \end{aligned}$$

This completes the proof. □

Now we establish a lower bound for the radio number of $G_{m,n}$.

Theorem 3.1 *Let $G_{m,n}$ be a stacked-book graph with $m \geq 4$ and $n \geq 5$. Then,*

$$rn(G_{m,n}) \geq \frac{mn^2 + m + 2n - 4}{2}.$$

Proof From Lemmas 3.3 and 3.4,

$$rn(G(*)) \geq m(n+2) - \frac{1}{2}(n-5) \text{ and } rn(G(**)) \geq \frac{1}{2}(mn^2 - 2mn - 3n + 2n - 12).$$

Now, since $G_{m,n} = G(*) \cup G(**)$, suppose that u_1 is the center vertex of $S_{m(\frac{n-1}{2})}$ and $v_1 \in S_{m(n)}$ is the center vertex of $S_{m(n)}$. Clearly $d(u_1, v_i) = \frac{n+1}{2}$. Now, $f(u_1) = f_{max}$ of $G(**)$ and

$$f(u_1) \geq \frac{1}{2}(mn^2 - 2mn - 3m + 2n - 12).$$

Therefore,

$$\begin{aligned} f(v_1) &\geq f(u_i) + n + 2 - \frac{(n+1)}{2} \\ &\geq \frac{1}{2}(mn^2 - 2mn - 3m + 2n - 12) + n + 2 - \frac{(n+1)}{2} \\ &\geq \frac{1}{2}(mn^2 - 2mn - 3m + n - 13) + n + 2. \end{aligned}$$

For $G(*)$, set $f(v_1) = f_{min}$. Thus, $rn(G_{m,n}) \geq f(v_1) + rn(G(*))$ and hence,

$$\begin{aligned} rn(G_{m,n}) &\geq \frac{1}{2}(mn^2 - 2mn - 3m + n - 13) + n + 2 + m(n+2) - \frac{1}{2}(n-5) \\ &\geq \frac{mn^2 + m + 2n - 4}{2}. \end{aligned}$$

This completes the proof. □

Next, we investigate the upper bound of a stacked-book graph. The technique involves manual radio labeling of subgraphs $G(*)$ and $G(**)$ and merging the results.

Lemma 3.5 *Let $G(**) \subset G_{m,n}$, with n-odd. Then, $rn(G(**)) \leq \frac{1}{2}(mn^2 - 2mn + 2n - 3m - 12)$.*

Proof From earlier definition, if n is odd, then, $G_{m,n} = G(*) \cup G(**)$, where $G(**)$ contains $\frac{n-3}{2}$ $G''(t)$ graphs. Let $G''(\frac{n-1}{2})$ be induced by $S_{m(\frac{n-1}{2})}$ and $S_{m(n-1)}$, n-odd. Let $V(S_{m(\frac{n+1}{2})}) = \left\{v_{\frac{n-1}{1}(i)}\right\}_{i=1}^{m}$, $V(S_{m(n-1)}) = \{v_{n-1}\}_{i=1}^{m}$, where $v_{\frac{n-1}{2}(1)}$, $v_{n-1(1)}$ are center vertices and $d(v_{\frac{n-1}{2}(j)}, v_{n-1(j)}) = \frac{n-1}{2}$, for all $1 \leq j \leq m$, $d(v_{\frac{n-1}{2}(1)}), v_{n-1(j)} = d(v_{n-1(1)}, v_{\frac{n-1}{2}(j)}) = \frac{n+1}{2}$ and $d(v_{\frac{n-1}{2}(j)}, v_{n-1(k)}) = \frac{n+3}{2}$. Now, let $f(v_{\frac{n-1}{2}(1)}) = 0$. Since $d(v_{\frac{n-1}{2}(1)}, v_{n-1(2)}) = \frac{n+1}{2}$, then set $f(v_{n-1(k)}) = \frac{n+3}{2}$. Let $f(v_{\frac{n-1}{2}(1)}) = 0$. Since $d(v_{\frac{n-1}{2}(1), v_{n-1(2)}}) = \frac{n+1}{2}$, then set $f(v_{n-1(2)}) = \frac{n+3}{2}$, $d(v_{n-1(2)}, v_{\frac{n-1}{2}(3)}) = \frac{n+3}{2}$ and thus,

