International J.Math. Combin. Vol.1(2020), 62-69

Common Fixed Point of Five Self Maps for a Class of A-Contraction on 2-Metric Space

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Abstract: In this paper we prove some common fixed points for self mappings in a complete 2-metric space using A-contraction and weakly compatible mappings which is a generalization of many results and improve of earlier results in this literature.

Key Words: Common fixed point, self maps, A-contraction, weak compatible mapping, 2-metric space.

AMS(2010): 47H10,54H25.

§1. Introduction

Fixed point theory is an important part of mathematics. Moreover, it's well known that the contraction mapping principle, which is introduced by S. Banach in 1922. During the last few decades, this theorem has undergone various generalizations either by relaxing the condition on contractivity or withdrawing the requirement of completeness or sometimes even both.Gahler [4] first introduce the concept of 2-metric space. Then many authors like Iseki [6], Rhodes [11], Simoniya [13], etc. investigating the existence of fixed point and common fixed point for various contractive mappings. Naidu and Prasad [10] prove that every convergent sequences in a 2-metric space need not be a Cauchy sequence. Recently, M. Akram [2] defined A-contraction on a metric space and proved some common fixed points theorems. G. Akinbo [1] generalize the result using the concept of weakly compatible mapping. V.Gupta et al. [5] , M.Saha et al. [12] also proved fixed point theorems on A-contraction in the 2-metric space. In this paper we prove some common fixed point theorem for five self mappings by using A-contraction and weak compatibility.

§2. Definitions and Preliminaries

Definition 2.1([4]) *Gahler introduce 2-metric space as:*

Let X be a non-empty set and let $d : X \times X \times X \to [0, \infty)$ be such that

[1]Received October 21, 2019, Accepted March 10, 2020.

(1) *To each pair of point* $x, y \in X$ *with* $x \neq y$ *there exists a point* $z \in X$ *such that* $d(x, y, z) \neq 0$*;*

(2) $d(x, y, z) = 0$ *when at least two of the three points are equal;*

(3) *For any* $x, y, z \in X$, $d(x, y, z) = d(x, z, y) = d(y, z, x)$*;*

(4) *For any* x, y, z, w *in* X ,$d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$.

Then d *is called a 2-metric and* (X, d) *is called a 2-metric space.*

Let X denote a complete 2-metric space unless or otherwise stated instead of (X, d).

Definition 2.2 *A sequence* $\{x_n\}$ *in* X *is called a Cauchy sequence when* $d(x_n, x_m, a) \to 0$ *as* $n, m \to 0$.

Definition 2.3 *A sequence* $\{x_n\}$ *in* X *is said to be converge to an element* x *in* X *when* $d(x_n, x, a) \to 0$ *as* $n \to \infty$.

Definition 2.4 *A 2-metric space* (X, d) *is said to be complete if every Cauchy sequence in* X *converges to a point of* X.

Definition 2.5 *Two self maps* A *and* B *of a 2-metric space* (X, d) *are said to be compatible if* $\lim\limits_{n\to\infty} d(ABx_n, BAx_n, a) = 0$ *for all* $a \in X$ *, where* $\{x_n\}$ *is a sequence in*X *such that if* $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = x$ *for some* x *in* X.

Definition 2.6 *Let* A *and* B *be mappings from a metric space* (X, d) *into it-self.* A *and* B *are said to be weakly compatible if they commute at their coincidence point i.e,* $Ax = Bx$ *for some* x *in* X *implies* $ABx = BAx$.

Definition 2.7 *On the other hand Akram* [2] *defined* A*-contraction as follows:*
Let a non-empty set A *consisting of all functions* $\alpha : R_+^3 \to R_+$ *satisfying*

(1) A *is continuous on the set* R_+^3 *of all triplet of non negative real's (with respect to the Euclidean metric on* R^3*);*

(2) $a \leq kb$ *for some* $k \in [0, 1)$ *whenever* $a \leq \alpha(a, b, b)$*or* $a \leq \alpha(b, a, b)$ *or* $a \leq \alpha(b, b, a)$ *for all* $a, b \in R_+$.

Definition 2.8 *A self map* T *on a 2-metric space* X *is said to be A-contractions if for each* $u \in X$,

$$d(Tx, Ty, u) \leq \alpha\{d(x, y, u), d(x, Tx, u), d(y, Ty, u)\}$$

holds for all $x, y \in X$ *and* $\alpha \in A$.

