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Classification of the Defining Equations of Flag Varieties $\mathcal{F}\ell_n(\mathbb{C})$

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Abstract: Using only combinatorial technique, we give a formula for classification of the defining equation of flag variety $\mathcal{F}\ell_n(\mathbb{C})$. The formula uses the theory of complete geometric graph based on the indexing set of the monomials of the ideal. In particular, we give a generating function to count the number of classes. The size of each class is also determined. We describe the procedure of obtaining the equations using a complete geometric graph and lastly, we give a formula to count these equations.

Key Words: Flag variety, Plücker coordinate, geometric graph, spanning tree.

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§1. Introduction

Let V be an n-dimension vector space over the field of complex numbers. By a flag F in V, we mean a sequence of subspaces:

$$F_\bullet : \{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_n = V \ \text{ such that } \dim F_i = i.$$

The set of all such flags in V is called the flag variety and denoted by $\mathcal{F}\ell_n(\mathbb{C})$. By fixing a basis $e_1, e_2, \ldots, e_n$, we let $E_\bullet$ to denote the standard flag spanned by

$$E_\bullet = \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \subset \langle e_1, e_2, \ldots, e_n\rangle.$$

The variety can also be described by considering the general linear group $GL(n, \mathbb{C})$ consisting of all non-singular $n \times n$ matrices and let $\mathcal{B}$ be the subset of all invertible upper triangular matrices. A flag $F_\bullet$ can be constructed by allowing F_i be the span of the first i columns of a given matrix Z in $GL(n, \mathbb{C})$. The matrices Z_1 and Z_2 are equivalent, that is, give the same flag if and only if there is an upper triangular matrix Y in $\mathcal{B}$ such that $Z_2 = YZ_1$. This defines an equivalence relation on $GL(n, \mathbb{C})$. Thus $\mathcal{F}\ell_n(\mathbb{C}) = GL(n, \mathbb{C})/B$. The precise implication is that the general linear group $GL(n, \mathbb{C})$ acts transitively on $\mathcal{F}\ell_n(\mathbb{C})$ and the stabilizer of standard flag is the Borel subgroup and hence the identification of $\mathcal{F}(n)$ with G/B. Therefore, $\mathcal{F}\ell_n(\mathbb{C})$ is viewed as a homogeneous space. More is true $\mathcal{F}\ell_n(\mathbb{C})$ is a smooth projective variety being a closed subvariety of the product of Grassmanians $\prod_{k=1}^{n-1} Gr(k, n)$. This gives rise to the Plücker

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embedding

$$\mathcal{F}\ell_n(\mathbb{C}) \hookrightarrow \mathbb{P}^{\binom{n}{1}-1} \times \mathbb{P}^{\binom{n}{2}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}.$$

The image of $\mathcal{F}\ell_n(\mathbb{C})$ via the embedding is cut out by Plücker relations (see [1]). These relations generate the homogeneous ideal of $\mathcal{F}\ell_n(\mathbb{C})$ which we denote by $\mathcal{I}$, indeed $\mathcal{I}$ is minimally generated by these quadrics. It is well known that each flag $F_\bullet$ can be represented by $n \times n$-matrix $A = (a_{ij})$ in which the subspace F_i is spanned by the first i rows. The relations that a point must satisfy in order to lie in the image of $\mathcal{F}\ell_n(\mathbb{C})$ via the embedding are called the Plücker relations. This is achieved by defining the map

$$\phi_n : \mathbb{K}[p_\alpha : \emptyset \neq \alpha \subseteq \{1, \ldots, n\}] \longrightarrow \mathbb{K}[a_{ij} : 1 \leq i \leq n-1, 1 \leq j \leq n]$$

sending each variable p_α to the determinant submatrix of A with row indices $1, \ldots, \mid \alpha \mid$ and column indices in α. It turns out that the ideal I_n of $\mathcal{F}\ell_n(\mathbb{C})$ is the kernel of ϕ_n. This homogeneous ideal is minimally generated by the Plücker relations. These relations which are quadrics are the equations defining the variety $\mathcal{F}\ell_n(\mathbb{C})$ (See [9],[1]).

