D-homothetic Deformations of Lorentzian Para-Sasakian Manifold

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Abstract: The aim of the present paper is to prove some results on the properties of LP-Sasakian manifolds under D-homothetic deformations. In the later sections we give several results on some properties which are conformal under the mentioned deformations. Lastly, we illustrate the main theorem by giving a detailed example.

Key Words: D-homothetic deformation, LP-Sasakian manifold, \(\phi\)-section, sectional curvature.


§1. Introduction

The notion of Lorentzian almost para-contact manifolds was introduced by K. Matsumoto [3]. Later on, a large number of geometers studied Lorentzian almost para-contact manifold and their different classes, viz., Lorentzian para-Sasakian manifolds and Lorentzian special para-Sasakian manifolds [4], [5], [6], [7]. In brief, Lorentzian para-Sasakian manifolds are called LP-Sasakian manifolds. The study of LP-Sasakian manifolds has vast applications in the theory of relativity.

In an \(n\)-dimensional differentiable manifold \(M\), \((\phi, \xi, \eta)\) is said to be an almost paracontact structure if it admits a \((1, 1)\) tensor field \(\phi\), a timelike contravariant vector field \(\xi\) and a 1-form \(\eta\) which satisfy the relations:

\[
\eta(\xi) = -1, \tag{1.1}
\]

\[
\phi^2 X = X + \eta(X)\xi, \tag{1.2}
\]

for any vector field \(X\) on \(M\). In an \(n\)-dimensional almost paracontact manifold with structure \((\phi, \xi, \eta)\), the following conditions hold:

\[
\phi \xi = 0, \tag{1.3}
\]

\[
\eta \circ \phi = 0, \tag{1.4}
\]

\[
\text{rank } \phi = n - 1. \tag{1.5}
\]

Let \(M^n\) be differentiable manifold with an almost paracontact structure \((\phi, \xi, \eta)\). If there exists a Lorentzian metric which makes \(\xi\) a timelike unit vector field, then there exists a

\[\text{Received September 11, 2018, Accepted May 24, 2019.}\]
Lorentzian metric $g$ satisfying

$$g(X, \xi) = \eta(X),$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

$$\nabla_X \phi Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

for all vector fields $X, Y$ on $\tilde{M}$ [2].

If a differentiable manifold $M$ admits the structure $(\phi, \xi, \eta, g)$ such that $g$ is an associated Lorentzian metric of the almost paracontact structure $(\phi, \xi, \eta, g)$ then we say that $M^n$ has a Lorentzian almost paracontact structure $(\phi, \xi, \eta, g)$ and $M^n$ is said to be Lorentzian almost paracontact manifold (LAP) with structure $(\phi, \xi, \eta, g)$.

In a LAP-manifold with structure $(\phi, \xi, \eta, g)$ if we put

$$\Omega(X, Y) = g(\phi X, Y),$$

then the tensor field $\Omega$ is a symmetric $(0, 2)$ tensor field [2], that is

$$\Omega(X, Y) = \Omega(Y, X),$$

for all vector fields $X, Y$ on $M^n$. A LAP-manifold with structure $(\phi, \xi, \eta, g)$ is said to be Lorentzian paracontact manifold if it satisfies

$$\Omega(X, Y) = \frac{1}{2}\{((\nabla_X \eta)Y + (\nabla_Y \eta)X\}$$

and $(\phi, \xi, \eta, g)$ is said to be Lorentzian paracontact structure. Here $\nabla$ denotes the operator of covariant differentiation w.r.t the Lorentzian metric $g$.

In a LP-Sasakian manifold we have the following results from [9]:

$$\nabla_X \xi = \phi X,$$

$$\nabla_X \eta Y = \Omega(X, Y) = g(\phi X, Y),$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y),$$

$$R(\xi, X)\xi = \eta(X)\xi - \eta(\xi)X = X + \eta(X)\xi,$$

$$S(X, \xi) = (n - 1)\eta(X),$$

$$S(\xi, \xi) = -(n - 1),$$

$$Q\xi = -(n - 1),$$

where $R$ is the curvature tensor of manifold of type $(1, 3)$, $S$ is Ricci tensor of type $(0, 2)$ and $Q$ being the Ricci operator. An example of a five-dimensional Lorentzian para-Sasakian manifold has been given by Matsumoto, Mihai and Rosaca in [5].
§2. **D-homothetic Deformations of LP-Sasakian Manifolds**

