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Further Results on Super (a, d) Edge-Antimagic Graceful Labeling of Graphs

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Abstract: An (a, d)-edge-antimagic graceful labeling is a bijection g from $V(G) \cup E(G)$ into $\{1, 2, \cdots, |V(G)| + |E(G)|\}$ such that for each edge $xy \in E(G)$, $|g(x) + g(y) - g(xy)|$ form an arithmetic progression starting from a and having a common difference d. An (a, d)-edge-antimagic graceful labeling is called super (a, d)-edge-antimagic graceful if $g(V(G)) = \{1, 2, \cdots, |V(G)|\}$. A graph that admits an super (a, d)-edge-antimagic graceful labeling is called a super (a, d)-edge-antimagic graceful graph. In this paper, we prove the super (a, d) edge antimagic gracefulness of regular graphs. Later, we study the non-regular graph is super $(a, 1)$-edge-antimagic graceful graph. Finally, we find super edge-antimagic graceful labeling of some classes of graphs.

Key Words: Labelling, (a, d)-edge-antimagic total labeling, Smarandachely edge-antimagic total labeling, (a, d)-edge-antimagic graceful labeling, super(a, d)-edge-antimagic graceful labeling.

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§1. Introduction

Throughout this paper, we only concern with connected, undirected simple graphs of order p and size q. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively.

Let $|V(G)| = p$ and $|E(G)| = q$ be the number of vertices and the number of edges of G, respectively. General references for graph-theoretic notions are [1,10].

A labeling of a graph is any map that carries some set of graph elements to numbers. Hartsfield and Ringel [4] introduced the concept of an antimagic labeling and they defined an antimagic labeling of a (p, q) graph G as a bijection f from $E(G)$ to the set $\{1, 2, \cdots, q\}$ such that the sums of label of the edges incident with each vertex $v \in V(G)$ are distinct.

An (a, d)-edge-antimagic total labeling was introduced by Simanjuntak, Bertault and Miller in [9]. This labeling is the extension of the notions of edge-magic labeling, see [5,6].

For a graph $G = (V, E)$, a bijection g from $V(G) \cup E(G)$ into $\{1, 2, \cdots, |V(G)| + |E(G)|\}$ is called an (a, d)-edge-antimagic total labeling of G if the edge-weights $w(xy) = g(x) + g(y) + g(xy), xy \in E(G)$, form an arithmetic progression starting from a and having a common differ-

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ence d. Generally, let $H \prec G$ be a typical subgraph of G with $|V(G-H)| = a'$, $|E(G-H)| = b'$. If there is an (a', d')-edge-antimagic total labeling g' on $G - H$, such a labeling g' is called a Smarandachely edge-antimagic total labeling. Particularly, let $H = \emptyset$ or a typical graph in K_2, P_3, C_3 or $S_{1,3}$. We get the (a, d)-edge-antimagic total labeling or nearly (a, d)-edge-antimagic total labeling of G.

The $(a, 0)$-edge-antimagic total labelings are usually called edge-magic in the literature. An (a, d)-edge antimagic total labeling is called super if the smallest possible labels appear on the vertices.

In [7] Marimuthu et al. introduced an edge magic graceful labeling of a graph. They presented some properties of super edge magic graceful graphs and proved some classes of graphs are super edge magic graceful. In [8] Marimuthu and Krishnaveni introduced super edge antimagic graceful labeling.

An (a, d)-edge-antimagic graceful labeling is defined as a one-to-one mapping from $V(G) \cup E(G)$ into the set $\{1, 2, 3, \cdots, p + q\}$ so that the set of edge-weights of all edges in G is equal to $\{a, a + d, a + 2d, \cdots, a + (q - 1)d\}$, for two integers $a \geq 0$ and $d > 0$.