Our interest is in the classification of these equations using complete geometric graphs. Specifically, we give a formula that partitions the equations by exploiting some similar properties shared by them. This ultimately allows us to know the number of equations in each subdivision thereby counts the generators for each ideal I_n. We plan a sequel paper to exploit this technique to give the degeneration of flag variety $\mathcal{F}\ell_n(\mathbb{C})$. Let $\mathbb{T}$ be a collection of points in the plane in general position. By geometric graph on $\mathbb{T}$, we mean a graph G whose vertices are the elements of $\mathbb{T}$ in which two are said to be adjacent if they are joined by a line segment. Our interest is in a graph where every pair of vertices is adjacent. This is called a complete geometric graph and is denoted by $\mathcal{K}_n$, n is the number of vertices. The number of edges of $\mathcal{K}_n$ is $\frac{n(n-1)}{2}$ which turns out to be the dimension of flag variety $\mathcal{F}\ell_n(\mathbb{C})$. In section 2, we give some background and results relevant to our discussion. In section 3, we describe the procedure to obtain the relations in the complete geometric digraph, $\mathcal{K}_n$ and also compute the relations in $\mathcal{K}_3$ and $\mathcal{K}_4$. In section 4, we give the classifications of relations in $\mathcal{K}_n$ and the class size. We also give generating functions on the classifications and the number of classes in any $\mathcal{K}_n$. This gives the classification of the equations defining flag varieties $\mathcal{F}\ell_n(\mathbb{C})$.

§2. Complete Geometric Directed Graphs

In this section we give some definitions on geometric graphs and trees (see [5], [4], [2], [3], [6], [8], [7], [10] for details).

Definition 2.1 *Let $\mathcal{K}_n$ be a complete geometric digraph with a n points and let $\sigma \subset [n]$. x_σ is said to be a point if $|\sigma| = 1$, a line if $|\sigma| = 2$, a triangle if $|\sigma| = 3$ and so on.*

Remark 2.2 All the x_σ's for which $|\sigma| \geq 3$ are empty, that is, they have no interior points.

Example 2.3 (i) For $n = 3$, the complete geometric digraph is

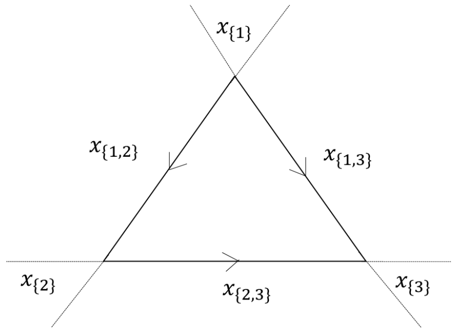

Figure 1

(ii) For $n = 4$, the complete geometric digraph is

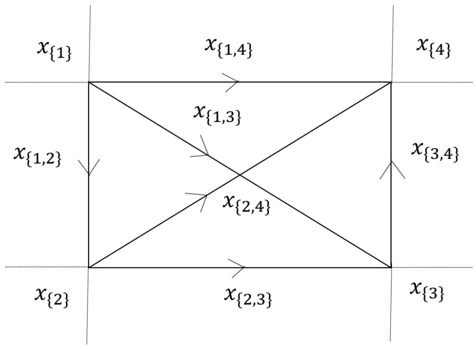

Figure 2

Given a complete geometric digraph $\mathcal{K}_n$, let $F_m = \{x_\sigma : |\sigma| = m, \sigma \subseteq [n]\}$, F_1 set of points, F_2 set of lines and so on. Let $f_m = \#F_m$.

Definition 2.4 (i) *A walk in $\mathcal{K}_n$ is a sequence of vertices $v_0, v_1, \cdots, v_k$ and sequence of edges $(v_i, v_{i+1}) \in F_2$. If v_i are distinct, then we have a path and if $(v_0, v_k) \in F_2$, then $v_0, v_1, \cdots, v_k, v_0$ is a cycle. The length of a path or cycle is the number of edges in it.*

(ii) *A tree is a connected graph without any cycles. The edges of a tree are called branches and the degree 1(number of edges incident with the vertex) vertex are called leaves.*

(iii) *A spanning tree T of a connected graph $\mathcal{K}_n$ is the subgraph of $\mathcal{K}_n$ containing all the vertices of $\mathcal{K}_n$. A chord is an edge of a graph that is not in a given spanning tree.*

(iv) *A rooted tree T with the vertex set V is the tree that has a specially designated vertex $v_1 \in V$. The root of any spanning tree is defined as the vertex with highest degree.*

Remark 2.5 (i) For any spanning tree T of $\mathcal{K}_n$, the number of branches is called the rank, r and the number of chords is called the nullity, μ (cyclomatic number or first Betti number). $r = n - 1$ and $\mu = \frac{(n-1)(n-2)}{2}$.

(ii) There are n^{n-2} spanning tree in a complete graph and n $(n-1)$-valent spanning trees since there are only n vertices with degree $n - 1$.

