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CR-Sub-Manifolds of (ϵ, δ)-Trans-Sasakian Manifolds Admitting Generalized Symmetric Metric Connection

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Abstract: A new connection on (ϵ, δ)-trans-Sasakian manifolds is introduced in this paper, which is the generalization both of the semi-symmetric and quarter-symmetric connection. We discuss the properties of CR-Sub-manifold of an (ϵ, δ)-trans-Sasakian manifolds with respect to a generalized symmetric metric connection and also presented the integrability conditions of distributions on CR-Sub-manifolds in this paper.

Key Words: CR-Sub-manifold, (ϵ, δ)-trans-Sasakian manifolds, ϵ-Kenmotsu manifolds, integrability condition.

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§1. Introduction

The metric connection with a torsion different from zero was introduced by Hayden [3] on a Riemannian manifold. In [2], Golab mentioned that the quarter symmetric connections, being more generalized form of semi-symmetric connections on a differentiable manifold. Tripathi [10] introduced and studied many types of connections which includes the semi-symmetric and quarter symmetric connections.

CR-Sub-manifolds of Sasakian manifold were studied by Kobayashi [6] and Hasan Shahid et al. [9]. Moreover, Kenmotsu [5] studied new class of almost contact Riemannian manifolds, known as Kenmotsu manifolds. CR-Sub-manifolds of such manifolds was studied by Papaghuic [7]. More general, one has the notion of α-Sasakian structure and β-Kenmotsu structure (See [14]). Motivating by these results, in this paper we studied the concept of (ϵ, δ)-trans-Sasakian manifolds.

A linear connection on a Riemannian manifold M is suggested to be a generalized sym-

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metric connection if its torsion tensor T is presented as follows:

$$T(U,V) = \alpha(u(V)U - u(U)V) + \beta(u(V)\phi U - u(U)\phi Y), \tag{1.1}$$

for all vector fields U and V on M, where α and β are smooth functions on M, ϕ is of tensor type (1,1) and u is regarded as a 1-form connected with the vector field. The connection mentioned here is a generalized metric one when a Riemannian metric g in M is available as $\overline{\nabla} g = 0$ or else it is non-metric.

In (1.1), if $\alpha = 0$, $\beta \neq 0$; $\alpha \neq 0$, $\beta = 0$, then the generalized symmetric connection is called β-quarter-symmetric connection and α-semi-symmetric connection respectively. Therefore, generalizing semi-symmetric and quarter-symmetric connection gives the generalized symmetric metric connection.

In this paper, we define a new connection on (ϵ,δ)-trans-Sasakian manifolds which is the generalization of semi-symmetric and quarter-symmetric connection. Nagaraja et al. [8] introduced (ϵ,δ)- trans-Sasakian manifold, which generalizes both ϵ-Sasakian and ϵ-Kenmotsu manifolds. In Section 2, the preliminaries of (ϵ,δ)-trans-Sasakian manifolds discussed. Section 3, illustrates generalized symmetric connection on an (ϵ,δ)-trans-Sasakian manifolds. In Section 4, we study the properties of CR-Sub-manifold of an (ϵ,δ)-trans-Sasakian manifolds with respect to a generalized symmetric metric connection. We also presented the integrability conditions of distributions on CR-Sub-manifolds.

§2. **Preliminaries**

Let M be a differentiable manifold of dimension n endowed with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η and metric g, which satisfies

$$\phi^2 X_1 = -X_1 + \eta(X_1)\xi, \;\; \phi\xi = 0, \;\; \eta(\phi X_1) = 0, \;\;\; \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \epsilon\eta(X_1)\eta(X_2), \;\; \eta(X_1) = \epsilon g(X,\xi) \tag{2.2}$$

$$\begin{aligned}(\nabla_{X_1}\phi)X_2 &= \alpha(g(X_1,X_2)\xi - \epsilon\eta(X_2)X_1) \\ &\quad + \beta(g(\phi X_1, X_2)\xi - \delta\eta(X_2)\phi(X_1))\end{aligned} \tag{2.3}$$

