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Cube Sum Labeling (Taxi-Cab Labeling) of Graphs

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Abstract: Let $G(V,E)$ be a graph with order p and size q. A bijection $f : V(G) \to \{0,1,2,\cdots,p-1\}$ is said to be a cube sum labeling if the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = [f(u)]^3 + [f(v)]^3$ is injective. Such a function f is said to be a cube sum labeling and the graph G is a cube sum graph. In this paper we discuss some algebraic properties and evaluate some families of cube sum graph.

Key Words: Smarandachely cube sum H labeling, cube sum graph, complete graph, tree, wheel graph, helm graph, Fermat's Last Theorem.

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§1. Introduction

Labeling of graphs is one of the emerging topics in graph theory. The credit goes to Rosa [1] to explore this innovative idea. If the vertices or edges or both of the graph are assigned values subject to certain condition(s) then it is known as graph labeling. The idea of graph labeling was originated in 1967. Till then graph labeling has attracted many researchers and due to the wholehearted efforts for research in this field, more than 200 graph labeling techniques and more than 2500 research papers are available. A dynamic survey on graph labeling is regularly updated by Gallian [6] and it is published by *The Electronic Journal of Combinatorics*.

In this paper we consider simple, finite, undirected and connected graph. A graph $G(V,E)$ with p vertices and q edges is also denoted as $G(p,q)$ graph. We refer to Bondy and Murty [5] for the standard terminology and notations related to graph theory and Burton [2] for the terms related to number theory. We denote an edge with end vertices u and v by uv.

A square sum labeling is one of the graph labeling techniques, where edge label is obtained by sum of squares of labels of end vertices of the corresponding edge. The square sum labeling was introduced by Ajitha, Arumugam and Germina.

Definition 1.1(Ajitha et al., [10]) *A graph $G=(V,E)$ with p vertices and q edges is said to be a square sum graph, if there exists a bijection $f : V(G) \to \{0,1,2,\cdots,p-1\}$ such that the induced function $f^* : E(G) \to \mathbb{N}$, defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, is injective.*

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Many interesting results are carried out for square sum labeling of graphs. The literature on square sum labeling is accessible in electronic form in different research papers such as [3], [4], [7], [10] etc.

A cube sum labeling was introduced by Vediyappan Govindan, Sandra Pinelas and S.Dhivya [9] as follow and they proved that paths, cycle, stars, wheel graph, fan graphs are cube sum graph.

Definition 1.2(Vediyappan Govindan et al., [9]) *A graph $G = (V, E)$ with p vertices and q edges is said to be a cube sum graph, if there exists a bijection $f : V(G) \to \{0, 1, 2, \cdots, p-1\}$ such that the induced function $f^* : E(G) \to \mathbb{N}$, defined by $f^*(uv) = (f(u))^3 + (f(v))^3$, is injective.*

Notice that 1729 is the smallest natural number expressible as a sum of two cubes in two different ways as $12^3 + 1^3$ and $9^3 + 10^3$. From the story of G.H. Hardy and Srinivasa Ramanujan, 1729 is known as *Ramanujan number* or *Taxi-cab number* [2]. Other numbers which can expressed as sum of two cubes in two different ways are $4104 = 2^3 + 16^3 = 9^3 + 15^3, 13832 = 2^3 + 24^3 = 18^3 + 20^3, 20683 = 10^3 + 27^3 = 19^3 + 24^3$, etc. Taxi-cab number is related with sum of cube of two numbers. So, we also refer cube sum labeling as *Taxi-cab labeling* as well.

In this paper we have used the Fermat's Last Theorem [2] which states that *No three positive integers a, b and c satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than* 2.

§2. Cube Sum Labeling

Definition 2.1 *A bijective function $f : V(G) \to \{0, 1, 2, \cdots, p-1\}$ is said to be a cube sum labeling if the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = [f(u)]^3 + [f(v)]^3$ is injective.*

Generally, let $H \prec G$ be a typical subgraph of G such as those of path, cycle. If such an induced function f^* is injective on $E(G) \backslash E(H)$ but not on $E(G)$, such a labeling f is said to be a Smarandachely cube sum H labeling. Particularly, if $H = \emptyset$, then such a Smarandachely cube sum H labeling is nothing else but a cube sum labeling.

