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Graph Coloring, Types and Applications: A Survey

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Abstract: In recent decade, Graph Theory has many applications in problems like security key generation, brain MRI segmentation and tumor detection by cut sets, virus graph and its application during COVID-19 pandemic. The color assignment to various graph's elements is a significantly important topic for research in graph theory. It has a wide-ranging applications in sciences, medical sciences, computer engineering, electronics and telecommunication, electrical engineering, network theory, artificial intelligence and machine learning, psychology and economics, to name a few. Many conjectures are remains open problems and many researchers and mathematicians from around the world are working on it. In this paper, we review the graph's coloring, the types of coloring, theorems and axioms related to the graph-coloring, and applications.

Key Words: Graph coloring, Smarandachely Λ-coloring, vertex Smarandachely Λ-coloring, edge Smarandachely Λ-coloring, Smarandachely total coloring, total coloring, face Smarandachely Λ-coloring, perfect coloring, list coloring, strong edge coloring, acyclic coloring.

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§1. Introduction

The famous Konigsberg seven-bridge problem launched graph theory [85, 86]. The task was to begin at any point, walk through all seven bridges on the Pregel river just one time, and then came back to the initial point. In 1736, Euler [54] used a graph to resolve this issue. He represented lands with vertices and a bridge connecting two lands is an edge between them. In this manner, the problem was represented in a graph. Euler found that there is no such closed walk exists for this problem. As a result in graph theory, the Eulerian circuit concept was introduced and "*A connected graph is Eulerian if the degree of all vertices is an even number and vice versa.*" This was the first paper considered, and so the evolution of graph theory began.

Later, L.Euler [50,54,120] developed the planer graph formula based on the invariant of polyhedron in algebraic topology. If polyhedron P has n vertices, f faces and m edges then

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$n+f-m$ is invariant and $n+f-m=2$. L.Euler made significant contributions to the field of mathematics and physics; especially for the advancement in graph theory.

In 1850, legendary four-color map brought graph theory to the forefront. For nearly 127 years, this was an unsolved difficult problem. Many mathematicians and researchers attempted to solve this problem but were unsuccessful.

§2. **Graph Coloring**

Coloring a graph began in the mid-nineteenth century with the legendary four-color conjecture (4CC). Francis Guthrie discovered that all the nations on the administrative map of England in 1850 were painted in only four colors, with every two adjacent nation-states painted in a different color. He talked about it with his brother, Frederick Guthrie. Later, Frederick discussed this problem with his professor, Augustus De-Morgan but he was unable to answer. Morgan questioned William Hamilton concerning 4CC in 1852. Following that, in 1878, Arthur Cayley worked on the problem and posted a question in the London mathematical society. May [105] quotes in an article given by Harary [69] that "*Any map on a plane or the surface of a sphere can be colored with on four colors so that no two adjacent nations have the same color?*" The 4CC was first proved by Kemp [91] in 1879. But there was an error in the proof shown by Heawood [80] and demonstrated that conjecture was correct for five colors. Many mathematicians are worked to prove 4CC for more than 100 years. In 1969, Ore and Stemple [111] revealed proof of 4CC with the numerical method for all maps with less than 40 countries. Meanwhile the work on coloring of graph elements started. Finally, in 1977, Apple and Haken [84,90,94] demonstrated 4CC using a computer with 1200 hours of computer time. First time in history, a famous mathematics problem was solved extensively by using the computer.

After this many mathematicians verified the proof of 4CC in various ways. Robertson, Sanders, Seymour and Thomas [117] proved 4CC with 633 unavoidable reducible configurations. H. R. Bhapkar [14] proved this by PNR of a graph. Birkhoff [17] proposed a Chromatic polynomial in 2012, which is based on Gauss [55] fundamental algebraic theorem, which states that each n-degree polynomial with a complex coefficient has precisely n zeros.

Graphs have the ability to exemplify a wide variety of practical problems. The solutions to these problems are given by graph theory. For example, the road network problems, electrical networks consisting of resisters, capacitors and the inductors, maximum flow problems, optimal path and minimized cost for transportation problems, the communication network, social media networks, time-table scheduling of flights, trains and buses, signal flow problems in signal transmission, representation of the structure of an organic molecule in chemistry, etc. These structures can be represented as graphs, which are collections of points and lines connecting some or all pairs of points and are known as vertices and edges, respectively.