$$f(v_{n-1}(4)) = \frac{n+3}{2} + 2\frac{n+1}{2}.$$

Thus, by continuously alternating the process, it gets to the case where $d(v_{\frac{n-1}{2}(m)}, d(v_{n-1(m-1)})) = \frac{n+3}{2}$. Thus,

$$f(\frac{n-1}{2}(m)) = \frac{n+3}{2} + m - 2(\frac{n+1}{2})$$

and since $d(v_{\frac{n-1}{2}(m)}, v_{n-1(3)}) = \frac{n+3}{2}$,

$$f(v_{n-1(3)}) = \frac{n+3}{2} + (m-1)(\frac{n+1}{2}), \quad f(v_{\frac{n-1}{2}(2)}) = \frac{n+3}{2} + m(\frac{n+1}{2}).$$

Depending on the size of m, the labeling continues until

$$f(v_{\frac{n-1}{2}}) = \frac{n+3}{2} + 2m - 3\frac{(n+3)}{2} + 2(2m-3)(\frac{n+1}{2})$$

is attained and finally, $d(v_{\frac{n-1}{2}(m-1)}, v_{n-1(1)}) = \frac{n+1}{2}$ and thus, $f(v_{n-1(1)}) = \frac{2(n+3)}{2} + 2m - 3 + \frac{n+1}{2}$. (By following the same argument, it is easy to obtain similar result for m-even.) Now, $d(v_{n-1(1)}, v_{\frac{n-3}{2}(1)}) = \frac{n+1}{2}$, where $v_{\frac{n-3}{2}(1)}$ is the center vertex of $G''_{m(\frac{n-3}{2})}$. Therefore,

$$\begin{aligned} f_{min}(G''(\frac{n-3}{2})) &= f(v_{\frac{n-3}{2}(1)}) = f(v_{n-1(1)}) + n + 2 - \frac{n+2}{2} \\ &= f(v_{n-1(1)}) + \frac{n+3}{2} = \frac{3(n+3)}{2} + 2m - 3(\frac{n+1}{2}) \end{aligned}$$

and

$$\begin{aligned} f_{max}(G''(\frac{n-3}{2}) &= f(v_{n-2(1)}) = f(v_{\frac{n-3}{2}(1)}) + \frac{2(n+3)}{2} + \frac{(2m-3)(n+1)}{2} \\ &= \frac{5(n+3)}{2} + 2(2m-3)\frac{(n+1)}{2}, \end{aligned}$$

which is $f_{max}(G''(\frac{n-3}{2}))$. Now, the process is extended to $G''(2)$, for which

$$\begin{aligned} f(\frac{n+3}{2}) &= \frac{(n-5)(n+3)}{4} + \frac{(n-3)(n+3)}{2} + \frac{(n-2)(2m-3)(n+1)}{4} \\ &= \frac{1}{2}(mn^2 - 2mn + 2n - 3m - 12). \end{aligned}$$

□

Remark 3.8 It can be observed that for the optimal radio labeling of $G(*)$, $f_{max}(G(*))$ is $f(v_{\frac{n+1}{2}(1)})$, the label on the center vertex of $S_{m(\frac{n+1}{2})}$. Since for v_α, v_β in $S_{m(1)}$ and $S_{m(n)}$ respectively, $\alpha, \beta \neq 1$, $d(v_{\frac{n+1}{2}(1)}, v_\alpha) = d(v_{\frac{n+1}{2}(1)}, v_\beta) = \frac{n+1}{2}$, which is less than $\frac{n+3}{2}$, the value of $d(v_{\frac{n+1}{2}(k)}, v_\alpha)$, where $k \neq \alpha$, $k, \alpha \neq 1$, and v_α either belongs to $S_{m(1)}$ or $S_{m(n)}$. Thus, we manually label $G(*)$, such that $v_{\frac{n+1}{2}(1)}$ gets the last label and thus, $f(v_{\frac{n+1}{2}(1)}) = f_{max}(G(*))$.