A graph G with cube sum labeling is called a cube sum graph.

Lemma 2.2(Burton, [2]) *The cube of any integer is one of the form $9k, 9k + 1$ or $9k + 8$.*

Theorem 2.3 *Let G be a cube sum graph with cube sum labeling f. Then, for any edge $e \in E(G)$, $f^*(e) \not\equiv 3, 4, 5, 6 (mod\ 9)$.*

Proof Let $u, v \in V(G)$, $f(u) = a$ and $f(v) = b$. Then, for edge $e = uv \in E(G)$, $f^*(uv) = a^3 + b^3$.

Since a and b are integers, from Lemma 3.1, $a^3 \equiv 0, 1$ or $8 (mod\ 9)$ and $b^3 \equiv 0, 1$ or $8 (mod\ 9)$. But then $a^3 + b^3 \equiv 0, 1, 2, 7$ or $8 (mod\ 9)$. Hence, the result is proved. □

Lemma 2.4(Burton, [2]) *The cube of any integer is one of the form $7k$ or $7k \pm 1$.*

Theorem 2.5 *Let G be a cube sum graph with cube sum labeling f. Then for any edge $e \in E(G)$, $f^*(e) \not\equiv 3, 4(mod\ 7)$.*

Proof Let $f(u) = a$ and $f(v) = b$. Then, for edge $e = uv \in E(G), f^*(uv) = a^3 + b^3$. From lemma 3.2, since a and b are integers, $a^3 \equiv 0, 1$ or $6(mod\ 7)$, and $b^3 \equiv 0, 1$ or $6(mod\ 7)$. But then $a^3 + b^3 \equiv 0, 1, 2, 5$ *or* $6(mod\ 7)$. This completes the proof. □

Theorem 2.6 *If $G(p, q)$ is cube sum graph with cube sum labeling f, then*

$$\sum_{uv \in E(G)} f^*(uv) = \sum_{v \in V(G)} [f(v)]^3 d(v)$$

where $d(v)$ is the degree of vertex v in G.

Proof Let $f : V(G) \to \{0, 1, 2, \cdots, p-1\}$ be a cube sum labeling of a graph G with each edge uv is assigned the label $f^*(uv) = [f(u)]^3 + [f(v)]^3$.

Now every edge is incident with exactly two vertices and degree of a vertex is the number of edges incident with that vertex. Then, while counting the total sum of edge labels, the number of times of repetition (occurrence) of each vertex label is equal to the number of edges incident to the corresponding vertex. Then the sum of $f^*(e)$ count $[f(v)]^3$ at total number of times an edge is incident with a vertex v. So

$$\sum_{uv \in E(G)} f^*(uv) = \sum_{v \in V(G)} [f(v)]^3 d(v). \qquad \square$$

Corollary 2.7 *If $G(p, q)$ is an r-regular cube sum graph, then*

$$\sum_{uv \in E(G)} f^*(uv) = \frac{r(p-1)^2 p^2}{4}.$$

Proof From Theorem 3.3, we have

$$\sum_{uv \in E(G)} f^*(uv) = \sum_{v \in V(G)} [f(v)]^3 d(v). \tag{1}$$

Here, $G(p, q)$ is an r-regular cube sum graph, i.e. $d(v) = r,\ \forall v \in V(G)$.

$$\begin{aligned} \sum_{uv \in E(G)} f^*(uv) &= r \sum_{v \in V(G)} [f(v)]^3 \quad \{\text{from } (1)\} \\ &= r\big(0^3 + 1^3 + \cdots + (p-1)^3\big) = \frac{r(p-1)^2 p^2}{4}. \end{aligned} \qquad \square$$

§3. Some Cube Sum Graphs

Theorem 3.1 *A complete graph K_n is a cube sum graph if and only if $n \leq 11$.*

Proof Let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(K_n) = \{v_i v_j \mid 1 \leq i, j \leq n, i \neq j\}$. Here, $|V(K_n)| = n$ and $|E(K_n)| = \frac{n(n-1)}{2}$.