An (a, d)-edge-antimagic graceful labeling g is called super (a, d)-edge-antimagic graceful if $g(V(G)) = \{1, 2, \cdots, p\}$ and $g(E(G)) = \{p+1, p+2, \cdots, p+q\}$. A graph G is called (a, d)-edge-antimagic graceful or super (a, d)-edge-antimagic graceful if there exists an (a, d)-edge-antimagic graceful or a super (a, d)-edge-antimagic graceful labeling of G.

Baca et al. [2] proved super $(a, 1)$-edge-antimagic total labeling of regular graphs. In [3] Baca et.al proved some classes of graphs like Frienship graphs, Fan graphs and Wheel graphs has super edge-antimagic graceful labeling. In this paper, we study super (a, d)-edge-antimagic graceful labeling of regular graphs. We also prove some classes of graphs, including friendship graphs, cycles and fan graphs has super (a, d)-edge-antimagic graceful labeling.

§2. Main Results

Theorem 2.1 *If G is a connected super (a, d)-edge-antimagic graceful graph, then $d \leq 2$.*

Proof Let G be a connected super (a, d)-edge-antimagic graceful graph. Suppose that $d \geq 3$. There exists a bijection $g : V(G) \cup E(G) \to \{1, 2, \cdots, p + q\}$ which is a super (a, d)-edge-antimagic graceful labeling with the set of edge-weights.

$$\begin{aligned} W &= \{w(xy) : w(xy) = |g(x) + g(y) - g(xy)|, xy \in E(G)\} \\ &= \{a, a + d, a + 2d, \cdots, a + (q - 1)d\}. \end{aligned}$$

It is easy to see that the minimum possible edge-weight in a super (a, d)-edge-antimagic graceful labeling is at least $|1 + p - (p + 1)| = 0$.

We observe that $a \geq 0$. On the other hand, the maximum edge-weight is no more than $|1+2-(p+q)| = p+q-3$. Therefore, $a+(q-1)d \leq p+q-3$. This shows that $(q-1)d \leq p+q-3$.

Hence,

$$\begin{aligned}
&d \leq \frac{p+q-3}{q-1} \Rightarrow 3 \leq d \leq \frac{p+q-3}{q-1}\\
&\Rightarrow 3 \leq \frac{p+q-3}{q-1} \Rightarrow 3 \leq \frac{p-2}{q-1}+1 \Rightarrow 2 \leq \frac{p-2}{q-1}\\
&\Rightarrow 2 \leq \frac{p-2}{p-1-1} \text{ (since the size of every connected graph of order } p \text{ is at least } p-1)\\
&\Rightarrow 2 \leq 1,
\end{aligned}$$

a contradiction. Hence, $d \leq 2$. □

Theorem 2.2 *Let G be a connected $(p,q)-$graph which is not a tree. If G has a super (a,d)-edge-antimagic graceful labeling then $d=1$.*

Proof Assume that G has a super (a,d)-edge-antimagic graceful labeling $f : V(G) \cup E(G) \longrightarrow \{1,2,\cdots,p+q\}$ and $\{w(uv) : uv \in E(G)\} = \{a, a+d, a+2d, \cdots, a+(q-1)d\}$ is the set of edge-weights. The minimum possible edge-weight $a \geq 0$. The maximum edge-weight is no more than $p+q-3$. Thus $a+(q-1)d \leq p+q-3$. and

$$d \leq \frac{p+q-3}{q-1}. \tag{2.1}$$

But, $p \leq q$ (Since G is not a tree T). Then, (2.1) gives $d < 2$. □

§3. Super (a,d)-Edge-Antimagic Graceful Labeling of Regular Graphs

Proposition 3.1(Petersen theorem) *Let G be a $2r-$regular graph. Then there exists a $2-factor$ in G.*

Notice that after removing edges of the $2-factor$ guaranteed by the Petersen theorem we have again an even regular graph. Thus, by induction, an even regular graph has a $2-$factorization.

