Further Results on Super Geometric Mean Graphs

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Abstract: Let $G$ be a graph and $f : V(G) \to \{1, 2, 3, \ldots, p + q\}$ be an injection. For each edge $uv$, the induced edge labeling $f^*$ is defined as $f^*(uv) = \lceil \sqrt{f(u)f(v)} \rceil$. Then $f$ is called a super geometric mean labeling if $f(V(G)) \cup \{f^*(uv) : uv \in E(G)\} = \{1, 2, 3, \ldots, p + q\}$. A graph that admits a super geometric mean labeling is called a super geometric mean graph. In this paper, we have discussed the super geometric meanness of the graphs $P_n \cup C_m$, $T_n \cup C_m$, $mC_n$, the complete graph $K_n$, $[P_n; S_m]$, subdivision of $P_n \odot K_1$, $TW(P_n)$, middle graph of a path, triangular ladder, $C_n \odot K_1$, duplication of a vertex of the cycle, duplication of an edge of the cycle, triangular grid graph and edge identification of two cycles.

Key Words: Labeling, super geometric mean labeling, super geometric mean graph.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with $p$ vertices and $q$ edges. For notations and terminology, we follow [5]. For a detailed survey on graph labeling, we refer [4].

A path on $n$ vertices is denoted by $P_n$ and a cycle on $n$ vertices is denoted by $C_n$. The union of $m$ copies of a graph $G$ is denoted by $mG$. A complete graph $K_n$ is a graph on $n$ vertices in which every pair of distinct vertices are joined by an edge. A star graph $S_n$ is a complete bipartite graph $K_{1,n}$. Let $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots, v_{m+1}^{(i)}$ and $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the $i^{th}$ copy of the star graph $S_m, 1 \leq i \leq n$ and the path $P_n$ respectively. The graph $[P_n; S_m]$ is obtained from $n$ copies of $S_m$ and the path $P_n$ by joining $u_i$ with the central vertex $v_1^{(i)}$ of the $i^{th}$ copy of $S_m$ by means of an edge, for $1 \leq i \leq n$. For a graph $G$, the graph $S(G)$ is obtained by subdividing each edge of $G$ by a vertex. A twig $TW(P_n), n \geq 3$ is a graph obtained from a path by attaching exactly two pendant vertices to each internal vertices to each internal vertices of the path.

The middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $\{v : v \in V(G)\} \cup \{e : e \in E(G)\}$ and the edge set is $\{e_1e_2 : e_1, e_2 \in E(G)\}$ and $e_1$ and $e_2$ are adjacent edges of $G\} \cup \{ve : v \in V(G), e \in E(G)\}$ and $e$ is incident with $v$.

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A ladder $L_n$ is a graph $P_2 \times P_n$ with $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n - 1\}$. A triangular ladder $TL_n, n \geq 2$ is a graph obtained by completing the ladder $L_n \cong P_2 \times P_n$ by adding the edges $u_i v_{i+1}$ for $1 \leq i \leq n - 1$. $G \odot K_1$ is the graph obtained from $G$ by attaching a new pendant vertex at each vertex of $G$.

Duplication of a vertex $v_k$ of a graph $G$ produces a new graph $G'$ by adding a vertex $v'_k$ with $N(v_k) = N(v'_k)$. Duplication of an edge $e = uv$ of a graph $G$ by adding an edge $e' = u'v'$ such that $N(u') = N(u) \cup \{v'\} - \{v\}$ and $N(v') = N(v) \cup \{u'\} - \{u\}$.

In [6], S.K. Vaidya et al. discussed the harmonic mean labeling of duplication of a vertex and edge of a cycle. In [7], R. Vasuki et al. discussed the super mean labeling of some standard graphs. A. Durai Baskar et al. [1,2] discussed the geometric mean labeling some standard graphs. Motivated by these works, the concept of super geometric mean labeling was introduced and studied in [3].

A vertex labeling of $G$ is an assignment $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ be an injection. For a vertex labeling $f$, the induced edge labeling $f^*$ is defined as $f^*(uv) = \lceil \sqrt{f(u)f(v)} \rceil$. Then $f$ is called a super geometric mean labeling if $f(V(G)) \cup \{f^*(uv) : uv \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. A graph that admits a super geometric mean labeling is called a super geometric mean graph.

The graph shown in Figure 1 is a super geometric mean graph.

![Figure 1](image_url)

In this paper, we have established the super geometric meanness of the graphs $P_n \cup C_m$ for $n \geq 1$ and $m \geq 3$, $T_n \cup C_m$ for $n \geq 4$ and $m \geq 3$, $mC_n$, the complete graph $K_n$, $[P_n; S_m]$ for $n \geq 1$ and $m \leq 2$, subdivision of $P_n \odot K_1$, $TW(P_n)$ for $n \geq 3$, middle graph of a path, triangular ladder, $C_n \odot K_1$ for $n \geq 3$, duplication of a vertex of the cycle, duplication of an edge of the cycle, triangular grid graph and edge identification of two cycles.

§2. Main Results

**Theorem 2.1** $P_n \cup C_m$ is a super geometric mean graph, for $n \geq 1$ and $m \geq 3$.

**Proof** Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be the vertices of the cycle $C_m$ and the path $P_n$ respectively.

**Case 1.** $m \geq 4$. 

We define \( f : V(P_n \cup C_m) \cup E(P_n \cup C_m) \to \{1, 2, 3, \ldots, 2m + 2n - 1\} \) as follows:

\[
\begin{align*}
  f(u_i) = & \begin{cases} 
    1 & i = 1 \\
    4i - 4 & 2 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \\
    2m - 3 & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is odd} \\
    2m & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is even} \\
    2m - 3 & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is even} \\
    4m + 5 - 4i & \left\lfloor \frac{m}{2} \right\rfloor + 3 \leq i \leq m
  \end{cases} \\
  f(v_i) = & 2m + 2i - 1 \text{ for } 1 \leq i \leq n.
\end{align*}
\]

The induced edge labeling is as follows:

\[
\begin{align*}
  f^*(u_iu_{i+1}) = & \begin{cases} 
    4i - 2 & 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \\
    2m - 1 & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \\
    2m - 2 & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is odd} \\
    2m - 5 & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is even} \\
    4m + 3 - 4i & \left\lfloor \frac{m}{2} \right\rfloor + 3 \leq i \leq m - 1,
  \end{cases} \\
  f^*(u_1u_m) = & 3 \text{ and } f^*(v_iv_{i+1}) = 2m + 2i \text{ for } 1 \leq i \leq n - 1.
\end{align*}
\]

Case 2. \( m = 3 \).