Case 1. $n \leq 11$.

Let us define a function $f : V(K_n) \to \{0, 1, 2, \cdots, n-1\}$ as

$$f(v_i) = i - 1 \ ; \ 1 \leq i \leq n.$$

It is obvious that f is bijective and the induced function $f^* : E(K_n) \to \mathbb{N}$ defined by

$$f^*(uv) = (f(u))^3 + (f(v))^3,$$

for every $uv \in E(K_n)$ is injective. Hence, K_n is a cube sum graph for $n \leq 11$.

Case 2. $n > 11$.

Notice that every two vertices are adjacent to each other in a complete graph. So, defining a mapping $f : V(K_n) \to \{0, 1, 2, \cdots, n-1\}$ in any form, we have two edges e_i and e_j such that $f^*(e_i) = 12^3 + 1^3 = 1729$ and $f^*(e_j) = 9^3 + 10^3 = 1729$. Thus, the induced function f^* is not injective. Hence, K_n is not a cube sum graph for $n > 11$. □

Theorem 3.2 *A complete bipartite graph $K_{2,n}$ is a cube sum graph for any integer $n \geq 1$.*

Proof Let $V_1 = \{v_1, v_{n+2}\}$ and $V_2 = \{v_2, v_3, \cdots, v_{n+1}\}$ be bipartition of $V(K_{1,n}) = \{v_1, v_2, v_3, \cdots, v_{n+1}, v_{n+2}\}$ and $E(K_{1,n}) = \{v_i v_j \mid i = 1, n+2 \text{ and } j = 2, 3, \cdots, n+1\}$. Here, $|V(K_{2,n})| = n+2$ and $|E(K_{2,n})| = 2n$. Let us define a function $f : V(K_{2,n}) \to \{0, 1, 2, \cdots, n+1\}$ as

$$f(v_i) = i - 1 \ ; \ 1 \leq i \leq n+2.$$

It is obvious that f is bijective.

Furthermore, one can observe that

$$\begin{aligned} f^*(v_1v_2)(= 1) &< f^*(v_1v_3)(= 2^3) < f^*(v_1v_4)(= 3^3) \\ &< \cdots < f^*(v_1v_{n+1})(= n^3) < f^*(v_{n+2}v_2)(= (n+1)^3 + 1) \\ &< f^*(v_{n+2}v_3)(= (n+1)^3 + 2^3) < \cdots < f^*(v_{n+2}v_{n+1})(= (n+1)^3 + n^3). \end{aligned}$$

Then, the induced function $f^* : E(K_{2,n}) \to \mathbb{N}$ defined by

$$f^*(uv) = (f(u))^3 + (f(v))^3,$$

for every $uv \in E(K_{2,n})$ is injective. Hence, $K_{2,n}$ is a cube sum graph. □

Theorem 3.3 *Every tree is a cube sum graph.*

Proof Let $v_{0,0}$ be a vertex with maximum degree in a tree T. Choose $v_{0,0}$ as a root vertex of T (say zero level vertex). Let l be the height of T. Consider $n_0 = 0$.

Let n_1 be the number of vertices at distance one from $v_{0,0}$ and let us denote these vertices by $v_{1,1}, v_{1,2}, \cdots v_{1,n_1}$. These vertices are first level vertices. Let n_2 be the number of vertices at distance two from $v_{0,0}$ which are denoted by $v_{2,1}, v_{2,2}, \cdots v_{2,n_2}$. These vertices are second level vertices. We give priority as in ascending order.

Repeating this way, let n_l be the number of vertices at distance l from $v_{0,0}$ which are denoted by $v_{l,1}, v_{l,2}, \cdots v_{l,n_l}$. These are l^{th} level vertices.