Next, we consider some necessary conditions for establishing the upper bound of $G(*)$.

Lemma 3.6 *Let $G(*) \subset G_{m,n}$ be a subset of $G_{m,n}$, induced by $S_{m(1)}, S_{m(\frac{n+1}{2})}$ and $S_{m(n)}$. If $v_{1(1)}$ (or $v_{n(1)}$) and $v_{\frac{n+2}{2}(1)}$ are the center vertices of $S_{m(1)}$ (or $S_{m(n)}$) and $S_{m(\frac{n+1}{2})}$ respectively, and $f_{min}(G(*)) \neq f(v_{1(1)})$ (or $f(v_{n(1)})$), and $f_{max}(G(*)) \neq f(v_{\frac{n+1}{2}})$ (or vice versa), then, $|f_{min}(G(*)) - f_{max}(G(*))| \neq rn(G(*))$.*

Proof Without loss of generality, select $v_{1(1)}$ over $v_{n(1)}$. Suppose that $f(v_{1,1})$ and $f(v_{\frac{n+1}{2}(1)})$ are not $f_{min}(G(*))$ and $f_{max}(G(*))$ respectively. Let $v_\alpha \in V(S_{m(1)})$, $v_\beta \in V(S_{m(\frac{n+1}{2})})$, and $v_\gamma \in V(S_{m(n)})$ be non-center vertices, and let the set of the following vertices, $\{v_\alpha, v_\beta, v_\gamma, v_{1(1)}, v_{\frac{n+1}{2}(1)}\}$ be X, and let $H = V(G(*))\backslash X$ be the subgraph of $G(*)$ induced by $V(G(*)) - X$, and such that the radio number of H is positive integer p. Without loss of generality, let there be some $v_k \in V(H)$, where $v_k = v_{n(i)} \in S_{m(n)}, \gamma \neq i$ and $d(v_k, v_\beta) = \frac{n+3}{2}$, there exist a radio numbering sequence $v_k \to v_\beta, \to v_{1(1)} \to v_\gamma \to v_{\frac{n+1}{2}(1)} \to v_\alpha$. Suppose that $f(v_k)$ is the $f_max(H)$, that is, $f(v_k) = p$. Since $d(v_k, v_\beta) = \frac{n+3}{2}$, then $f(v_\beta) = p + \frac{n+1}{2}$ and likewise, it is observed that the radio labeling sequence yields $f_{max}(G(*)) = p{+}2n{+}7$. Now, suppose on the contrary, that $f(v_{1(1)})$ and $f(v_{\frac{n+1}{2}(1)})$ are $f_{min}(G(*))$ and $f_{max}(G(*))$ respectively. Let $v_{k(0)}$ be the vertex in H, which holds the least radio label. Obviously $v_{k(0)} \neq v_k$ and since $|V(G(*))| - |V(H)| \equiv 3 \text{mod } 1$, then $v_{k(0)}$ is a is also a vertex on the same star as v_k, this time, $S_{m(n)}$. Thus,if $v_{k(0)}$ is also not a center vertex, then, $d(v_{1(1)}, v_{k(0)}) = n$. Let $f(v_{1(1)}) = 0$. Now, we have the radio labeling sequence: $v_{1(1)} \to (v_{k(0)} \to \cdots \to v_k) \to v_\beta \to v_\alpha \to v_\gamma \to v_{\frac{n+1}{2}(1)}$. Since $d(v_{k(0)}, v_{1(1)}) = n$, then, $f(v_{k(0)}) = 2$ and since $|f_{min}(H) - f_{max}(H)| = p$, then $f(v_k) = 2 + p$. Labeling the sequence, afterwards, we have

$$f_{max}(G(*)) = f(v_{\frac{n+1}{2}(1)}) = p + \frac{3n+11}{2},$$

which is less than $p + 2n + 7$. □

Remark 3.9 It is noted that $v_{1(1)}$ (or $v_{n(1)(1)}$) and $v_{\frac{n+1}{2}}$ can be $f_{min}(G(*))$ and $f_{max}G(*)$ interchangeably. However, they both will have to be used for these roles. It is trivial to show that optimal radio labeling will not be attained if just one of them is used.