Let \( M(\phi, \xi, \eta, g) \) be a Lorentzian almost paracontact structure. By D-homothetic deformation [8], we mean a change of structure tensors of the form

\[
\overline{\eta} = a\eta, \quad \overline{\xi} = \frac{1}{a}\xi, \quad \overline{\phi} = \phi, \quad \overline{g} = ag + a(a-1)\eta \otimes \eta,
\]

where \( a \) is a positive constant.

**Theorem 2.1** Under D-homothetic deformation \( M(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g}) \) is also an LP-Sasakian manifold \( M(\phi, \xi, \eta, g) \).

**Proof** Calculation shows that

\[
\begin{align*}
\overline{\eta}(\overline{\xi}) &= \eta(1/a\xi) = \eta(\xi) = -1, \\
\overline{\phi}^2(X) &= \phi^2(X) = X + \eta(X)\xi, \\
\overline{\phi} \circ \overline{\xi} &= \phi(1/a\xi) = \phi(\xi) = 1/a \phi \xi = 0, \\
\overline{\eta} \circ \overline{\phi} &= \eta(\phi(X)) = a\eta(\phi(X)) = 0, \\
\text{rank } \overline{\phi} &= \text{rank } \phi = n-1, \\
\overline{\eta}(X) &= \eta(X) = ag(X, \xi), \\
\overline{\eta}(\phi X, \phi Y) &= \eta(\phi X, \phi Y) = (ag + a(a-1)\eta \otimes \eta)(\phi X, \phi Y) = ag(\phi X, \phi Y), \\
(\nabla_X \phi)(Y) &= (\nabla_X \phi)(Y) = g(\phi X, Y) + c\eta(Y)X + 2\eta(X)\eta(Y)\xi.
\end{align*}
\]

\( \square \)

**Theorem 2.2** Under D-homothetic deformation of a LP Sasakian manifold the following relation holds

\[
(L_\xi \overline{g})(X, Y) = a(L_\xi g)(X, Y),
\]

where \( L_\xi \) is the Lie derivative.

**Proof** For an LP-Sasakian manifold we know \((L_\mu g)(X, Y) = 2g(\phi X, Y)\) since \(g(\phi X, Y) = g(X, \phi Y)\). Under D-homothetic deformation

\[
\begin{align*}
(L_\xi \overline{g})(X, Y) &= 2\overline{\eta}(\phi X, Y) \\
&= a(L_\xi g)(X, Y) + 2(a^2 - a)\eta(\phi X)\eta(Y) \\
&= a(L_\xi g)(X, Y).
\end{align*}
\]

\( \square \)

§3. **D-homothetic Deformations of Curvature Tensors on LP-Sasakian Manifolds**

In this section we consider conformally flat LP-Sasakian manifold \( M^n(\phi, \xi, \eta, g) \) \( (n > 3) \). The
Weyl conformal curvature tensor $C$ is given by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX$$

$$- g(X, Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y]. \tag{3.1}$$

For conformally flat manifold we have $C(X,Y)Z = 0$. So from (3.1) we have

$$R(X,Y)Z = \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]$$

$$- \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y]. \tag{3.2}$$

Putting $Z = \xi$ in (3.2), we obtain from (1.14)

$$\eta(Y)X - \eta(X)Y = \frac{1}{n-2}[S(Y, \xi)X - S(X, \xi)Y + S(Y, \xi)QX - g(X, \xi)QY]$$

$$- \frac{r}{(n-1)(n-2)}[g(Y, \xi)X - g(X, \xi)Y]. \tag{3.3}$$

Putting $Y = \xi$ in (3.3) we calculate

$$\eta(\xi)X - \eta(X)\xi = \frac{1}{n-2}[S(\xi, \xi)X - S(X, \xi)\xi + S(\xi, \xi)QX - g(X, \xi)Q\xi]$$

$$- \frac{r}{(n-1)(n-2)}[g(\xi, \xi)X - g(X, \xi)\xi]. \tag{3.4}$$

After some steps of calculations we obtain

$$QX = (-1 + \frac{r}{n-1})X + (-1 + \frac{r}{n-1})\eta(X)\xi - (n-1)\eta(X). \tag{3.5}$$

