

Comparing the Number of Acyclic and Totally Cyclic Orientations with That of Spanning Trees of a Graph

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Abstract: A Smarandache k -orientation of G for an integer $k \geq 0$ is such an orientation on G with exactly k oriented cycles. If $k = 0$, then it is the common acyclic orientation. In C. Merino and D.J.A.Welsh, *Forests, colourings and acyclic orientations of the square lattice*, Annals of Combinatorics, 3 (1999), pp. 417 C 429, the following conjecture appears: *If G is a 2-connected graph with no loops, then either the number of acyclic orientations or the number of the totally cyclic orientations of G is bigger than the number of spanning trees of G .* In this paper we examine this conjecture for threshold graphs, which includes complete graphs, and complete bipartite graphs. Also we show the results of our computational search for a counterexample to the conjecture.

Key Words: Smarandache k -tree, Smarandache k -orientation, Tutte polynomial, spanning tree, acyclic orientation.

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§1. Preliminaries

The graph terminology that we use is standard and we follow Diestel's notation, see [7]. We consider throughout labelled connected graphs which may have loops and multi-edges. For a graph G we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. We denote by $G \setminus e$ the deletion of e from G and by G/e the contraction of e in G .

A Smarandache k -tree \mathcal{T}^k for an integer $k \geq 0$ is a connected spanning subgraph of a connected graph G with exactly k cycles. Particularly, if $k = 0$, i.e., \mathcal{T}^0 is the commonly spanning tree which is a connected acyclic subgraph of G . A Smarandache k -orientation of G is such an orientation on G with exactly k oriented cycles for an integer $k \geq 0$. Particularly, a

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Smarandache 0-orientation is the common acyclic orientation on G which contains no oriented cycle, and it is totally cyclic if every arc is part of an oriented cycle. When G is connected, the totally cyclic orientations correspond to the orientations which make G strongly connected. The number of spanning trees, acyclic and totally cyclic orientations of a connected G are denoted by $\tau(G)$, $\alpha(G)$ and $\alpha^*(G)$, respectively.

1.1 Tutte Polynomial

As all the invariants mentioned here are evaluations of the Tutte polynomial, let us define it and rephrase the conjecture in this setting.

The Tutte polynomial $T(G; x, y)$ of a graph G in variables x, y is $T(G; x, y) = 1$ if G has no edges, otherwise for any $e \in E(G)$:

- \mathbf{R}_1 $T(G; x, y) = xT(G/e; x, y)$, whenever e is a bridge;
- \mathbf{R}_2 $T(G; x, y) = yT(G \setminus e; x, y)$, whenever e is a loop;
- \mathbf{R}_3 $T(G; x, y) = T(G \setminus e; x, y) + T(G/e; x, y)$, otherwise.

In other words, T may be calculated recursively by choosing the edges in any order and repeatedly using the relations \mathbf{R}_1 – \mathbf{R}_3 to evaluate T . The resulting polynomial T is well defined in the sense that it is independent of the order in which the edges are chosen, see [3].

When evaluating the Tutte polynomial along different curves and points we get several interesting invariants of graphs. Among them we have the chromatic and flow polynomials of a graph; the all terminal reliability probability of a network; the partition function of a Q -state Potts model. But here we are interested in the evaluations at the three points $T(G; 1, 1) = \tau(G)$, $T(G; 2, 0) = \alpha(G)$ and $T(G; 0, 2) = \alpha^*(G)$. Further details of many of the invariants given by evaluations of the Tutte polynomial can be found in [21] and [5].

