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Decomposition of Tensor Product of Complete Graphs into Connected Unicyclic Bipartite Graphs with Eight Edges

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Abstract: In this paper, we obtain necessary and sufficient conditions for decomposing tensor product of complete graphs into some connected unicyclic bipartite graphs with eight edges.

Key Words: Decomposition, Smarandache decomposition, wreath product, tensor product, unicyclic graph.

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§1. Introduction

All the graphs considered here are loopless and finite. For a given graph G and an integer $\lambda \geq 1$, we use the notation $G(\lambda)$ to represent the multigraph obtained from G by replacing each of its edges with λ parallel edges. Similarly, λG denotes the graph consisting of λ edge-disjoint copies of G. The notations P_t, C_t, K_t, and $\overline{K}_t$ represents the path, cycle, complete graph, and complement of the complete graph, each with t vertices, respectively. Also, we denote the induced subgraph H of G induced by S as $\langle S\rangle$. Consider a *complete bipartite graph* $K_{t,t}$ with bipartition (X, Y), where $X = \{x_0, x_1, \cdots, x_{t-1}\}$ and $Y = \{y_0, y_1, \cdots, y_{t-1}\}$. We define the spanning subgraph $F_i(X, Y)$ of $K_{t,t}$ as $\langle\{x_j y_{j+i} : 0 \leq j \leq t-1\}\rangle$, where addition in the subscripts are taken modulo t. It is clear that $F_i(X, Y)$ is a 1-factor of $K_{t,t}$ with a distance i from X to Y. Moreover, $K_{t,t} = \bigoplus_{i=0}^{t-1} F_i(X, Y)$, where $\oplus$ denotes the edge-disjoint union of graphs, also called a *Smarandache decomposition* if $K_{t,t}$ is labeled.

For two graphs G and H, their *lexicographic product* $G \otimes H$ has the vertex set $V(G \otimes H) = V(G) \times V(H)$ and the edge set $E(G \otimes H) = \{(g_1, h_1)(g_2, h_2) : g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)\}$. Similarly, the *tensor product* $G \times H$ of two graphs G and H has the vertex set $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H) = \{(g_1, h_1)(g_2, h_2) : g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)\}$. Note that, the tensor product is commutative and distributive over edge-disjoint union of graphs, that is, if $G = G_1 \oplus G_2 \oplus \cdots \oplus G_u$, then $G \times H = (G_1 \times H) \oplus \cdots \oplus (G_u \times H)$. One can easily observe that $(K_u \otimes \overline{K}_g) - gK_u \cong K_u \times K_g$, where gK_u denotes g disjoint copies of K_u.

For some integer $r \geq 1$, we say that the graph G has a *decomposition* into the subgraphs

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$G_1, G_2, \cdots, G_r$ if $G = \oplus_{i=1}^{r} G_i$, and $G_1, G_2, \cdots, G_r$ are pairwise edge-disjoint subgraphs of G. For each i, $1 \le i \le r$, if $G_i \cong H$, then we say that G has an H-*decomposition* and we denote such decomposition by $H|G$. A graph G is said to be *unicyclic* if it has exactly one cycle.

Decomposition of graphs into subgraphs has been an interesting research area in graph theory since 1950s. Adams et al. [1] published an excellent survey on decomposing complete graphs into subgraphs containing up to six vertices. Tian et al. [17] established the decomposition of complete graphs into unicyclic graphs with six vertices and seven edges, while Froncek et al. [10] proved the decomposition of complete graphs into connected unicyclic bipartite graphs with seven edges. In recent studies, Froncek et al. [11,12] proved the decomposition of complete graphs into tri-cyclic and bi-cyclic graphs, each with eight edges. Furthermore, Fahnenstiel et al. [5] established the necessary and sufficient conditions for the existence of a decomposition of complete graphs into connected unicyclic bipartite graphs with eight edges. Huang et al. [13] proved the decomposition of complete equipartite graphs into connected unicyclic graphs, each having a size of five vertices. Similarly, Paulraja et al. [14] established the decomposition of certain regular graphs into unicyclic graphs of order five. Sowndhariya et al. [15] proved the decomposition of product graphs into sunlet graphs of order eight. Aspenson et al. [3], proved the decomposition K_{18n} and K_{18n+1} into connected unicyclic graphs with nine edges. Similarly, Bonhert et al. [4], proved the decompositions of complete graphs into unicyclic disconnected bipartite graphs with nine edges. Recently, we have proved the existence of decomposition of λ-fold complete equipartite graphs into connected unicyclic bipartite graphs with eight edges in [6] and the general problem is open for other classes of product of graphs. In this paper, we show the existence of such decomposition in tensor product of complete graphs.