We define \( f : V(P_n \cup C_3) \cup E(P_n \cup C_3) \to \{1, 2, 3, \ldots, 2n + 5\} \) as follows \( f(u_1) = 1, f(u_2) = 4, f(u_3) = 6 \) and \( f(v_i) = 5 + 2i \) for \( 1 \leq i \leq n \). The induced edge labeling is as follows:

\[
\begin{align*}
  f^*(u_1u_2) = & 2, f^*(u_2u_3) = 5, f^*(u_3u_1) = 3 \text{ and } f^*(v_iv_{i+1}) = 6 + 2i \text{ for } 1 \leq i \leq n - 1.
\end{align*}
\]

Hence, \( f \) is a super geometric mean labeling of \( P_n \cup C_m \). Thus the graph \( P_n \cup C_m \) is a super geometric mean graph for \( n \geq 1 \) and \( m \geq 3 \).

The super geometric mean labeling of \( P_5 \cup C_6 \) and \( P_4 \cup C_3 \) are shown in Figure 2.
Theorem 2.2 For a T-graph $T_n, T_n \cup C_m$ is a super geometric mean graph, for $n \geq 4$ and $m \geq 3$.

Proof Let $u_1, u_2, \cdots, u_m$ be the vertices of the cycle $C_m$ and $v_1, v_2, \cdots, v_{n-1}$ be the vertices of the path $P_{n-1}$ and let $v_n$ be the pendant vertex identified with $v_{n-2}$ in $T_n$.

Case 1. $m \geq 4$.

We define $f : V(T_n \cup C_m) \cup E(T_n \cup C_m) \rightarrow \{1, 2, 3, \cdots, 2m + 2n - 1\}$ as follows:

$$f(u_i) = \begin{cases} 
1 & i = 1 \\
4i - 4 & 2 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \\
2m - 3 & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is odd} \\
2m & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is even} \\
2m & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is odd} \\
2m - 3 & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is even} \\
4m + 5 - 4i & \left\lfloor \frac{m}{2} \right\rfloor + 3 \leq i \leq m,
\end{cases}$$

$$f(v_i) = 2m + 2i - 1 \text{ for } 1 \leq i \leq n - 3, \quad f(v_{n-2}) = 2m + 2n - 3, \quad f(v_{n-1}) = 2m + 2n - 6 \text{ and } f(v_n) = 2m + 2n - 1.$$ 

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 
4i - 2 & 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \\
2m - 1 & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \\
2m - 2 & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is odd} \\
2m - 5 & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is even} \\
4m + 3 - 4i & \left\lfloor \frac{m}{2} \right\rfloor + 3 \leq i \leq m - 1,
\end{cases}$$

$$f^*(u_1u_m) = 3, \quad f^*(v_iv_{i+1}) = 2m + 2i \text{ for } 1 \leq i \leq n - 4,$$

$$f^*(v_{n-3}v_{n-2}) = 2m + 2n - 5, \quad f^*(v_{n-2}v_{n-1}) = 2m + 2n - 4 \text{ and}$$

$$f^*(v_{n-2}v_n) = 2m + 2n - 2.$$ 

Case 2. $m = 3$.

We define $f : V(T_n \cup C_3) \cup E(T_n \cup C_3) \rightarrow \{1, 2, 3, \cdots, 2n + 5\}$ as follows:

$$f(u_1) = 1, \quad f(u_2) = 4, \quad f(u_3) = 6, \quad f(v_i) = 5 + 2i \text{ for } 1 \leq i \leq n - 3, \quad f(v_{n-2}) = 2n + 3,$$

$$f(v_{n-1}) = 2n \text{ and } f(v_n) = 2n + 5.$$
The induced edge labeling as follows:

\[ f^*(u_1u_2) = 2, \quad f^*(u_2u_3) = 5, \quad f^*(u_3u_1) = 3, \]
\[ f^*(v_i;v_{i+1}) = 6 + 2i \text{ for } 1 \leq i \leq n - 4, \]
\[ f^*(v_{n-3}v_{n-2}) = 2n + 1, \quad f^*(v_{n-2}v_{n-1}) = 2n + 2 \text{ and} \]
\[ f^*(v_{n-2}v_n) = 2n + 4. \]

Hence, \( f \) is a super geometric mean labeling of \( T_n \cup C_m \). Thus the graph \( T_n \cup C_m \) is a super geometric mean graph for \( n \geq 4 \) and \( m \geq 3 \).

The super geometric mean labeling of \( T_6 \cup C_7 \) and \( T_7 \cup C_3 \) are shown in Figure 3.

\[ \text{Figure 3} \]

**Theorem 2.3** \( mC_n \) is a super geometric mean graph, for any \( m \) and \( n \).

**Proof** Let \( \{v_{j}^{(i)} : 1 \leq j \leq n\} \) be the vertices of the \( i^{th} \) copy of the cycle \( C_n \), \( 1 \leq i \leq m \).

**Case 1.** \( n \geq 5 \).

We define \( f : V(mC_n) \cup E(mC_n) \rightarrow \{1, 2, 3, \cdots, 2mn\} \) as follows:
and \( f(v_j^{(1)}) = \begin{cases} 1 & j = 1 \\ 4j - 4 & 2 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 2n - 3 & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\ 2n & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\ 2n & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd} \\ 2n - 3 & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\ 4n + 5 - 4i & \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq i \leq n, \end{cases} \)

and \( f(v_j^{(2)}) = \begin{cases} 2n + 1 & j = 1 \\ 2n + 4j - 5 & 2 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 4n - 3 & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\ 4n & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\ 4n & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd} \\ 4n - 3 & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\ 6n + 6 - 4j & \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq j \leq n \end{cases} \)

and \( f(v_j^{(i)}) = 2n + f(v_j^{(i-1)}) \) for \( 3 \leq i \leq m \) and \( 1 \leq j \leq n \).

The induced edge labeling is as follows:

\[
f^*(v_j^{(1)}v_{j+1}^{(1)}) = \begin{cases} 4j - 2 & 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 2n - 1 & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 2n - 2 & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd} \\ 2n - 5 & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\ 4n + 3 - 4j & \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq j \leq n - 1, \end{cases} \]

\[
f^*(v_1^{(1)}v_n^{(1)}) = 3, \]

\[
f^*(v_j^{(2)}v_{j+1}^{(2)}) = \begin{cases} 2n + 2 & j = 1 \\ 2n + 4j - 3 & 2 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ 4n - 5 & j = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is odd} \\ 4n - 2 & j = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is even} \\ 4n - 1 & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 6n + 4 - 4j & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq j \leq n - 1, \end{cases} \]

\[
f^*(v_1^{(2)}v_n^{(2)}) = 2n + 4, \]

\[
f^*(v_j^{(i)}v_{j+1}^{(i)}) = 2n + f^*(v_j^{(i-1)}v_{j+1}^{(i-1)}) \text{ for } 3 \leq i \leq m \text{ and } 1 \leq j \leq n - 1 \]

and \( f^*(v_1^{(i)}v_n^{(i)}) = 2n + f^*(v_1^{(i-1)}v_n^{(i-1)}) \) for \( 3 \leq i \leq m \).
Case 2. \( n = 4 \).