2.1. Graph

Definition 3.1([48,69,144,145]) *A graph H made up of two sets, $V(H)$-nonempty set of elements called vertices of graph H represented by point, $E(H)$-set of unordered pairs of vertices joined by an arc or a line called an edge-set of graph H. It is symbolized with $H(V,E)$. Please* *refer to books for a basic understanding of graphs.*

2.2. Graph's Coloring

Graph coloring, it is procedure of giving colors to graph components like vertices, edges, regions in a manner that separates the colors of nearby elements. This term described as a proper coloring in graph. The significant work is done on vertex coloring because the graph's edge and region coloring is identical to its line and dual graph respectively. But many problems of coloring are studied in their original form in order for getting better results and applications.

Generally, a *Smarandachely* Λ*-coloring* of a graph G on a surface $\mathcal{S}$ by colors in $\mathscr{C}$ is a mapping $\varphi_\Lambda : \mathscr{C} \to V(G) \cup E(G) \cup F(G)$ such that $\varphi(u) \neq \varphi(v)$ if u and v are elements of a subgraph isomorphic to $\Lambda \prec G$, where $F(G)$ is the face set of 2-cell embedding of G on $\mathcal{S}$ ([95, 154]).

2.3. Proper Coloring

Definition 2.2([48,144]) *A proper coloring is the process of color allocating to graph's element so that neighboring elements colored differently. If k different colors are required, it is known as k-proper coloring or k-colorable.*

The primary categories of graph coloring and some other special types of coloring are surveyed in the follows sections.

§3. Vertex Coloring

Definition 3.1([48,95,144,154]) *A vertex coloring means adjoining vertices of a graph colored differently. If we needed k colors, then it is known as k-proper vertex coloring.*

Generally, a Smarandachely Λ-coloring $\varphi_\Lambda|_{V(G)} : \mathscr{C} \to V(G)$ is called a vertex Smarandachely Λ-coloring.

Definition 3.2 *A chromatic number is the number k of least set of distinct colors required for a graph to be k-properly vertex colorable. It is represented by χ.*

The cycle graph C_4 is 2-colorable, 3-colorable and 4-colorable. The bare minimum, though, is 2-colorable. Consequently, the chromatic number $\chi(C_4)$ is 2.

The following are some standard graphs and their chromatic numbers.

(1) A complete graph K_n is n-vertex colorable;
(2) A null graph is 1-vertex colorable;
(3) A bipartite graph $(K_{m,n})$ is 2-vertex colorable;
(4) A cycle C_n is 3 or 2-vertex colorable if n is odd or even, respectively;
(5) The chromatic number of tree is 2;
(6) The chromatic number of star graph $S_{1,n}$ is 2;
(7) The chromatic number of path P_n is 2.

Many researchers contributed to the understanding of vertex coloring. In 1936, Konig [85] characterized two-colorable graphs as below.

Theorem 3.1 *If a graph G is 2-vertex colorable then G is without an odd cycle and vice versa.*

Theorem 3.2 *A graph is bipartite if and only if it is without an odd cycle and vice versa.*

He also demonstrated how to divide any k-regular bipartite graph into one factor. In 1941, Brooks [5] characterized vertex coloring for connected graphs as below.

Theorem 3.3 *Let H be a connected graph with a largest degree Δ. Then,*

(A) $\chi(H)$ *is at most Δ, excluding a complete graph or/and an odd cycle;*
(B) $\chi(H)$ *is Δ+1 for a complete graph and an odd cycle.*

In other words, Brook's theorem is equivalent to this: *An odd cycles and complete graph are both $(k-1)$-regular, k-critical graphs.* There is no way to compute chromatic number of any random graph. Greedy color algorithm is one of the most important graph color algorithm for getting this. The vertex coloring has many bounds. Thus, a clique number is one among them. Szekeres and Wilf [129] provide the upper bound.

Theorem 3.4 $\chi(H) \leq 1 + max[\ \delta(K)]$*, for a graph H and for all induced subgraphs $K \subseteq H$.*

And then, Berge C. [8, pp. 37] and Ore O. [110, pp. 225] provide a lower bound. Harary, Hedetniemi [70] provide an upper bound in terms of independent number, as below.