§2. Introduction

In 1999, Merino and D.J.A. Welsh were working in the asymptotic behavior of the number of spanning trees and acyclic orientations of the square lattice L_n . They foresaw that

$$\lim_{n \rightarrow \infty} (\tau(L_n))^{\frac{1}{n^2}} < \lim_{n \rightarrow \infty} (\alpha(L_n))^{\frac{1}{n^2}}.$$

(the result was later proved in [6]). So, for some big N and all $n \geq N$, $\tau(L_n) < \alpha(L_n)$. They also checked this for some small values of n . No proof of the inequality $\tau(L_n) < \alpha(L_n)$ for all n has been given yet. It is obvious that $\alpha(G) \geq \tau(G)$ is not true for general graphs, like for example the complete graphs K_n for $n \geq 5$, but it is true for the cycles C_n for all n . By considering the dual concept (more than the dual matroids, as it was important to stay in the class of graphic matroids), namely the quantity α^* , they convinced themselves that $\alpha^*(K_n) \geq \tau(K_n)$ for $n \geq 5$ was true but did not prove it formally. This small piece of evidence was enough to pose the following

Conjecture 2.1 *Let G be a 2-connected graph with no loops, then*

$$\max\{\alpha(G), \alpha^*(G)\} \geq \tau(G).$$

The conjecture can be stated in terms of the Tutte polynomial: *let G be a 2-connected graph with no loops, then*

$$\max\{T(G; 2, 0), T(G; 0, 2)\} \geq T(G; 1, 1).$$

First let us consider why the conditions are necessary. Even though the conjecture is true for trees, since $\tau(T_n) = 1 < 2^n = \alpha(T_n)$, where T_n is a tree with n edges, in general we need the condition of G being 2-connected. Take two graphs G_1 and G_2 with the properties that $0 < \alpha(G_1) < \frac{1}{\sqrt{2}}\tau(G_1)$, like K_6 , and $0 < \alpha^*(G_2) < \frac{1}{\sqrt{2}}\tau(G_2)$, like C_5 , to have a couple of counterexamples. One, by taking two copies of G_1 and joining them by an edge; this new graph G_3 has a bridge and it is such that $\alpha(G_3) = 2\alpha^2(G_1) < \tau^2(G_1) = \tau(G_1)$ and $\alpha^*(G_3) = 0$. The other one, by attaching a loop to G_2 to obtain G_4 ; in this case $\alpha^*(G_4) = 2\alpha(G_2) < \tau(G_2) = \tau(G_4)$ and $\alpha(G_4) = 0$. Also, the conjecture can be strengthened in several ways.

Conjecture 2.2 *Let G be a 2-connected graph with no loops, then*

$$T(G; 2, 0)T(G; 0, 2) \geq T^2(G; 1, 1).$$

If true, this conjecture would imply Conjecture 2.1. We denote the quantity $T(G; 2, 0)T(G; 0, 2)$ by $\gamma(G)$. We use this conjecture for the class of threshold graphs. Another such strengthened is the following:

Conjecture 2.3 *Let G be a 2-connected graph with no loops, then*

$$T(G; 2, 0) + T(G; 0, 2) \geq 2T(G; 1, 1).$$

The interest in the conjecture goes beyond being an interesting combinatorial problem. As seen by Conjecture 2.3 the problem is a first step towards checking the convexity of the Tutte polynomial along the line from $(2, 0)$ to $(0, 2)$. Here is a rather more daring strengthening of the conjecture.

Conjecture 2.4 *Let G be a 2-connected simple graph with no loops, then $T(G; x, y)$ is convex along the the line from $(2, 0)$ to $(0, 2)$.*

Our interest in the conjecture is in trying to understand the structure of the Tutte polynomials, in particular, we believe there is a combinatorial structure in the graph G corresponding to the evaluation along the segment $x = 1 + t$, $y = 1 - t$ for $-1 \leq t \leq 1$.

This article tries to provide further evidence for the truth of Conjecture 2.1 to make it more plausible. The rest of the paper contains 3 more sections. In Section 3 we provide different classes of graphs which satisfy the conjecture. Then, we explain some experimental results that validate the conjecture for graphs with a small number of vertices. The final section contains our conclusions.

§3. Theoretical evidence of the conjecture

Now, let us give some infinite families of graphs for which the conjecture is true. Evidently for the examples of cycles C_n we have that $\alpha^*(C_n) = 2^n - 2 \geq n = \tau(C_n)$ for $n \geq 2$. By duality the conjecture is also true for the graphs P_2^n with 2 vertices and n edges in parallel.