We define \( f : V(mC_4) \cup E(mC_4) \rightarrow \{1, 2, 3, \cdots, 8m\} \) as follows:

\[
f(v_j^{(1)}) = \begin{cases} 
1 & j = 1 \\
5j - 6 & 2 \leq j \leq 3 \\
5 & j = 4
\end{cases}
\]

and \( f(v_j^{(2)}) = \begin{cases} 
8 & j = 1 \\
6j & 2 \leq j \leq 3 \\
13 & j = 4.
\end{cases} \)

The induced edge labeling is as follows:

\[
f^*(v_j^{(1)}v_{j+1}^{(1)}) = \begin{cases} 
4j - 2 & 1 \leq j \leq 2 \\
7 & j = 3
\end{cases}, \quad f^*(v_1^{(1)}v_4^{(1)}) = 3
\]

\[
f^*(v_j^{(2)}v_{j+1}^{(2)}) = \begin{cases} 
5j + 5 & 1 \leq j \leq 2 \\
16 & j = 3
\end{cases} \quad \text{and} \quad f^*(v_1^{(2)}v_4^{(2)}) = 11.
\]

Subcase 2.1 \( m \) is odd and \( m \geq 3 \).

\[
f(v_j^{(3)}) = \begin{cases} 
14 & j = 1 \\
2j + 16 & 2 \leq j \leq 4,
\end{cases}
\]

\[
f(v_j^{(4)}) = \begin{cases} 
2j + 23 & 1 \leq j \leq 3 \\
34 & j = 4,
\end{cases}
\]

\[
f(v_j^{(5)}) = \begin{cases} 
31 & j = 1 \\
3j + 29 & 2 \leq j \leq 3 \\
40 & j = 4
\end{cases}
\]

\[
f(v_j^{(i)}) = f(v_j^{(i-2)}) + 16 \quad \text{for} \ 6 \leq i \leq m \text{ and } 1 \leq j \leq 4
\]

The induced edge labeling is as follows

\[
f^*(v_j^{(3)}v_{j+1}^{(3)}) = \begin{cases} 
17 & j = 1 \\
2j + 17 & 2 \leq j \leq 3
\end{cases}, \quad f^*(v_1^{(3)}v_4^{(3)}) = 19,
\]

\[
f^*(v_j^{(4)}v_{j+1}^{(4)}) = \begin{cases} 
26 & j = 1 \\
4j + 20 & 2 \leq j \leq 3
\end{cases}, \quad f^*(v_1^{(4)}v_4^{(4)}) = 30,
\]

\[
f^*(v_j^{(5)}v_{j+1}^{(5)}) = \begin{cases} 
33 & j = 1 \\
2j + 33 & 2 \leq j \leq 3
\end{cases}, \quad f^*(v_1^{(5)}v_4^{(5)}) = 36,
\]

\[
f^*(v_j^{(i)}v_{j+1}^{(i)}) = f^*(v_j^{(i-2)}v_{j+1}^{(i-2)}) + 16 \quad \text{for} \ 6 \leq i \leq m \text{ and } 1 \leq j \leq 3
\]

and

\[
f^*(v_j^{(i)}v_4^{(i)}) = f^*(v_j^{(i-2)}v_4^{(i-2)}) + 16 \quad \text{for} \ 6 \leq i \leq m.
\]
Subcase 2.2 \( m \) is even and \( m \geq 4 \).

\[
\begin{align*}
\text{Subcase 2.2} & \quad m \text{ is even and } m \geq 4. \\
\end{align*}
\]

\[
\begin{align*}
f(v_j^{(3)}) &= \begin{cases} 
14 & j = 1 \\
3j + 13 & 2 \leq j \leq 3 \\
26 & j = 4
\end{cases}, \\
\text{and} \\
f(v_j^{(5)}) &= \begin{cases} 
2j + 31 & 1 \leq j \leq 3 \\
42 & j = 4
\end{cases}
\end{align*}
\]

\[
\begin{align*}
f(v_j^{(i)}) &= f(v_j^{(i-2)}) + 16 \text{ for } 6 \leq i \leq m \text{ and } 1 \leq j \leq 4.
\end{align*}
\]

The induced edge labeling is as follows

\[
\begin{align*}
f^*(v_j^{(3)}v_{j+1}^{(3)}) &= \begin{cases} 
17 & j = 1 \\
3j + 15 & 2 \leq j \leq 3 \\
20 & j = 4
\end{cases}, \\
f^*(v_j^{(4)}v_{j+1}^{(4)}) &= \begin{cases} 
25 & j = 1 \\
2j + 25 & 2 \leq j \leq 3 \\
28 & j = 4
\end{cases}, \\
f^*(v_j^{(5)}v_{j+1}^{(5)}) &= \begin{cases} 
34 & j = 1 \\
4j + 28 & 2 \leq j \leq 3 \\
38 & j = 4
\end{cases},
\end{align*}
\]

\[
\begin{align*}
f^*(v_j^{(i)}v_{j+1}^{(i)}) &= f^*(v_j^{(i-2)}v_{j+1}^{(i-2)}) + 16 \text{ for } 6 \leq i \leq m \text{ and } 1 \leq j \leq 3. \\
\text{and} \\
f^*(v_1^{(i)}v_4^{(i)}) &= f^*(v_1^{(i-2)}v_4^{(i-2)}) + 16 \text{ for } 6 \leq i \leq m.
\end{align*}
\]

Case 3. \( n = 3 \).

