International J.Math. Combin. Vol.3(2023), 66-71

Edge C_k Symmetric n-Sigraphs

P. Somashekar

(Department of Mathematics, Government First Grade College, Nanjangud-571 301, India)

S. Vijay

(Department of Mathematics, Government Science College, Hassan-573 201, India)

C. N. Harshavardhana

(Department of Mathematics, Government First Grade College for Women, Holenarasipur-573 211, India)

E-mail: somashekar2224@gmail.com, vijayshivanna82@gmail.com, cnhmaths@gmail.com

Abstract: An n-tuple $(a_1, a_2, \cdots, a_n)$ is *symmetric* if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n), a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \rightarrow H_n$ $(\mu : V \rightarrow H_n)$ is a function. In this paper, we introduced a new notion edge C_k symmetric n-sigraph of a symmetric n-sigraph and its properties are obtained. Also, we obtained the structural characterization of edge C_k symmetric n-signed graphs.

Key Words: Symmetric n-sigraph, Smarandachely symmetric n-marked graph, symmetric n-marked graph, Smarandachely symmetric n-marked graph, balance, switching, edge C_k symmetric n-sigraph, complementation.

AMS(2010): 05C22.

§1. **Introduction**

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [1]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric* if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \rightarrow H_n$ $(\mu : V \rightarrow H_n)$ is a function. Generally, a *Smarandachely symmetric n-sigraph* (*Smarandachely symmetric n-marked graph)* for a subgraph $H \prec G$ is such a graph that $G - E(H)$ is *symmetric n-sigraph (symmetric n-marked graph)*. For example, let H be an edge $e \in E(G)$, a path $P_s \succ G$

[1]Received April 21, 2023, Accepted August 18, 2023.
[2]Corresponding author: somashekar2224@gmail.com

for an integer $s \geq 2$ or a claw $K_{1,3} \prec G$. Certainly, if $H = \emptyset$, a Smarandachely symmetric n-sigraph (or Smarandachely symmetric n-sigraph) is nothing else but a symmetric n-sigraph (or symmetric n-marked graph).

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

An n-tuple $(a_1, a_2, \cdots, a_n)$ is an *identity n-tuple* if $a_k = +$ for $1 \leq k \leq n$. Otherwise, it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*. Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [9], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P.S.K.Reddy [5]:

Definition 1.1 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then,*

(i) S_n is identity balanced (or i-balanced), if product of n-tuples on each cycle of S_n is the identity n-tuple, and

(ii) S_n is balanced, if every cycle in S_n contains an even number of non-identity edges.

Note 1.1 An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [9].

Theorem 1.1 (E. Sampathkumar et al. [9]) *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

In [9], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ (See also [2, 6 - 8, 11 - 20, 22]) as follows:

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, \cdots, a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, \cdots, a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*. Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n.

We make use of the following known result (see [9]).

Theorem 1.2 (E. Sampathkumar et al. [9]) *Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S defined as follows: each vertex $v \in V$, $\mu(v)$ is the product of the n-tuples on the edges incident at v. *Complement* of S is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma')$, where for any edge $e = uv \in \overline{G}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Theorem 1.1.

§2. Edge C_k Symmetric n-Sigraph of an n-Sigraph

The *edge C_k graph $E_k(G)$* of a graph G is defined in [4] as follows:

The edge C_k graph of a graph $G = (V, E)$ is a graph $E_k(G) = (V', E')$, with vertex set $V' = E(G)$ such that two vertices e and f are adjacent if, and only if, the corresponding edges in G either incident or opposite edges of some cycle C_k. In this paper, we extend the notion of $E_k(G)$ to realm of symmetric n-sigraphs: Given an n-sigraph $S_n = (G, \sigma)$ its *edge C_k n-sigraph* $E_k(S_n) = (E_k(G), \sigma')$ is that n-sigraph whose underlying graph is $E_k(G)$, the edge C_k graph of G, where for any edge e_1e_2 in $E_k(S_n)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. When $k = 3$, the definition coincides with triangular line n-sigraph of a graph [2], and when $k = 4$, the definition coincides with the edge E_4 n-sigraph of an n-sigraph [12].

