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Complementary Distance Energy of Complement of Line Graphs of Regular Graphs

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Abstract: The complementary distance (CD) matrix of a graph G is defined as $CD(G) = [c_{ij}]$, where $c_{ij} = 1 + D - d_{ij}$ if $i \neq j$ and $c_{ij} = 0$, otherwise, where D is the diameter of G and d_{ij} is the distance between the vertices v_i and v_j in G. The CD-energy of G is defined as the sum of the absolute values of the eigenvalues of CD-matrix. Two graphs are said to be CD-equienergetic if they have same CD-energy. In this paper we obtain the CD-energy of the complement of line graphs of certain regular graphs in terms of the order and regularity of a graph and thus construct pairs of CD-equienergetic graphs of same order and having different CD-eigenvalues.

Key Words: Complementary distance eigenvalues, energy, equienergetic graphs.

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§1. Introduction

Let G be a simple, undirected, connected graph with n vertices and m edges. Let the vertex set of G be $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of a graph G is the square matrix $A = A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The eigenvalues of $A(G)$ are the *adjacency eigenvalues* of G, and they are labeled as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. These form the *adjacency spectrum* of G [4].

The *distance* between the vertices v_i and v_j, denoted by d_{ij}, is the length of the shortest path joining v_i and v_j. The *diameter* of a graph G, denoted by $diam(G)$, is the maximum distance between any pair of vertices of G [3]. A graph G is said to be *r-regular graph* if all of its vertices have same degree equal to r.

The *complementary distance* between the vertices v_i and v_j, denoted by c_{ij} is defined as

$$c_{ij} = 1 + D - d_{ij},$$

where D is the diameter of G and d_{ij} is the distance between v_i and v_j in G.

The *complementary distance matrix* or *CD-matrix* [7] of a graph G is an $n \times n$ matrix

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$CD(G) = [c_{ij}]$, where

$$c_{ij} = \begin{cases} 1 + D - d_{ij}, & \text{if} \quad i \neq j \\ \\ 0, & \text{if} \quad i = j. \end{cases}$$

The complementary distance matrix is an important source of structural descriptors in the quantitative structure property relationship (QSPR) model in chemistry [7,9].

The eigenvalues of $CD(G)$ labeled as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are said to be the *complementary distance eigenvalues* or *CD-eigenvalues* of G and their collection is called *CD-spectra* of G. Two non-isomorphic graphs are said to be *CD-cospectral* if they have same CD-spectra.

The *complementary distance energy* or *CD-energy* of a graph G denoted by $CDE(G)$ is defined as [10]

$$CDE(G) = \sum_{i=1}^{n} |\mu_i| \ . \tag{1}$$

The Eq. (1) is defined in full analogy with the *ordinary graph energy* $E(G)$, defined as [5]

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \ . \tag{2}$$

Two connected graphs G_1 and G_2 are said to be *complementary distance equienergetic* or *CD-equienergetic* if $CDE(G_1) = CDE(G_2)$. The CD-equienergetic graphs are reported in [10]. In this paper we obtain the CD-energy of the complement of line graphs of certain regular graphs and thus construct CD-equienergetic graphs having different CD-spectra.

The *line graph* of G, denoted by $L(G)$ is the graph whose vertices corresponds to the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G [6].

For $k = 1, 2, \cdots$ the k-th iterated line graph of G is defined as $L^k(G) = L(L^{k-1}(G))$, where $L^0(G) = G$ and $L^1(G) = L(G)$ [6]. The line graph of a regular graph G of order n_0 and of degree r_0 is a regular graph of order $n_1 = (n_0 r_0)/2$ and of degree $r_1 = 2r_0 - 2$. Consequently the order and degree of $L^k(G)$ are [1,2]

$$n_k = \frac{r_{k-1} n_{k-1}}{2} \tag{3}$$

and

$$r_k = 2r_{k-1} - 2, \tag{4}$$

where n_i and r_i stands for order and degree of $L^i(G)$, $i = 0, 1, \cdots$.

Therefore

$$r_k = 2^k r_0 - 2^{k+1} + 2 \tag{5}$$

and

$$n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2) \tag{6}$$

We need following results.

Theorem 1.1([4]) *If G is an r-regular graph, then its maximum adjacency eigenvalue is equal to r.*

Theorem 1.2([12]) *If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r, then the adjacency eigenvalues of $L(G)$ are*

$$\begin{array}{lll} \lambda_i + r - 2, & i = 1, 2, \cdots, n, & \textit{and} \\ -2, & (n(r-2)/2 \ \textit{times}) \ . & \end{array}$$

Theorem 1.3([11]) *Let G be an r-regular graph of order n. If $r, \lambda_2, \cdots, \lambda_n$ are the adjacency eigenvalues of G, then the adjacency eigenvalues of $\overline{G}$, the complement of G, are $n - r - 1$ and $-\lambda_i - 1$, $i = 2, 3, \cdots, n$.*

Theorem 1.4([10]) *Let G be an r-regular graph on n vertices and $diam(G) = 2$. If $r, \lambda_2, \cdots, \lambda_n$ are the adjacency eigenvalues of G, then CD-eigenvalues of G are $n + r - 1$ and $\lambda_i - 1$, $i = 2, 3, \cdots, n$.*