Next we obtain an upper bound for $G(*)$, based on Lemma 3.6.

Theorem 3.2 *For $G(*) \subset G_{m,n}$, $m \geq 5$, $rn(G(*)) \leq \frac{1}{2}(2mn + 4m - n + 7)$.*

Proof From Lemma 3.6, for $v_{1(1)} \in S_{m(1)}$, let $f(v_{1(1)}) = 0$. There exist $m-1$ vertices of $S_{m(n)}$, such that for each $v_{n(i)} \in V(S_{m(n)})$, $i \neq 1$, $d(v_{1(1)}, v_{n(i)}) = n$. Thus, without loss of generality, let the first vertex be $v_{n(2)}$. Then, $f(v_{n(2)}) = 2$. Likewise, there exists $m-1$, non-center vertex on $S_{m(\frac{n+1}{2})}$, and for each $v_{\frac{n+1}{2}(j)}$, $j \neq 1$, $d(v_{n(2)}, v_{\frac{n+1}{2}(j)}) = \frac{n+3}{2}$, where $j \neq 2$. So, now, let $j = 3$, then,

$$f(v_{\frac{n+1}{2}(3)}) = 2 + n + 2 - \frac{n+3}{2} = 2 + \frac{n+1}{2}.$$

In similar way,

$$f(v_{1(4)}) = 2 + \frac{n+1}{2} + \frac{n+1}{2}.$$

Now, we label $v_{n(1)}$, which is at distance n from $v_{1(4)}$ as $f(v_{n(1)}) = 4 + \frac{n+1}{2} + \frac{n+1}{2}$. Now, two of the center vertices are labeled. For, say, $v_{\frac{n+1}{2}(5)}$,

$$f(v_{\frac{n+1}{2}(5)}) = 4 + \frac{n+1}{2} + \frac{n+1}{2} + \frac{n+3}{2}.$$

It can be seen that for each of $S_{m(1)}$, $S_{m(\frac{n+1}{2})}$ and $S_{m(n)}$, there are $m-2$ vertices left to be labeled. This is now done by adding $\frac{(n+1)}{2}$ and 1 in alternating manner to the cumulative label values, such that we have

$$f(v_{1(6)}) = 4 + 3(\frac{n+1}{2}) + \frac{n+3}{2} \text{ and } f(v_{n(7)}) = 5 + 3(\frac{n+1}{2}) + \frac{n+3}{2}.$$

Thus by continuing the iteration until it gets to

$$f(v_{\frac{n+1}{2}(1)}) = (m+2) + (2m-3)(\frac{n+1}{2}) + 2(\frac{n+3}{2}) = \frac{1}{2}(2mn + 4m - n + 7). \qquad \square$$

Next, we merged the last results to obtain an upper bound for the radio number of a stacked-book graph $G_{m,n}$, where $m \geq 5$.

Theorem 3.3 *Let $m \geq 5$. Then, $rn(G_{m,n}) \leq \frac{1}{2}(mn^2 + 2n + m - 2)$.*

Proof Recall that $G = G(*) \cup G(**)$. From Lemma 3.5, where $G(**)$ is labeled, we see that for $G(**)$, $f_{max}(G(**)) = f(v_{\frac{n+3}{2}(1)})$. For $G(*) \in G_{m,n}$, we see in Lemma 3.2 that $f(v_{1(1)}) = f_{min}$. Clearly, $d(v_{\frac{n+3}{2}(1)}, v_{1(1)}) = \frac{n+1}{2}$. Thus, for $v_{1(1)} \in G_{m,n}$,

$$f(v_{1(1)}) = f(v_{\frac{n+3}{2}(1)}) + \frac{n+3}{2} = \frac{1}{2}(mn^2 - 2mn + 2n - 3m - 12) + \frac{n+3}{2} = \frac{1}{2}(mn^2 - 2mn + 3n - 3m - 9).$$

Thus by Lemma 3.2,

$$f_{max}(G_{m,n}) = f(v_{1(1)}) + f_{max}(G(**)) = \frac{1}{2}(mn^2 + 2n + m - 2). \qquad \square$$

Remark 3.10 We observe that the result in Theorem 3.3 that the there is just a difference of of 1 between this upper bound and the lower bound established earlier in the work. It is believed that the lower bound can be improved to coincide with the upper bound.