Hence, we shall call a given n-sigraph an edge C_k n-sigraph if there exists an n-sigraph S'_n such that $S_n \cong E_k(S'_n)$. In the following subsection, we shall present a characterization of edge C_k n-sigraphs.

The following result indicates the limitations of the notion of edge C_k n-sigraphs as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be edge C_k n-sigraphs.

Theorem 2.1 *For any n-sigraph $S_n = (G, \sigma)$, its edge C_k n-sigraph $E_k(S_n)$ is i-balanced.*

Proof Since the n-tuple of any edge uv in $E_k(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n, by Theorem 1, $E_k(S_n)$ is i-balanced. □

When $k = 3$ and $k = 4$, we can deduce the following results.

Corollary 2.1 (Lokesha et al. [2]) *For any n-sigraph $S_n = (G, \sigma)$, its triangular line n-sigraph $\mathcal{T}(S_n)$ is i-balanced.*

Corollary 2.2 (P.S.K.Reddy et al. [12]) *For any n-sigraph $S_n = (G, \sigma)$, its edge C_4 n-sigraph $E_4(S_n)$ is i-balanced.*

For any positive integer i, the i^{th} iterated edge C_k n-sigraph, $E^i_k(S_n)$ of S_n is defined as follows:

$$E^0_k(S_n) = S_n,\ E^i_k(S_n) = E_k(E^{i-1}_k(S_n)).$$

Corollary 2.3 *For any n-sigraph $S_n = (G, \sigma)$ and any positive integer m, $E^m_k(S_n)$ is i-balanced.*

In [21], the authors obtained the characterizations for the edge C_k graph of a graph G is connected, complete, bipartite etc. The authors have also proved that the edge C_k graph has no forbidden subgraph characterization. The dynamical behavior such as convergence, periodicity, mortality and touching number of $E_k(G)$ are also discussed.

Recall that, the edge C_k graph coincides with the line graph for any acyclic graph. As a case, for a connected graph G, $E_k(G) = G$ if, and only if $G = C_n, n \neq k$ ([4]).

We now characterize n-sigraphs that are switching equivalent to their the edge C_k n-sigraphs.

Theorem 2.2 *For any n-sigraph $S_n = (G, \sigma)$, $S_n \sim E_k(S_n)$ if and only if $G \cong C_n$, where $n \geq 5$ and S_n is i-balanced.*

Proof Suppose $S_n \sim E_k(S_n)$. This implies, $G \cong E_k(G)$ and hence G is isomorphic to C_n, where $n \geq 5$, Theorem 3 implies that $E_k(S_n)$ is i-balanced and hence if S_n is i-unbalanced and its $E_k(S_n)$ being i-balanced can not be switching equivalent to S_n in accordance with Theorem 1.2. Therefore, S_n must be i-balanced.

Conversely, suppose that S_n is an i-balanced n-sigraph and its undrelying G is isomorphic to C_n, where $n \geq 5$. Then, since $E_k(S_n)$ is i-balanced as per Theorem 3 and since $G \cong E_k(G)$, the result follows from Theorem 1.2 again. $\square$

In [21], we obtained the following result.

Theorem 2.3 (P.S.K.Reddy et al. [21]) *For a graph $G = (V, E)$, $E_k(G) \cong L(G)$ if, and only if G is C_k-free.*

In view of the above result, we have the following characterization.

Theorem 2.4 *For any n-sigraph $S_n = (G, \sigma)$, $E_k(S_n) \cong L(S_n)$ if, and only if G is C_k-free.*

For any $m \in H_n$, the *m-complement* of $a = (a_1, a_2, \cdots, a_n)$ is $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the *m-complement* of M is $M^m = \{a^m : a \in M\}$.