We define \( f : V(mC_3) \cup E(mC_3) \rightarrow \{1, 2, 3, \ldots, 6m\} \) as follows:

\[
\begin{align*}
f(v_1^{(i)}) &= 6i - 5 \text{ for } 1 \leq i \leq m, \\
f(v_2^{(i)}) &= \begin{cases} 
4 & i = 1 \\
6i - 3 & 2 \leq i \leq m
\end{cases}, \\
\text{and} \\
f(v_3^{(i)}) &= 6i \text{ for } 1 \leq i \leq m. 
\end{align*}
\]

The induced edge labeling is as follows:

\[
\begin{align*}
f^*(v_1^{(i)}v_2^{(i)}) &= 6i - 4 \text{ for } 1 \leq i \leq m, \\
f^*(v_2^{(i)}v_3^{(i)}) &= 6i - 1 \text{ for } 1 \leq i \leq m \text{ and} \\
f^*(v_3^{(i)}v_1^{(i)}) &= \begin{cases} 
3 & i = 1 \\
6i - 2 & 2 \leq i \leq m
\end{cases}.
\end{align*}
\]

Hence, \( f \) is super geometric mean labeling of \( mC_n \). Thus the graph \( mC_n \) is a super geometric mean graph for any \( m \) and \( n \). \( \Box \)

The super geometric mean labeling of \( 4C_6, 7C_4 \) and \( 5C_3 \) are shown in Figure 4.
Corollary 2.4 \( mC_n \cup P_k \) is a super geometric mean graph for any \( m,n \) and \( k \).

Proof By the above Theorems 2.1 and 2.3 the results follows. \( \square \)

Theorem 2.5 \( K_n \) is a super geometric mean graph if and only if \( n \leq 3 \).

Proof Based on the definition of super geometric mean labeling, 1 and \( p + q \) should be the vertex labels.

For all \( p \geq 5 \), the edge having the end vertices whose labels are 1 and \( p + q \) is less than or equal to \( p - 1 \). So we cannot have distinct edge labels for the edges incident with a vertex whose vertex label is 1.

When \( p = 4 \), \( 1, p + q = 10 \) and \( p + q - 2 = 8 \) are to be the vertex labels whose induced edge labels are 3, 4 and 9. So we cannot label for the 4th vertex in which the edge label is 2. Also 2 cannot be the vertex label. \( \square \)

The super geometric mean labeling of \( K_1, K_2 \) and \( K_3 \) are shown in Figure 5.

Theorem 2.6 \( [P_n; S_m] \) is a super geometric mean graph, for \( n \geq 1 \) and \( m \leq 2 \).
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Proof Let \( u_1, u_2, \ldots, u_n \) be the vertices of the path \( P_n \) and \( v_1^{(i)}, v_2^{(i)}, \ldots, v_m^{(i)} \) be the pendant vertices at each vertex \( u_i \) of the path \( P_n \), for \( 1 \leq i \leq n \).

Case 1. \( m = 1 \).

We define \( f : V([P_n; S_1]) \cup E([P_n; S_1]) \rightarrow \{1, 2, 3, \ldots, 6n - 1\} \) as follows:

\[
f(u_i) = \begin{cases} 
5 & i = 1 \\
6i - 5 & 2 \leq i \leq n 
\end{cases};
\]

\[
f(v_2^{(i)}) = \begin{cases} 
1 & i = 1 \\
6i & 2 \leq i \leq n-1 
\end{cases} \quad \text{and} \quad f(v_2^{(n)}) = 6n - 1.
\]

The induced edge labeling is as follows:

\[
f^*(u_i u_{i+1}) = \begin{cases} 
6 & i = 1 \\
6i - 2 & 2 \leq i \leq n-1, 
\end{cases}
\]

\[
f^*(u_i v_1^{(i)}) = \begin{cases} 
4 & i = 1 \\
6i - 4 & 2 \leq i \leq n, 
\end{cases}
\]

\[
f^*(v_1^{(i)} v_2^{(i)}) = \begin{cases} 
2 & i = 1 \\
6i - 1 & 2 \leq i \leq n-1
\end{cases} \quad \text{and} \quad f^*(v_1^{(n)} v_2^{(n)}) = 6n - 2.
\]

Case 2. \( m = 2 \).

We define \( f : V([P_n; S_2]) \cup E([P_n; S_2]) \rightarrow \{1, 2, 3, \ldots, 8n - 1\} \) as follows:

\[
f(u_i) = \begin{cases} 
2i + 5 & 1 \leq i \leq 2 \\
8i - 8 & 3 \leq i \leq n, 
\end{cases}
\]

\[
f(v_1^{(i)}) = \begin{cases} 
5 & i = 1 \\
8i - 5 & 2 \leq i \leq n-1, 
\end{cases}
\]

\[
f(v_1^{(n)}) = 8n - 3,
\]

\[
f(v_2^{(i)}) = \begin{cases} 
1 & i = 1 \\
8i - 1 & 2 \leq i \leq n-1, 
\end{cases} \quad \text{and} \quad f(v_2^{(n)}) = 8n - 6,
\]

\[
f(v_3^{(i)}) = \begin{cases} 
2 & i = 1 \\
8i + 1 & 2 \leq i \leq n-1
\end{cases} \quad \text{and} \quad f(v_3^{(n)}) = 8n - 1.
\]
The induced edge labeling is as follows

\[
f^*(u_i u_{i+1}) = \begin{cases} 
8 & i = 1 \\
8i - 4 & 2 \leq i \leq n - 1, 
\end{cases}
\]

\[
f^*(u_i v_1^{(i)}) = \begin{cases} 
6 & i = 1 \\
8i - 6 & 2 \leq i \leq n - 1, 
\end{cases}
\]

\[
f^*(u_n v_1^{(n)}) = 8n - 5, \quad f^*(v_1^{(i)} v_2^{(i)}) = \begin{cases} 
3 & i = 1 \\
8i - 3 & 2 \leq i \leq n - 1, 
\end{cases}
\]

\[
f^*(v_1^{(n)} v_2^{(n)}) = 8n - 4 \quad \text{and} \quad f^*(v_1^{(i)} v_3^{(i)}) = \begin{cases} 
4 & i = 1 \\
8i - 2 & 2 \leq i \leq n. 
\end{cases}
\]

Hence, \( f \) is a super geometric mean labeling of \([P_n; S_m]\). Thus the graph \([P_n; S_m]\) is a super geometric mean graph, for \( n \geq 1 \) and \( m \leq 2 \).

The super geometric mean labeling of \([P_6; S_1]\) and \([P_5; S_2]\) are shown in Figure 6.

---

**Theorem 2.7** \( S(P_n \odot K_1) \) is a super geometric mean graph, for \( n \geq 1 \).

**Proof** Let \( V(P_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\} \). Let \( x_i \) be the vertex which divides the edge
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We define $f : V(S(P_n \odot K_1)) \cup E(S(P_n \odot K_1)) \to \{1, 2, 3, \ldots, 8n - 3\}$ as follows:

$$f(u_i) = \begin{cases} 
5 & i = 1 \\
8i - 7 & 2 \leq i \leq n 
\end{cases}, f(y_i) = 8i - 1 \text{ for } 1 \leq i \leq n - 1,$$

$$f(x_i) = 8i - 5 \text{ for } 1 \leq i \leq n, f(v_i) = \begin{cases} 
1 & i = 1 \\
8i - 2 & 2 \leq i \leq n - 1 
\end{cases},$$

and $f(v_n) = 8n - 3$.