Taking inner product with $Y$, above equation can be written as

$$S(X, Y) = (1 + \frac{r}{n-1})g(X, Y) + (-1 + \frac{r}{n-1})\eta(Y)g(Y, \xi) - (n-1)\eta(X). \tag{3.6}$$

In view of (3.5), (3.6) equation (3.2) takes the form

$$R(X,Y)Z = [g(Y, Z)X - g(X, Z)Y] \left[ (1 + \frac{r}{n-1}) \frac{1}{n-2} ight]$$

$$+ \frac{1}{n-2} \left( 1 + \frac{r}{n-1} - \frac{r}{(n-1)(n-2)} \right)$$

$$+ g(Y, Z)\eta(X) \left[ (\frac{r}{n-1} - 1) \frac{1}{n-2} \xi - (n-1) \right] + g(X, Z)\eta(Y)$$

$$\times \left[ (\frac{r}{n-1} - 1) \frac{1}{n-2} \xi - (n-1) \right] + X\eta(Y) \left[ (\frac{r}{n-1} - 1) \frac{1}{n-2} \eta(Z) - \frac{n-1}{n-2} \right]$$

$$+ Y\eta(X) \left[ (\frac{r}{n-1} - 1) \frac{1}{n-2} \eta(Z) - \frac{n-1}{n-2} \right]. \tag{3.7}$$

For a conformally flat LP-Sasakian manifold, $R(X,Y)Z$ is given by the equation (3.7). Again in
a LP-Sasakian manifold the following relation holds [9]

\[ R(X, Y)\phi Z = \phi(R(X, Y)Z) + 2\{\eta(X)Y - \eta(Y)X\}\eta(Z) + 2\{g(Y, Z)\eta(X)
- g(X, Z)\eta(Y)\}\xi - g(\phi X, Z)\phi Y + g(\phi Y, Z)\phi X - g(Y, Z)X
+ g(X, Z)Y. \] (3.8)

Again, on using equations (1.15), (1.18) and (1.4) in (3.8) we calculate

\[ g(\phi R(\phi X, \phi Y)Z, \phi W) = g(R(Z, W)\phi X, \phi Y). \]

Using (3.8) and then (1.7), (1.15) in the above equation we obtain

\[ g(\phi R(\phi X, \phi Y)Z, \phi W) = g(R(X, Y)Z, W) - g(W, X)\eta(Z)\eta(Y) + g(Z, X)\eta(W)\eta(Y)
+ 2\eta(Z)\eta(X)g(W, \phi Y) - 2\eta(W)\eta(X)g(Z, \phi Y) - g(\phi Z, X)g(\phi W, \phi Y)
+ g(\phi W, X)g(\phi Z, \phi Y) - g(W, X)g(Z, \phi Y) + g(Z, X)g(W, \phi Y). \] (3.9)

Replacing \(X, Y\) by \(\phi X\) and \(\phi Y\) respectively in (3.8) and taking inner product with \(\phi W\) we obtain

\[ g(\phi R(\phi X, \phi Y)\phi Z, \phi W) = g(R(X, Y)\phi Z, W) - g(W, X)\eta(Z)\eta(Y) + g(Z, X)\eta(W)\eta(Y)
+ 3g(\phi Y, \phi W)\eta(Z)\eta(X) - 3g(\phi Z, \phi Y)\eta(W)\eta(X) + 2g(\phi W, X)g(Z, Y)
+ 2g(\phi W, X)\eta(Z)\eta(Y) - 2g(W, X)g(Z, \phi Y). \] (3.10)

Now we shall recall the definition of \(\phi\)-section. A plane section in the tangent space \(T_p(M)\) is called a \(\phi\)-section if there exists a unit vector \(X\) in \(T_p(M)\) orthogonal to \(\xi\) such that \(\{X, \phi X\}\) is an orthonormal basis of the plane section. Then the sectional curvature

\[ K(X, \phi X) = g(R(X, \phi X)X, \phi X) \] (3.11)

is called a \(\phi\)-sectional curvature. A contact metric manifold \(M(\phi, \xi, \eta, g)\) is said to be of constant \(\phi\)-sectional curvature if at any point \(P \in M\), the sectional curvature \(K(X, \phi X)\) is independent of the choice of non-zero \(X \in D_p\), where \(D\) denotes the contact distributions of the contact metric manifold defined by \(\eta = 0\). The definition is valid for Lorentzian manifolds also [10].