Let G_1, G_2, G_3, G_4 and G_5 be the graphs shown in Figure 1. We assume that these graphs have the vertex set $\{v_1, v_2, \cdots, v_8\}$. The edge set of the unicyclic graphs G_1, G_2, G_3, G_4, and G_5 are denoted by $(v_1v_2v_3v_4)$ $[v_1v_5v_6v_7v_8]$, $(v_1v_2v_3v_4)$ $[v_1v_5v_7v_8]$ $[v_5v_6]$, $(v_1v_2v_3v_4)$ $[v_2v_6v_7v_8]$ $[v_1v_5]$, $(v_1v_2v_3v_4)$ $[v_1v_5v_6]$ $[v_3v_7v_8]$, and $(v_1v_2v_3v_4)[v_1v_5v_6]$ $[v_4v_7]$ $[v_3v_8]$, respectively. Clearly, each G_i, $1 \le i \le 5$, is a connected unicyclic bipartite graph with eight edges.

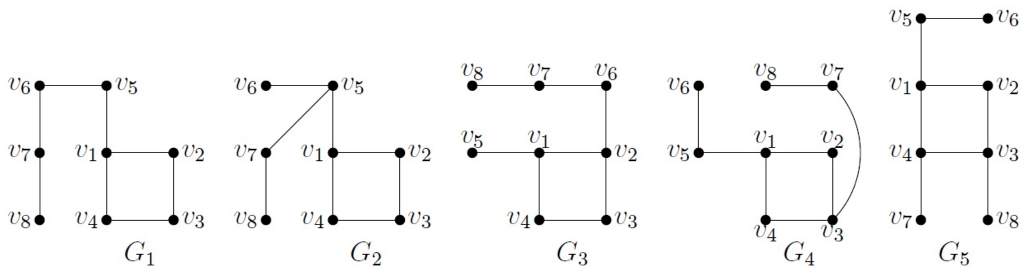

Figure 1. Connected unicyclic bipartite graphs with eight edges

To prove our results we state the following:

Theorem 1.1([16]) *There exists a P_{m+1}-decomposition of $K_u(\lambda)$ if and only if $\lambda u(u-1) \equiv 0 \pmod{2m}$, $u \ge m+1$.*

Theorem 1.2([2]) *For all positive odd integers m and n with $3 \le m \le n$, there exists a C_m-decomposition of K_n if and only if $n(n-1) \equiv 0 \pmod{2m}$.*

Theorem 1.3([6]) *There exists a G_i-decomposition of $K_{4x,4y}$, $1 \le i \le 5$.*

§2. G_i-Decomposition of Base Graphs

In this part, we have established some crucial lemmas to prove our main results.

Lemma 2.1 *The graphs $K_{4,2}$, $K_{4,4}$ and $K_{4,6}$ admits a P_3-decomposition.*

Proof Our proof is divided into two cases.

Case 1. $P_3|K_{4,4}$

Let $V(K_{4,4}) = (U, V)$, where $U = \{u_0, u_1, u_2, u_3\}$ and $V = \{v_0, v_1, v_2, v_3\}$. Let $P_3^{j,1} = [v_j u_j v_{j+1}]$ and $P_3^{j,2} = [u_j v_{j+2} u_{j+3}]$, $j \in \mathbb{Z}_4$ and additions in the subscripts of u and v are taken modulo 4. When j varies, $\{P_3^{j,1}, P_3^{j,2}\}$ gives a required P_3-decomposition of $K_{4,4}$.

Case 2. $P_3|K_{4,6}$

Let $V(K_{4,4}) = (U, V)$, where $U = \{u_0, u_1, u_2, u_3\}$ and $V = \{v_0, v_1, \cdots, v_5\}$. Let $P_3^{j,1} = [v_j u_j v_{j+1}]$, $P_3^{j,2} = [u_j v_{j+2} u_{j+3}]$, and $P_3^{j,3} = [v_4 u_j v_5]$, $j \in \mathbb{Z}_4$ and additions in the subscripts of u and v are taken modulo 4. When j varies, $\{P_3^{j,1}, P_3^{j,2}, P_3^{j,3}\}$ gives a required P_3-decomposition of $K_{4,6}$. □

Lemma 2.2 *There exists a G_i-decomposition of $P_3 \times K_5$, $1 \le i \le 5$.*

Proof Let $V(P_3 \times K_5) = \cup_{i \in \mathbb{Z}_3} X_i$, where $X_0 = \{u_0, u_1, \cdots, u_4\}$, $X_1 = \{v_0, v_1, \cdots, v_4\}$ and $X_2 = \{w_0, w_1, \cdots, w_4\}$. The required G_i-decomposition of $P_3 \times K_5$ is shown below.

Let $G_1^j = (u_{j+1} v_{j+3} u_{j+4} v_{j+2})[v_{j+2} w_{j+3} v_{j+4} w_{j+2} v_j]$,
$G_2^j = (u_{j+1} v_{j+3} u_{j+4} v_{j+2})[v_{j+2} w_{j+3} v_{j+4} w_{j+2}][w_{j+3} v_{j+1}]$,
$G_3^j = (u_{j+1} v_{j+3} w_{j+1} v_{j+2})[u_{j+1} v_j w_{j+2} v_{j+1}][v_{j+3} u_j]$,
$G_4^j = (u_{j+1} v_{j+3} u_{j+4} v_{j+2})[v_{j+3} w_j v_{j+1}][v_{j+2} w_{j+3} v_j]$, and
$G_5^j = (u_{j+1} v_{j+3} w_{j+1} v_{j+2})[w_{j+1} v_{j+4} w_j][v_{j+2} u_{j+4}][u_{j+1} v_j], j \in \mathbb{Z}_5$, where the additions

in the subscripts of u, v, and w are taken modulo 5. Clearly, $G_i^j \cong G_i, i = 1, 2, 3, 4, 5$, $j \in \mathbb{Z}_5$ shown in Figure 1. When j varies we get the required decomposition of $P_3 \times K_5$. □