For any $m \in H_n$, the *m-complement* of an n-sigraph $S_n = (G, \sigma)$, written (S_n^m), is the same graph but with each edge label $a = (a_1, a_2, \cdots, a_n)$ replaced by a^m.

For an n-sigraph $S_n = (G, \sigma)$, the $R(S_n)$ is i-balanced. We now examine, the condition under which m-complement of $E_k(S_n)$ is i-balanced, where for any $m \in H_n$.

Theorem 2.5 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $m \in H_n$, if $E_k(G)$ is bipartite then $(E_k(S_n))^m$ is i-balanced.*

Proof Since, by Theorem 2.1, $E_k(S_n)$ is i-balanced, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $E_k(S_n)$ whose k^{th} co-ordinate are $-$ is even. Also, since $E_k(G)$ is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $E_k(S_n)$ whose k^{th} co-ordinate are $+$ is also even. This implies that the same thing is true in any m-complement, where for any $m, \in H_n$. Hence $(E_k(S_n))^t$ is i-balanced. $\square$

In [3], the authors proved that for a connected complete multipartite graph G, $E_k(G)$ is complete. The following result follows from the above observation and Theorem 2.1.

Theorem 2.6 *For a connected n-sigraph $S_n = (G, \sigma)$, $E_k(S_n)$ is complete i-balanced signed* *graph if, and only if G is complete multipartite graph.*

In [21], the authors proved that: For a connected graph $G = (V, E)$, $E_k(G)$ is bipartite if, and only if, G is either a path or an even cycle of length $r \neq k$. The following result follows from the above result and Theorem 2.1.

Theorem 2.7 *For a connected n-sigraph $S_n = (G, \sigma)$, $E_k(S_n)$ is bipartite i-balanced signed graph if, and only if G is isomorphic to either path or C_{2n}, where $n \geq 3$.*

§3. Characterization of Edge C_k Signed Graphs

The following result characterize n-sigraphs which are edge C_k n-sigraphs.

Theorem 3.1 *An n-sigraph $S_n = (G, \sigma)$ is an edge C_k n-sigraph if, and only if S_n is i-balanced n-sigraph and its underlying graph G is an edge C_k graph.*

Proof Suppose that S_n is i-balanced and G is an edge C_k graph. Then there exists a graph Γ' such that $E_k(G') \cong G$. Since S_n is i-balanced, by Theorem 1, there exists an n-marking ζ of G such that each edge uv in S_n satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the n-sigraph $S'_n = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $E_k(S'_n) \cong S_n$. Hence S_n is an edge C_k n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is an edge C_k n-sigraph. Then there exists an n-sigraph $S'_n = (G', \sigma')$ such that $E_k(S'_n) \cong S_n$. Hence G is the edge C_k graph of G' and by Theorem 3, S_n is i-balanced. □

If we take $k = 3$ and $k = 4$ in $E_k(S_n)$, then we can deduce the triangular line n-sigraph and edge C_4 n-sigraph respectively. In [2,12], the authors obtained structural characterizations of triangular line n-sigraphs and edge C_4 n-sigraphs and clearly Theorem 2.7 is the generalization of above said notions.

Acknowledgements

The authors gratefully thank to the Referee for the constructive comments and recommendations which definitely help to improve the readability and quality of the paper.

References

[1] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., 1969.

[2] V. Lokesha, P.S.K.Reddy and S. Vijay, The triangular line n-sigraph of a symmetric n-sigraph, *Advn. Stud. Contemp. Math.*, 19(1) (2009), 123-129.

[3] Manju K Menon and A. Vijayakumar, The edge C_4 graph of a graph, *Proc. of ICDM (2006), Ramanujan Math. Soc., Lecture Notes Series*, Number 7, (2008), 245-248.

[4] E. Prisner, *Graph Dyanamics*, Longman, 1995.