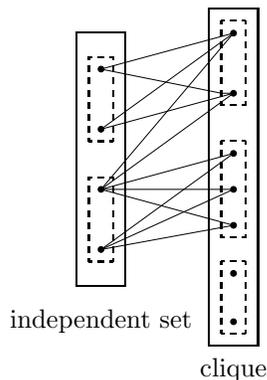


Fig.1 A threshold graph with $\omega(G) = 7$

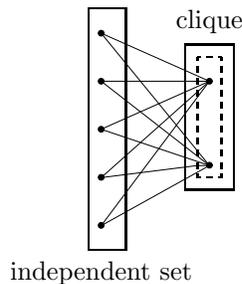


Fig.2 The case of Lemma 3.3, $K_{2,m}^+$

3.1 Threshold graphs

Threshold graphs are an important class of perfect graphs, see [16], and they can be considered a generalization of complete graphs with which they share many properties. A threshold graph is a simple graph $G = (V, E)$ with $V = \{1, \dots, n\}$ such that

- (1) for $\omega(G) \geq 2$, the clique number of G , we have that the first $\omega(G)$ vertices of G induce a $K_{\omega(G)}$ while the last $n - \omega(G)$ vertices form an independent set;
- (2) for $1 \leq i \leq j \leq n$, we have that $N(j) \subset N(i)$, where $N(i)$ denotes the neighborhood of i .

Thus, a threshold graph is a complete graph $K_{\omega(G)}$ to which we add $n - \omega(G)$ simplicial vertices, one by one, preserving the nested neighborhood. Threshold graphs are a kind of chordal graph. Observe that, if G is not the complete graph, $N(\omega(G) + 1) \subset N(\omega(G))$.

We assume that $d_1 \geq d_2 \geq \dots \geq d_n$, where d_i is the degree of vertex i . We are concerned just with $d_n \geq 2$ which is the same as requiring the graph to be 2-connected. When $\omega(G) = n$, we have a complete graph.

One of the nice properties of threshold graphs is that the number of spanning trees and the chromatic polynomial can be computed easily. In fact, for G , a connected threshold graph, we have that

$$\tau(G) = \prod_{i=2}^{\omega(G)-1} (d_i + 1) \prod_{i=\omega(G)+1}^n d_i. \quad (3-1)$$

The formula follows directly from a result due to R. Merris in [14]. Observe that when $\omega(G) = n$, the formula is Cayley's formula for the number of spanning trees of K_n . The chromatic polynomial can be computed by using a classical result of Read on quasi-separations, see [17], we have that

$$\chi(G; x) = \prod_{i=1}^{\omega(G)} (x - i + 1) \prod_{i=\omega(G)+1}^n (x - d_i).$$

Now, applying the famous result in [19] that relates the number of acyclic orientations and the chromatic polynomial, namely $\alpha(G) = |\chi(G; -1)|$, we get that

$$\alpha(G) = \omega(G)! \prod_{i=\omega(G)+1}^n (d_i + 1). \quad (3-2)$$

We start by giving some useful lemmas.

Lemma 3.1 For $x \geq 2$,

$$\ln(x+1) - \ln(x) < \frac{1}{x + \frac{1}{3}}.$$

Lemma 3.2 If G is a 2-connected graph with a vertex v of degree d , then $(2^d - 2)\alpha^*(G - v) \leq \alpha^*(G)$.

Proof If $\alpha^*(G - v) = 0$, the result is trivial. Otherwise, choose any totally cyclic orientation θ_{G-v} of $G - v$. Now, choose θ_v as any of the $2^d - 2$ orientations of the edges adjacent to v that do not make it a source or a sink. We extend θ_{G-v} to an orientation θ_G of G by oriented the edges on v according to θ_v . We prove that this orientation is totally cyclic. Any arc $u \rightarrow w$ with $u \neq v \neq w$ is in a oriented cycle of θ_{G-v} . To an arc $\vec{a} = v \rightarrow u$ corresponds an arc $\vec{b} = w \rightarrow v$ by the choice of θ_v . The orientation θ_{G-v} makes $G - v$ strongly connected, so there is a directed path from u to w ; that together with $w \rightarrow v \rightarrow u$ form a directed cycle. Thus, both \vec{a} and \vec{b} are in an oriented cycle. \square

We prove that threshold graphs satisfy Conjecture 2.2 in three stages.