Lemma 2.3 *There exists a G_i-decomposition of $P_3 \times K_8$, $1 \le i \le 5$.*

Proof Let $V(P_3 \times K_8) = \cup_{i \in \mathbb{Z}_3} X_i$, where $X_0 = \{u_0, u_1, \cdots, u_7\}$, $X_1 = \{v_0, v_1, \cdots, v_7\}$ and $X_2 = \{w_0, w_1, \cdots, w_7\}$. The required G_i-decomposition of $P_3 \times K_8$ is shown below.

Let $G_1^{j,1} = (u_{j+5} v_7 w_{j+6} v_{j+4})[w_{j+6} v_{j+3} u_{j+2} v_j w_7]$,
$G_1^{j,2} = (u_j v_{j+2} w_{j+1} v_{j+3})[u_j v_{j+4} w_{j+5} v_{j+1} u_7]$,
$G_2^{j,1} = (u_{j+6} v_7 w_{j+6} v_{j+5})[v_{j+5} w_{j+4} v_{j+2} u_7][w_{j+4} v_{j+1}]$,
$G_2^{j,2} = (u_{j+5} v_j w_{j+5} v_{j+1})[v_{j+1} u_j v_{j+5} w_7][u_j v_{j+4}]$,

$G_3^{j,1} = (u_{j+6}v_7w_{j+6}v_{j+5})[u_{j+6}v_{j+4}w_{j+3}v_{j+1}][v_{j+5}u_7]$,
$G_3^{j,2} = (u_{j+5}v_jw_{j+5}v_{j+1})[v_{j+1}u_jv_{j+4}w_7][w_{j+5}v_{j+2}]$,
$G_4^{j,1} = (u_{j+6}v_7w_{j+6}v_{j+5})[u_{j+6}v_{j+4}u_7][w_{j+6}v_{j+3}w_7]$,
$G_4^{j,2} = (u_jv_{j+2}w_jv_{j+1})[u_jv_{j+3}u_{j+6}][w_jv_{j+5}w_{j+2}]$,
$G_5^{j,1} = (u_{j+6}v_7w_{j+6}v_{j+5})[w_{j+6}v_{j+4}u_7][v_{j+5}w_7][u_{j+6}v_{j+3}]$ and
$G_5^{j,2} = (u_jv_{j+2}w_jv_{j+1})[u_jv_{j+3}u_{j+5}][v_{j+1}w_{j+5}][w_jv_{j+4}]$, $j \in \mathbb{Z}_7$, where the additions in

the subscripts of u, v, and w are taken modulo 7. Clearly, $G_i^{j,l} \cong G_i, i = 1,2,3,4,5, \ j \in \mathbb{Z}_7, l \in \{1,2\}$.When j and l varies, we get the required decomposition of $P_3 \times K_8$. □

Lemma 2.4 *There exists a G_1-decomposition of $P_3 \times K_{12}$.*

Proof Let $V(P_3 \times K_{12}) = \cup_{i\in\mathbb{Z}_3} X_i$, where $X_0 = \{u_0, u_1, \cdots, u_{11}\}$, $X_1 = \{v_0, v_1, \cdots, v_{11}\}$ and $X_2 = \{w_0, w_1, \cdots, w_{11}\}$. The required G_1-decomposition of $P_3 \times K_{12}$ is given below.

Let $G_1^{j,1} = (u_{j+10}v_{11}w_{j+10}v_{j+9})[w_{j+10}v_{j+8}u_{j+7}v_{j+5}w_{11}]$,
$G_1^{j,2} = (u_{j+10}v_{j+6}w_{j+10}v_{j+7})[u_{j+10}v_{j+5}w_{j+4}v_{j+10}u_{11}]$ and
$G_1^{j,3} = (u_jv_{j+2}w_{j+9}v_{j+3})[u_jv_{j+5}w_{j+3}v_{j+6}u_{j+2}]$, $j \in \mathbb{Z}_{11}$, where the additions in the

subscripts of u, v, and w are taken modulo 11. Clearly, $G_1^{j,l} \cong G_1, j \in \mathbb{Z}_{11}, l \in \{1,2,3\}$. When j and l varies, we get the required decomposition of $P_3 \times K_{12}$. □

Lemma 2.5 *There exists a G_i-decomposition of $P_5 \times K_6$, $1 \le i \le 5$.*

Proof Let $V(P_5 \times K_6) = \cup_{i\in\mathbb{Z}_5} X_i$, where $X_0 = \{u_0, u_1, \cdots, u_5\}$, $X_1 = \{v_0, v_1, \cdots, v_5\}$, $X_2 = \{w_0, w_1, \cdots, w_5\}$, $X_3 = \{x_0, x_1, \cdots, x_5\}$, and $X_4 = \{y_0, y_1, \cdots, y_5\}$. The required G_i-decomposition of $P_5 \times K_6$ is given below.