Theorem 3.5 *For graph H on $n-$ vertices, $\frac{n}{\beta} \leq \chi \leq n - \beta + 1$, where β is the cardinality of maximal independent subset of H.*

So, *is there any graph that has no triangles but a very high Chromatic number?* Dirac [49] posed this question, and Descartes [46] and Mycielski [104] responded positively. The result was proved for $n \geq 2$ by Kelly and Kelly [87], i.e., *there exists n-chromatic graph with a girth is more than 5.* Later, Erdos [51] and Lavasz [96] established the high Chromatic-number result as shown below.

Theorem 3.6 *For integers $p > 0$ and $q > 0$, there is a q-chromatic graph with a girth greater than p.*

In 1912, Birkhoff [17] introduced the Chromatic polynomial. It is an $n-$degree polynomial that give us count of vertex coloring for set of m colors $1, 2, \cdots, m$, where $m \geq \chi(H)$ is a positive integer. It is clear that it has integer roots, namely $1, 2, \cdots, m-1$ if $\chi = m$. There are many properties of chromatic polynomials that are explained by Birkhoff [17,6], Whitney [147], Rota [121], Read [118], and many other authors, as follows:

Theorem 3.7 *If a graph H has p vertices, q edges and k components with chromatic number χ, then the chromatic polynomial $f(H, x)$ of graph H yields the following results.*

(1) *The coefficients are alternate in sign;*
(2) *The polynomial $f(H, x)$ has degree p;*
(3) *The coefficient of x^n is 1;*
(4) *The coefficient of x^{n-1} is $-q$;*
(5) *The polynomial's constant term is zero;*

(6) *The chromatic polynomial with* k *components is*

$$f(H,x)=\prod_{i=1}^{k} f(H_i,x);$$

(7) *The smallest exponent of* x *is* q*;*

(8) *The numbers* $1,2,3,\cdots,\chi-1$ *are zeros of chromatic polynomial.*

Theorem 3.8 *4CC is equivalent to* $f(H,x)=0$ *for m=1,2,3 and 4.*

As a result of Birkhoff's attempt to solve the graph theory coloring problem using an algebraic method, more than 600 papers have been published since the advent of the chromatic polynomial until today. Benzer [7] identified the linear structure of the DNA molecule in 1955; as a result, Hajnal and Suranyi [72] introduced and studied interval graphs, a subclass of chordal graphs, in 1958. Chordal graphs has all of roots from a set $\{1,2\cdots,(\chi-1)\}$. Dmitriev [47] discovered the characterization of chordal graphs and chromatic polynomials, i.e., "*A graph H is chordal if and only if all the roots of the chromatic polynomial for every induced subgraph* H' *are integers from the set* $\{1,2,\cdots,(\chi-1)\}$*, which are roots of chromatic polynomial of a graph H.* There was lots of research going on chordal graphs and chromatic polynomials.

The complement graph's chromatic number is given as $\overline{\chi(H)}=\chi(\overline{H})$. It is clear that $\chi(\overline{H})=\beta$. Nordhaus and Gaddum [107] provide bounds in terms of sum and product.

Theorem 3.9 *The chromatic numbers* χ*, satisfy inequalities for* n *vertices graph,*

(A) $2\sqrt{n}\le\chi+\overline{\chi}\le n+1$;

(B) $n\le\chi\overline{\chi}\le\left(\frac{n+1}{2}\right)^2$.

In general, the disjoint union (addition) of two graphs's chromatic number is largest among both chromatic numbers. The results of the graph operations on two or more graphs are discussed by Vizing [134] in 1963 and Aberth [3] in 1964, as below.

Theorem 3.10 *Let* P *and* Q *be two graphs. Then, the chromatic number of their Cartesian product is maximum from* $\chi(P)$ *or* $\chi(Q)$*.*

Theorem 3.11 *Let* P *and* Q *be two graphs. Then, the chromatic number of their join is a sum of* $\chi(P)$ *and* $\chi(Q)$*.*

Chvatal [44] verified the below result for graphs without triangles in 1970.

Theorem 3.12 *If a graph has no triangles and it is four regular, then it is a* 4*-chromatic graph.*

In 1970, Thatcher et al. [132] proved the complexity of vertex coloring as below.