Lemma 2.6 *Let $\mathcal{C}$ be the set of flag varieties and $\mathcal{B}$ be the set of complete geometric digraphs, there is a bijection*

$$\alpha : \mathcal{C} \longrightarrow \mathcal{B}$$

$$\mathcal{F}l_n(\mathbb{C}) \longmapsto \mathcal{K}_n.$$

Theorem 2.7 *Given a complete geometric digraph $\mathcal{K}_n$, then f_m is given by the coefficient of*

$$P_n(t) = \sum_{|\sigma|=1}^{n} \binom{n}{|\sigma|} t^{|\sigma|}.$$

Proof Given a complete geometric digraph, $\mathcal{K}_n$ with points indexed by $[n]$, let $\sigma \subseteq [n]$ and $|\sigma| = r$. For $r = 1$, we have a point and the number of choice of selection is $\binom{n}{1}$ and for $r = 2$ we have a line and the number of choice of selection is $\binom{n}{2}$. Continuing until $r = n$, we have $\binom{n}{n}$. Then this can be generalised as

$$\binom{n}{1}t + \binom{n}{2}t^2 + \cdots + \binom{n}{r}t^r + \cdots + \binom{n}{n}t^n$$

where the power of t is $|\sigma|$ and the coefficient of t is the number of such σ. □

Theorem 2.7 gives the size of F_m for $1 \leq m \leq n$ in $\mathcal{K}_n$.

n	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
1	1									
2	2	1								
3	3	3	1							
4	4	6	4	1						
5	5	10	10	5	1					
6	6	15	20	15	6	1				
7	7	21	35	35	21	7	1			
8	8	28	56	70	56	28	8	1		
9	9	36	84	126	126	84	36	9	1	
10	10	45	120	210	252	210	120	45	10	1

Table 1. Statistics of f_m in $\mathcal{K}_n$

Let Ω be the union of all F_m, we defined an ordering on Ω as follows:

Given $\sigma, \rho \subseteq [n]$ such that $\sigma = \{a_1 < \cdots < a_m\}$ and $\rho = \{b_1 < \cdots < b_r\}$. Let $x_\sigma \leq x_\rho$ in the poset $\mathcal{P}$ if $m \geq r$ and $\sigma_i \leq \rho_i$ for all $i = 1, \cdots, r$.

Let $\nabla = \{x_\sigma x_\tau + lower\ terms : 1 \leq |\sigma| \leq n-2\ and\ 2 \leq |\tau| \leq n-1\}$ be the set of relations between x_σ's and x_τ's.

Theorem 2.8 *Given x_σ in $\mathcal{K}_n$ such that $|\sigma| = 3$ (i.e x_σ is a triangle), then x_σ can be expressed as a linear combination of x_{τ_i} which sum to zero for $|\tau_i| = 2$, $\tau_i \subset \sigma$ and $\bigcap \tau_i = \emptyset$. Moreover, the number of summands is $|\sigma|$.*

Proof Given a complete geometric digraph, $\mathcal{K}_n$ with points indexed by $[n]$. Suppose $\sigma \subset [n]$ with $|\sigma| > 2$, x_σ is a subgraph of $\mathcal{K}_n$, there is a closed path in x_σ (x_σ are line segments), which is the sum of x_{τ_i} and $\bigcap \tau_i = \emptyset$ and the number of such τ_i is $|\sigma|$. □

Remark 2.9 (i) The sign of x_{τ_i} in Theorem 2.8 is negative if the distance of τ is $|\sigma| - 1$, otherwise positive.

(ii) Theorem 2.8 gives the relation of the paths in x_σ.

Example 2.10 Given the complete geometric digraph $\mathcal{K}_3$. Then triangle, $x_{\{1,2,3\}}$ with lines $x_{\{1,2\}}$, $x_{\{2,3\}}$ and $x_{\{1,3\}}$, can be expressed as

$$x_{\{1,2,3\}} = x_{\{1,2\}} + x_{\{2,3\}} - x_{\{1,3\}} = 0.$$

Remark 2.11 From Example 2.10, $x_{\{1,3\}}$ is called the equivalent path and can be expressed as $x_{\{1,3\}} = x_{\{1,2\}} + x_{\{2,3\}}$.