Let $G_1^{j,1} = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_5w_{j+4}v_{j+3}u_5]$,
$G_1^{j,2} = (v_jw_{j+3}x_{j+1}w_{j+2})[w_{j+3}v_{j+4}w_5x_{j+4}y_5]$,
$G_1^{j,3} = (w_{j+4}x_5y_{j+4}x_{j+1})[y_{j+4}x_jy_{j+1}x_{j+4}w_{j+3}]$,
$G_2^{j,1} = (u_jv_{j+1}u_{j+3}v_{j+2})[v_{j+2}w_{j+3}v_5u_{j+4}][w_{j+3}x_{j+4}]$,
$G_2^{j,2} = (v_jw_{j+3}x_{j+1}w_{j+2})[w_{j+3}v_{j+4}w_5x_{j+4}][v_{j+4}u_5]$,
$G_2^{j,3} = (w_{j+4}x_5y_{j+4}x_{j+1})[y_{j+4}x_jy_{j+1}x_{j+4}][x_jy_5]$,
$G_3^{j,1} = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_5w_{j+3}x_{j+4}][v_{j+1}w_{j+2}]$,
$G_3^{j,2} = (v_jw_{j+3}x_{j+1}w_{j+2})[w_{j+3}v_{j+4}w_5x_{j+4}][v_ju_5]$,
$G_3^{j,3} = (w_{j+4}x_5y_{j+4}x_{j+1})[y_{j+4}x_jy_{j+1}x_{j+4}][x_{j+1}y_5]$,
$G_4^{j,1} = (u_{j+4}v_5w_{j+4}v_{j+3})[u_{j+4}v_{j+2}u_5][w_{j+4}v_{j+1}w_{j+3}]$,
$G_4^{j,2} = (v_{j+4}w_5x_{j+4}w_{j+3})[v_{j+4}u_{j+3}v_j][x_{j+4}y_{j+3}x_j]$,
$G_4^{j,3} = (w_{j+4}x_5y_{j+4}x_{j+3})[w_{j+4}x_{j+1}w_{j+3}][y_{j+4}x_{j+2}y_5]$,
$G_5^{j,1} = (u_{j+4}v_5w_{j+4}v_{j+3})[u_{j+4}v_{j+2}u_5][v_{j+3}w_j][w_{j+4}v_{j+1}]$,

$G_5^{j,2} = (v_{j+4}w_5x_{j+4}w_{j+3})[v_{j+4}u_{j+3}v_j][w_{j+3}x_{j+1}][x_{j+4}y_{j+3}]$ and
$G_5^{j,3} = (w_{j+4}x_5y_{j+4}x_{j+3})[y_{j+4}x_{j+2}y_5][x_{j+3}y_{j+1}][w_{j+4}x_{j+1}]$, $j \in \mathbb{Z}_5$, where the

additions in the subscripts of u, v, w, x, and y are taken modulo 5. Clearly, $G_i^{j,l} \cong G_i, i = 1, 2, 3, 4, 5,\ j \in \mathbb{Z}_5, l \in \{1, 2, 3\}$. When j and l varies, we get the required decomposition of $P_5 \times K_6$. □

Lemma 2.6 *There exists a G_i-decomposition of $K_9 \times K_2$, $1 \le i \le 5$.*

Proof Let $V(K_9 \times K_2) = (U, V)$, where $U = \{u_0, u_1, \cdots, u_8\}$ and $V = \{v_0, v_1, \cdots, v_8\}$. The required G_i-decomposition of $K_9 \times K_2$ is given below.

Let $G_1^j = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_{j+6}u_{j+2}v_{j+7}u_{j+1}]$,
$G_2^j = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_{j+6}u_{j+2}v_{j+8}][v_{j+6}u_{j+1}]$,
$G_3^j = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_{j+6}u_{j+2}v_{j+8}][v_{j+2}u_{j+6}]$,
$G_4^j = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_{j+6}u_{j+2}][u_jv_{j+5}u_{j+8}]$, and
$G_5^j = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_{j+6}u_{j+2}][v_{j+2}u_{j+5}][u_jv_{j+5}]$, $j \in \mathbb{Z}_9$, where the additions

in the subscripts of u and v are taken modulo 9. Clearly, $G_i^j \cong G_i, i = 1, 2, 3, 4, 5,\ j \in \mathbb{Z}_9$. When j varies, we get the required decomposition of $K_9 \times K_2$. □

Lemma 2.7 *There exists a G_i-decomposition of $C_3 \times K_8$, $1 \le i \le 5$.*

Proof Let $V(C_3 \times K_8) = \cup_{i\in\mathbb{Z}_3} X_i$, where $X_0 = \{u_0, u_1, \cdots, u_7\}$, $X_1 = \{v_0, v_1, \cdots, v_7\}$ and $X_2 = \{w_0, w_1, \cdots, w_7\}$. The required G_i-decomposition of $C_3 \times K_8$ is given below.