$$\nabla_{X_1}\xi = -\epsilon\alpha\phi X_1 - \beta\delta\phi^2 X_1, \;\;\; rank \;\; \phi = n-1, \tag{2.4}$$

for any $X_1, Y_1 \in TM$, where ∇ denotes the Levi-Civita connection with respect to the (ϵ,δ)-trans-Sasakian manifolds metric g. Such manifold (M,ϕ,ξ,η,g) is called (ϵ,δ)-trans-Sasakian manifolds. In addition, if η is closed on an (ϵ,δ)-trans-Sasakian manifolds then we have

$$(\nabla_{X_1}\eta)X_2 = -\alpha g(\phi X_1, X_2) + \epsilon\delta\beta g(\phi X_1, \phi X_2) \tag{2.5}$$

for any vector field X_1 and X_2.

The Gauss and Weingarten formulae give by

$$\nabla_{X_1} X_2 = \nabla'_{X_1} X_2 + h(X_1, X_2), \quad \forall X_1, X_2 \in \Gamma(TM'), \tag{2.6}$$

$$\nabla_{X_1} N = -A_N X_1 + \nabla^{\perp}_{X_1} N, \ \ \forall N \in \Gamma(T^{\perp} M'), \tag{2.7}$$

where $(\nabla_{X_1} X_2, A_N X_1)$ and $(h(X_1, X_2), \nabla^{\perp}_{X_1} N)$ belong to $\Gamma(TM')$ and $\Gamma(T^{\perp} M')$, respectively.

§3. (ϵ, δ)-Trans-Sasakian Manifold with Generalized Symmetric Metric Connection

We have $\overline{\nabla}$ as a linear connection and ∇ as a Levi-Civita connection of (ϵ, δ)-trans-Sasakian manifold M, in such a way that

$$\overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + H(X_1, X_2), \tag{3.1}$$

for all the vector field X_1 and X_2. Since $\overline{\nabla}$ is a generalized symmetric metric connection of ∇, where H is $(1, 2)$ tensor type.

$$H(X_1, X_2) = \frac{1}{2}[T(X_1, X_2) + T'(X_1, X_2) + T'(X_2, X_1)] \tag{3.2}$$

$$g(T'(X_1, X_2), W) = g(T(W, X_1), X_2). \tag{3.3}$$

Therefore, from (1.1) and (3.3), we have:

$$T'(X_1, X_2) = \alpha(\eta(X_1)X_2 - g(X_1, X_2)\xi) + \beta(\eta(X_1)\phi X_2 - g(\phi X_1, X_2)\xi) \tag{3.4}$$

now taking (1.1), (3.2) and (3.4), we get:

$$H(X_1, X_2) = \alpha(\eta(X_2)X_1 - g(X_1, X_2)\xi) + \beta(\eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi) \tag{3.5}$$

Corollary 3.1 *For an (ϵ, δ)-trans-Sasakian manifold, the generalized symmetric metric connection $\overline{\nabla}$ of type (α, β) is given by*

$$\begin{aligned} \overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 &+ \alpha(\eta(X_2)X_1 - g(X_1, X_2)\xi) \\ &+ \beta(\eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi). \end{aligned} \tag{3.6}$$

If $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, the generalized metric connection is declined to a semi-symmetric metric and a quarter-symmetric metric one as given below

$$\overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \eta(X_2)X_1 - g(X_1, X_2)\xi \tag{3.7}$$

$$\overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi. \tag{3.8}$$

Using (2.3), (2.5) and (3.6), we have the following lemma.

Lemma 3.1 *If M is (ϵ,δ)-trans-Sasakian manifold with the generalized symmetric metric connection then the following relations holds*

$$(\overline{\nabla}_{X_1}\phi)X_2 = [(\alpha-\beta)g(X_1,X_2)+(\alpha+\beta)g(\phi X_1,X_2) + \beta\eta(X_1)\eta(X_2)(\epsilon-1)]\xi - \alpha\epsilon\eta(X_2)X_1 - \eta(X_2)\phi X_1(\alpha+\delta\beta), \tag{3.9}$$

$$\overline{\nabla}_{X_1}\xi = -\alpha\epsilon\phi X_1 + (\alpha+\delta\beta)(X_1-\eta(X_1)\xi), \tag{3.10}$$

$$(\overline{\nabla}_{X_1}\eta)X_2 = (\alpha+\delta\beta)g(\phi X_1,\phi X_2) - \alpha\epsilon g(\phi X_1,X_2). \tag{3.11}$$

for every $X_1, X_2 \in \Gamma(TM)$.