§3. Main Results

Theorem 3.1 *Let* F, G, S, T *and* h *be five continuous self mappings of a complete 2-metric space* (X, d)*, such that* $T(X) \subset G(X)$ *and* $S(X) \subset F(X)$*. Assume* h *is an injective mapping.*

If $S(X)$ or $T(X)$ is a complete subspace of X and satisfy

$$d(hSx, hTy, u) \leq \alpha\{d(hGx, hFy, u), d(hGx, hSx, u), d(hFy, hTy, u)\} \tag{3.1}$$

where $\alpha \in A$ and for all $x, y, u \in X$. Suppose further that (T, F) and (S, G) are weakly compatible subspace of X, then (S, G, h) and (T, F, h) have a coincidence point in X. Also, F, G, S, T and h have a uniquely common fixed point in X.

Proof Here, F, G, S, T and h be self maps of 2-metric space . Let x_0 be any point in X and as $S(X) \subset FX), T(X) \subset G(X)$ then there exists x_1, x_2 in X such that $Sx_0 = Fx_1$, $Tx_1 = Gx_2$, $Sx_2 = Fx_3 \cdots$. Inductively, we can construct sequences $\{x_n\}$ and$\{y_n\}$ in X such that

$y_n = hSx_n = hFx_{n+1}$ when n is even,

$y_n = hTx_n = hGx_{n+2}$, when n is odd.

Now, we will prove that $\{y_n\}$ is a Cauchy sequence. Assuming $n \in \mathbb{N}$ is even, then

$$\begin{aligned} d(y_n, y_{n+1}, u) &= d(hSx_n, hTx_{n+1}, u) \\ &\leq \alpha\{d(hGx_n, hFx_{n+1}, u), d(hGx_n, hSx_n, u), d(hFx_{n+1}, hTx_{n+1}, u)\} \ (by\ (3.1)) \\ &= \alpha\{d(y_{n-1}, y_n, u), d(y_{n-1}, y_n, u), d(y_n, y_{n+1}, u)\} \leq kd(y_{n-1}, y_n, u), \end{aligned} \tag{3.2}$$

where $k \in [0, 1)$ as $\alpha \in A$.

When, $n \in \mathbb{N}$ is odd, then

$$\begin{aligned} d(y_n, y_{n+1}, u) &= d(hTx_n, hSx_{n+1,}u) = d(hSx_{n+1}, hTx_n, u)) \\ &\leq \alpha\{d(hGx_{n+1}, hFx_n, u), d(hGx_{n+1}, hSx_{n+1}, u), d(hFx_n, hTx_n, u)\} \ (by\ (3.1)) \\ &= \alpha\{d(y_n, y_{n-1}, u), d(y_n, y_{n+1}, u), d(y_{n-1}, y_n, u)\} \leq kd(y_{n-1}, y_n, u), \end{aligned} \tag{3.3}$$

where $k \in [0, 1)$ (as $\in A$).

Thus, whether n is even or odd, we have

$$d(y_n, y_{n+1}, u) \leq kd(y_{n-1}, y_n, u)$$

for some $k \in [0, 1)$.

Inductively,

$$\begin{aligned} d(y_n, y_{n+1}, u) &\leq kd(y_{n-1}, y_n, u) \leq k^2 d(y_{n-2}, y_{n-1}, u) \\ &\leq \cdots \leq k^n d(y_0, y_1, u)) \end{aligned} \tag{3.4}$$

for some $k \in [0, 1)$.

Now,

$$\begin{aligned} d(y_n, y_{n+2}, u) &\le d(y_n, y_{n+2}, y_{n+1}) + d(y_n, y_{n+1}, u) + d(y_{n+1}, y_{n+2}, u) \\ &= d(y_n, y_{n+2}, y_{n+1}) + \sum_{r=0}^{1} d(y_{n+r}, y_{n+r+1}, u). \end{aligned} \tag{3.5}$$

When n is even,

$$\begin{aligned} d(y_n, y_{n+2}, y_{n+1}) &= d(y_{n+1}, y_{n+2}, y_n) = d(hTx_{n+1}, hSx_{n+2,}y_n) \\ &= d(hSx_{n+2,}hTx_{n+1,}y_n) \\ &\le \alpha\{d(hGx_{n+2}, hFx_{n+1}, y_n), d(hGx_{n+2}, hSx_{n+2}, y_n), d(hFx_{n+1}, hTx_{n+1}, y_n)\} \\ &\quad (by\ (3.1)) \\ &= \alpha\{d(y_{n+1}, y_n, y_n), d(y_{n+1}, y_{n+2}, y_n), d(y_n, y_{n+1}, y_n)\} \\ &= \alpha\{0, d(y_{n+1}, y_{n+2}, y_n), 0\} \le k \cdot 0 = 0. \end{aligned}$$

Similarly, when n is odd we can find $d(y_n, y_{n+2}, y_{n+1}) = 0$.