We give the following theorem.

**Theorem 3.1** In a LP-Sasakian manifold \(M(\phi, \xi, \eta, g)\) the relation \((Q\phi - \phi Q)X = 4\eta\phi X\) holds for any vector field \(X\) on \(M\).

**Proof** Let \(\{X_i, \phi X_i, \xi\} (i = 1, 2, \cdots, m)\) be a local \(\phi\)-basis at any point of the manifold. Now putting \(Y = Z = X_i\) in (3.10) and taking summation over \(i\), we obtain by virtue of \(\eta(X_i) = 0\),

\[ \sum \phi R(\phi X_i, \phi X_i)\phi X_i = \Sigma R(X, X_i)X_i + 2\phi X g(X_i, X_i). \] (3.12)

Again setting \(Y = Z = \phi X_i\) in (3.10) we have

\[ \sum \phi R(\phi X, \phi^2 X_i)\phi^2 X_i = \Sigma R(X, \phi X_i)\phi X_i + 2\phi X g(X_i, X_i). \] (3.13)
Adding (3.12) and (3.13) and using the definition of Ricci operator, we calculate

\[ \phi(Q(\phi X) - R(\phi X, \xi)\xi) = QX - R(X, \xi)\xi + 4n\phi X. \]  \tag{3.14} \]

We can write from (1.16)

\[ R(\phi X, \xi)\xi = \phi X. \]  \tag{3.15} \]

Using (3.13) and (3.14)

\[ \phi(Q(\phi X)) = QX + 4n\phi X. \]  \tag{3.16} \]

Operating \( \phi \) on both sides and using (1.17)

\[ Q(\phi X) - \phi(QX) = 4n\phi X. \]  \tag{3.17} \]

By virtue of (3.17) theorem (3.1) is proved.

For the next proof we consider the symbol \( W_{ijk} \) where \( W_{ijk} \) denotes the difference \( \Gamma_{ijk} - \Gamma'_{ijk} \) of Christoffel symbols in an LP-Sasakian manifold [8]. In global notation we can write

\[ W(Y, Z) = (1 - a)[\eta(Y)\phi Y + \eta(Y)\phi Z] + \frac{1}{2}(1 - \frac{1}{a})[(\nabla_Y \eta)Z + (\nabla_Z \eta)Y]\xi, \]  \tag{3.18} \]

for all \( Y, Z \in \chi(M) \). We state our next theorem.

**Theorem 3.2** Under a D-homothetic deformation, the operator \( Q\phi - \phi Q \) of a LP-Sasakian manifold \( M(\phi, \xi, \eta, g) \) is conformal.

**Proof** If \( R \) and \( \overline{R} \) denote the curvature tensors of the LP-Sasakian manifold \( M(\phi, \xi, \eta, g) \) and \( M(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g}) \) respectively then we know from [8]

\[ \overline{R}(X, Y)Z = R(X, Y)Z + (\nabla_X W)(Y, Z) - (\nabla_Y W)(Z, X) \]
\[ + W(W(Y, Z), X) - W(W(Z, X), Y). \]  \tag{3.19} \]

Using (1.13) in (3.18) we calculate

\[ W(Y, Z) = (1 - a)[\eta(Y)\phi Y + \eta(Y)\phi Z] + (1 - \frac{1}{a})g(\phi Y, Z)\xi. \]  \tag{3.20} \]

Taking covariant differentiation w.r.t. \( X \) and after using (1.8), (3.2), we obtain,

\[ (\nabla_X W)(Y, Z) = (1 - a)[g(\phi X, Z)\phi Y + g(X, Y)\eta(Z)\xi + 2\eta(Z)\eta(Y)X \]
\[ + 4\eta(X)\eta(Y)\eta(Z)\xi + g(\phi X, Y)\phi Z + g(X, Z)\eta(Y)\xi] \]
\[ + (1 - \frac{1}{a})g(\phi Y, Z)\phi X. \]  \tag{3.21} \]