Theorem 3.13 *The vertex coloring is NP-complete problem.*

In 2007, W. Klotz and T. Sander [93] gave a result on a unitary Cayley graph.

Theorem 3.14 *If* X_n *is unitary Cayley graph such that* p *is the minimum prime divisor of* n*, then its vertex coloring number of* X_n *is* p *and the vertex coloring number of its complementary graph is* n/p*.*

§4. **Edge Coloring**

Definition 4.1([48,69,95,144,145,154]) *The edge coloring means proper-coloring of edges of a graph. If we needed k colors, then it is known as k-proper edge coloring.*

Generally, a Smarandachely Λ*-coloring* $\varphi_\Lambda|_{E(G)} : \mathscr{C} \to E(G)$ *is called an edge Smarandachely* Λ*-coloring.*

Definition 4.2 *The chromatic Index means least k different colors needed such that graph is k-proper edge colorable. We symbolized this by* χ'*.*

Here are some examples of standard graphs and their edge chromatic numbers.

(1) For a complete graph K_n, $\chi'(K_n) = (n-1)$, where n is a positive integer;
(2) For a complete-bipartite graph $(K_{m,n})$, $\chi'(K_{m,n})$ = maximum of $\{m, n\}$;
(3) For a cycle C_n, $\chi'(C_n) = 2$ or 3 if n is even or odd positive integer, respectively;
(4) For a tree T, $\chi'(T) = 2$;
(5) For a star graph $S_{1,n}$, $\chi'(S_{1,n}) = 2$;
(6) For a path P_n, $\chi'(P_n) = 2$.

In 1890, Peter Tait [130] proved the result of edge coloring for a planar cubic map.

Theorem 4.1 *A cubic planar map with four colors is equivalent to 3 edge coloring and vice versa.*

Claude Shannon [122] published results on the tight bounds of lines in any electrical network colored differently for identification in 1949.

Theorem 4.2 *A multipartite graph H having highest degree* Δ*, satisfies* $\Delta \le \chi'(H) \le \lfloor \frac{3}{2}\Delta \rfloor$*.*

In 1964, Vizing [139] characterized tight bonds for edge coloring of simple connected graphs as below.

Theorem 4.3 *If graph H is a simple connected with highest degree is* Δ*, then* $\Delta \le \chi'(H) \le \Delta + 1$*.*

If $\chi'(H) = \Delta$, the graph H is classified as Class-I and if $\chi'(H) = \Delta + 1$, it is classified as Class-II.

Example 4.1 A complete K_{2n} is of class-I and K_{2n+1} is of class II.

After this, Vizing [133] worked on a simple planar cubic graph and its edge coloring in 1965 and gave the following characterization.

Theorem 4.4 *If a simple cubic planar graph H with an extreme degree is* $\Delta \ge 8$*, then* $\chi'(H) = \Delta$*.*

The above result was enhanced for $\Delta \ge 7$ by Grunewald [61], Zhang [151], and Sanders [123] independently in the years 2000 and 2001. The problem is still open for $\Delta \ge 6$. In general, for multigraph with a largest vertex of degree Δ having multiplicity μ, Vizing and

Gupta [58] proved the following theorem for edge coloring. This is known as the Vizing and Gupta Conjecture.

Theorem 4.5 *A connected multigraph H with the largest degree is Δ and having multiplicity μ, then $\chi'(H)$ is Δ or $\Delta + \mu$.*

Vizing [133] introduced critical graph concept in terms of an edge coloring of a graph and its deleted subgraph, if $\chi'(H - e) < \chi'(H)$, every edge e and proved the result.

Theorem 4.6 *Every critical graph has 3 and more vertices of maximum degree.*

In 1970, there was lots of research on critical graphs and the main critical graph conjecture is as below.

Theorem 4.7 *There is no critical graph of class-II exists for an even number of vertices.*

This result was proved for$n = 4, 6, 8,$ and 10 by Jakobsen [82,83]. Later, Fiorini and Beineke [21] extended to $n = 14$ and Lars Andersen [1,2,18] to $n = 16$ (with Fiorini).

In 1973, Mark Goldberg [62] observed that for graph $H, \omega(H)$ is a density function where

$$\omega(H) = \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil .$$

Then, $\chi'(H) = \omega$ is possibly the best lower bound for edge coloring, and he published the following conjecture.