Lemma 3.3 If G is a threshold graph with degree sequence (d_1, \dots, d_n) , $d_n = \dots = d_3 = 2$, and $\omega(G) = 3$, then $\gamma(G) \geq \tau^2(G)$.

Proof The graph G is $K_{2,m}$ plus the edge e_0 joining the two vertices of degree m , where $m = n - 2$. We denote this graph by $K_{2,m}^+$. By the deletion and contraction relation \mathbf{R}_3 applied to e_0 , we get that the Tutte polynomial of G is the sum of the Tutte polynomials of the graphs St_m^2 and $K_{2,m}$, where St_m^2 is the graph obtained from a star with m edges by replacing every edge by a 2-cycle. The first graph has Tutte polynomial $(x + y)^m$. The Tutte polynomial of $K_{2,m}$ could be easily computed by using the general formula for the Tutte polynomial of the tensor product of graphs G_1 and G_2 , where, G_1 is the graph P_2^m , that is m parallel edges, and G_2 is the path P_3 of length 2. This particular case of the tensor product is also referred to as the stretching of G_1 , see [10]. We get that

$$T(K_{2,m}; x, y) = (x + 1)^{m-1} \left(\sum_{i=1}^{m-1} \left(\frac{x + y}{x + 1} \right)^i + x^2 \right). \quad (3-3)$$

By evaluating the Tutte polynomial of G at $(1, 1)$, $(0, 2)$ and $(2, 0)$ we obtain $\tau(G) = 2^{m-1}(2+m)$, $\alpha^*(G) = 2^{m+1} - 2$ and $\alpha(G) = 2 \times 3^m$. So, we want to prove that $6^m - 1 \geq 2^{2m}(\frac{m+2}{4})^2$. It is enough to prove that $m \ln(3/2) > 2\ln(\frac{m+2}{4})$. But basic calculus shows that the function $\ln(3/2)x - 2\ln(\frac{x+2}{4})$ is positive in the interval $[1, 1)$. \square

Let $n \geq 4$, $m \geq 0$ and let G be a threshold graph with degree sequence (d_1, \dots, d_{n+m}) such that $d_1 = d_2 = n + m - 1$, $d_3 = \dots = d_n = n - 1$, and $d_{n+1} = \dots = d_{n+m} = 2$. So, the graph G is the parallel connection of K_n and the graph $K_{2,m}^+$ along e_0 . We denote these graphs by $\Pi(n, m)$.

Lemma 3.4 For $n \geq 4$ and $m \geq 0$, $\gamma(\Pi(n, m)) \geq \tau^2(\Pi(n, m))$.

Proof For $n = 4$ and $m \geq 0$ we can use the formula for computing the Tutte polynomial of the parallel connection given in [2] to get an expression for the number of spanning trees, acyclic orientations and totally cyclic orientations. Thus, $\tau(\Pi(4, m)) = 2^{m+2}(4+m)$, $\alpha(\Pi(4, m)) = 8 \times 3^{m+1}$, $\alpha^*(\Pi(4, m)) = 9 \times 2^{m+2} - 12$. Then, $\gamma(\Pi(4, m)) > \tau^2(\Pi(4, m))$ for $m \geq 0$. By formulae (3-1) and (3-2) and Lemma 3.2, we have the following recursive relations for acyclic and totally cyclic orientations: $\alpha(\Pi(n+1, m)) = (n+1)\alpha(\Pi(n, m))$, $\alpha^*(\Pi(n+1, m)) \geq (2^n - 2)\alpha^*(\Pi(n, m))$ and for trees we have

$$\tau(\Pi(n+1, m)) = \left(\frac{n+m+1}{n+m}\right) \left(\frac{n+1}{n}\right)^{n-3} (n+1)\tau(\Pi(n, m)).$$