Corollary 2.12 *Every x_τ such that $|\tau| > 3$ can be expressed as a linear combination of x_{α_i} such that $|\alpha_i| = 3$ and $\alpha_i \subset \tau$.*

Example 2.13 Given the complete geometric digraph $\mathcal{K}_4$. Then $x_{\{1,2,3,4\}}$ with lines $x_{\{1,2\}}$, $x_{\{2,3\}}$, $x_{\{3,4\}}$ and $x_{\{1,4\}}$. Then we have $x_{[4]} = x_{\{1,2\}} + x_{\{2,3\}} + x_{\{3,4\}} - x_{\{1,4\}} = 0$.
$x_{[4]}$ can be decompose into triangles as follows:

$$x_{[4]} = x_{\{1,2,3\}} - x_{\{1,2,4\}} + x_{\{1,3,4\}} - x_{\{2,3,4\}}.$$

The branches (for $|\alpha_i| = 2$) in the spanning trees of $\mathcal{K}_n$ are related. The relation is given by the theorem below which generalizes for $|\alpha_i| \geq 2$.

Theorem 2.14 *Given $\mathcal{K}_n$ and $\sigma \subset [n]$ such that $|\sigma| \geq 3$, then x_σ^τ, the linear combination of x_{α_i} such that $\tau \subset \alpha_i \subset \sigma$ is given by*

$$x_\sigma^\tau = \sum_{i=1}^{|\sigma|-1} (-1)^{i+1} x_{\alpha_i}$$

for $2 \leq |\alpha_i| \leq |\sigma|$ and $1 \leq |\tau| \leq |\sigma| - 1$.

Proof Given a complete geometric digraph, $\mathcal{K}_n$ with vertices indexed by $[n]$. Since $\sigma \bigcap \alpha_i = \tau$, then x_σ^τ is the sum of all subgraphs of x_σ containing the subgraph x_τ. □

Remark 2.15 Theorem 2.14 gives the relation of the branches in the spanning trees of $\mathcal{K}_n$.

Example 2.16 In a complete geometric digraph $\mathcal{K}_n$ with points indexed $[4] = \{1, 2, 3, 4\}$, then $x_{[3]}^{\{1\}} = x_{\{1,2\}} - x_{\{1,3\}}$.

§3. Computation of the Relations in $\mathcal{K}_n$

In this section we give the procedure for computing the relations in a complete geometric digraph $\mathcal{K}_n$ for $n \leq 6$. Given a complete geometric digraph $\mathcal{K}_n$, the order is n and size is $\frac{rn}{2}$, where r is the rank of $\mathcal{K}_n$. The relations in $\mathcal{K}_n$ is defined by its complete subgraphs, that is, the cycle, C_3 and the spanning trees in the complete subgraphs, $K_n(n \geq 4)$ of $\mathcal{K}_n$. Since $\mathcal{K}_1$ and $\mathcal{K}_2$ has no cycles, they have no relation. So $3 \leq n \leq 6$, given $\mathcal{K}_n$ and $\Lambda_{\sigma,\tau} \in \nabla$ as follows:

(1) For $|\sigma| = 2$ and $|\tau| = 1$, any cycle C_3 in $\mathcal{K}_n$ contains three binary spanning trees and each has exactly one chord. These chords are the paths in C_3 which are linearly related as defined by Theorem 2.8 and each chord in the relation is multiplied by the root of its tree.

(2) For $|\sigma| = 2$ and $|\tau| = 2$, any complete geometric subgraph K_4 of $\mathcal{K}_n$ contains four 3-valent spanning trees and each has three chords. The branches are linearly related as defined by Theorem 2.14 and each branch in the relation is multiplied by the chord not adjacent to it. Any of the four 3-valent spanning tree of a K_4 gives the same relation.

(3) For $|\sigma| = 3$ and $|\tau| = 1$, any complete geometric subgraph K_4 of $\mathcal{K}_n$ contains four 3-valent spanning trees and each has three chords which formed a triangle. These triangle are linearly related as defined by Theorem 2.8 and is multiplied by the root of its spanning tree.

(4) For $|\sigma| = 3$, $|\tau| = 2$ and any branch in any 3-valent spanning tree of K_4, there exist a C_3 formed by a chord and the other branches. The branches are linearly related as defined by Theorem 2.14 and each is multiplied by its corresponding C_3. This is repeated for each 3-valent spanning tree.

(5) For integers $n \geq 5$, consider all the complete geometric subgraphs, K_5 of $\mathcal{K}_n$. For any branch in any 4-valent spanning tree of K_5, there is exactly one C_3 formed by the chords not adjacent to the branch. Applying to Theorem 2.14 to the branches and multiplying each branch by these C_3, we realize the graph relation.

(6) For $|\sigma| = 3$ and $|\tau| = 3$, consider all the complete geometric subgraphs, K_5 of $\mathcal{K}_n$. There are six chords in any 4-valent spanning tree of K_5 with three pairs of non-adjacent chords. For any pair, we have two C_3 formed by the chords with the branches. Applying to Theorem 2.14 to the C_3 in each pair containing the branch highest leave (label-wise), we multiply each C_3 in the relation with it corresponding pair.