Theorem 4.8(Goldberg conjecture) *A simple connected graph H has an edge coloring number χ' is Max $\{\Delta, \omega\}$ or $\Delta + 1$.*

In 1974, Paul Seymour worked on the graph density function and edge coloring of graphs and derived the same result as Goldberg's conjecture. Later, it was known as the Seymour-Goldberg conjecture. In 1977, Seymour [124] published the following results from his work with planar multigraphs and edge coloring.

Theorem 4.9 *For a planar multigraph H, $\chi'(H) = \omega$ or Δ, where Δ is largest degree and ω is density of graph H.*

R. P. Gupta [63] also worked on Goldberg's conjecture in 1978 and published a result known as Gupta's conjecture, which is an equivalent form of Goldberg's conjecture.

Theorem 4.10 *For a planar multigraph H, the chromatic index*

$$\chi' > \Delta + 1 + \left(\frac{\Delta - 2}{2t} \right)$$

with $t \geq 1$ is a fixed number and encloses a sub-multigraph K of H having $2t' + 1$ vertices such that

$$\chi' = \left\lceil \frac{|E(K)|}{t'} \right\rceil$$

if $1 \leq t' < t$.

For $t = 1$, it is Claude Shannon's bound [122]. For $t = 2$, it is the Goldberg conjecture [62]. Many researchers worked on this conjecture, and finally, in 1990, the famous 1.1-theorem was proved by Nishizeki and Kashiwagi [108] as below.

Theorem 4.11 *For a graph H, if $\chi' > 1.1\Delta(H) + 0.8$, then $\chi' = \omega$.*

Goldberg's conjecture is published in parameterized form by using Gupta's conjecture [63] as below.

Theorem 4.12 *For each graph H, it's chromatic index is $\chi' > \Delta(H) + 1 + \frac{(\Delta(H)-2)}{(m-1)}$, odd integer number $m > 4$ is an elementary-graph.*

Several researchers have demonstrated this result for m, which is from 5 to 38. For $m = 39$ the result was solved by Chen and Jing [34] in 2017.

In 1977, Erdos and Wilson [52] discussed the chromatic index in Combinatorial Theory Journal.

Theorem 4.13 *Almost all graphs have a distinctive vertex of maximum degree, and hence almost all graphs are of class-I.*

Garey [59] and Holyer [73] discussed about NP completeness of edge coloring of graphs. In 1981, Holyer demonstrated that it is NP-hard problem and proved result as below.

Theorem 4.14 *The chromatic index is NP-hard problem to decide for any arbitrary graph. Cubic graph is NP-complete to conclude whether chromatic index is 3 or 4.*

Chudnovsky [38,39] discussed about r-regular planar graphs and edge coloring in 2011.

Theorem 4.15 *All 7-regular planar graphs with oddly seven-edge connected are seven edge-colorable.*

Theorem 4.16 *Eight-regular planar graph is 8-edge-colorable if and only if graph is oddly 8-edge connected.*

In 2012, Huang and Wang [76] proved the result for a planar graph.

Theorem 4.17 *A planar graph not having seven-cycle with largest degree six is class-I graph.*

In 2013, Machadoa et al. [102] worked on chordless graphs coloring, their time complexity in polynomial time.

Theorem 4.18 *The chordless graph H having maximum degree $d > 2$, graph H is d-edge colorable and its time complexity is $O(|V(H)|^3|E(H)|)$.*

§5. Face Coloring

Definition 5.1([95,130,154]) *A face coloring means proper face F coloring of a planar graph. If it requires k colors, so-called k-proper region coloring. This is also known as face coloring or map coloring.*

Generally, a Smarandachely Λ-coloring $\varphi_\Lambda|_{F(G)} : \mathscr{C} \rightarrow V(G)$ is called a face Smarandachely Λ-coloring.

We know that graph coloring was started with four map color and hence four colorable; so region chromatic number is $1, 2, 3$ or 4.

Theorem 5.1 *All planar graphs are four-colorable.*

The result was proved for the first time by Kemp [91] in 1879. However, there was an error in the proof shown by Heawood [80] who proved the conjecture for five colors in 1890.