Lemma 1.5([8]) *Let G be an r-regular graph on n vertices. If $r \leq \frac{n-1}{2}$ then*

$$diam\left(\overline{L^k(G)}\right) = 2, \quad k \geq 1.$$

§2. CD-Energy

Theorem 2.1 *Let G be an r-regular graph of order n. If $r \leq \frac{n-1}{2}$, then*

$$CDE\left(\overline{L(G)}\right) = 2r(n-2).$$

Proof Let the adjacency eigenvalues of G be $r, \lambda_2, \cdots, \lambda_n$. From Theorem 1.3 , the adjacency eigenvalues of $L(G)$ are

$$\left.\begin{array}{lll} 2r-2, & \text{and} & \\ \lambda_i + r - 2, & i = 2, 3, \cdots, n, & \text{and} \\ -2, & n(r-2)/2 \ \text{ times.} & \end{array}\right\} \qquad (7)$$

From Theorem 1.4 and the Eq. (3), the adjacency eigenvalues of $\overline{L(G)}$ are

$$\left.\begin{array}{lll} (nr/2)-2r+1, & \text{and} & \\ -\lambda_i-r+1, & i=2,3,\cdots,n, & \text{and} \\ 1, & n(r-2)/2 \ \text{times}. & \end{array}\right\} \tag{8}$$

The graph $\overline{L(G)}$ is a regular graph of order $nr/2$ and of degree $(nr/2)-2r+1$. Since $r \leq \frac{n-1}{2}$, by Lemma 1.5, $diam\left(\overline{L(G)}\right) = 2$. Therefore by Theorem 1.4 and Eq. (8), the CD-eigenvalues of $\overline{L(G)}$ are

$$\left.\begin{array}{lll} nr-2r, & \text{and} & \\ -\lambda_i-r, & i=2,3,\cdots,n, & \text{and} \\ 0, & n(r-2)/2 \ \text{times}. & \end{array}\right\} \tag{9}$$

All adjacency eigenvalues of a regular graph of degree r satisfy the condition $-r \leq \lambda_i \leq r$ [4]. Therefore $\lambda_i + r \geq 0$, $i=1,2,\cdots,n$. Hence from Eq. (9),

$$\begin{aligned} CDE\left(\overline{L(G)}\right) &= nr-2r+\sum_{i=2}^{n}(\lambda_i+r)+|0|\times\frac{n(r-2)}{2} \\ &= 2r(n-2) \end{aligned}$$

because of

$$\sum_{i=2}^{n}\lambda_i = -r.$$

This completes the proof. □

Corollary 2.2 *Let G be a regular graph of order n_0 and of degree r_0. Let n_k and r_k be the order and degree respectively of the k-th iterated line graph $L^k(G)$, $k \geq 1$. If $r_0 \leq \frac{n_0-1}{2}$, then*

$$CDE\left(\overline{L^k(G)}\right) = 2r_{k-1}(n_{k-1}-2).$$

Proof If $r_0 \leq \frac{n_0-1}{2}$, then by Eqs. (3) and (4), we have

$$r_1 = 2r_0-2 \leq n_0-3 \leq \frac{1}{2}\left(\frac{n_0r_0}{2}-1\right) = \frac{n_1-1}{2}.$$

Hence

$$r_{k-1} \leq \frac{n_{k-1}-1}{2}.$$

Therefore, by Theorem 2.1,

$$CDE\left(\overline{L^k(G)}\right) = CDE\left(\overline{L(L^{k-1}(G))}\right) = 2r_{k-1}(n_{k-1}-2). \qquad \square$$

Corollary 2.3 *Let G be a regular graph of order n_0 and of degree r_0. Let n_k and r_k be the order and degree respectively of the k-th iterated line graph $L^k(G)$, $k \geq 1$. If $r_0 \leq \frac{n_0-1}{2}$, then*

$$CDE\left(\overline{L^k(G)}\right) = \left[\frac{2n_0}{2^{k-1}}\prod_{i=0}^{k-1}(2^i r_0 - 2^{i+1} + 2)\right] - 4(2^{k-1}r_0 - 2^k + 2).$$

§3. CD-Equienergetic Graphs

If G_1 and G_2 are the regular graphs of same order and of same degree. Then $L(G_1)$ and $L(G_2)$ are of the same order and of same degree. Further their complements are also of same order and of same degree.

Lemma 3.1 *Let G_1 and G_2 be regular graphs of the same order n and of the same degree r. If $r \leq \frac{n-1}{2}$, then $\overline{L(G_1)}$ and $\overline{L(G_2)}$ are CD-cospectral if and only if G_1 and G_2 are cospectral.*

Proof The result follows from Eqs. (7), (8) and (9). $\square$

Lemma 3.1 can be extended for k-iterated line graph as given below.

Lemma 3.2 *Let G_1 and G_2 be regular graphs of the same order n and of the same degree r. If $r \leq \frac{n-1}{2}$, then for $k \geq 1$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are CD-cospectral if and only if G_1 and G_2 are cospectral.*

Theorem 3.3 *Let G_1 and G_2 be regular, non CD-cospectral graphs of the same order n and of the same degree r. If $r \leq \frac{n-1}{2}$, then for $k \geq 1$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ form a pair of non CD-cospectral, CD-equienergetic graphs of equal order and of equal number of edges.*

Proof The result follows from Lemma 3.2 and Corollary 2.3. $\square$

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