Proof We know that $(\overline{\nabla}_{X_1}\phi)X_2 = \overline{\nabla}_{X_1}\phi X_2 - \phi(\overline{\nabla}_{X_1}X_2)$. Replacing X_2 with ϕX_2 in (3.6) we have

$$\overline{\nabla}_{X_1}\phi X_2 = (\alpha-\beta)g(X_1,X_2)+(\alpha+\beta)g(\phi X_1,X_2)\xi + \beta\epsilon\eta(X_1)\eta(X_2)\xi - \alpha\epsilon\eta(X_2)X_1 - \beta\delta\eta(X_2)\phi X_1 + \phi(\nabla_{X_1}X_2) \tag{3.12}$$

Substituting (3.12) in $(\overline{\nabla}_{X_1}\phi)X_2$, we obtain (3.9), and put $X_2=\xi$ in (3.9) we get (3.10). Similarly, taking $(\overline{\nabla}_{X_1}\eta)X_2 = g(X_2,\overline{\nabla}_{X_1}\xi)$, we get (3.11). Hence, the proof is completes. □

§4. CR-Sub-Manfolds of (ϵ,δ)-Trans-Sasakian Manifold with Generalized Symmetric Metric Connection

An n-dimensional Riemannian manifold M of an (ϵ,δ)-trans-Sasakian manifold M' is called a CR-sub-manifold if ξ tangent to M and there exists on M a differentiable distribution $D: x \to D_x \subset T_x(M)$ such that

(i) D is invariant under ϕ, that is, $\phi D \subset D$;

(ii) The orthogonal complement distribution $D^\perp : x \to D_x^\perp \subset T_x(M)$ of the distribution D on M is totally real, that is, $\phi D^\perp \subset T^\perp M$.

Here, the distribution D is called horizontal distribution. The pair $(D, D^\perp)$ is called ξ-horizontal if $\xi \in \Gamma(D)$. The CR-sub-manifold is also called ξ-horizontal if $\xi \in \Gamma(D)$.

The orthogonal component $\phi D^\perp$ in $T^\perp M$ is given by

$$TM = D \oplus D^\perp, \quad T^\perp M = \phi D^\perp \oplus \mu,$$

where $\phi\mu = \mu$.

Let M be a CR-sub-manifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection $\overline{\nabla}$. For any $X \in \Gamma(TM')$ and $X \in \Gamma(T^\perp M')$, we have

$$X_1 = PX_1 + QX_1, PX_1 \in \Gamma(D), \quad QX_1 \in \Gamma(D^\perp), \tag{4.1}$$

$$\phi N = BN + CN, \quad BN \in \Gamma(D^\perp), CN \in \Gamma(\mu). \tag{4.2}$$

The Gauss and Weingarten formulae with respect to $\overline{\nabla}$ are as follows

$$\overline{\nabla}_{X_1}X_2 = \overline{\nabla}'_{X_1}X_2 + \overline{h}(X_1, X_2) \tag{4.3}$$

$$\overline{\nabla}_{X_1}N = -\overline{A}_N X_1 + \overline{\nabla}^{\perp}_{X_1}N \tag{4.4}$$

for any $X_1, X_2 \in \Gamma(TM^{\perp})$. Now, the above equation becomes

$$P\overline{\nabla}'_{X_1}X_2 = P\nabla'_{X_1}X_2 + \alpha\eta(X_2)PX - \alpha g(X_1, X_2)P\xi + \beta\eta(X_2\phi PX - \beta(g(\phi X, X_2)P\xi)) \tag{4.5}$$

$$\overline{h}(X_1, X_2) = h(X_1, X_2) + \beta(\eta(X_2)\phi QX_1 \tag{4.6}$$

$$Q\overline{\nabla}'_{X_1}X_2 = \nabla'_{X_1}X_2 + \alpha(\eta(X_2)QX_1 - \alpha g(X_1, X_2)Q\xi - \beta g(\phi X_1, X_2)Q\xi) \tag{4.7}$$

for any $X_1, X_2 \in \Gamma(TM')$.