Using these relations we get the following inequalities:

$$\begin{aligned} \ln(\tau^2(\Pi(n+1, m))) &= \ln(\tau^2(\Pi(n, m))) + 2\ln\left(\frac{n+m+1}{n+m}\right) \\ &\quad + 2(n-3)\ln\left(\frac{n+1}{n}\right) + 2\ln(n+1) \\ &\leq \ln(\tau^2(\Pi(n, m))) + 2(n-2)\ln\left(\frac{n+1}{n}\right) + 2\ln(n+1) \\ &\leq \ln(\tau^2(\Pi(n, m))) + \ln(2^n - 2) + 2\ln(n+1). \end{aligned}$$

The last inequality can be proved using Lemma 3.1 for $n \geq 5$ and by direct computation for $n = 4$. Now, by using induction we have that the last term is at most $\ln((\Pi(n, m))) + \ln(2^n - 2) + \ln(n+1)$. Algebraic manipulation and the recursive relations mentioned above give that the last quantity is at most $\ln((\Pi(n+1, m)))$. \square

For $m = 0$, a similar proof as above gives

Theorem 3.5 For $n \geq 4$, $\alpha^*(K_n) > \tau(K_n)$.

The quantity $\alpha^*(K_n)$ has also the interpretation of being the number of strongly connected labeled tournaments on n nodes, and its exponential generating function is given by $1 - 1/(1 + f(x))$ where

$$f(x) = \sum_{m \geq 1} 2^{\binom{n}{2}} x^m / m!,$$

see [11] for details.

Finally, if G is a threshold graph with degree sequence $(d_1, \dots, d_n, 2, \dots, 2)$ and we delete the vertex labeled $n \geq 3$ with degree $d_n = p$, we obtain the (unique up to isomorphisms) threshold graph $G' = G - \{n\}$ with degree sequence $(d_1 - 1, \dots, d_p - 1, d_{p+1}, \dots, d_{n-1}, 2, \dots, 2)$. If G is not the complete graph, then $\omega(G) = \omega(G - \{n\})$.

Lemma 3.6 *If G is a threshold graph with degree sequence $(d_1, \dots, d_n, 2, \dots, 2)$ and $d_n \geq 3$, then $\gamma(G) \geq \tau^2(G)$.*

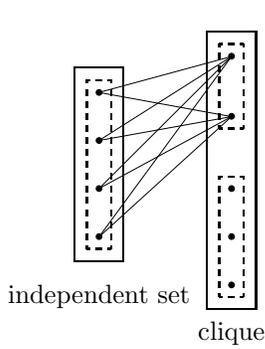


Fig.3 The case of Lemma 3.4.

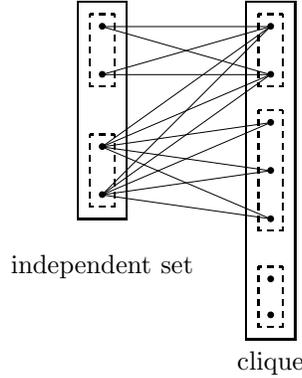


Fig.4 The case of Lemma 3.6.

Proof If $n = 2$, and thus $\omega(G) = 3$, then the result is true by Lemma 3.3. If $d_n = n - 1$, and thus $\omega(G) = n$, the result follows by Lemma 3.4. We assume that $d_n = p < n - 1$.