The above process is possible because there is one and only one path between any pair of vertices in any tree. Here, $|V(T)| = \sum_{i=1}^{l}(n_i) + 1 = n$ and $|E(T)| = \sum_{i=1}^{l}(n_i) = n - 1$.

Let us define a function $f : V(T) \to \{0, 1, 2, 3, \cdots, n-1\}$ as

$$f(v_{i,j}) = \begin{cases} 0; & i = 0, j = 0. \\ f(v_{i-1,n_{i-1}}) + j; & 1 \leq i \leq l, 1 \leq j \leq \ n_i. \end{cases}$$

Here, vertex labels are in ascending order from zero level vertex to l level vertices. So, it is obvious that f is bijective and for edge labels we have following arguments. We have following two cases for edge labels. Without loss of generality, let e_1 and e_2 be any two arbitrary edges of tree T.

Case 1. Let e_1 and e_2 be two incident edges. Then obviously $f^*(e_1) \neq f^*(e_2)$.

Case 2. Let $e_1 = v_1v_2$ and $e_2 = v_3v_4$ be the edges such that e_1 and e_2 have no common vertex. Here $\{f(v_1), f(v_2), f(v_3), f(v_4)\}$ is non-empty subset of $\mathbb{N}$. So, by well ordering principle, $\{f(v_1), f(v_2), f(v_3), f(v_4)\}$ has a least element, say $f(v_1)$.

Since T is a tree, at least one of the vertex v_3 or v_4 is not adjacent to v_1. If v_4 is not adjacent to v_1, then $f(v_4) > f(v_2)$ and if v_3 is not adjacent to v_1 then $f(v_3) > f(v_2)$. So, $f^*(e_1) \neq f^*(e_2)$. Thus, the induced function $f^* : E(G) \to \ \mathbb{N}$ defined by

$$f^*(uv) = (f(u))^3 + (f(v))^3,$$

for every $uv \in E(G)$ is injective. Hence, tree T is a cube sum graph. □

Theorem 3.4 *A cycle C_n is a cube sum graph.*

Proof Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(C_n) = \{v_iv_{i+1} \mid 1 \leq i \leq n-1\} \bigcup \{v_nv_1\}$. Here, $|V(C_n)| = n$ and $|E(C_n)| = n$.

Let us define a function $f : V(C_n) \to \{0, 1, 2, \cdots, n-1\}$ as per subsequent two cases.

Case 1. n is even.

In this case, define

$$f(v_i) = \begin{cases} 0; & i = 1. \\ 2i - 3; & 2 \leq i \leq \ \frac{n+2}{2}. \\ n - 2(i - \frac{n+2}{2}); & \frac{n+2}{2} < i \leq \ n. \end{cases}$$

It is obvious that f is bijective and we can observe that

$$\begin{aligned} f^*(v_1v_2)(=1) &< f^*(v_1v_n)(=2^3) < f^*(v_2v_3)(=1^3+3^3) \\ &< \cdots < f^*(v_{\frac{n}{2}}v_{\frac{n+2}{2}}) < f^*(v_{\frac{n+4}{2}}v_{\frac{n+2}{2}}). \end{aligned}$$

Case 2. n is odd.

In this case, define

$$f(v_i) = \begin{cases} 0; & i = 1. \\ 2i-3; & 2 \leq i \leq \frac{n+1}{2}. \\ n+1-2(i-\frac{n+1}{2}); & \frac{n+1}{2} < i \leq n. \end{cases}$$

It is obvious that f is bijective and we can observe that

$$\begin{aligned} f^*(v_1v_2)(=1) &< f^*(v_1v_n)(=2^3) < f^*(v_2v_3)(=1^3+3^3) \\ &< \cdots < f^*(v_{\frac{n}{2}}v_{\frac{n+2}{2}}) < f^*(v_{\frac{n+4}{2}}v_{\frac{n+2}{2}}). \end{aligned}$$