(7) For integers $n \geq 6$, consider all the complete geometric subgraphs, K_6 of $\mathcal{K}_n$. For any two branches in any 5-valent spanning tree of K_6, there is exactly one C_3(non-adjacent C_3) formed by the chords not adjacent to these branches. There are exactly five of such relations in any of the 5-valent spanning tree, applying to Theorem 2.14 to the C_3 formed by a chord with these branches and multiplying each by the non-adjacent C_3, we realize the graph relation.

(8) For $|\sigma| = 4$ and $|\tau| = 1$, consider all the complete geometric subgraphs, K_5 of $\mathcal{K}_n$. Taking root of each 4-valent spanning tree of K_5 to multiply the C_4 formed by the chords with the leaves. The C_4 in each 4-valent spanning tree are linearly related as defined by Theorem 2.8.

(9) For $|\sigma| = 4$ and $|\tau| = 2$, consider all the complete geometric subgraphs, K_5 of $\mathcal{K}_n$. For any branch in any 4-valent spanning tree of K_5, there is a C_4 formed by the chords with the leaves. These branches are linearly related as defined by Theorem 2.14 and each is multiplied

it corresponding C_4. This is repeated for each 4-valent spanning tree.

(10) For integers $n \geq 6$, consider all the complete geometric subgraphs, K_6 of $\mathcal{K}_n$. For any branch in any 5-valent spanning tree of K_5, there is exactly one C_4 formed by the chords not adjacent to the branch. Applying to Theorem 2.14 to the branches and multiplying each branch by this C_4, we realize the graph relation.

(11) For $|\sigma| = 4$ and $|\tau| = 3$, consider all the complete geometric subgraphs, K_5 of $\mathcal{K}_n$. For any branch in any 4-valent spanning tree of K_5, we have three C_3 containing that branch, a chord and one other branch. For each C_3, there is a C_4 containing that branch, two chords and one other branch These C_3 are linearly related as defined by Theorem 2.14 and each C_3 is multiplied by the corresponding C_4. This is repeated for each branch in all the 4-valent spanning tree.

(12) For integers $n \geq 6$, consider all the complete geometric subgraphs, K_6 of $\mathcal{K}_n$. For any branch in any 5-valent spanning tree of K_6, we have four C_3 containing that branch, a chord and one other branch. For each C_3, there is a C_4 formed by four chords in the leave of that branch and other three branches not in the C_3. These triangles are linearly related as defined by Theorem 2.14 and each C_3 is multiplied by the corresponding C_4. Also for each C_3, there is a C_4 formed by two branches and two chords in the leave of other branches not in the triangle. These C_3 are linearly related as defined by Theorem 2.14 and each C_3 is multiplied by the corresponding C_4. This is repeated for each branch in all the 5-valent spanning tree.

We also consider all the complete geometric subgraphs, K_7 of $\mathcal{K}_n$. For any branch in any 6-valent spanning tree of K_7, we have five C_3 containing that branch, a chord and one other branch. For each C_3, there is a C_4 formed by four chords not adjacent to any of the branches in the C_3. These C_3 are linearly related as defined by Theorem 2.14 and each C_3 is multiplied by the corresponding C_4. This is repeated for each branch in all the 6-valent spanning tree.

Example 3.1 For $n = 3$, the relation is define by the points and lines of the graph in Figure 1. The relation is derived as follows:

Since $\mathcal{K}_3$ is a C_3, then it contains three binary spanning trees and each has exactly one chord. These chords are the paths in C_3 which are linearly related as defined by Theorem 2.8.

$$x_{\{1,3\}} = x_{\{1,2\}} + x_{\{2,3\}}$$

each chord in the relation is multiplied by the root of its tree, we have

$$x_{\{1,2\}}x_{\{3\}} - x_{\{1,3\}}x_{\{2\}} + x_{\{2,3\}}x_{\{1\}} = 0$$

The equations above give the relation for $\mathcal{K}_3$.