The induced edge labeling is as follows

$$f^*(u_iy_i) = \begin{cases} 
6 & i = 1 \\
8i - 4 & 2 \leq i \leq n - 1 
\end{cases}, f^*(y_iu_{i+1}) = 8i \text{ for } 1 \leq i \leq n - 1,$$

$$f^*(u_ix_i) = \begin{cases} 
4 & i = 1 \\
8i - 6 & 2 \leq i \leq n 
\end{cases}, f^*(x_iv_i) = \begin{cases} 
2 & i = 1 \\
8i - 3 & 2 \leq i \leq n - 1 
\end{cases},$$

and $f^*(x_nv_n) = 8n - 4$.

Hence, $f$ is a super geometric mean labeling of $S(P_n \odot K_1)$. Thus the graph $S(P_n \odot K_1)$ is a super geometric mean graph, for $n \geq 1$. \qed

A super geometric mean labeling of $S(P_4 \odot K_1)$ is shown in Figure 7.

---

**Theorem 2.8** $TW(P_n)$ is a super geometric mean graph, for $n \geq 3$.

**Proof** Let $u_1, u_2, \ldots, u_n$ be the vertices of the path $P_n$ and $v_1^{(i)}, v_2^{(i)}$ be the pendant vertices
at each vertex \( u_i \) of the path \( P_n \), for \( 2 \leq i \leq n - 1 \). Then
\[
V(TW(P_n)) = V(P_n) \cup \{v_1^{(i)}, v_2^{(i)} : 2 \leq i \leq n - 1\} \quad \text{and} \quad E(TW(P_n)) = E(P_n) \cup \{u_iv_1^{(i)}, u_iv_2^{(i)} : 2 \leq i \leq n - 1\}.
\]

We define \( f : V(TW(P_n)) \cup E(TW(P_n)) \rightarrow \{1, 2, 3, \ldots, 6n - 9\} \) as follows
\[
f(u_i) = \begin{cases} 
1 & i = 1 \\
6i - 7 & 2 \leq i \leq n - 2,
\end{cases}
\]
\[
f(u_{n-1}) = 6n - 11, f(u_n) = 6n - 9,
\]
\[
f(v_1^{(i)}) = \begin{cases} 
2 & i = 2 \\
6i - 9 & 3 \leq i \leq n - 2,
\end{cases}
\]
\[
f(v_2^{(i)}) = 6i - 5 \quad \text{for} \ 2 \leq i \leq n - 2 \quad \text{and} \quad f(v_2^{(n-1)}) = 6n - 14.
\]

The induced edge labeling is as follows
\[
f^*(u_iu_{i+1}) = \begin{cases} 
3 & i = 1 \\
6i - 4 & 2 \leq i \leq n - 3,
\end{cases}
\]
\[
f^*(u_{n-1}u_n) = 6n - 10,
\]
\[
f^*(u_iv_1^{(i)}) = 6i - 8 \quad \text{for} \ 2 \leq i \leq n - 2, \quad f^*(u_{n-1}v_1^{(n-1)}) = 6n - 13 \quad \text{and}
\]
\[
f^*(u_iv_2^{(i)}) = 6i - 6 \quad \text{for} \ 2 \leq i \leq n - 1.
\]

Hence, \( f \) is a super geometric mean labeling of \( TW(P_n) \). Thus the graph \( TW(P_n) \) is a super geometric mean graph, for \( n \geq 3 \). \( \square \)

A super geometric mean labeling of \( TW(P_8) \) is shown in Figure 8.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{Figure 8}
\end{figure}

**Theorem 2.9** \( M(P_n) \) is a super geometric mean graph, for \( n \geq 4 \).

**Proof** Let \( V(P_n) = \{v_1, v_2, \ldots, v_n\} \) and \( E(P_n) = \{e_i = v_iv_{i+1} : 1 \leq i \leq n - 1\} \) be the
vertex set and edge set of the path $P_n$. Then
\[
V(M(P_n)) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}\} \quad \text{and} \quad E(M(P_n)) = \{v_ie_i, e_iv_{i+1} : 1 \leq i \leq n - 1\} \cup \{e_ie_{i+1} : 1 \leq i \leq n - 2\}.
\]

We define $f: V(M(P_n)) \cup E(M(P_n)) \rightarrow \{1, 2, 3, \ldots, 5n-5\}$ as follows:
\[
f(v_i) = \begin{cases} 1 & i = 1 \\ 2i + 1 & 2 \leq i \leq 3 \\ 5i - 5 & 4 \leq i \leq n \end{cases} \quad \text{and} \quad f(e_i) = \begin{cases} 8i - 5 & 1 \leq i \leq 2 \\ 5i - 2 & 3 \leq i \leq n - 1. \end{cases}
\]

The induced edge labeling is as follows:
\[
f^*(e_ie_{i+1}) = \begin{cases} 6i & 1 \leq i \leq 2 \\ 5i + 1 & 3 \leq i \leq n - 2 \end{cases} \quad \text{and} \quad f^*(e_iv_i) = \begin{cases} 2 & i = 1 \\ 2i + 4 & 2 \leq i \leq 3 \\ 5i - 3 & 4 \leq i \leq n - 1 \end{cases}
\]

and $f^*(e_iv_{i+1}) = 5i - 1$ for $1 \leq i \leq n - 1$.

Hence, $f$ is a super geometric mean labeling of $M(P_n)$. Thus the graph $M(P_n)$ is a super geometric mean graph for $n \geq 4$.

A super geometric mean labeling of $M(P_3)$ is shown in Figure 9.

![Figure 9](image_url)

**Theorem 2.10** $TL_n$ is a super geometric mean graph, for $n \geq 3$.

**Proof** Let the vertex set of $TL_n$ be $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and the edge set of $TL_n$ be $\{u_iv_{i+1}, u_iv_{i+1}, v_iv_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_iv_i : 1 \leq i \leq n\}$. Then $TL_n$ has $2n$ vertices and $4n - 3$ edges. We define $f: V(TL_n) \cup E(TL_n) \rightarrow \{1, 2, 3, \ldots, 6n-3\}$ as follows:
\[
f(v_i) = \begin{cases} 1 & i = 1 \\ 6i - 6 & 2 \leq i \leq n \end{cases} \quad \text{and} \quad f(u_i) = 6i - 2 \text{ for } 1 \leq i \leq n - 1.
\]

and $f(u_n) = 6n - 3$. The induced edge labeling is as follows:
\[
f^*(u_iv_{i+1}) = 6i - 3 \text{ for } 1 \leq i \leq n - 1, \quad f^*(u_iv_i) = 6i + 1 \text{ for } 1 \leq i \leq n - 1, \\
f^*(u_{i+1}v_i) = 6i - 4 \text{ for } 1 \leq i \leq n \text{ and } f^*(u_iv_{i+1}) = 6i - 1 \text{ for } 1 \leq i \leq n - 1.
\]

Hence, $f$ is a super geometric mean labeling of $TL_n$. Thus the graph $TL_n$ is a super geometric mean graph for $n \geq 3$. \qed
A super geometric mean labeling of $TL_7$ are shown in Figure 10.