A radio labeling of $G_{5,5}$ is shown in Figure 3, where it is demonstrated that $rn(G_{5,5}) \leq 69$.

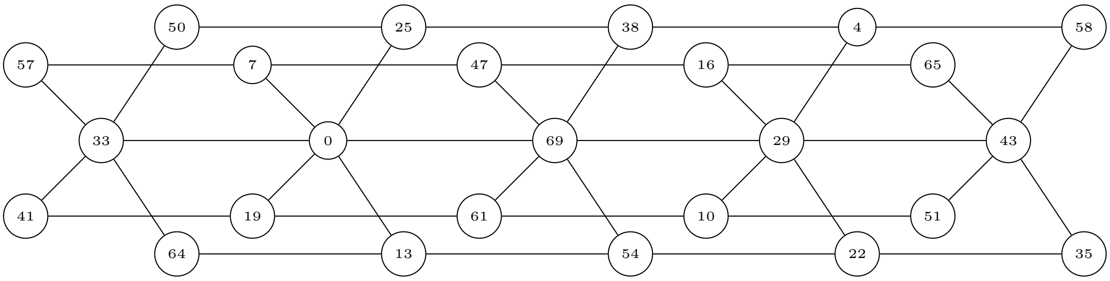

Figure 3 A $G_{5,5}$ stacked-book graph

§4. Conclusion

This work has greatly improved results obtained in [1] and extended the outcomes of [1] to the odd-path factor of the stacked-book graph class. It is safe now to say that this work and [2] have provided a tight bounds for the radio number of the general stacked-book graph. Further work to obtain the exact value of the radio number for stacked-book graph should be considered.

References

[1] D.O.A. Ajayi and T.C. Adefokun, On bounds of radio number of certain product graphs, *Jour Nigeria Math Soc.*, 37(2) (2018) 71-78.

[2] T.C. Adefokun and D.O.A. Ajayi, On radio number of stacked-book graph, *arXiv:1901.00355.*, Jan 2019.

[3] D. Bantva, S Vaidya and S. Zhou, Radio numbers of trees, *Electron. Notes in Discrete Math.*, 48 (2015) 135-141.

[4] D. Bantva, Radio numbers of middle graph of paths, *Electron. Notes in Discrete Math.*, 66 (2017) 93-100.

[5] G. Chartrand, D. Erwin, F Harary and P. Zhang, Radio Labelings of Graphs, *Bull. Inst. Combin. Appl.*, 33 (2001) 77-85.

[6] G. Chartrand, D. Erwin and P. Zhang, A graph labeling problem suggested by FM channel restrictions, *Bull. Inst. Combin. Appl.*, 43 (2005) 43-57.

[7] Hale W. K. Hale, Frequency assignment theory and applications, *Proc. IEEE*, 68 (1980) 1497 - 1514

[8] T-S Jiang, The radio number of grid graphs, *arXiv: 1401.658v1.*, 2014.

[9] D.D.-F. Liu and M. Xie, Radio number for square paths, *Ars Combin.*, 90 (2009) 307-319.

[10] D.D.-F. Liu and M. Xie, Radio number for square cycles, *Congr. Numer.*, 169 (2004) 105-125.

[11] D. Liu and X. Zhu, Multilevel distance labelings for paths and cycles, *SIAM J. Discrete Math.*, 19 (2005) 610-621.

[12] A. Naseem, K. Shabbir and H. Shaker, The radio number of edge-joint graphs, *Ars Combin.*, 139 (2018) 337-351.

[13] L. Saha and P. Panigrahi, On the radio numbers of toroidal grid, *Aust. Jour. Combin.*, 55 (2013) 273-288.