The Gauss and Weingarten formulae with respect to generalized symmetric metric connection is of the form (See [1]):

$$\overline{\nabla}_{X_1}X_2 = \overline{\nabla}'_{X_1}X_2 + h(X_1, X_2) + \beta\eta(X_2)\phi QX_1 \tag{4.8}$$

$$\overline{\nabla}_{X_1}N = -A_N X_1 + \nabla^{\perp}N + \alpha\eta(N)X_1 + \beta\eta(N)\phi X_1 - \beta g(\phi X_1, N)\xi. \tag{4.9}$$

Theorem 4.1 *Let M be a CR-submanifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then*

$$h(X_1, \phi PX_2) + \nabla^{\perp}_{X_1}\phi QX_2 = Ch(X_1, X_2) - (\alpha + \delta\beta)\eta(X_2)\phi QX_1 + \phi Q\overline{\nabla}' X_1X_2, \tag{4.10}$$

$$\begin{aligned} P\overline{\nabla}' X_1\phi PX_2 - PA_{\phi}QX_2X_1 - \beta g(\phi X_1, \phi QX_2)P\xi &= (\alpha - \beta)g(X_1, X_2)P\xi \\ &+ (\alpha + \beta)g(\phi X_1, X_2)P\xi + (\epsilon - 1)\beta\eta(X_1)\eta(X_2)P\xi - \alpha\epsilon\eta(X_2)P(X_1) \\ &- (\alpha + \delta\beta)\eta(X_2)\phi PX_1 + \phi P\overline{\nabla}' X_1X_2 + \beta\eta(X_2)\eta(QX_1)P\xi, \end{aligned} \tag{4.11}$$

$$\begin{aligned} Q\overline{\nabla}'\phi PX_2 - QA_{\phi QX_2}X_1 - \beta g(\phi X_1, \phi QX_2)Q\xi &= (\alpha - \beta)g(X_1, X_2)Q\xi \\ &+ (\alpha + \beta)g(\phi X_1, X_2)Q\xi + (\epsilon - 1)\beta\eta(X_1)\eta(X_2)Q\xi - \alpha\epsilon\eta(X_2)Q(X_1) \\ &+ Bh(X_1, X_2) + \beta\eta(X_2)QX_1 + \beta\eta(X_2)\eta(QX_1)Q\xi. \end{aligned} \tag{4.12}$$

for any $X_1, X_2 \in \Gamma(TM)$.

Proof We know that $\overline{\nabla}_{X_1}\phi X_2 = (\overline{\nabla}_{X_1}\phi)X_2 + \phi(\overline{\nabla}_{X_1}X_2)$. Consider LHS of the above equation and using (3.12) we have

$$\overline{\nabla}_{X_1}\phi X_2 = \overline{\nabla}_{X_1}\phi PX_2 + \overline{\nabla}_{X_1}\phi QX_2 \tag{4.13}$$

Now using (4.8) for the tangential part and (4.9) for the normal part of the above equation we get

$$\overline{\nabla}_{X_1}\phi X_2 = \overline{\nabla}'_{X_1}\phi PX_2 + h(X_1, \phi PX_2) - A_{\phi QX_2}X_1 + \nabla^{\perp}_{X_1}\phi QX_2 - \beta g(\phi Q, \phi QX_2)\xi \tag{4.14}$$

Again by using (3.12) we obtain

$$\overline{\nabla}_{X_1}\phi X_2 = P\overline{\nabla}'_{X_1}\phi PX_2 + h(X_1,\phi PX_2) + \nabla^{\perp}_{X_1}\phi QX_2 - \beta g(\phi X_1,\phi QX_2)P\xi \\ - \beta g(\phi X_1,\phi QX_2)Q\xi + Q\overline{\nabla}'_{X_1}\phi QX_2 - PA_{\phi QX_2} - QA_{\phi QX_2}. \tag{4.15}$$

