Computing the Number of
Distinct Fuzzy Subgroups for the Ninpotent $p$-Group of $D_{2n} \times C_4$

S. A. Adebisi¹, M. Ogiugo ² and M. EniOluwafe³

1. Department of Mathematics, Faculty of Science, University of Lagos, Nigeria
2. Department of mathematics, School of Science, Yaba College of Technology, Lagos, Nigeria
3. Department of mathematics, Faculty of Science, University of Ibadan, Nigeria

E-mail: adesinasunday@yahoo.com, ekpenogiugo@gmail.com, michael.enioluwafe@gmail.com

Abstract: In this paper, the explicit formulae is given for the number of distinct fuzzy subgroups of the cartesian product of the dihedral group of order $2^n$ with a cyclic group of order four, where $n > 3$.

Key Words: Finite $p$-Groups, nilpotent Group, fuzzy subgroups, dihedral Group, inclusion-exclusion principle, maximal subgroups.

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§1. Introduction

In the fuzzy group theory, the classification of the fuzzy subgroups, most especially the finite $p$-groups cannot be underestimated. This aspect of pure Mathematics has undergone a dynamic developments over the years. For instance, many researchers have treated cases of finite abelian groups (see [2], [3]). The starting point for this concept all started as presented in [5] and [6]. Since then, the study has been extended to some other important classes of finite abelian and nonabelian groups such as the dihedral, quaternion, semidihedral, and hamiltonian groups.

Although, the natural equivalence relation was introduced in [7], where a method to determine the number and nature of fuzzy subgroups of a finite group $G$ was developed with respect to the natural equivalence. In [1] and [3], a different approach was applied for the classification. In this work, an essential role in solving counting problems is played by adopting the Inclusion-Exclusion Principle. The process leads to some recurrence relations from which the solutions are then finally computed with ease.

§2. Preliminaries

Suppose that $(G, \cdot, e)$ is a group with identity $e$. Let $S(G)$ denote the collection of all fuzzy subsets of $G$. An element $\lambda \in S(G)$ is said to be a fuzzy subgroup of $G$ if

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(i) $\lambda(ab) \geq \min\{\lambda(a), \lambda(b)\}$, $\forall a, b \in G$;
(ii) $\lambda(a^{-1}) \geq \lambda(a)$ for any $a \in G$.

And, since $(a^{-1})^{-1} = a$, we have that $\lambda(a^{-1}) = \lambda(a)$, for any $a \in G$. Also, by this notation and definition, $\lambda(e) = \sup \lambda(G)$ (see Marius [6]), which implies

**Theorem 2.1** The set $FL(G)$ possessing all fuzzy subgroups of $G$ forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of $G$.

We define the level subset

$$\lambda G_{\beta} = \{a \in G/\lambda(a) \geq \beta\} \text{ for each } \beta \in [0, 1]$$

The fuzzy subgroups of a finite $p$-group $G$ are thus, characterized, based on these level subsets. In the sequel, $\lambda$ is a fuzzy subgroup of $G$ if and only if its level subsets are subgroups in $G$. Theorem 2.1 gives a link between $FL(G)$ and $L(G)$, the classical subgroup lattice of $G$.

Moreover, some natural relations on $S(G)$ can also be used in the process of classifying the fuzzy subgroups of a finite $p$-group $G$ (see [6]). One of them is defined by: $\lambda \sim \gamma$ if and only if $(\lambda(a) > \lambda(b) \iff v(a) > v(b), \forall a, b \in G)$. Also, two fuzzy subgroups $\lambda, \gamma$ of $G$ and said to be distinct if $\lambda \neq \gamma$.

As a result of this development, let $G$ be a finite $p$-group and suppose that $\lambda : G \rightarrow [0, 1]$ is a fuzzy subgroup of $G$. Put $\lambda(G) = \{\beta_1, \beta_2, \ldots, \beta_k\}$ with the assumption that $\beta_1 < \beta_2 > \cdots > \beta_k$. Then, ends in $G$ is determined by $\lambda$.

$$\lambda G_{\beta_1} \subset \lambda G_{\beta_2} \subset \cdots \subset \lambda G_{\beta_k} = G \quad (2-1)$$

Also, we have that

$$\lambda(a) = \beta_t \iff t = \max\{r/a \in \lambda G_{\beta_r}\} \iff a \in \lambda G_{\beta_t} \setminus \lambda G_{\beta_{t-1}},$$

for any $a \in G$ and $t = 1, \cdots, k$, where by convention, set $\lambda G_{\beta_0} = \phi$.

§3. Methodology

In the sequel, the method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite $p$-group $G$ is described. Suppose that $M_1, M_2, \ldots, M_t$ are the maximal subgroups of $G$, and denote by $h(G)$ the number of chains of subgroups of $G$ which ends in $G$.