Example 3.2 For $n = 4$, the relations are define by the points, lines and triangles of the graph in Figure 2. The set of points, F_1 is $\{x_{\{1\}}, x_{\{2\}}, x_{\{3\}}, x_{\{4\}}\}$, the set of lines, F_2 is $\{x_{\{1,2\}}, x_{\{1,3\}}, x_{\{1,4\}}, x_{\{2,3\}}, x_{\{2,4\}}, x_{\{3,4\}}\}$ and the set of C_3, F_3 is

$$\{x_{\{1,2,3\}}, x_{\{1,2,4\}}, x_{\{1,3,4\}}, x_{\{2,3,4\}}\}.$$

The relations are given below

$$
\begin{aligned}
&x_{\{1,2\}}x_{\{3\}} - x_{\{1,3\}}x_{\{2\}} + x_{\{2,3\}}x_{\{1\}} = 0,\\
&x_{\{1,2\}}x_{\{4\}} - x_{\{1,4\}}x_{\{2\}} + x_{\{2,4\}}x_{\{1\}} = 0,\\
&x_{\{1,3\}}x_{\{4\}} - x_{\{1,4\}}x_{\{3\}} + x_{\{3,4\}}x_{\{1\}} = 0,\\
&x_{\{2,3\}}x_{\{4\}} - x_{\{2,4\}}x_{\{3\}} + x_{\{3,4\}}x_{\{2\}} = 0,\\
&x_{\{2,3\}}x_{\{1,4\}} - x_{\{2,4\}}x_{\{1,3\}} + x_{\{3,4\}}x_{\{1,2\}} = 0,\\
&x_{\{2,3,4\}}x_{\{1\}} - x_{\{1,3,4\}}x_{\{2\}} + x_{\{1,2,4\}}x_{\{3\}} - x_{\{1,2,3\}}x_{\{4\}} = 0,\\
&x_{\{1,3,4\}}x_{\{1,2\}} - x_{\{1,2,4\}}x_{\{1,3\}} + x_{\{1,2,3\}}x_{\{1,4\}} = 0,\\
&x_{\{2,3,4\}}x_{\{1,2\}} - x_{\{1,3,4\}}x_{\{2,3\}} + x_{\{1,2,3\}}x_{\{2,4\}} = 0,\\
&x_{\{2,3,4\}}x_{\{1,3\}} - x_{\{1,3,4\}}x_{\{2,3\}} + x_{\{1,2,3\}}x_{\{3,4\}} = 0,\\
&x_{\{2,3,4\}}x_{\{1,4\}} - x_{\{1,3,4\}}x_{\{2,4\}} + x_{\{1,2,4\}}x_{\{3,4\}} = 0.
\end{aligned}
$$

§4. Classifications of the Equations Defining Flag Varieties

In this section, we give the classifications of the relations in a complete geometric graphs.

Theorem 4.1 *Given $\Lambda_{\sigma,\tau} \in \nabla$, if $\sigma \cap \tau \neq \emptyset$, then σ and τ have at most $n-3$ points of intersection and* $3 \leq |\sigma| + |\tau| \leq 2n-3$.

Proof Given any relation in $\Lambda_{\sigma,\tau}$ such $\alpha, \tau \subset [n]$ then $1 \leq |\sigma| \leq n-2$ and $2 \leq |\tau| \leq n-1$. If $\alpha \cap \tau \neq \emptyset$ and $\tau \nsubseteq \sigma$, then there is at least one point in σ not in τ. Therefore $n-3$ possible points of intersection. It also follows from the bound on $|\sigma|$ and $|\tau|$ that $3 \leq |\sigma| + |\tau| \leq 2n-3$. □

The number of terms in any relation in $\mathcal{K}_n$ is bounded below by the size of C_3 and above by n, which is capture in Theorem 4.2 following.

Theorem 4.2 *In a complete geometric digraph $\mathcal{K}_n$, there are at least three terms and at most n terms in any relations.*

Proof This follows from Theorems 2.8 and 2.14. □

$\mathcal{K}_3$ has one relation which contain three terms, $\mathcal{K}_4$ has ten relations out of which nine relations have three terms each and one relation has four terms and $\mathcal{K}_5$ has sixty-six relations out of which forty-five relations have three terms each, fifteen relation have four terms each and one relation has five terms.

Theorem 4.3 *In a complete geometric digraph $\mathcal{K}_n$, if the elements of F_m form a relation then*

$$f_m \geq \binom{n}{n-2}.$$

Proof Given a complete geometric digraph, $\mathcal{K}_n$. Suppose the elements of F_m form relations then by Theorem 4.1, $m \leq n-2$, which implies that

$$f_m \geq \binom{n}{n-2}. \qquad \square$$

Consider the complete geometric digraph, $\mathcal{K}_3$, it contains no relations between lines and lines since $f_2 = 3$ but $\mathcal{K}_4$ contains one relation between lines and lines since $f_2 = 6$.

Suppose we wish to classify the relations in $\mathcal{K}_n$ as points and lines relations, lines and lines relations, points and C_3 relations, lines and C_3 relations, and so on. For any $\mathcal{K}_n$, the number of relations in any of such classification is given by the following theorem.