Now consider RHS of $\overline{\nabla}_{X_1}\phi X_2 = (\overline{\nabla}_{X_1}\phi)X_2 + \phi(\overline{\nabla}_{X_1}X_2)$ and using (3.9) we obtain

$$(\overline{\nabla}_{X_1}\phi)X_2 + \phi(\overline{\nabla}_{X_1}X_2) = [(\alpha-\beta)g(X_1,X_2) + (\alpha+\beta)g(\phi X_1,X_2) \\ + \beta\eta(X_1)\eta(X_2)(\epsilon-1)]\xi - \alpha\epsilon\eta(X_2)X_1 \\ - \eta(X_2)\phi X_1(\alpha+\delta\beta) + \phi[\overline{\nabla}'_{X_1}X_2 + h(X_1,X_2) \\ + \beta\eta(X_2)\phi QX_1] \tag{4.16}$$

Now using (4.7) and (4.8) in the above equation we obtain

$$(\overline{\nabla}_{X_1}\phi)X_2 + \phi(\overline{\nabla}_{X_1}X_2) = (\alpha-\beta)g(X_1,X_2)P\xi + (\alpha-\beta)g(X_1,X_2)Q\xi \\ + (\alpha+\beta)g(\phi X_1,X_2)P\xi + (\alpha+\beta)g(\phi X_1,X_2)Q\xi \\ + (\epsilon-1)\beta\eta(X_1)\eta(X_2)P\xi + (\epsilon-1)\beta\eta(X_1)\eta(X_2)Q\xi \\ - \alpha\epsilon\eta(X_2)X_1P\xi - \alpha\epsilon\eta(X_2)X_1Q\xi - (\alpha+\delta\beta)\eta(X_2)\phi PX_1 \\ - (\alpha+\delta\beta)\eta(X_2)\phi QX_1 + \phi P\overline{\nabla}'_{X_1}X_2 + \phi Q\overline{\nabla}'_{X_1}X_2 + Bh(X_1,X_2) \\ + Ch(X_1,X_2) + \beta\eta(X_2)QX_1 + \beta\eta(X_2)\eta(QX_1)\xi$$

Now on comparing LHS and RHS the normal,horizontal and vertical components we obtain (4.9), (4.10) and (4.11). The proof is completes. □

Theorem 4.2 *Let M be a ξ -vertical CR-submanifold of a (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then*

$$\phi[X_2,X_3] = A_{\phi X_3}X_2 - A_{\phi X_2}X_3 + 2(\alpha+\beta)g(X_3,\phi X_2)\xi. \tag{4.17}$$

for any $X_2, X_3 \in (\Gamma D^{\perp})$.

Proof Consider

$$g(\phi([X_2,X_3]),V) = g(\phi(\overline{\nabla}'_{X_2}X_3 - \overline{\nabla}'_{X_3}X_2),V),$$

Using (4.7) we take

$$g(\phi([X_2,X_3]),V) = -g(\overline{\nabla}'_{X_3}\phi X_2,V) + g((\overline{\nabla}'_{X_3}\phi)X_2,V) + g(\overline{\nabla}'_{X_2}\phi X_3,V) \\ - g((\overline{\nabla}'_{X_2}\phi)X_3,V) + \beta\eta(X_3)g(QX_2,V).$$

Now using (4.8) and also (3.9) in the above equation we have

$$\begin{aligned} g(\phi([X_2,X_3]),V) &= g(A_{\phi X_2}X_3,V) - g(\nabla^{\perp}_{X_3}\phi X_2,V) + (\alpha+\beta)g(\phi X_3,X_2)\epsilon\eta(V) \\ &\quad - g(A_{\phi X_3}X_2,V) + g((\nabla^{\perp}_{X_2}\phi)X_3,V) - (\alpha+\beta)g(\phi X_2,X_3)\epsilon\eta(V) \\ &\quad + \alpha\epsilon\eta(X_3)g(X_2,V) + \eta X_3 g(X_2,V) + \eta(X_3)g(\phi X_2,V)(\alpha+\delta\beta) \\ &= -g(\nabla^{\perp}_{X_3}\phi X_2,V) - g(A_{\phi X_3}X_2,V) + g(A_{\phi X_2}X_3,V) \\ &\quad + g(\nabla^{\perp}_{X_2}\phi X_3,V) + 2\epsilon(\alpha+\beta)g(\phi X_3,X_2)\eta(V). \end{aligned}$$

This completes the proof. □

Hence this theorem is verifying the following corollary.