Let $G_1^{j,1} = (u_{j+5}v_{j+6}u_7w_{j+6})[u_{j+5}v_ju_{j+4}v_{j+1}u_{j+3}]$,
$G_1^{j,2} = (u_{j+6}v_7w_{j+6}v_{j+5})[v_{j+5}w_jv_{j+4}w_{j+1}v_{j+3}]$,
$G_1^{j,3} = (u_{j+6}w_7v_{j+6}w_{j+5})[w_{j+5}u_jw_{j+4}u_{j+1}w_{j+3}]$,
$G_2^{j,1} = (u_{j+5}v_{j+6}u_7w_{j+6})[u_{j+5}v_ju_{j+4}v_{j+1}][v_ju_{j+2}]$,
$G_2^{j,2} = (u_{j+6}v_7w_{j+6}v_{j+5})[v_{j+5}w_jv_{j+4}w_{j+1}][w_jv_{j+2}]$,
$G_2^{j,3} = (u_{j+6}w_7v_{j+6}w_{j+5})[w_{j+5}u_jw_{j+4}u_{j+1}][u_jw_{j+2}]$,
$G_3^{j,1} = (u_{j+5}v_{j+6}u_7w_{j+6})[u_{j+5}v_ju_{j+4}v_{j+1}][v_{j+6}w_{j+4}]$,
$G_3^{j,2} = (u_{j+6}v_7w_{j+6}v_{j+5})[v_{j+5}w_jv_{j+4}w_{j+1}][w_{j+6}u_{j+4}]$,
$G_3^{j,3} = (u_{j+6}w_7v_{j+6}w_{j+5})[w_{j+5}u_jw_{j+4}u_{j+1}][u_{j+6}v_{j+4}]$,
$G_4^{j,1} = (u_{j+5}v_{j+6}u_7w_{j+6})[v_{j+6}w_{j+1}v_{j+5}][w_{j+6}u_{j+1}w_{j+5}]$,
$G_4^{j,2} = (u_{j+6}v_7w_{j+6}v_{j+5})[u_{j+6}v_{j+1}u_{j+5}][w_{j+6}u_{j+4}w_j]$,
$G_4^{j,3} = (u_{j+6}w_7v_{j+6}w_{j+5})[u_{j+6}v_{j+3}u_{j+5}][v_{j+6}w_{j+3}v_{j+5}]$,
$G_5^{j,1} = (u_{j+5}v_{j+6}u_7w_{j+6})[w_{j+6}u_{j+3}w_{j+5}][u_{j+5}v_j][v_{j+6}w_{j+1}]$,
$G_5^{j,2} = (u_{j+6}v_7w_{j+6}v_{j+5})[u_{j+6}v_{j+2}u_{j+5}][v_{j+5}w_{j+1}][w_{j+6}u_{j+2}]$ and
$G_5^{j,3} = (u_{j+6}w_7v_{j+6}w_{j+5})[v_{j+6}w_{j+2}v_{j+5}][w_{j+5}u_j][u_{j+6}v_{j+4}]$, $j \in \mathbb{Z}_7$,

where the additions in the subscripts of u, v, and w are taken modulo 7. Clearly, $G_i^{j,l} \cong G_i, i = 1,2,3,4,5,\ j \in \mathbb{Z}_7, l \in \{1,2,3\}$. When j and l varies, we get the required decomposition of $C_3 \times K_8$. □

Lemma 2.8 *There exists a G_i-decomposition of $K_4 \times K_4,\ 2 \le i \le 5$.*

Proof Let $V(K_4 \times K_4) = \cup_{i \in \mathbb{Z}_4} X_i$, where $X_0 = \{u_0, u_1, u_2, u_3\},\ X_1 = \{v_0, v_1, v_2, v_3\},\ X_2 = \{w_0, w_1, w_2, w_3\}$, and $X_3 = \{x_0, x_1, x_2, x_3\}$. The required G_i-decomposition of $K_4 \times K_4$ is given below.

$$\begin{aligned}
\text{Let}\quad G_2^{j,1} &= (u_3v_{j+2}w_3x_{j+2})[u_3w_{j+1}x_3u_{j+1}][w_{j+1}u_j],\\
G_2^{j,2} &= (u_jv_{j+1}w_jx_{j+1})[w_ju_{j+1}v_3w_{j+2}][u_{j+1}w_3],\\
G_2^{j,3} &= (u_{j+1}v_jw_{j+1}x_j)[v_jx_{j+1}v_{j+2}x_3][x_{j+1}v_3],\\
G_3^{j,1} &= (u_3v_{j+2}w_3x_{j+2})[v_{j+2}x_3w_{j+2}v_3][u_3w_{j+1}],\\
G_3^{j,2} &= (u_jv_{j+1}w_jx_{j+1})[v_{j+1}x_{j+2}v_3u_{j+2}][u_jw_3],\\
G_3^{j,3} &= (u_{j+1}v_jw_{j+1}x_j)[u_{j+1}w_ju_{j+2}x_3][v_jx_{j+2}],\\
G_4^{j,1} &= (u_3v_{j+2}w_3x_{j+2})[u_3w_{j+1}u_j][w_3u_{j+1}x_3],\\
G_4^{j,2} &= (u_jv_{j+1}w_jx_{j+1})[u_jw_{j+2}x_3][w_jv_3u_{j+2}],\\
G_4^{j,3} &= (u_{j+1}v_jw_{j+1}x_j)[v_jx_{j+1}v_3][x_jv_{j+1}x_3],\\
G_5^{j,1} &= (u_3v_{j+2}w_3x_{j+2})[u_3w_{j+1}u_j][v_{j+2}x_{j+1}][w_3u_{j+2}],\\
G_5^{j,2} &= (u_jv_{j+1}w_jx_{j+1})[w_jx_3u_{j+2}][v_{j+1}x_{j+2}][u_jv_3] \text{ and}\\
G_5^{j,3} &= (u_{j+1}v_jw_{j+1}x_j)[w_{j+1}v_3x_{j+2}][v_jx_3][u_{j+1}w_j],\ j \in \mathbb{Z}_3,
\end{aligned}$$

