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# Computing the Number of Distinct Fuzzy Subgroups for the Ninpotent $p$-Group of $D_{2^n} \times C_4$

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**Abstract**: In this paper, the explicit formulae is given for the number of distinct fuzzy subgroups of the cartesian product of the dihedral group of order $2^n$ with a cyclic group of order four, where $n > 3$.

**Key Words**: Finite $p$-Groups, nilpotent Group, fuzzy subgroups, dihedral Group, inclusion-exclusion principle, maximal subgroups.

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## §1. Introduction

In the fuzzy group theory , the classification of the fuzzy subgroups, most especially the finite $p$-groups cannot be underestimated. This aspect of pure Mathematics has undergone a dynamic developments over the years. For instance, many researchers have treated cases of finite abelian groups (see [2], [3]). The starting point for this concept all started as presented in [5] and [6] . Since then, the study has been extended to some other important classes of finite abelian and nonabelian groups such as the dihedral , quaternion, semidihedral, and hamiltonian groups.

Although, the natural equivalence relation was introduced in [7], where a method to determine the number and nature of fuzzy subgroups of a finite group $G$ was developed with respect to the natural equivalence. In [1] and [3], a different approach was applied for the classification. In this work , an essential role in solving counting problems is played by adopting the Inclusion-Exclusion Principle. The process leads to some recurrence relations from which the solutions are then finally computed with ease.

## §2. Preliminaries

Suppose that $(G, \cdot, e)$ is a group with identity $e$. Let $S(G)$ denote the collection of all fuzzy subsets of $G$. An element $\lambda \in S(G)$ is said to be a fuzzy subgroup of $G$ if

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$(i)$ $\lambda(ab) \geq min\{\lambda(a), \lambda(b)\}, \ \forall\, a, b \in G$;
$(ii)$ $\lambda(a^{-1}) \geq \lambda(a)$ for any $a \in G$.

And, since $(a^{-1})^{-1} = a$, we have that $\lambda(a^{-1}) = \lambda(a)$, for any $a \in G$. Also, by this notation and definition, $\lambda(e) = \sup \lambda(G)$ (see Marius [6]), which implies

**Theorem** 2.1 *The set $FL(G)$ possessing all fuzzy subgroups of $G$ forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of $G$.*

We define the level subset

$$\lambda G_\beta = \{a \in G/\lambda(a) \geq \beta\} \text{ for each } \beta \in [0, 1]$$

The fuzzy subgroups of a finite $p$-group $G$ are thus, characterized, based on these level subsets. In the sequel, $\lambda$ is a fuzzy subgroup of $G$ if and only if its level subsets are subgroups in $G$. Theorem 2.1 gives a link between $FL(G)$ and $L(G)$, the classical subgroup lattice of $G$.

Moreover, some natural relations on $S(G)$ can also be used in the process of classifying the fuzzy subgroups of a finite $p$-group $G$ (see [6]). One of them is defined by: $\lambda \sim \gamma$ if and only if $(\lambda(a) > \lambda(b) \Longleftrightarrow v(a) > v(b), \ \forall\, a, b \in G)$. Also, two fuzzy subgroups $\lambda, \gamma$ of $G$ and said to be distinct if $\lambda \times v$.

As a result of this development, let $G$ be a finite $p$-group and suppose that $\lambda : G \longrightarrow [0, 1]$ is a fuzzy subgroup of $G$. Put $\lambda(G) = \{\beta_1, \beta_2, \ldots, \beta_k\}$ with the assumption that $\beta_1 < \beta_2 > \cdots > \beta_k$. Then, ends in $G$ is determined by $\lambda$.

$$\lambda G_{\beta_1} \subset \lambda G_{\beta_2} \subset \cdots \subset \lambda G_{\beta_k} = G \tag{2 − 1}$$

Also, we have that

$$\lambda(a) = \beta_t \Longleftrightarrow t = \max\{r/a \in \lambda G_{\beta_r}\} \Longleftrightarrow a \in \lambda G_{\beta_t} \backslash \lambda G_{\beta_{t-1}},$$

for any $a \in G$ and $t = 1, \cdots, k$, where by convention, set $\lambda G_{\beta_0} = \phi$.