By simply applying the technique of computing $h(G)$, using the application of the inclusion-exclusion principle, we have that:

$$h(G) = 2 \left( \sum_{r=1}^{t} h(M_r) - \sum_{1 \leq r_1 < r_2 \leq t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h \left( \bigcap_{r=1}^{t} M_r \right) \right). \quad (3-1)$$

In [5], the formula (3-1) was used to obtain the explicit formulas for some positive integers $n$. 
Theorem 3.1([6]) The number of distinct fuzzy subgroups of a finite $p$-group of order $p^n$ which have a cyclic maximal subgroup is

(i) $h(\mathbb{Z}_{p^n}) = 2^n$;

(ii) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^n-1}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$.

§4. The number of Fuzzy Subgroups for $\mathbb{Z}_4 \times \mathbb{Z}_4$

Lemma 4.1 Let $G$ be abelian such that $G = \mathbb{Z}_4 \times \mathbb{Z}_4$. Then, $h(G) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_2^2) = 48$.

Proof By the use of GAP (Group Algorithms and Programming), $G$ has three maximal subgroups in which each of them is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2^2$. Hence, we have that

$$\frac{1}{2} h(G) = 3h(\mathbb{Z}_2 \times \mathbb{Z}_2^2) - 3h(\mathbb{Z}_2 \times \mathbb{Z}_2^2) + h(\mathbb{Z}_2 \times \mathbb{Z}_2^2) = h(\mathbb{Z}_2 \times \mathbb{Z}_4).$$

Applying Theorem 3.1, $h(\mathbb{Z}_2 \times \mathbb{Z}_2^2) = 24 \Rightarrow h(\mathbb{Z}_4 \times \mathbb{Z}_4) = 48$. □

Corollary 4.2 Following Lemma 4.1, $h(\mathbb{Z}_4 \times \mathbb{Z}_2^2), h(\mathbb{Z}_4 \times \mathbb{Z}_4^2), h(\mathbb{Z}_4 \times \mathbb{Z}_2^2) = 1536, 4096, 10496$ and 26112, respectively.

Proposition 4.3 Suppose that $G = \mathbb{Z}_4 \times \mathbb{Z}_{2^n}, n \geq 2$. Then, $h(G) = 2^n [n^2 + 5n - 2]$.

Proof $G$ has three maximal subgroups of which two are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ and the third is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^n-1}$. Hence,

$$h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^n}) + 2^4h(\mathbb{Z}_2 \times \mathbb{Z}_{2^n-1}) + 2^2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^n-2}) + 2^3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^n-3}) + 2^4h(\mathbb{Z}_2 \times \mathbb{Z}_{2^n-4}) + \cdots + 2n-2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^n})$$

$$= 2^{n+1}[2(n+1) + \sum_{j=1}^{n-2}((n+1) - j)$$

$$= 2^{n+1}[2(n+1) + \frac{1}{2}(n-2)(n+3)] = 2^n(n^2 + 5n - 2)$$

for $n \geq 2$. We therefore know that

$$h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n-1}) = 2^{n-1}((n-1)^2 + 5(n-1) - 2) = 2^{n-1}(n^2 + 3n - 6)$$

for $n > 2$. This completes the proof. □

Theorem 4.4([4]) Let $G = D_{2^n} \times C_2$, the nilpotent group formed by the cartesian product of the dihedral group of order $2^n$ and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of $G$ is given by $h(G) = 2^{2n}(2n + 1) - 2^{n+1}$ for $n > 3$. 
§5. The Number of Fuzzy Subgroups for $D_{2n} \times C_4$

**Proposition 5.1** Suppose that $G = D_{2n} \times C_4$. Then, the number of distinct fuzzy subgroups of $G$ is given by

$$2^{2(n-2)}(64n + 173) + 3 \sum_{j=1}^{n-3} 2^{(n-1+j)}(2n + 1 - 2j)$$

for $n \geq 3$.

*Proof* Calculation shows that

$$\frac{1}{2}h(D_{2n} \times C_4) = h(D_{2n} \times C_2) + 2h(D_{2n-1} \times C_4) - 4h(D_{2n-1} \times C_2) + h(Z_4 \times Z_{2n-1})$$

$$- 2h(Z_2 \times Z_{2n-1}) - 2h(Z_4 \times Z_{2n-2}) + 8h(Z_2 \times Z_{2n-2}) + h(Z_{2n-1}) - 4h(Z_{2n-1})$$

$$= (n-3)2^{2n+2} + 2^{2(n-3)}(1460) + 3(2^n(2n - 1) + 2^{n+1}(2n - 3) + 2^{n+2}(2n - 5)$$

$$+ \ldots + 7(2^{2(n-2)})$$

$$= (n-3)2^{2n+2} + 2^{2(n-3)}(1460) + 3 \sum_{j=1}^{n-3} 2^{n-1+j}(2n + 1 - 2j)$$

$$= 2^{2(n-2)}(64n + 173) + 3 \sum_{j=1}^{n-3} 2^{n-1+j}(2n + 1 - 2j).$$

This completes the proof. $\square$

**References**