where the additions in the subscripts of u, v, w, and x are taken modulo 3. Clearly, $G_i^{j,l} \cong G_i, i = 1,2,3,4,5,\ j \in \mathbb{Z}_3, l \in \{1,2,3\}$. When j and l varies, we get the required decomposition of $K_4 \times K_4$. □

Lemma 2.9 *For $g \equiv 0 \pmod 8$, there exists a G_i-decomposition of $K_6 \times K_g,\ 1 \le i \le 5$.*

Proof Let $g = 8x,\ x \ge 1$. We can write $K_{8x} = (K_x \otimes \overline{K}_8) \oplus xK_8 = \binom{x}{2}(K_2 \otimes \overline{K}_8) \oplus xK_8 \cong \binom{x}{2}K_{8,8} \oplus xK_8$ and hence $K_{8x} \times K_6 = \binom{x}{2}(K_{8,8} \times K_6) \oplus x(K_8 \times K_6) = 15\binom{x}{2}(K_{8,8} \times K_2) \oplus x(K_8 \times K_6)$. By Theorem 1.3, $G_i | K_{8,8}$, since G_i is bipartite, $G_i \times K_2 = 2G_i$. By Theorem 1.1, $P_5 | K_8$ and hence $G_i | P_5 \times K_6$ by Lemma 2.5. Therefore, the graph $K_6 \times K_{8x}$ has a required G_i-decomposition. □

Lemma 2.10 *For $g \equiv 0 \pmod 8$, there exists a G_i-decomposition of $P_3 \times K_g,\ 1 \le i \le 5$.*

Proof Let $g = 8x,\ x \ge 1$. We can write $P_3 \times K_{8x} = ((P_3 \times K_x) \otimes \overline{K}_8) \oplus x(P_3 \times K_8) = ((P_3 \times \binom{x}{2}K_2) \otimes \overline{K}_8) \oplus x(P_3 \times K_8) = \left(\binom{x}{2}(P_3 \times K_2) \otimes \overline{K}_8\right) \oplus x(P_3 \times K_8) = 4\binom{x}{2}(K_2 \otimes \overline{K}_8) \oplus x(P_3 \times K_8) = 4\binom{x}{2}K_{8,8} \oplus x(P_3 \times K_8)$. By Theorem 1.3 and Lemma 2.3, the graph $P_3 \times K_{8x}$ has a required G_i-decomposition. □

Lemma 2.11 *For $u \equiv 0, 4 \pmod 8$ and $g \equiv 0 \pmod 4$, G_1-decomposition of $K_u \times K_g$ exists.*

Proof Let $u = 8x + t,\ x \ge 1$ and $t \in \{0, 4\}$. We can write $K_{8x+t} = K_{8+t} \oplus (x-1)K_8 \oplus$

$(x-1)K_{8,8+t} \oplus (K_{x-1} \otimes \overline{K}_8) = K_{8+t} \oplus (x-1)K_8 \oplus (x-1)K_{8,8+t} \oplus (\binom{x-1}{2}(K_2 \otimes \overline{K}_8)) = K_{8+t} \oplus (x-1)K_8 \oplus (x-1)K_{8,8+t} \oplus \binom{x-1}{2}K_{8,8}$. By Theorem 1.1, $P_3|K_g$ and $G_1|K_{8+t} \times P_3$, by Lemmas 2.3 and 2.4. By Theorem 1.3, $G_1|K_{8,8+t}$ and hence $G_1 \times K_g = G_1 \times (\binom{g}{2}K_2) = \binom{g}{2}(G_1 \times K_2)$, since G_1 is bipartite, $G_1 \times K_2 = 2G_1$. Therefore, G_1-decomposition of $K_u \times K_g$ exists. □

§3. G_i-Decomposition of $K_u \times K_g$

Theorem 3.1 *Let $u, g \geq 4$. For $1 \leq i \leq 5$, $G_i|K_u \times K_g$ if and only if $ug(u-1)(g-1) \equiv 0 \pmod{16}$, except possibly $(u, g, G_i) = (4, 4, G_1)$.*

Proof *Necessity:* The number of edges in $K_u \times K_g$ are $\binom{u}{2}(g^2 - g)$ and G_i has 8 edges. If $G_i|K_u \times K_g$, then $8|\binom{u}{2}(g^2 - g)$. Hence $ug(u-1)(g-1) \equiv 0 \pmod{16}$.