So, from (3.5) we get

$$d(y_n, y_{n+2}, u) \le \sum_{r=0}^{1} d(y_{n+r}, y_{n+r+1}, u).$$

Similarly proceeding as above we will get

$$\begin{aligned} d(y_n, y_{n+p}, u) &\le \sum_{r=0}^{p-1} d(y_{n+r}, y_{n+r+1}, u) \\ &= (k^n + k^{n+1} + \cdots + k^{n+p-1}) d(y_{n+r}, y_{n+r+1}, u) \\ &= k^n(1-k^p)/(1-k) d(y_{n+r}, y_{n+r+1}, u) \\ &\le k^n/(1-k) d(y_{n+r}, y_{n+r+1}, u). \end{aligned}$$

Taking $\lim\limits_{n\to\infty}$ on the above inequality we get, $\lim_{n\to\infty} d(y_n, y_{n+p}, u) \le 0$ as $k \in [0,1)$, Which shows that $\{y_n\}$ is a Cauchy sequence in X.

Since, $T(X)$ is complete then $\{y_n\}$ converges to a point z in $T(X)$, and since, $T(X) \subset G(X)$, then there exists a point q in X such that $Gq = z$. and as h is injective so

$$hGq = hz. \tag{3.6}$$

Now, let $hSq \neq hz$ and

$$\lim_{n\to\infty} y_n = z \text{ (as } y_n \text{ converges to } z). \tag{3.7}$$

Now,

$$d(hSq, hz, u) \le d(hSq, hz, hTy_m) + d(hSq, hTy_m, u) + d(hTy_m, hz, u). \tag{3.8}$$

Now,

$$\begin{aligned} d(hSq, hTy_m, u) &\leq \alpha\{d(hGq, hFy_m, u), d(hGq, hSq, u), d(hFy_m, hTy_m, u)\} \ (by\ (3.1)) \\ &= \alpha\{d(hz, hz, u), d(hz, hSq, u), d(hz, hz, u)\} \\ &= \alpha\{0, d(hz, hSq, u), 0\} \leq k \cdot 0 = 0. \end{aligned}$$

Using the above value and taking $\lim\limits_{n\to\infty}$ we get from (3.8)

$$d(hSq, hz, u) \leq d(hSq, hz, hz) + 0 + d(hz, hz, u) = 0 + 0 + 0 = 0.$$

So,

$$hSq = hz. \tag{3.9}$$

Now, from (3.6) and (3.8) we get that

$$hGq = hz = hSq. \tag{3.10}$$

Since $S(X) \subset F(X)$, we know that there exists a point $v \in X$ such that $Fv = z$ or $hFv = hz$, i.e.,

$$hFv = hSq = hz = hGq. \tag{3.11}$$

Now,

$$\begin{aligned} d(hz, hTv, u) &= d(hSq, hTv, u) \\ &\leq \alpha\{d(hGq, hFv, u), d(hGq, hSq, u), d(hFv, hTv, u)\} \ (by\ (3.1)) \\ &= \alpha\{d(hz, hz, u), d(hz, hz, u), d(hz, hTv, u)\} \\ &= \alpha\{0, 0, d(hz, hTv, u)\} \leq k \cdot 0 = 0. \end{aligned}$$

So, $hTv = hz$ as u is a arbitrary point in X. Thus,

$$hFv = hTv = hz = hGq = hSq. \tag{3.12}$$

As, (F, T) and (S, G) are weakly compatible pair, then F, T are commute at v and S, G are commute at q. So that,

$$hFz = F(hz) = F(hTv) = T(hFv) = T(hz) = hTz$$

and

$$hSz = S(hz) = S(hGq) = G(hSq) = G(hz) = hGz(\text{ by } (3.12)). \tag{3.13}$$

Now,

$$\begin{aligned} d(hSz, hz, u) &= d(hSz, hTv, u) \leq \alpha\{d(hGz, hFv, u), d(hGz, hSz, u), d(hFv, hTv, u)\} \ (by\ (3.1)) \\ &= \alpha\{d(hsz, hz, u), d(hSz, hSz, u), d(hz, hz, u)\} \ (by\ (3.12)\ \&\ (3.13)) \\ &= \alpha\{d(hsz, hz, u), 0, 0\} \leq k \cdot 0 = 0. \end{aligned}$$

So, $hSz = hz$, i.e., $Shz = hz$..