The construction in the following theorem allows us to find a super $(a,1)-$ edge-antimagic graceful labeling of any even regular graph. Notice that the construction does not require the graph to be connected. In the following theorem we denote $[a,b]$ is the set of consecutive integers $\{a, a+1, \cdots, b\}$.

Theorem 3.2 *Let G be a graph on p vertices that can be decomposed into two factors G_1 and G_2. If G_1 is edge-empty or if G_1 is a super (0,1)-edge-antimagic graceful graph and G_2 is a 2r-regular graph then G is super (0,1)-edge-antimagic graceful.*

Proof First we start with the case when G_1 is not edge-empty. Since G_1 is a super (0,1)-edge-antimagic graceful graph with p vertices and q edges, there exists a total labeling $f : V(G_1) \cup E(G_1) \longrightarrow [1, p+q]$ such that $\{\left|f(x)+f(y)-f(xy)\right| : xy \in E(G)\} = \{0,1,2\cdots,q-1\}$.

By the Petersen theorem there exists a $2-$factorization of G_2. We denote the $2-$factors by F_j, $j=1,2,\cdots,r$. Let $V(G)=V(G_1)=V(F_j)$ for all j and $E(G)= U_{j=1}^{r}E(F_j)\bigcup E(G_1)$. Each factor F_j is a collection of cycles. We order and orient the cycles arbitrarily. Now by the symbol $e_j^{out}(v_i)$ we denote the unique outgoing arc from the vertex v_i in the factor F_j. We define a total labeling g of G in the way that $g(v)=f(v)$ for $v\in V(G)$, $g(e)=f(e)$ for $e\in E(G_1)$ and $g(e)=q+jp+f(v_i)$ for $e=e_j^{out}(v_i)$. Then, the vertices are labeled by the first p integers. The edges of G_1 by the next q labels and the edges of G_2 by consecutive integers starting at $p+q+1$. Thus g is a bijection $V(G)\cup E(G)\longrightarrow\{1,2\cdots,p+q+pr\}$ Since $|E(G)|= q+pr$. It is not difficult to verify that g is a super $(0,1)$-edge-antimagic graceful labeling of G. The weights of the edges e in $E(G_1)$ is $w_g(e)= w_f(e)$. The weights form the progression $0,1,2,\cdots,q-1$. For convenience, we denote by v_k the unique vertex such that $v_iv_k=e_j^{out}(v_i)$ in F_j. The weights of the edges in F_j, $j=1,2,\cdots,r$ are

$$\begin{aligned} w_g(e_j^{out}(v_i)) &= w_g(v_iv_k)=\big|g(v_i)-(q+jp+f(v_i))+g(v_k)\big| \\ &= \big|f(v_i)-(q+jp+f(v_i))+f(v_k)\big| = \big|-q-jp+f(v_k)\big| \\ &= \big|-(q+jp-f(v_k))\big| \end{aligned}$$

for all $i=1,2,\cdots,p$ and $j=1,2,\cdots,r$. Since F_j is a factor, the set $\{f(v_k):v_k\in F_j\}=[1,p]$. Hence we have that the set of edge-weights in the factor F_j is $[q+(j-1)p,q+jp-1]$ and thus the set of all edge-weights in G is $[0,q+rp-1]$. If G_1 is edge-empty it is enough to take $q=0$. and proceed with the labeling of factors F_j. □

By taking an edge-empty graph G_1 we have the following theorem.

Theorem 3.3 *All even regular graphs of order p with at least one edge are super* $(0,1)$*-edge-antimagic graceful.*

The disjoint union of $m\geq 1$ copies of a graph G is denoted by mG.