We have, by formula (3 - 1), the recursive relation $\tau(G) = p \left(\frac{n}{n-1}\right)^{p-1} \tau(G')$. Thus we obtained the following inequalities:

$$\begin{aligned} \ln(\tau^2(G)) &= \ln(\tau^2(G')) + 2(p-1) \ln\left(\frac{n}{n-1}\right) + \ln(p^2) \\ &< \ln(\tau^2(G')) + \frac{6(p-1)}{3n-2} + \ln(p^2) \\ &< \ln(\tau^2(G')) + \frac{6(p-1)}{3p+4} + \ln(p^2) \\ &< \ln(\tau^2(G')) + \ln(2^p - 2) + \ln(p+1), \end{aligned}$$

where the second inequality follows by Lemma 3.1, the third inequality is because $p \leq n - 2$, and the last inequality is because the function $\ln(2^x - 2) + \ln(p + 1) - \frac{6(x-1)}{3x+4} - \ln(x^2)$ is positive in $[3, \infty)$ which can be proved by basic calculus. Using induction we have that the last expression is at most $\ln(\gamma(G')) + \ln(2^p - 2) + \ln(p + 1)$. But by formulae (3 - 2) and Lemma 3.2 we have

$\alpha(G) = (p+1)\alpha(G')$ and $\alpha^*(G) \geq (2^p - 2)\alpha^*(G')$. Therefore, $\ln(\gamma(G')) + \ln(2^p - 2) + \ln(p+1) \leq \ln\gamma(G)$. \square

Theorem 3.7 *If G is a threshold graph, then $\max\{\alpha(G), \alpha^*(G)\} \geq \tau(G)$.*

It is not difficult to modify the proof of the previous theorem to get the following

Theorem 3.8 *If G is a threshold graph with degree sequence (d_1, \dots, d_n) and $d_n \geq 3$, then $\alpha^*(G) \geq \tau(G)$.*

We believe that the same techniques can be used to prove that the conjecture is true for chordal graphs.

3.2 Complete bipartite graphs

The complete bipartite graphs $K_{n,m}$ also satisfy the conjecture. Using the matrix-tree theorem it is easy to prove that the number of spanning trees of $K_{n,m}$ is $n^{m-1}m^{n-1}$. For $K_{2,m}$ equation (3-3) can be used to compute the number of acyclic orientations, and it can be checked that $\tau(K_{2,m}) = 2^{m-1}m \leq 2 \times 3^m - 2^m = \alpha(K_{2,m})$ for $m \geq 2$.

The exponential generating function for the chromatic polynomial of $K_{n,m}$ appears in [20]. Using the same technique the authors in [13] gave the exponential generation function for the Tutte polynomial of $K_{n,m}$. From that we get an expression for the exponential generating function for totally cyclic orientations,

$$1 - \sum_{(n,m) \neq (0,0) \in \mathbb{N}^2} \alpha^*(K_{n,m}) \frac{x^n}{n!} \frac{y^m}{m!} = \left(\sum_{(n,m) \in \mathbb{N}^2} 2^{nm} \frac{x^n}{n!} \frac{y^m}{m!} \right)^{-1}.$$

By grouping terms, the right hand side can be written as the inverse of the exponential generating function $e^y + e^{2y}x + e^{4y}x^2/2! + \dots$. By solving we get $e^{-y} - x + (2e^y - e^{2y})x^2/2! + (6e^{3y} - e^{6y} - 6e^{2y})x^3/3! + \dots$. Thus, $\alpha^*(K_{1,m}) = 0$ for all $m \geq 1$; $\alpha^*(K_{2,m}) = 2^m - 2$ for $m \geq 1$ and $\alpha^*(K_{3,m}) = 6^m - 6 \times 3^m + 6 \times 2^m$ for $m \geq 1$. We conclude that $\alpha^*(K_{3,m})$ is bigger than $\tau(K_{3,m})$ for $m \geq 4$.

For $n \geq 3$ we have the following recursive relations:

$$\tau(K_{n+1,m}) = m \left(1 + \frac{1}{n}\right)^{m-1} \tau(K_{n,m})$$

and

$$\alpha^*(K_{n+1,m}) \geq (2^m - 2)\alpha^*(K_{n,m}).$$

As $n \geq 3$, $\ln(1 + 1/n) \leq 3/(3n + 1) \leq 3/10$ by Lemma 3.1. Thus, $\tau(n + 1, m) \leq \alpha^*(n + 1, m)$ follows by induction as $\ln m + 3/10(m - 1) \leq \ln(2^m - 2)$ for $m \geq 4$. Putting these together we have the following.