Theorem 4.4 *In a complete geometric digraph $\mathcal{K}_n$, for any $E_{\sigma,\tau} \subset \nabla$, the cardinality of $E_{\sigma,\tau}$ ($E_{i,j} = \#E_{\sigma,\tau}$) is given by*

$$E_{i,j} = \begin{cases} \binom{n}{i-1}\binom{n}{j+1}, & if\ i < j \\ \\ \binom{n}{i-2}\binom{n}{j+2}, & if\ i = j \end{cases}$$

for $|\sigma| = i$ and $|\tau| = j$.

Proof Given $E_{\sigma,\tau} \subset \nabla$ in $\mathcal{K}_n$ such that $\sigma = \{\sigma_1, \cdots, \sigma_i\}$ and $\tau = \{\tau_1, \cdots, \tau_i\}$ for $\sigma, \tau \subseteq [n]$ and $\sigma \nsubseteq \tau$. Let $E_{i,j} = \#E_{\sigma,\tau}$, there exist two cases for $E_{i,j}$.

Case 1. If $i < j$, then either $\sigma \cap \tau \neq \emptyset$ or $\sigma \cap \tau = \emptyset$. Suppose $\sigma \cap \tau \neq \emptyset$, then $i + j \geq n$. Since $\sigma \nsubseteq \tau$, then there is a distinct element in σ not in τ. This element is moved to τ, thereby increasing $|\tau|$ by 1 and reducing $|\sigma|$ by 1. Then the choice of selection of $\sigma\tau$ is $\binom{n}{i-1}\binom{n}{j+1}$. But if $\sigma \cap \tau = \emptyset$, then $i + j \leq n$. So a distinct element of σ is moved to τ, thereby increasing $|\tau|$ by 1 and reducing $|\sigma|$ by 1. Hence the choice of selection of $\sigma\tau$ is $\binom{n}{i-1}\binom{n}{j+1}$.

Case 2. If $i = j$, then either $\sigma \cap \tau \neq \emptyset$ or $\sigma \cap \tau = \emptyset$. Suppose $\sigma \cap \tau \neq \emptyset$, then $i + j \geq n$. Since $\sigma \nsubseteq \tau$, then there are two distinct elements in σ not in τ. These elements are moved to τ, thereby increasing $|\tau|$ by 2 and reducing $|\sigma|$ by 2. Then the choice of selection of $\sigma\tau$ is $\binom{n}{i-2}\binom{n}{j+2}$. But if $\sigma \cap \tau = \emptyset$, then $i + j \leq n$. So, the two distinct elements of σ are moved to τ, thereby increasing $|\tau|$ by 2 and reducing $|\sigma|$ by 2. Hence the choice of selection of $\sigma\tau$ is $\binom{n}{i-2}\binom{n}{j+2}$. This completes the proof. $\square$

Example 4.5 Consider relations of $\mathcal{K}_4$, $E_{\{1,2\}} = \binom{4}{0}\binom{4}{3} = 4$, $E_{\{2,2\}} = \binom{4}{0}\binom{4}{4} = 1$, $E_{\{1,3\}} = \binom{4}{0}\binom{4}{4} = 1$ and $E_{\{2,3\}} = \binom{4}{1}\binom{4}{4} = 4$.

Theorem 4.4 gives the number of relations in any class ($E_{\sigma,\tau}$). The following theorem gives a generating functions classifying the relations in $\mathcal{K}_n$.

Theorem 4.6 *In a complete geometric digraph $\mathcal{K}_n$, for any $E_{\sigma,\tau} \subset \nabla$ such that $|\sigma| = r$ and*

$|\tau| = m$, then the cardinality of $E_{\sigma,\tau}$ in ∇ for a fixed m is given by

$$\gamma_n^{\{m\}}(q) = \sum_{r=1}^{m-1} \binom{n}{m+1}\binom{n}{r-1} q^{(r,m)} + \binom{n}{m+2}\binom{n}{m-2} q^{(m,m)}$$

for $2 \leq m \leq n-1$, $n \geq 3$.