Theorem 3.4 *Let* k,m *be positive integers. Then the graph* $kP_2\cup mK_1$ *is super* $(0,1)$*-edge-antimagic graceful.*

Proof We denote the vertices of the graph $G\cong kP_2\cup mK_1$ by the symbols $v_1,v_2,\cdots,v_{2k+m}$ in such a way that $E(G)= \{v_iv_{k+m+i}:i=1,2,\cdots,k\}$ and the remaining vertices are denoted arbitrarily by the unused symbols. We define the labeling $f:V(G)\cup E(G)\longrightarrow\{1,2,\cdots,3k+m\}$ in the following way $f(v_j)=j$ for $j=1,2,\cdots,2k+m$, $f(v_iv_{k+m+i})=2k+m+i$ for $i=1,2,\cdots,k$. It is easy to see that f is a bijection and that the vertices of G are labeled by the smallest possible numbers. For the edge-weights we get $w_f(v_iv_{k+m+i})=|f(v_i)+f(v_{k+m+i})-f(v_iv_{k+m+i})|=$ k-i for $i=1,2,\cdots,k$. Thus, f is a super $(0,1)$-edge-antimagic graceful labeling of G. □

Now by taking $m=0$ and observing that the number of vertices in kP_2 is $2k$, then we immediately obtain the following corollary.

Corollary 3.5 *If* G *is an odd regular graph on p vertices that has a* $1-factor$ *then* G *is super*

$(0,1)$-*edge-antimagic graceful.*

§4. Friendship Graphs

The friendship graph $\mathbf{F_n}$ is a set of n triangles having a common center vertex and otherwise disjoint. Let c denote the center vertex. For the i^{th} triangle, let x_i and y_i denote the other two vertices.

Theorem 4.1 *Every friendship graph* $\mathbf{F_n}, n \geq 1$, *has super* $(a,1)$-*edge-antimagic graceful labeling.*

Proof Label the vertices and edges of $\mathbf{F_n}$ by the following functions g_1 and g_2 respectively.

$$g_1(c) = n+1, \quad g_1(x_i) = i, \quad g_1(y_i) = 2n+2-i \text{ for } 1 \leq i \leq n,$$
$$g_2(x_i c) = 3n+2i, \quad g_2(y_i c) = 5n+3-2i, \quad g_2(x_i y_i) = 2n+1+i.$$

Notice that, in this labeling $a = 0$. It is easy to verify that the set of edge-weights consists of the consecutive integers $\{0, 1, 2, \cdots, 3n-1\}$ and we arrive at the desired result. □

Figure 1 illustrates the proof of Theorem 4.1.

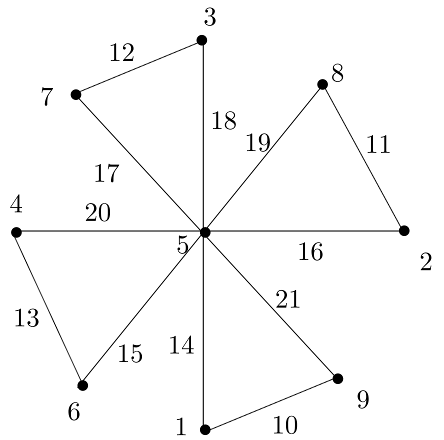

Figure 1. A $(0,1)$-super edge-antimagic graceful labeling of $\mathbf{F_4}$.

§5. Cycles

Theorem 5.1 *For* $n \geq 3$, *the cycle* C_n *has super* $(a,1)$-*edge-antimagic graceful labeling.*

Proof Let a cycle C_n be defined as follows:

$$V(C_n) = \{p_1, p_2, \cdots, p_n\} \text{ and}$$
$$E(C_n) = \{p_i p_{i+1} : i = 1, 2, \cdots, n-1\} \cup \{p_n p_1\}.$$

Also, define the vertex labeling $f_1 : V(C_n) \to \{1, 2, \cdots, n\}$ and the edge labeling $f_2 : E(Cn) \to \{n+1, n+2, \cdots, n+n\}$ in the following way.

$$f_1(v_i) = i, \quad 1 \le i \le n;$$
$$f_2(v_iv_{i+1}) = n + 1 + i \text{ for } 1 \le i \le n-1;$$
$$f_2(v_nv_1) = n + 1.$$

Combining the vertex labeling f_1 and the edge labeling f_2 given above, we obtain a total labeling. The set of edge-weights consists of the consecutive integers $\{0, 1, 2, \cdots, n-1\}$. □

An illustration of Theorem 5.1 is given in Figure 2.