Theorem 3.9 *For all $m \geq n \geq 2$, $\max\{\alpha(K_{n,m}), \alpha^*(K_{n,m})\} \geq \tau(K_{n,m})$.*

§4. Experimental data

Our experiments are based on two computer programs. One is **TuLiC** written in Java to compute the Tutte polynomial for moderate-sized graphs. The program was developed by Conde using the algorithm given in [18]. The program can be found at

<http://ada.fciencias.unam.mx/~rconde/tulic/>

The second one is the **Nauty** package of Brendan McKay which we use to generate all non-isomorphic 2-connected simple graphs, see [12]. We automatized the search for a counterexample to Conjecture 2.1 by the following steps:

1. We generated eight files, each one containing all the non-isomorphic 2-connected simple graphs of 3, 4, 5, 6, 7, 8, 9 and 10 vertices respectively, in a simple format that is easily parsed.
2. We wrote a program in the Java language that uses **TuLiC**'s libraries to evaluate the Tutte polynomial of each graph at the points $(1, 1)$, $(2, 0)$ and $(0, 2)$. The program enumerates all the graphs and the output for each graph is of the form *graph-number* $\tau(G)\alpha(G)\alpha^*(G)$ [*true|false*].
3. Finally, we made a shell script that uses the program described above to execute all the experiments. The script needs at least two parameters: The input file, containing the graphs to be tested, and the output file, where it puts the results.

For example, the output file at the end of the process for all the 2-connected simple graphs on 5 vertices will look like this:

```

1 :  12  46   6  true
2 :  20  54  14  true
3 :   5  30   2  true
4 :  11  42   6  true
5 :  21  54  18  true
6 :  40  72  60  true
7 :  24  60  24  true
8 :  45  78  78  true
9 :  75  96 204  true
10 : 125 120 544  true

```

To find a counterexample for the conjecture, we only needed to search these files for a line containing the word *false*. With the graph number, the graph can be extracted from the original file. After we ran the experiments on all the graphs, in total 9,945,269, no counterexample was found.

§5. Conclusion

We proved Conjecture 2.1 for some infinite families of graphs and so we have provided evidence that makes the conjecture more plausible. There are some more families; for example Marc Noy

(private communication) proved that $\tau(G) \leq \alpha(G)$ when G is a maximal outerplanar graph by using contraction-deletion.

However, we were unable to prove anything relevant for graphs with multiple edges or, the dual concept, with edges in series. It remains to be discovered if by checking graphs with a good mixed of multiple edges and edges in series a counterexample can be found.

There are several related problems. For example, the complexity of deciding if $\alpha(G)$ is bigger than $\tau(G)$ is not clear. Computing exactly $\alpha(G)$ is #P-complete, see [10], but this is much more than is needed. In [4] an FPRAS for $\alpha(G)$ is given provided G has girth at least $(5 + \delta)\log_2 n$. So, for this class of graphs, there is an FPRAS for the quantity $\alpha(G) - \tau(G)$.

Our results about complete bipartite graphs suggest that Conjecture 2.1 may be true for graphs that either contain two edge-disjoint spanning trees or in which the edge-set is the union of two spanning trees. Probing the conjecture for 4-edge-connected graphs would be a good first step but we were not able to do that. However, this can be proved for a similar class of graphs. If G is an n -vertex m -edge graph with edge-connectivity $\Omega(\log(n))$, then $\alpha^*(G) = 2^m(1 - O(1/n))$ by a result in [9]. We also have a general upper bound for $\tau(G)$ given in [8]; that is,

$$\tau(G) \leq \left(\frac{n}{n-1}\right)^{n-1} \left(\frac{\prod_i d_i}{2m}\right)$$

where $\prod_i d_i$ is the product of the vertex-degrees. Since 2^m is bigger than the last quantity, this class of graphs also satisfies the conjecture.

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