Using (3.21) in (3.19) we obtain

\[ \overline{R}(X, Y)Z = R(X, Y)Z + (1 - a)\eta(Y)g(X, Z)\xi \]
\[ + 2(1 - a)\eta(Y)(1 - a)g(X, Z)\phi Y + (1 - \frac{1}{a})g(\phi Z, Z)\phi X \]
\[ - (1 - a)g(Y, Z)\eta(X)\xi \]
\[ - 2(1 - a)\eta(X)\eta(Z)Y - (1 - a)g(\phi Y, Z)\phi X - (1 - \frac{1}{a})g(\phi Z, X)\phi Y. \]
\[ + (1 - a) \eta(Y) [(1 - \frac{1}{a}) g(\phi^2 Z, X) \xi] + (1 - a) \eta(Z) [(1 - a) \eta(X) \phi^2 Y \]
\[ + (1 - \frac{1}{a}) g(\phi^2 Y, X) \xi] + (1 - \frac{1}{a}) g(\phi Z, X) [- (1 - a) \phi X] \]
\[ - (1 - a) \eta(X) [(1 - \frac{1}{a}) g(\phi^2 Z, Y) \xi] - (1 - a) \eta(Z) [(1 - a) \eta(Y) \phi^2 X \]
\[ + (1 - \frac{1}{a}) g(\phi^2 X, Y) \xi] - (1 - \frac{1}{a}) g(\phi Z, X) [- (1 - a) \phi Y]. \]

From (3.22) we get
\[ a \mathcal{S}(Y, Z) = \mathcal{S}(Y, Z) + \frac{(1 - a)^2}{a}. \]

Using the properties of Ricci operator
\[ a \mathcal{Q} Y = \mathcal{Q} Y + \frac{(1 - a)^2}{a}. \]

Operating \( \phi = \phi \) on both sides from left hand side
\[ a \phi \mathcal{Q} Y = \phi \mathcal{Q} Y + \frac{(1 - a)^2}{a}. \]

Operating \( \phi = \phi \) on both sides from right hand side
\[ a \mathcal{Q} \phi Y = \mathcal{Q} \phi Y + \frac{(1 - a)^2}{a}. \]

Subtracting the above two equations we obtain
\[ a (\phi \mathcal{Q} - \mathcal{Q} \phi) = (\phi \mathcal{Q} - Q \phi). \]

The equation (3.24) proves our theorem. \( \square \)

We can also prove the following theorems as a consequence of D-homothetic deformation.

**Theorem 3.3** Under D-homothetic deformation, an \( \eta \)-Einstein LP-Sasakian manifold remains invariant.

**Proof** In an \( \eta \)-Einstein LP-Sasakian manifold [9]
\[ S(X, Y) = \frac{r}{n-1} - 1 \] \[ g(X, Y) + \frac{r}{n-1} - 1 \eta(X) \eta(Y). \]

Under D-homothetic deformation we get
\[ \mathcal{S}(X, Y) = a \left( \frac{r}{n-1} - 1 \right) g(X, Y) + a (a - 1) \left( \frac{r}{n-1} - 1 \right) + a^2 \left( \frac{r}{n-1} - 1 \right) \eta(X) \eta(Y). \]

Hence the result is proved. \( \square \)

**Theorem 3.4** Under D-homothetic deformation, the \( \phi \)-sectional curvature of a LP-Sasakian manifold is conformal.