Thus,

$$hz \text{ is a fixed point of } S. \tag{3.14}$$

From (3.13) we get, $hSz = hz = hGz = Ghz$, i.e.,

$$hz \text{ is a fixed point of } G. \tag{3.15}$$

Now,

$$\begin{aligned} d(hz, hTz, u) &= d(hSz, hTz, u) \\ &\le \alpha\{d(hGz, hFz, u), d(hGz, hSz, u), d(hFz, hTz, u)\}\ (by(3.1)) \\ &= \alpha\{d(hz, hTz, u), d(hz, hz, u), d(hTz, hTv, u)\} \\ &= \alpha\{d(hz, hTz, u), 0, 0\} \le k \cdots 0 = 0. \end{aligned}$$

So, $hTz = hz$ and $Thz = hz = hFz = Fhz$ (by (3.13)). Thus,

$$hz \text{ is a fixed point of } T \text{ and } F. \tag{3.16}$$

As h is an injective function so, $hz = z$ i.e.,

$$z \text{ is a fixed point of } h. \tag{3.17}$$

From (3.14), (3.15), (3.16) and (3.17), we get that z is a fixed point of S, G, T, F and h.

To prove the uniqueness, let r be another fixed point of S, G, T, F and h such that $r \neq z$. Then,

$$\begin{aligned} d(z, r, u) &= d(hSz, hTr, u) \le \alpha\{d(hGz, hFr, u), d(hGz, hSz, u), d(hFr, hTr, u)\} \\ &= \alpha\{(z, r, u), (z, z, u), (r, r, u)\} = \alpha\{(z, r, u), 0, 0\} \le k \cdot 0 = 0. \end{aligned}$$

So $z = r$, i.e.,z is a unique common fixed point of S, G, T, F and h. □

Theorem 3.2([1],[5]) *Let F, G, S and T be continuous self mappings of a complete 2-metric space (X, d), such that $T(X) \subset G(X)$ and $S(X) \subset F(X)$. If $S(X)$ or $T(X)$ is a complete subspace of X and satisfy*

$$d(Sx, Ty, u) \le \alpha\{d(Gx, Fy, u), d(Gx, Sx, u), d(Fy, Ty, u)\}, \tag{3.18}$$

where$\alpha \in A$ and for all $x, y, u \in X$. Suppose further that (T, F) and (S, G) are weakly compatible subspace of X, then (S, G) and (T, F) have a coincidence point in X. Also, F, G, S and T have a common unique fixed point in X.

Proof If we put $h = I$, the identity mapping in our main results, then the theorem immediately follows. □

Theorem 3.3 *LetF, G, S, T and h be five continuous self mappings of a complete 2-metric space (X, d) and let$\{S_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ be sequence on S and T such that $T_n(X) \subset G(X)$ and $S_n(X) \subset F(X)$. Assume h is an injective mapping. If $S(X)$ or $T(X)$ is a complete subspace of X and satisfy*

$$d(hS_ix, hT_jy, u) \leq \alpha\{d(hGx, hFy, u), d(hGx, hS_ix, u), d(hFy, hT_jy, u)\}, \tag{3.19}$$

where $\alpha \in A$ and for all $x, y, u \in X$. Suppose further that (T_n, F) and (S_n, G) are weakly compatible subspace of X, then (S_n, G, h) and (T_n, F, h) have a coincidence point in X. Also, F, G, S_n, T_n and h have a common unique fixed point in X.

Proof For any arbitrary $x_0 \in X$ and $n = 0, 1, 2, 3 \cdots$ following a similar argument as in Theorem 3.1 we can define a sequence $\{y'_n\}$ in X such that

$$y'_n = hS_nx_n = hFx_{n+1}$$

when n is even and

$$y'_n = hT_nx_n = hGx_{n+2},$$

when n is odd.

Now, for each $i = 1, 3, 5, \cdots$ and $j = 2, 4, 6, \cdots$, we get from (3.19)

$$d(y'_i, y'_{i+1}, u) \leq kd(y'_{i-1}, y'_i, u)$$

and

$$d(y'_j, y'_{j+1}, u) \leq kd(y'_{j-1}, y'_j, u),$$

i.e.,

$$d(y'_n, y'_{n+1}, u) \leq kd(y'_{n-1}, y'_n, u), \ n = 1, 2, \cdots$$

By induction (as in the proof of Theorem 3.1) we have

$$d(y'_n, y'_{n+1}, u) \leq k^n d(y'_0, y'_1, u)$$

for some $k \in [0, 1)$. Consequently sequence $\{y'_n\}$ is Cauchy in X. The rest of the proof is similar to the corresponding part of the proof of Theorem 3.1. □

Conclusion Our main result is a generalization and improve result of the existing results in this literature. We generalize the results of G. Akinbo [1], M.Saha and D. Dey [2], and many others.

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