So, in both the cases the induced function $f^* : E(C_n) \to \mathbb{N}$ defined by

$$f^*(uv) = (f(u))^3 + (f(v))^3,$$

is injective. Hence, C_n is a cube sum graph. □

Theorem 3.5 *A wheel W_n is a cube sum graph.*

Proof Let $V(W_n) = \{v_0, v_1, \cdots, v_n\}$ and $E(W_n) = \{v_0v_i \mid 1 \leq i \leq n\} \bigcup \{v_iv_{i+1} \mid 1 \leq i \leq n-1\} \bigcup \{v_nv_1\}$, where v_0 is apex and $v_1, v_2, \cdots, v_n$ are rim vertices of W_n. Clearly, $|V(W_n)| = n+1$ and $|E(W_n)| = 2n$.

Let us define a function $f : V(W_n) \to \{0, 1 \cdots, n\}$ as follows.

$$f(v_i) = \begin{cases} 0\ ; i = 0. \\ 1\ ; i = 1. \\ 2(i-1)\ ; 1 < i \leq \lfloor \frac{n+2}{2} \rfloor. \\ 2n-2i+3\ ; \lfloor \frac{n+2}{2} \rfloor < i \leq n. \end{cases}$$

It is obvious that f is bijective. We consider the following two cases for the edge labels.

Case 1. n is odd.

From above vertex labels, one can observe that labels of rim edges are in ascending order as

$$\begin{aligned} f^*(v_1v_2)(=1^3+2^3) &< f^*(v_1v_n)(=1^3+3^3) < f^*(v_2v_3)(=2^3+4^3) \\ &< f^*(v_nv_{n-1})(=3^3+5^3) < \cdots < f^*(v_{\frac{n+1}{2}}v_{\frac{n+3}{2}})(=(n-1)^3+n^3). \end{aligned}$$

From Fermat's Last Theorem, $f^*(v_0v_i)$ is never equal to any of above edge labels for integers $1 \leq i \leq n$.

Case 2. n is even.

From above vertex labels, one can observe that labels of rim edges are in ascending order as

$$\begin{aligned} f^*(v_1v_2)(=1^3+2^3) &< f^*(v_1v_n)(=1^3+3^3) < f^*(v_2v_3)(=2^3+4^3) \\ &< f^*(v_nv_{n-1})(=3^3+5^3) < \cdots < f^*(v_{\frac{n+2}{2}}v_{\frac{n+4}{2}})(=(n-1)^3+n^3). \end{aligned}$$

From Fermat's Last Theorem, $f^*(v_0v_i)$ $(1 \leq i \leq n)$ is never equal to any one of above edge labels. So, in both the cases the induced function $f^* : E(W_n) \to \mathbb{N}$ defined by

$$f^*(uv) = (f(u))^3 + (f(v))^3,$$

for every $uv \in E(W_n)$ is injective. Hence, W_n is a cube sum graph. $\square$

Corollary 3.6 *A gear G_n is a cube sum graph.*

Corollary 3.7 *A shell S_n is a cube sum graph.*

Theorem 3.8 *A helm H_n is a cube sum graph.*

Proof Let $V(H_n) = \{v_0, v_i, u_i \mid 1 \leq i \leq n\}$ and $E(H_n) = \{v_0v_i \mid 1 \leq i \leq n\} \bigcup \{v_iu_i \mid 1 \leq i \leq n\} \bigcup \{v_iv_{i+1} \mid 1 \leq i \leq n-1\} \bigcup \{v_nv_1\}$, where v_0 is apex, $v_1, v_2, \cdots, v_n$ are rim vertices and $u_1, u_2 \cdots, u_n$ are pendant vertices of helm H_n. Obviously, $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$.

Let us define a function $f : V(H_n) \to \{0, 1 \cdots, 2n\}$ as follows.

Case 1. n is even.