Sufficiency: To prove the sufficiency, from the edge divisibility condition, it is enough to discuss the following cases.

- $u \equiv 0 \pmod 4$ and $g \equiv 0 \pmod 4$;
- $u \equiv 0 \pmod 4$ and $g \equiv 1 \pmod 4$;
- $u \equiv 2 \pmod 4$ and $g \equiv 0 \pmod 8$;
- $u \equiv 2 \pmod 4$ and $g \equiv 1 \pmod 8$;
- $u \equiv 3 \pmod 4$ and $g \equiv 0 \pmod 8$;
- $u \equiv 3 \pmod 4$ and $g \equiv 1 \pmod 8$;
- $u \equiv 1 \pmod 4$ and $g \equiv 1 \pmod 4$.

Case 1. $u \equiv 0 \pmod 4$ and $g \equiv 0 \pmod 4$

By Lemma 2.11, $G_1|K_u \times K_g$ exists and hence it is enough to prove $G_i|K_u \times K_g$, $2 \leq i \leq 5$. Let $u = 4x$ and $g = 4y$, $x, y \geq 1$. We can write $K_{4x} = (K_x \otimes \overline{K}_4) \oplus xK_4 = \binom{x}{2}(K_2 \otimes \overline{K}_4) \oplus xK_4 = \binom{x}{2}K_{4,4} \oplus xK_4$ and $K_{4y} = \binom{y}{2}K_{4,4} \oplus yK_4$. Then $K_{4x} \times K_{4y} = \big(\binom{x}{2}K_{4,4} \oplus xK_4\big) \times \big(\binom{y}{2}K_{4,4} \oplus yK_4\big) = \binom{x}{2}\binom{y}{2}(K_{4,4} \times K_{4,4}) \oplus y\binom{x}{2}(K_{4,4} \times K_4) \oplus x\binom{y}{2}(K_{4,4} \times K_4) \oplus xy(K_4 \times K_4) = 16\binom{x}{2}\binom{y}{2}(K_{4,4} \times K_2) \oplus 6y\binom{x}{2}(K_{4,4} \times K_2) \oplus 6x\binom{y}{2} \oplus (K_{4,4} \times K_2) \oplus xy(K_4 \times K_4)$. By Theorem 1.3, $G_i|K_{4,4}$ since G_i is bipartite, $G_i \times K_2 = 2G_i$. By Lemma 2.8, $G_i|K_4 \times K_4$, $2 \leq i \leq 5$. Therefore, the graph $K_{4x} \times K_{4y}$ has a required G_i-decomposition.

Case 2. $u \equiv 0, 1 \pmod 4$ and $g \equiv 1 \pmod 4$

Let $g = 4x + 1$, $x \geq 1$. We can write $K_{4x+1} = (K_x \otimes \overline{K}_4) \oplus xK_5 = \binom{x}{2}K_{4,4} \oplus xK_5$ and hence $K_u \times K_{4x+1} = \binom{x}{2}(K_u \times K_{4,4}) \oplus x(K_u \times K_5) = \binom{u}{2}\binom{x}{2}(K_2 \times K_{4,4}) \oplus x(K_u \times K_5)$. By Theorem 1.3, $G_i|K_{4,4}$, since G_i is bipartite, $G_i \times K_2 = 2G_i$. By Theorem 1.1, $P_3|K_u$ and hence $G_i|P_3 \times K_5$ by Lemma 2.2. Therefore, the graph $K_u \times K_{4x+1}$ has a required G_i-decomposition.

Case 3. $u \equiv 2 \pmod 4$ and $g \equiv 0 \pmod 8$

Let $u = 4x+2$, $x \geq 1$. We can write $K_{4x+2} = K_6 \oplus (x-1)K_4 \oplus (x-1)K_{4,6} \oplus (K_{x-1} \otimes \overline{K}_4) = K_6 \oplus (x-1)K_4 \oplus (x-1)K_{4,6} \oplus \binom{x-1}{2}K_{4,4}$ and hence $K_{4x+2} \times K_g = (K_6 \times K_g) \oplus (x-1)(K_4 \times K_g) \oplus (x-1)(K_{4,6} \times K_g) \oplus \binom{x-1}{2}(K_{4,4} \times K_g)$. By Lemma 2.9, the graph $K_6 \times K_g$ has a G_i-decomposition. By Theorem 1.1 and Lemma 2.1, $P_3|K_4$, $P_3|K_{4,4}$, and $P_3|K_{4,6}$ and hence $G_i|P_3 \times K_g$ by Lemma 2.10. Therefore, the graph $K_{4x+2} \times K_g$ has a required G_i-decomposition.