![Figure 10](image)

**Theorem 2.11** $C_n \odot K_1$ is a super geometric mean graph.

**Proof** Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle $C_n$ and $u_1, u_2, \ldots, u_n$ be the pendant vertices of the cycle $C_n$.

**Case 1.** $n \geq 7$.

We define $f : V(C_n \odot K_1) \cup E(C_n \odot K_1) \rightarrow \{1, 2, 3, \ldots, 4n\}$ as follows:

$$f(v_i) = \begin{cases} 
3 & i = 1 \\
8i - 11 & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
4n - 7 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\
4n - 2 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\
8n + 12 - 8i & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n 
\end{cases}$$

$$f(u_i) = \begin{cases} 
7i - 6 & 1 \leq i \leq 3 \\
8i - 9 & 4 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
4n - 5 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\
4n - 2 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\
4n - 7 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\
8n + 10 - 8i & \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq i \leq n.
\end{cases}$$

The induced edge labeling is as follows:

$$f^*(v_i v_{i+1}) = \begin{cases} 
4 & i = 1 \\
8i - 7 & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\
4n - 11 & i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is odd} \\
4n - 6 & i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is even} \\
4n - 3 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\
8n + 8 - 8i & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n - 1,
\end{cases}$$

$f^*(v_1 v_n) = 6$ and
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\[ f^*(u_iv_i) = \begin{cases} 
5i - 3 & 1 \leq i \leq 2 \\
8i - 10 & 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
4n - 6 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\
4n - 1 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\
8n + 11 - 8i & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n
\end{cases} \]

Case 2. \( n = 3, 4, 5, 6. \)

In this case, the super geometric mean labelings are given in Figure 11.

Hence, \( f \) is a super geometric mean labeling of \( C_n \circ K_1 \). Thus the graph \( C_n \circ K_1 \) is a super geometric mean graph.

Figure 11

A super geometric mean labeling of \( C_9 \circ K_1 \) is shown in Figure 12.
Theorem 2.12 The graph obtained by duplication of an arbitrary vertex in cycle \( C_n \) is a super geometric mean graph, for \( n \geq 4 \).

Proof Let \( v_1, v_2, v_3, \ldots, v_n \) be the vertices of the cycle \( C_n \), for \( n \geq 4 \). Without loss of generality we duplicate the vertex \( v = v_1 \) and its duplicated vertex is \( v'_1 \). Then the resultant graph \( G \) will have \( n + 1 \) vertices and \( n + 2 \) edges.

We define \( f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, 2n + 3\} \) as follows:

\[
f(v'_1) = 1, \\
f(v_i) = \begin{cases} 
8 - 2i & 1 \leq i \leq 2 \\
4i & 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
2n + 3 & i = \left\lceil \frac{n}{2} \right\rceil + 1 \\
2n & i = \left\lceil \frac{n}{2} \right\rceil + 2 \text{ and } n \text{ is odd} \\
2n - 1 & i = \left\lceil \frac{n}{2} \right\rceil + 2 \text{ and } n \text{ is even} \\
4n + 7 - 4i & \left\lceil \frac{3n}{2} \right\rceil + 3 \leq i \leq n - 1 
\end{cases}
\]

and 

\[
f(v_n) = 9. 
\]

The induced edge labeling is as follows

\[
f^*(v'_1v_2) = 2, \quad f^*(v'_1v_n) = 3, 
\]
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\[
f^*(v_i, v_{i+1}) = \begin{cases} 
2i + 3 & 1 \leq i \leq 2 \\
4i + 2 & 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\
2n + 1 & i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is odd} \\
2n + 2 & i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is even} \\
2n + 2 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\
2n + 1 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\
4n + 5 - 4i & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n - 2,
\end{cases}
\]

\[f^*(v_{n-1}v_n) = 10 \text{ and } f^*(v_1v_n) = 8.\]

Hence, \(f\) is a super geometric mean labeling of \(G\). Thus the graph obtained by duplication of an arbitrary vertex in the cycle \(C_n\) is a super geometric mean graph, for \(n \geq 4\).

The graph obtained by duplication of a vertex in \(C_9\) and its super geometric mean labeling is shown in Figure 13.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure13.png}
\caption{Figure 13}
\end{figure}

**Theorem 2.13** The graph obtained by duplication of an arbitrary edge in cycle \(C_n\) is a super geometric mean graph, for \(n \geq 3\).

**Proof** Let \(v_1, v_2, v_3, \ldots, v_n\) be the vertices of the cycle \(C_n\). Without loss of generality we duplication an edge \(e = v_1v_2\) and its duplicated edge is \(e' = v'_1v'_2\). Then the resultant graph \(G\) will have \(n + 2\) vertices and \(n + 3\) edges.

**Case 1.** \(n \geq 6\).

We define \(f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, 2n + 5\}\) as follows:

\[f(v'_1) = 1, f(v'_2) = 3\] and
\[
f(v'_i) = \begin{cases} 
9 & i = 1 \\
5i - 5 & 2 \leq i \leq 3 \\
4i - 2 & 4 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \\
2n + 5 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \\
2n + 2 & i = \left\lfloor \frac{n}{2} \right\rfloor + 3 \text{ and } n \text{ is odd} \\
2n + 1 & i = \left\lfloor \frac{n}{2} \right\rfloor + 3 \text{ and } n \text{ is even} \\
4n + 13 - 4i & \left\lfloor \frac{n}{2} \right\rfloor + 4 \leq i \leq n.
\end{cases}
\]

The induced edge labeling is as follows:
\[
f^*(v'_1v'_2) = 2, f^*(v'_1v'_n) = 4, f^*(v'_2v'_3) = 6,
\]
\[
f^*(v_i v_{i+1}) = \begin{cases} 
i + 6 & 1 \leq i \leq 2 \\
4i & 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
2n + 3 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\
2n + 4 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\
2n + 4 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd} \\
2n + 3 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\
4n + 11 - 4i & \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq i \leq n - 1
\end{cases}
\]

and \( f^*(v_1v_n) = 11. \)

**Case 2.** \( n = 3, 4, 5. \)

In this case, the super geometric mean labelings are given in Figure 14.