[5] R. Rangarajan and P.S.K.Reddy, Notions of balance in symmetric n-sigraphs, *Proceedings*

of the Jangjeon Math. Soc., 11(2) (2008), 145-151.

[6] R. Rangarajan, P.S.K.Reddy and M. S. Subramanya, Switching equivalence in symmetric n-sigraphs, *Adv. Stud. Comtemp. Math.*, 18(1) (2009), 79-85.

[7] R. Rangarajan, P.S.K.Reddy and N. D. Soner, Switching equivalence in symmetric n-sigraphs-II, *J. Orissa Math. Sco.*, 28 (1 & 2) (2009), 1-12.

[8] R. Rangarajan, P.S.K.Reddy and N. D. Soner, m^{th} power symmetric n-sigraphs, *Italian Journal of Pure & Applied Mathematics*, 29(2012), 87-92.

[9] E. Sampathkumar, P.S.K.Reddy, and M. S. Subramanya, Jump symmetric n-sigraph, *Proceedings of the Jangjeon Math. Soc.*, 11(1) (2008), 89-95.

[10] E. Sampathkumar, P.S.K.Reddy, and M. S. Subramanya, The Line n-sigraph of a symmetric n-sigraph, *Southeast Asian Bull. Math.*, 34(5) (2010), 953-958.

[11] P.S.K.Reddy and B. Prashanth, Switching equivalence in symmetric n-sigraphs-I, *Advances and Applications in Discrete Mathematics*, 4(1) (2009), 25-32.

[12] P.S.K.Reddy, S. Vijay and B. Prashanth, The edge C_4 n-sigraph of a symmetric n-sigraph, *Int. Journal of Math. Sci. & Engg. Appls.*, 3(2) (2009), 21-27.

[13] P.S.K.Reddy, V. Lokesha and Gurunath Rao Vaidya, The Line n-sigraph of a symmetric n-sigraph-II, *Proceedings of the Jangjeon Math. Soc.*, 13(3) (2010), 305-312.

[14] P.S.K.Reddy, V. Lokesha and Gurunath Rao Vaidya, The Line n-sigraph of a symmetric n-sigraph-III, *Int. J. Open Problems in Computer Science and Mathematics*, 3(5) (2010), 172-178.

[15] P.S.K.Reddy, V. Lokesha and Gurunath Rao Vaidya, Switching equivalence in symmetric n-sigraphs-III, *Int. Journal of Math. Sci. & Engg. Appls.*, 5(1) (2011), 95-101.

[16] P.S.K.Reddy, B. Prashanth and Kavita. S. Permi, A Note on Switching in Symmetric n-Sigraphs, *Notes on Number Theory and Discrete Mathematics*, 17(3) (2011), 22-25.

[17] P.S.K.Reddy, M. C. Geetha and K. R. Rajanna, Switching Equivalence in Symmetric n-Sigraphs-IV, *Scientia Magna*, 7(3) (2011), 34-38.

[18] P.S.K.Reddy, K. M. Nagaraja and M. C. Geetha, The Line n-sigraph of a symmetric n-sigraph-IV, *International J. Math. Combin.*, 1 (2012), 106-112.

[19] P.S.K.Reddy, M. C. Geetha and K. R. Rajanna, Switching equivalence in symmetric n-sigraphs-V, *International J. Math. Combin.*, 3 (2012), 58-63.

[20] P.S.K.Reddy, K. M. Nagaraja and M. C. Geetha, The Line n-sigraph of a symmetric n-sigraph-V, *Kyungpook Mathematical Journal*, 54(1) (2014), 95-101.

[21] P.S.K.Reddy, K. M. Nagaraja and and V. M. Siddalingaswamy, The Edge C_k graph of a graph, *Vladikavkaz Mathematical Journal*, 16(4) (2014), 61-64.

[22] P.S.K.Reddy, R. Rajendra and M. C. Geetha, Boundary n-Signed Graphs, *Int. Journal of Math. Sci.* & *Engg. Appls.*, 10(2) (2016), 161-168.