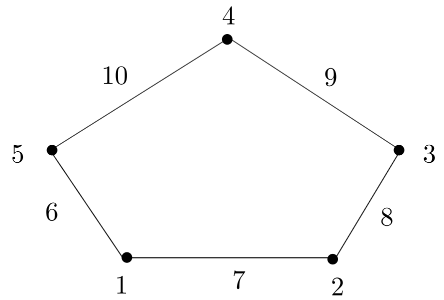

Figure 2. A $(0, 1)$-super edge-antimagic graceful labeling of C_5.

§6. Fans

A fan $\mathcal{F}_n, n \ge 2$ is a graph obtained by joining all vertices of path P_n to a further vertex called the center. Thus $\mathcal{F}_n$ contains $n+1$ vertices, say, $c, x_1, x_2, \cdots, x_n$ and $2n-1$ edges say $cx_i, 1 \le i \le n$ and $x_ix_{i+1}, 1 \le i \le n-1$.

Theorem 6.1 *The fan $\mathcal{F}_n$ is super $(a, 1)$-edge-antimagic graceful if $2 \le n \le 6$ and $d = 1$.*

Proof Label the vertices of $\mathcal{F}_n$ by $g : V(\mathcal{F}_n) \to \{1, 2, \cdots, n+1\}$ as follows:

If $n = 2$, let the labels of vertices be $g(x_1) = 1, g(x_2) = 2$ and $g(c) = 3$; If $n = 3$, let the labels be $g(x_1) = 1, g(x_2) = 2, g(x_3) = 4$ and $g(c) = 3$; If $n = 4$, let the labels be $g(x_1) = 1, g(x_2) = 2, g(x_3) = 4, g(x_4) = 5$ and $g(c) = 3$; If $n = 5$, let the labels be $g(x_1) = 2, g(x_2) = 1, g(x_3) = 3, g(x_4) = 5$, $g(x_5) = 6$ and $g(c) = 4$, and if $n = 6$, let the labels be $g(x_1) = 2, g(x_2) = 1, g(x_3) = 3, g(x_4) = 5$, $g(x_5) = 7, g(x_6) = 6$ and $g(c) = 4$. Generally, let $W_g = \{w_g(q_i) = 2 + i : 1 \le i \le 2n-1\}$ be the set of edge-weights of edges $q_i \in \mathcal{F}_n$ and label the edges of $\mathcal{F}_n$ by $g_1 : E(\mathcal{F}_n) \to \{n+2, n+3, \cdots, 3n\}$ where

$$g_1(q_i) = \begin{cases} n + 1 + \frac{i+1}{2} & \text{if } i \text{ is odd,} \\ 2n + 1 + \frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

Combining the vertex labeling g and the edge labeling g_1 gives a super $(a, 1)$-edge-antimagic graceful labeling where

$$W = \{|w_g(q_i) - g_1(q_i)| : 1 \leq i \leq 2n - 1\}$$

is the set of edge-weights. □

A $(0, 1)$-super edge-antimagic graceful labeling of the fan $\mathcal{F}_3$ is given in Figure 3.

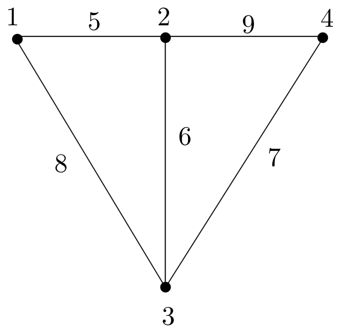

Figure 3. A $(0, 1)$-super edge-antimagic graceful labeling of $\mathcal{F}_3$.

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