Proof Given a complete geometric digraph, $\mathcal{K}_n$. For $E_{\sigma,\tau} \subset \nabla$ such that $|\sigma| = r$ and $|\tau| = m$. Then, either $r < m$ or $r = m$ in ∇. So, from Theorem 4.4, by fixing m and $1 \leq r \leq m$ we can express the number of relation $q^{(r,m)}$ as a generating function $\gamma_n^{\{m\}}(q)$ for integers $2 \leq m \leq n-1$. □

Example 4.7 In $\mathcal{K}_3$, $n = 3$, $m = 2$, then we have

$$\gamma_3^{\{2\}}(q) = q^{(1,2)}.$$

In $\mathcal{K}_4$, $n = 4$, $m = 2, 3$, then we have

$$\begin{aligned} \gamma_4^{\{2\}}(q) &= 4q^{(1,2)} + q^{(2,2)}, \\ \gamma_4^{\{3\}}(q) &= q^{(1,3)} + 4q^{(2,3)}. \end{aligned}$$

In $\mathcal{K}_5$, $n = 5$, $m = 2, 3, 4$, then we have

$$\begin{aligned} \gamma_5^{\{2\}}(q) &= 10q^{(1,2)} + 5q^{(2,2)}, \\ \gamma_5^{\{3\}}(q) &= 5q^{(1,3)} + 25q^{(2,3)} + 5q^{(3,3)}, \\ \gamma_5^{\{4\}}(q), &= q^{(1,4)} + 5q^{(2,4)} + 10q^{(3,4)}. \end{aligned}$$

Total number of relations in $\mathcal{K}_n$, for $n = 3, 4$ and 5 are $1, 10$ and 66 respectively.

Theorem 4.8 *In a complete geometric digraph $\mathcal{K}_n$, for any $E_{\sigma,\tau} \subset \nabla$ such that $|\sigma| = i$ and $|\tau| = j$, then the cardinality of $E_{\sigma,\tau}$ in ∇ is given by*

$$M_n(q) = \sum_{j=2}^{n-1}\sum_{i=1}^{j-1} \binom{n}{i-1}\binom{n}{j+1} q^{(i,j)} + \sum_{r=2}^{n-2} \binom{n}{r-2}\binom{n}{r+2} q^{(r,r)}$$

for $n \geq 3$.

Proof Given a complete geometric digraph, $\mathcal{K}_n$, for any $E_{\sigma,\tau} \subset \nabla$ and $n \geq 3$. By Theorem 4.6, the sum over all possible $\gamma_n^{\{i\}}(q)$ equals $M_n(q)$ for $n \geq 3$. □

Example 4.9 In $\mathcal{K}_3$, $n = 3$,

$$M_3(q) = q^{(1,2)}.$$

In $\mathcal{K}_4$, $n = 4$,

$$M_4(q) = 4q^{(1,2)} + q^{(1,3)} + 4q^{(2,3)} + q^{(2,2)}.$$

In $\mathcal{K}_5$, $n = 5$,

$$M_5(q) = 10q^{(1,2)} + 5q^{(1,3)} + 25q^{(2,3)} + q^{(1,4)} + 5q^{(2,4)} + 10q^{(3,4)} + 5q^{(2,2)} + 5q^{(3,3)}.$$

Theorem 4.10 *In a complete geometric digraph $\mathcal{K}_n$, the number of classes in $\mathcal{K}_n$ is two less than the size of $\mathcal{K}_n$ for $n \geq 3$.*

Proof Given $\mathcal{K}_n$, from Theorem 4.8 the number of terms in $M_n(q)$ gives the number of classes in $\mathcal{K}_n$. The number of terms is $\frac{n(n-1)}{2} - 2$ which is less than the size of $\mathcal{K}_n$. □

Remark 4.11 (i) The coefficient of $q^{(k,k)}$ equals $q^{(k-1,k+1)}$ for $k \geq 2$. Also $q^{(i+r,i+m)}$ and $q^{(r,m)}$ have equal coefficient for $m + r < n$ and $1 \leq i \leq n-3$.

(ii) The number of equations defining flag varieties $\mathcal{F}\ell_n(\mathbb{C})$ is given by

$$M_n = \sum_{j=2}^{n-1}\sum_{i=1}^{j-1}\binom{n}{i-1}\binom{n}{j+1} + \sum_{r=2}^{n-2}\binom{n}{r-2}\binom{n}{r+2}$$

with values for small number n in Table 2.

Order(n)	**Size**	**Number of relations(M_n)**	**Number of Classes**
3	3	1	1
4	6	10	4
5	10	66	8
6	15	365	13
7	21	1835	19
8	28	8705	26
9	36	39748	34
10	45	176740	43
11	55	770914	53
12	66	3314601	64
13	78	14094822	76
14	91	59418623	89
15	105	248756927	103
16	120	1035577973	118
17	136	4291186292	134
18	153	17713099208	151
19	171	72878464142	169
20	190	299021980928	188

Table 2. Statistics of a complete geometric digraph

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