![Figure 14](image)

Hence, \( f \) is a super geometric mean labeling of \( G \). Thus the graph obtained by duplication of an arbitrary edge in cycle \( C_n \) is a super geometric mean graph, for \( n \geq 3. \)

The graph obtained by duplication of an edge in \( C_8 \) and its super geometric mean labeling is shown in Figure 15.
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A triangular grid $T_n(G)$ with $n$ vertices in each side are constructed as follows: The vertices of $T_n(G)$ are $\{v^{(j)}_i : 1 \leq j \leq n, 1 \leq i \leq n + 1 - j\}$ and the edges are $\{v^{(j)}_i v^{(j)}_{i+1} : 1 \leq j \leq n - 1, 1 \leq i \leq n - j\}$ $\bigcup \{v^{(j)}_i v^{(j+1)}_i : 1 \leq j \leq n - 1, 1 \leq i \leq n - j\}$ $\bigcup \{v^{(j)}_{i+1} v^{(j+1)}_i : 1 \leq j \leq n - 1, 1 \leq i \leq n - j\}$. The triangular grid graph $T_n(G)$ is shown in Figure 16.

**Theorem 2.14** The triangular grid graph $T_n(G)$ is a super geometric mean graph.

**Proof** Let $\{v^{(j)}_i : 1 \leq j \leq n, 1 \leq i \leq n + 1 - j\}$ be the vertex set of $T_n(G)$. Then the edge set of $T_n(G)$ are $\{v^{(j)}_i v^{(j)}_{i+1} : 1 \leq j \leq n - 1, 1 \leq i \leq n - j\}$ $\bigcup \{v^{(j)}_i v^{(j+1)}_i : 1 \leq j \leq n - 1, 1 \leq i \leq n - j\}$ $\bigcup \{v^{(j)}_{i+1} v^{(j+1)}_i : 1 \leq j \leq n - 1, 1 \leq i \leq n - j\}$.
We define \( f : V(T_n(G)) \cup E(T_n(G)) \rightarrow \{1, 2, 3, \ldots, n(2n - 1)\} \) as follows

\[
f(v_i^{(1)}) = i(2i - 1) \text{ for } 1 \leq i \leq n \text{ and }
f(v_i^{(j)}) = f(v_{i+1}^{(j-1)}) - 2 \text{ for } 2 \leq j \leq n \text{ and } 1 \leq i \leq n - 1 - j.
\]

The induced edge labeling is as follows

\[
f^*(v_i^{(1)}v_{i+1}^{(1)}) = i(2i + 1) \text{ for } 1 \leq i \leq n - 1,
f^*(v_i^{(j)}v_{i+1}^{(j)}) = f^*(v_{i+1}^{(j-1)}v_{i+2}^{(j-1)}) - 2 \text{ for } 2 \leq j \leq n - 1 \text{ and } 1 \leq i \leq n - j,
f^*(v_i^{(1)}v_i^{(2)}) = (i + 1)(2i - 1) \text{ for } 1 \leq i \leq n - 1,
f^*(v_i^{(j)}v_{i+1}^{(j+1)}) = f^*(v_{i+1}^{(j-1)}v_{i+2}^{(j-1)}) - 2 \text{ for } 2 \leq j \leq n - 1 \text{ and } 1 \leq i \leq n - j,
f^*(v_{i+1}^{(1)}v_i^{(2)}) = i(2i - 1) + 4i \text{ for } 1 \leq i \leq n - 1 \text{ and }
f^*(v_i^{(j)}v_{i+1}^{(j+1)}) = f^*(v_{i+2}^{(j-1)}v_{i+1}^{(j)}) - 2 \text{ for } 2 \leq j \leq n - 1 \text{ and } 1 \leq i \leq n - j.
\]

Hence, \( f \) is a super geometric mean labeling of \( T_n(G) \). Thus the graph \( T_n(G) \) is a super geometric mean graph.

A super geometric mean labeling of \( T_7(G) \) is shown in Figure 17.

![Figure 17](image_url)

The graph \( G'(p_1, p_2, \ldots, p_n) \) is obtained from \( n \) cycles of length \( p_1, p_2, \ldots, p_n \) by identifying the \( j^{th} \) cycle and \( (j + 1)^{th} \) cycle by the edges \( v_i^{(j)}v_{i+1}^{(j)} \) and \( v_i^{(j+1)}v_{p_i+1}^{(j+1)} \), for \( 1 \leq j \leq n - 1 \).

**Theorem 2.15** The graph \( G'(p_1, p_2, \ldots, p_n) \) is a super geometric mean graph all \( p_j \)'s are odd or all \( p_j \)'s are even with \( p_j \neq 4 \) for \( 1 \leq j \leq n \).
Proof Let \( \{v_i^{(j)} : 1 \leq j \leq n \text{ and } 1 \leq i \leq p_j\} \) be the vertices of the number of cycles with \( p_j \neq 4 \). For \( 1 \leq j \leq n - 1 \), the \( j^{th} \) cycle and \((j + 1)^{th} \) cycle by the edges \( v_{p_{j+1}}^{(j)}v_{p_{j+1}}^{(j+1)} \) and \( v_{p_{j+1}}^{(j+1)}v_{p_{j+1}}^{(j+1)} \). We define \( f : V(G') \cup E(G') \to \{1, 2, 3, \ldots, \sum_{i=1}^{n} 2p_i - 3n + 3\} \) as follows.

**Case 1.** \( p_j \) is odd.

When \( p_1 = 5 \), define
\[
f(v_1^{(1)}) = 3, f(v_2^{(1)}) = 1, f(v_3^{(1)}) = 10, f(v_4^{(1)}) = 8 \text{ and } f(v_5^{(1)}) = 6.
\]
The induced edge labeling is as follows:
\[
f^*(v_1^{(1)}v_2^{(1)}) = 2, f^*(v_2^{(1)}v_3^{(1)}) = 4, f^*(v_3^{(1)}v_4^{(1)}) = 9,
\]
f\( f^*(v_4^{(1)}v_5^{(1)}) = 7 \) and \( f^*(v_1^{(1)}v_5^{(1)}) = 5. \)

When \( p_1 \geq 7 \), define
\[
f(v_1^{(1)}) = \begin{cases} 
4i - 3 & \text{for } 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\
4i - 2 & \text{for } i = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\
4i - 9 & \text{for } i = \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \\
4p_1 + 4 - 4i & \text{for } \left\lfloor \frac{p_1}{2} \right\rfloor + 3 \leq i \leq p_1
\end{cases}
\]