Corollary 4.1 *Let M be a ξ vertical CR-submanifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then the distribution $D^{\perp}$ is integrable iff*

$$A_{\phi X_2}X_3 - A_{\phi X_3}X_2 = 2(\alpha+\beta)g(\phi X_2,X_3)\xi \tag{4.19}$$

for any $X_2, X_3 \in \Gamma(D^{\perp})$.

Corollary 4.2 *Let M be a ξ vertical CR-submanifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then the distribution $D^{\perp}$ is integrable iff*

$$A_{\phi X_2}X_3 = A_{\phi X_3}X_2$$

for any $X_2, X_3 \in \Gamma(D^{\perp})$.

Corollary 4.3 *Let M be a ξ vertical CR-submanifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then the distribution $D^{\perp}$ is integrable iff*

$$A_{\phi X_2}X_3 - A_{\phi X_3}X_2 = g(\phi X_2,X_3)\xi$$

for any $X_2, X_3 \in \Gamma(D^{\perp})$.

Theorem 4.3 *Let M be a CR-submanifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then we have*

$$\nabla_{X_1}\xi = (\alpha+\delta\beta)PX_1 - \alpha\epsilon BX_1, \tag{4.20}$$
$$h(X_1,\xi) = -\alpha\epsilon CX_1 - (\alpha+\delta\beta)(QX_1+\eta(X_1)\xi), \tag{4.21}$$
$$\nabla_{X_2}\xi = (\alpha+\delta\beta)PX_2 - \alpha\epsilon BX_2, \tag{4.22}$$
$$h(X_2,\xi) = -\alpha\epsilon CX_2 - (\alpha+\delta\beta)(QX_2+\eta(X_2)\xi), \tag{4.23}$$
$$\nabla_{\xi}\xi = (\alpha+\delta\beta)P\xi, \tag{4.24}$$
$$h(\xi,\xi) = -(\alpha+\delta\beta)QX_2. \tag{4.25}$$

for any $X_2, X_3 \in \Gamma(D^\perp)$.

Proof The above theorem is proved from (3.10) by using (3.12), (4.1) and (4.2) considering

$$\begin{aligned}\nabla_{X_1}\xi + h(X_1,\xi) = -\alpha\epsilon BX_1 - \alpha\epsilon CX_1 + (\alpha+\delta\beta)PX_1 + (\alpha+\delta\beta)QX_1 \\ - (\alpha+\delta\beta)\eta(X_1)\xi \end{aligned} \tag{4.26}$$

On equating LHS and RHS we have (4.18) and (4.20). Now by replacing X_1 to X_2 in (4.25) and on equating LHS and RHS we get (4.21) and (4.22). Again by replacing X_1 to ξ in (4.25) and on equatingLHS and RHS we get (4.23) and (4.24). The proof is completed. □

Theorem 4.4 *Let M be a ξ horizontal CR-submanifold of (ϵ,δ)-trans-Sasakian manifold M' with generalized symmetric metric connection. Then the distribution D is integrable iff*

$$h(\phi X_1, X_2) = h(X_1, \phi X_2) \tag{4.27}$$

for any $X_1, X_2 \in \Gamma(D)$.

Proof Assuming M to be ξ horizontal we have from (4.9)

$$h(X_1, \phi PX_2) = Ch(X_1,X_2) - (\alpha+\delta\beta)\eta(X_2)\phi QX_1 + \phi Q\overline{\nabla}' X_1 X_2$$

for all $X_1, X_2 \in D$.

Since $[X_1, X_2] \in D$, we have D is integrable iff

$$h(\phi X_1, X_2) = h(X_1, \phi X_2) \tag{4.28}$$

This completes the proof. □

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