In this case, define

$$f(v_i) = \begin{cases} 0 \ ; i = 0. \\ 1 \ ; i = 1. \\ 4i-5 \ ; 2 \leq i \leq \frac{n+2}{2}. \\ 2n-3-4\big(i-\frac{n+4}{2}\big) \ ; \frac{n+2}{2} < i \leq n. \end{cases}$$

$$f(u_i) = \begin{cases} 4i-4 \ ; 2 \leq i \leq \frac{n+2}{2}. \\ 2n-2-4\big(i-\frac{n+4}{2}\big) \ ; \frac{n+2}{2} < i \leq n. \end{cases}$$

It is obvious that f is bijective and for the edge labels in the graph there are three possibilities as follows:

(1) Edge labels on rim edges are

$$\begin{aligned} f^*(v_1v_2)(=1^3+3^3) &< f^*(v_1v_n)(=1^3+5^3) \\ &< f^*(v_2v_3)(=3^3+7^3) < f^*(v_nv_{n-1})(=5^3+9^3) \\ &< \cdots < f^*(v_{\frac{n+2}{2}}v_{\frac{n+4}{2}})(=(2n-1)^3+(2n-3)^3). \end{aligned}$$

They are in ascending order of the form $2k$ ($k \in \mathbb{N}$) because the common end vertices of these edges are labeled by odd numbers (naturally distinct).

(2) Edge labels on edges incident to pendant vertices are

$$\begin{aligned} f^*(v_1u_1)(=1^3+2^3) &< f^*(v_2u_2)(=3^3+4^3) \\ &< f^*(v_nu_n)(=5^3+6^3) \\ &< \cdots < f^*(v_{\frac{n+2}{2}}u_{\frac{n+2}{2}})(=(2n-1)^3+(2n)^3). \end{aligned}$$

They are in ascending order of the form $2k+1$ ($k \in \mathbb{N}$) because common end vertices of these edges are labeled by consecutive numbers (naturally distinct).

(3) Edge labels on edges incident to apex are

$$\begin{aligned} f^*(v_0v_1)(=1^3) &< f^*(v_0v_2)(=3^3) < f^*(v_0v_n)(=5^3) \\ &< \cdots < f^*(v_0v_{\frac{n+2}{2}})(=(2n-1)^3). \end{aligned}$$

They are in ascending order of the form $2k+1$ ($k \in \mathbb{N}$) because common end vertices of these edges are labeled by 0 and other end vertices by odd numbers (naturally distinct).

It is clear that the labels of possibilities (1) and (2) are distinct.

From Fermat's Last Theorem, the edge labels in the possibilities (3) are distinct from the edge labels in the possibilities (1) and (2). So, the labels of above all possibilities are internally as well as externally distinct.

Case 2. n is odd.

In this case, define

$$f(v_i) = \begin{cases} 0 \ ; i = 0. \\ 1 \ ; i = 1. \\ 4i-5 \ ; 2 \le i \le \frac{n+1}{2}. \\ 2n-1-4\big(i-\frac{n+3}{2}\big) \ ; \frac{n+1}{2} < i \le n. \end{cases}$$

$$f(u_i) = \begin{cases} 4i-4 \ ; 2 \le i \le \frac{n+1}{2}. \\ 2n-4\big(i-\frac{n+3}{2}\big) \ ; \frac{n+1}{2} < i \le n. \end{cases}$$

Using the arguments similar to the case 1, one can observe that in this case the function f is bijective and for every $uv \in E(G)$ the induced edge labels $f^*(uv) = (f(u))^3 + (f(v))^3$ are all distinct.

So, in both the cases the induced function $f^* : E(H_n) \to \mathbb{N}$ defined by

$$f^*(uv) = (f(u))^3 + (f(v))^3,$$

is injective. Hence H_n is a cube sum graph. $\square$

§4. Concluding Remarks

Labeling of discrete structure is a potential area of research. We have discussed some algebraic properties of cube sum graph. We have also proved that the following graphs are cube sum graphs: a complete graph K_n if and only if $n \leq 11$, a complete bipartite graph $K_{2,n}$ for $n \geq 1$, every tree, cycle graph, wheel graph, gear graph, shell graph and helm graph. To investigate more results for various graphs as well as in the context of different graph operations is an open area of research.

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