The induced edge labeling is as follows:
\[
f^*(v_1^{(1)}v_{p_1}^{(1)}) = \begin{cases} 
4i - 1 & \text{for } 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor - 1 \\
4i & \text{for } i = \left\lfloor \frac{p_1}{2} \right\rfloor \\
4i - 3 & \text{for } i = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\
4p_1 + 2 - 4i & \text{for } \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \leq i \leq p_1 - 1
\end{cases}
\]
and
\[
f^*(v_1^{(1)}v_{p_1}^{(1)}) = 2.
\]

For \( 2 \leq j \leq n \), define
\[
f(v_i^{(j)}) = \begin{cases} 
\sum_{k=1}^{j-1} 2p_k - 3(j - 2) + 4i - 5 & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \\
\sum_{k=1}^{j-1} 2p_k + 4p_j - 3(j - 2) - 4i & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \leq i \leq p_j - 1
\end{cases}
\]

The induced edge labeling is as follows:
\[
f^*(v_i^{(j)}v_{i+1}^{(j)}) = \begin{cases} 
\sum_{k=1}^{j-1} 2p_k - 3(j - 2) + 2 & i = 1 \\
\sum_{k=1}^{j-1} 2p_k - 3(j - 2) + 4i - 3 & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor \\
\sum_{k=1}^{j-1} 2p_k + 4p_j - 3(j - 2) - 4i - 2 & \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \leq i \leq p_j - 2
\end{cases}
\]
and
\[
f^*(v_{p_j-1}^{(j)}v_{p_j}^{(j)}) = \sum_{k=1}^{j-1} 2p_k - 3(j - 2) + 1.
\]
Case 2. \( p_j \) is even.

When \( p_1 = 6 \), define
\[
f(v_1^{(1)}) = 6, f(v_2^{(1)}) = 8, f(v_3^{(1)}) = 10, f(v_4^{(1)}) = 12, f(v_5^{(1)}) = 1 \text{ and } f(v_6^{(1)}) = 3.
\]
The induced edge labeling is as follows:
\[
f^*(v_1^{(1)}v_2^{(1)}) = 7, f^*(v_2^{(1)}v_3^{(1)}) = 9, f^*(v_3^{(1)}v_4^{(1)}) = 11,
\]
\[
f^*(v_4^{(1)}v_5^{(1)}) = 4, f^*(v_5^{(1)}v_6^{(1)}) = 2 \text{ and } f^*(v_1^{(1)}v_6^{(1)}) = 5.
\]
When \( p_1 \geq 8 \), define
\[
f(v_1^{(1)}) = \begin{cases} 
4i - 3 & 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\
4p_1 + 4 - 4i & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \leq i \leq p_1.
\end{cases}
\]
The induced edge labeling is as follows:
\[
f^*(v_1^{(1)}v_{i+1}^{(1)}) = \begin{cases} 
4i - 1 & 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\
4p_1 + 2 - 4i & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \leq i \leq p_1 - 1
\end{cases}
\]
and \( f^*(v_1^{(1)}v_{p_1}^{(1)}) = 2. \)

Subcase 2.1 \( 2 \leq j \leq n \) and \( j \) is odd.

Let \( f(v_i^{(j)}) = \begin{cases} 
\sum_{k=1}^{j-1} 2p_k - 3(j-2) + 4i - 5 & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor - 1 \\
\sum_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 6 & i = \left\lfloor \frac{p_j}{2} \right\rfloor \\
\sum_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 3 & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \\
\sum_{k=1}^{j-1} 2p_k + 4p_j - 3(j-2) - 4i & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \leq i \leq p_j - 1.
\end{cases}\)

The induced edge labeling is as follows
\[
f^*(v_j^{(j)}v_{i+1}^{(j)}) = \begin{cases} 
\sum_{k=1}^{j-1} 2p_k - 3(j-2) + 2 & i = 1 \\
\sum_{k=1}^{j-1} 2p_k - 3(j-2) + 4i - 3 & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor - 1 \\
\sum_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 4 & i = \left\lfloor \frac{p_j}{2} \right\rfloor \\
\sum_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 5 & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \\
\sum_{k=1}^{j-1} 2p_k + 4p_j - 3(j-2) - 4i - 2 & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \leq i \leq p_j - 2
\end{cases}
\]
and \( f^*(v_{p_j-1}^{(j)}v_{p_j}^{(j)}) = \sum_{k=1}^{j-1} 2p_k - 3(j-2) + 1. \)

Subcase 2.2 \( 2 \leq j \leq n \) and \( j \) is even.
Let \( f(v^{(j)}_i) = \begin{cases} 
\sum_{k=1}^{j-1} 2p_k - 3(j - 2) + 4i - 4 & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor - 1 \\
\sum_{k=1}^{j-1} 2p_k + 2p_j - 3(j - 2) - 3 & i = \left\lfloor \frac{p_j}{2} \right\rfloor \\
\sum_{k=1}^{j-1} 2p_k + 2p_j - 3(j - 2) - 6 & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \\
\sum_{k=1}^{j-1} 2p_k + 4p_j - 3(j - 2) - 4i - 1 & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \leq i \leq p_j - 1.
\end{cases} \]

The induced edge labeling is as follows:

\[
f^*(v^{(j)}_i v^{(j)}_{i+1}) = \begin{cases} 
\sum_{k=1}^{j-1} 2p_k - 3(j - 2) + 1 & i = 1 \\
\sum_{k=1}^{j-1} 2p_k - 3(j - 2) + 4i - 2 & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor - 2 \\
\sum_{k=1}^{j-1} 2p_k + 2p_j - 3(j - 2) - 5 & i = \left\lfloor \frac{p_j}{2} \right\rfloor - 1 \\
\sum_{k=1}^{j-1} 2p_k + 2p_j - 3(j - 2) - 4 & i = \left\lfloor \frac{p_j}{2} \right\rfloor \\
\sum_{k=1}^{j-1} 2p_k + 4p_j - 3(j - 2) - 4i - 3 & \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \leq i \leq p_j - 2
\end{cases} \]

and \( f^*(v^{(j)}_{p_{j-1}} v^{(j)}_{p_j}) = \sum_{k=1}^{j-1} 2p_k - 3(j - 2) + 2. \)

Hence, \( f \) is a super geometric mean labeling of \( G'(p_1, p_2, \ldots, p_n) \). Thus it is a super geometric mean graph with \( p_j \neq 4 \) for \( 1 \leq j \leq n \). \( \square \)

A super geometric mean labeling of \( G'(7, 13, 11, 5) \) and \( G'(8, 10, 12, 8) \) are shown in Figure 18.
References