## §3. Methodology

In the sequel, the method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite $p$-group $G$ is described. Suppose that $M_1, M_2, \ldots, M_t$ are the maximal subgroups of $G$, and denote by $h(G)$ the number of chains of subgroups of $G$ which ends in $G$. By simply applying the technique of computing $h(G)$, using the application of the *inclusion-exclusion principle*, we have that:

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1 \leq r_1 < r_2 \leq t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right). \tag{3 − 1}$$

In [5], the formula (3-1) was used to obtain the explicit formulas for some positive integers $n$.

**Theorem** 3.1([6]) *The number of distinct fuzzy subgroups of a finite p-group of order $p^n$ which have a cyclic maximal subgroup is*

(*i*) $h(\mathbb{Z}_{p^n}) = 2^n$;

(*ii*) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$.

## §4. The number of Fuzzy Subgroups for $\mathbb{Z}_4 \times \mathbb{Z}_4$

**Lemma** 4.1 *Let $G$ be abelian such that $G = \mathbb{Z}_4 \times \mathbb{Z}_4$. Then, $h(G) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 48$.*

*Proof* By the use of GAP (Group Algorithms and Programming), $G$ has three maximal subgroups in which each of them is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^2}$. Hence, we have that

$$\begin{aligned} \frac{1}{2}h(G) &= 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) \\ &= h(\mathbb{Z}_2 \times \mathbb{Z}_4). \end{aligned}$$

Applying Theorem 3.1, $h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 24. \Rightarrow h(\mathbb{Z}_4 \times \mathbb{Z}_4) = 48.$ □

**Corollary** 4.2 *Following Lemma* 4.1, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^6}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^7})$ *and* $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^8}) = 1536, 4096, 10496$ *and* $26112$, *respectively.*

**Proposition** 4.3 *Suppose that $G = \mathbb{Z}_4 \times \mathbb{Z}_{2^n}, n \geq 2$. Then, $h(G) = 2^n[n^2 + 5n - 2]$.*

*Proof* $G$ has three maximal subgroups of which two are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ and the third is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}$. Hence,

$$\begin{aligned} h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) &= 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^n}) + 2^1 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) + 2^2 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}) + 2^3 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-3}}) \\ &\quad + 2^4 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-4}}) + \cdots + 2^{n-2} h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) \\ &= 2^{n+1}[2(n+1) + \sum_{j=1}^{n-2}((n+1) - j) \\ &= 2^{n+1}(2(n+1) + \frac{1}{2}(n-2)(n+3)) = 2^n(n^2 + 5n - 2) \end{aligned}$$

for $n \geq 2$. We therefore know that

$$\begin{aligned} h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) &= 2^{n-1}((n-1)^2 + 5(n-1) - 2) \\ &= 2^{n-1}(n^2 + 3n - 6) \end{aligned}$$

for $n > 2$. This completes the proof. □

**Theorem** 4.4([4]) *Let $G = D_{2^n} \times \mathbb{C}_2$, the nilpotent group formed by the cartesian product of the dihedral group of order $2^n$ and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of $G$ is given by $h(G) = 2^{2n}(2n+1) - 2^{n+1}$ for $n > 3$.*

## §5. The Number of Fuzzy Subgroups for $D_{2^n} \times C_4$

**Proposition** 5.1 *Suppose that $G = D_{2^n} \times C_4$. Then, the number of distinct fuzzy subgroups of $G$ is given by*

$$2^{2(n-2)}(64n+173) + 3\sum_{j=1}^{n-3} 2^{(n-1+j)}(2n+1-2j)$$

*for $n \geq 3$.*

*Proof* Calculation shows that

$$\begin{aligned}
\frac{1}{2}h(D_{2^n} \times C_4) &= h(D_{2^n} \times C_2) + 2h(D_{2^{n-1}} \times C_4) - 4h(D_{2^{n-1}} \times C_2) + h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) \\
&\quad -2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) + 8h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}) + h(\mathbb{Z}_{2^{n-1}}) - 4h(\mathbb{Z}_{2^{n-2}}) \\
&= (n-3)2^{2n+2} + 2^{2(n-3)}(1460) + 3(2^n(2n-1) + 2^{n+1}(2n-3) + 2^{n+2}(2n-5) \\
&\quad + \cdots + 7(2^{2(n-2)})) \\
&= (n-3).2^{2n+2} + 2^{2(n-3)}(1460) + 3\sum_{j=1}^{n-3} 2^{n-1+j}(2n+1-2j) \\
&= 2^{2(n-2)}(64n+173) + 3\sum_{j=1}^{n-3} 2^{n-1+j}(2n+1-2j).
\end{aligned}$$

This completes the proof. □

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