Case 4. $u \equiv 3 \pmod 4$ and $g \equiv 0 \pmod 8$

Let $u = 4x+3,\ x \geq 1$. We can write $K_{4x+3} = K_7 \oplus (x-1)K_5 \oplus (x-1)K_{4,6} \oplus (K_{x-1} \otimes \overline{K}_4) = K_7 \oplus (x-1)K_5 \oplus (x-1)K_{4,6} \oplus \binom{x-1}{2}K_{4,4}$ and hence $K_{4x+3} \times K_g = (K_7 \times K_g) \oplus (x-1)(K_5 \times K_g) \oplus (x-1)(K_{4,6} \times K_g) \oplus \binom{x-1}{2}(K_{4,4} \times K_g)$. By Theorem 1.2, $C_3|K_7$ and the graphs $K_5,\ K_{4,2},\ K_{4,4}$ has P_3-decomposition by Theorem 1.1 and Lemma 2.1. Then by Lemmas 2.7 and 2.3, the graphs $C_3 \times K_g$ and $P_3 \times K_g$ has G_i-decomposition. Therefore, the graph $K_{4x+3} \times K_g$ has a required G_i-decomposition.

Case 5. $u \equiv 2, 3 \pmod 4$ and $g \equiv 1 \pmod 8$

Let $g = 8x+1,\ x \geq 1$. We can write $K_{8x+1} = (K_x \otimes \overline{K}_8) \oplus xK_9 = \binom{x}{2}K_{8,8} \oplus xK_9$ and hence $K_u \times K_{8x+1} = \binom{x}{2}(K_u \times K_{4,4}) \oplus x(K_u \times K_9) = \binom{u}{2}\binom{x}{2}(K_2 \times K_{8,8}) \oplus x\binom{u}{2}(K_2 \times K_9)$. By Theorem 1.3, $G_i|K_{8,8}$, since G_i bipartite, $G_i \times K_2 = 2G_i$ and by Lemma 2.6, $G_i|K_9 \times K_2$. Therefore, the graph $K_u \times K_{8x+1}$ has a required G_i-decomposition. □

References

[1] P. Adams, D. Bryant, and M. Buchanan, A Survey on the Existence of G-Designs, *J. Combin. Des.*, 16 (2008), 373-410.

[2] B. Alspach and H. Gavlas, Cycle decompositions of K_n and K_{n-I}, *J. Combin. Theory Ser. B*, 81(2001), 77-99.

[3] G. Aspenson, D. Baker, B. Freyberg, and C. Schwieder, Decomposing K_{18n} and K_{18n+1} into connected unicyclic graphs with 9 edges, *Electron. J. Graph Theory Appl.* 11(1) (2023), 273 – 316.

[4] A. Bonhert, L. Branson, and PJ. Otto, On decompositions of complete graphs into unicyclic disconnected bipartite graphs on nine edges, *Electron. J. Graph Theory Appl*, 11(1) (2023), 329-341.

[5] J. Fahnenstiel and D. Froncek, Decomposition of complete graphs into connected unicyclic bipartite graphs with eight edges, *Electron. J. Graph Theory Appl*, 7(2)(2019), 235-250.

[6] S. Duraimurugan, S. Ganesamurthy, and A. Muthusamy, Decomposition of λ-fold complete equipartite graphs into connected unicyclic bipartite graphs with eight edges (Submitted).

[7] B. Freyberg and D. Froncek, Decomposition of complete graps into unicyclic graphs with eight edges, *J. Combin. Math. Combin. Comput.,* 114 (2020), 113-132.

[8] B. Freyberg and N. Tran, Decomposition of complete graps into unicyclic bipartite graphs with eight edges, *J. Combin. Math. Combin. Comput.,* 114 (2020), 133-142.

[9] D. Froncek, and O. Kingston, Decomposition of complete graphs into connected unicyclic graphs with eight edges and pentagon, *Indonesian J. Combin.*, 3(1) (2019), 24-33.

[10] D. Froncek and M. Kubesa, Decomposition of complete graphs into connected unicyclic bipartite graphs with seven edges, *Bull. Inst. Combin. Appl.*, 93 (2021), 52-80.

[11] D. Froncek and B. Kubik, Decomposition of complete graphs into tri-cyclic graphs with eight edges, *J. Combin. Math. Combin. Comput.,* 114 (2020), 143-166.

[12] D. Froncek and J. Lee, Decomposition of complete graphs into bi-cyclic graphs with eight edges, *Bull. Inst. Combin. Appl.*, 88(2020), 30-49.

[13] M. H. Huang, Decomposing complete equipartite graphs into connected unicyclic graphs

of size five, *Util. Math.*, 97 (2015), 109-117.

[14] P. Paulraja and T. Sivakaran, Decompositions of some regular graphs into unicyclic graphs of order five, *Discrete Math. Algorithms and Appl.*, 11(4) (2019), 1950042 (34 pages).

[15] K. Sowndhariya and A. Muthusamy, Decomposition of product graphs into sunlet graphs of order eight, *J. Algebra Comb. Discrete Appl.*, 8(1) (2021), 41-51.

[16] M.Tarsi, Decomposition of a Complete Multigraph into Simple Paths: Nonbalanced Handcuffed Designs *J. Combin. Theory Ser A.*, 34(1983), 60-70.

[17] J. Tian, Y. Du, and Q. Kang, Decomposing complete graphs into graphs with six vertices and seven edges, *Ars Combin.*, 81(2006), 257-279.