**Proof** Putting \( Y = \phi X, Z = X \) in (3.12) and taking inner product with \( \phi X \), we obtain on using (1.4) and the orthogonality property we get
\[ ag(\mathcal{R}(X, \phi X) X, \phi X) = g(R(X, \phi X) X, \phi X) + (a - \frac{1}{a}). \]

\[ (3.25) \]
\[ a\overline{K}(X,\phi X) - K(X,\phi X) = (a - \frac{1}{a}) \]

\[ \Box \]

**Theorem 3.5** There exists LP-Sasakian manifold with non-zero and non-constant \( \phi \)-sectional curvature.

**Proof** If the LP-Sasakian manifold satisfies \( R(X,Y)\xi = 0 \), then it can be proved easily that \( K(X,\phi X) = 0 \) and therefore from (3.25) we can conclude that \( \overline{K}(X,\phi X) \neq 0 \) for \( a \neq 1 \) where \( X \) is a unit vector field orthogonal to \( \xi \). Hence the result is proved. \( \Box \)

§4. An Example of a LP-Sasakian Manifold

In this section we shall prove the equality (3.25) by taking an example of LP-Sasakian manifold [1]. Let us consider a 5-dimensional manifold \( \tilde{M} = \{(x,y,z,u,v) \in R^5 : (x,y,z,u,v) \neq (0,0,0,0)\} \) where \( (x,y,z,u,v) \) are the standard coordinate in \( R^5 \). The vector fields

\[
\begin{align*}
e_1 &= -2\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \\
e_2 &= \frac{\partial}{\partial y}, \\
e_3 &= \frac{\partial}{\partial z}, \\
e_4 &= -2\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial z}, \\
e_5 &= \frac{\partial}{\partial v}
\end{align*}
\]

are linearly independent at each point of \( \tilde{M} \). Let \( g \) be the Lorentzian metric defined by

\[
\begin{align*}
g(e_i, e_j) &= 1, \text{ for } i = j \neq 3, \\
g(e_i, e_j) &= 0, \text{ for } i \neq j, \\
g(e_3, e_3) &= -1.
\end{align*}
\]

Here \( i \) and \( j \) runs from 1 to 5. Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \), for any vector field \( Z \) tangent to \( \tilde{M} \). Let \( \phi \) be the \((1,1)\) tensor field defined by

\[
\begin{align*}
\phi e_1 &= e_2, \\
\phi e_2 &= e_1, \\
\phi e_3 &= 0, \\
\phi e_4 &= e_5, \\
\phi e_5 &= e_4.
\end{align*}
\]

Then using the linearity of \( \phi \) and \( g \) we have

\[
\eta(e_3) = -1, \quad \phi^2 Z = Z + \eta(Z)e_3,
\]

for any vector fields \( Z, W \) tangent to \( \tilde{M} \). Thus for \( e_3 = \xi \), \( \tilde{M}(\phi, \xi, \eta, g) \) forms a LP-Sasakian manifold.

Let \( \nabla \) be the Levi-Civita connection on \( \tilde{M} \) with respect to the metric \( g \). Then the followings can be obtained

\[
[e_1, e_2] = -2e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.
\]

On taking \( e_3 = \xi \) and using Koszul’s formula for the metric \( g \), we calculate

\[
\begin{align*}
\nabla_{e_1} e_3 &= e_2, \\
\nabla_{e_2} e_3 &= -e_3, \\
\nabla_{e_3} e_1 &= 0, \\
\nabla_{e_2} e_3 &= e_1, \\
\nabla_{e_2} e_2 &= 0, \\
\nabla_{e_2} e_1 &= e_3, \\
\nabla_{e_3} e_3 &= 0, \\
\nabla_{e_3} e_2 &= e_1, \\
\nabla_{e_3} e_1 &= e_2.
\end{align*}
\]

Using the above relations, we can easily calculate the non-vanishing components of the curvature.
tensor as follows:

\[ R(e_1, e_2)e_2 = 3e_1, \quad R(e_1, e_2)e_1 = 3e_2, \quad R(e_2, e_3)e_3 = -e_2, \]
\[ R(e_1, e_3)e_2 = 0, \quad R(e_1, e_3)e_1 = -e_3, \quad R(e_2, e_3)e_2 = e_3, \]
\[ R(e_1, e_2)e_3 = 0. \]

In equation (3.22) we put \( X = e_1, Y = \phi e_1, Z = e_1. \) Taking inner product with \( \phi e_1 \) we obtain

\[ aK(e_1, \phi e_1) - K(e_1, \phi e_1) = a - \frac{1}{a}. \]

Hence, by this example Theorem 3.4 is verified.

References