Theorem 5.2 *Every planar graph is five-colorable.*

Ore and Stemple [111] demonstrated 4CC for maps with fewer than 40 countries using a numerical method in 1969.

Theorem 5.3 *All planar graphs upto thirty nine faces are four-colorable.*

After 87 years, Appel Kenneth and Haken Wolfgang [90,94] presented a proof of 4CC by verifying more than 1900 unavoidable reducible configurations of a planar graph with the help of 1200 computer hours in 1977 and proved that each planar graph can be colored with 4 or less colors. Robertson et al. [117] gave revised proof with less than 650 unavoidable reducible configurations. In 2014, Bhapkar [14] proved 4CC by using PRN (Pivot Region Number) of graph. There are various characterizations of 4CC demonstrated by many mathematicians, as below. In 1931, Whitney [146,147] proved the result on Hamiltonian planar graphs as below:

Theorem 5.4 *The 4CC holds iff all hamiltonian planar graphs are four-colorable.*

Vizing [139] described 4CC in the form of a chromatic index, as below.

Theorem 5.5 *The 4CC is true iff all cubic planar graphs not comprising bridge are 3 edge colorable.*

In 1943, Hadwiger [71] introduced the concept of contraction in graph theory and gave the famous conjecture below.

Theorem 5.6 *All n-chromatic connected graphs are contractible to complete graph K_n.*

The converse of this was proved by Wagner [140] in 1960.

Theorem 5.7 *4CC is the same to Hadwiger's conjecture for $n = 5$.*

Grötzsch's [56] characterized 3-colorable graphs in 1958 as below.

Theorem 5.8(Three color problem) *All triangle free planar graphs are three-colorable.*

Grünbaum [57] characterized 3-colorable graphs in 1963 as below.

Theorem 5.9 *All planar graphs having less than 4 triangles are three-colorable.*

Another Characterization of plane graph is given by Ore and Stemple [111] in 1969, as shown below.

Theorem 5.10 *The 4CC holds iff all bridgeless cubic plane graph are 4-colorable.*

In 1976, Steinberg raised the question below, which was proved by Gimbel [126] in 1993.

Theorem 5.11 *Every planar graph not comprising four-cycle and five-cycle is three-colorable.*

Later in 2005, Borodin et al. [19] improved this result as follows.

Theorem 5.12 *A planar graph not consists of cycles of* $4-7$ *length are three-colorable.*

After this, Borodin et al. [20] extended this result up to cycle $3-9$ length in 2006.

§6. Total Coloring

Definition 6.1([10,95,154]) *A total coloring means coloring vertices and edges both together properly. It we use k-colors, so-called k-total coloring of a graph.*

Generally, a Smarandachely total coloring of a graph G *by colors in* $\mathscr{C}$ *is a mapping* $\varphi_\Lambda : \mathscr{C} \to V(G) \cup E(G)$ *such that* $\varphi(u) \neq \varphi(v)$ *if* u *and* v *are elements of a subgraph isomorphic to* $\Lambda \prec G$.

Definition 6.2 *If we required least* k *different colors for coloring of vertices and edges then graph is known as k-total colorable or total chromatic number, denoted by* χ''.

In 1965, Mehdi Behzad [9] introduced the idea of total coloring. One of most the important results was total coloring conjecture. Mehdi Behzad [9,10] and Vizing [139] was discussed separately this result which is listed below.

Theorem 6.1 *For any graph H, with extreme degree is* Δ, *total chromatic number* $\chi''(H)$ *holds inequality* $\Delta + 1 \leq \chi''(H) \leq \Delta + 2$.

Therefore, graphs are characterized in two types according to their total coloring number. A graph H is called Type-I if $\chi''(H) = \Delta + 1$ and Type-II if $\chi''(H) = \Delta + 2$.

Example 6.1 A cycle C_{2n} is Type-I and C_{2n+1} is Type-II.

For graphs having very large maximum degree Δ, Reed and Molloy [115] proved by the probabilistic approach that its Total Chromatic numbers is at most $\Delta + 10^{26}$. To determine the TCC f is NP-hard problem for any arbitrary graph, which was proved by Sanchez-Arroyo [127] in 1989.

In 1971, Rosenfeld [116] discussed the results of TCC for cubic graphs.

Theorem 6.2 *The total coloring number of cubic graph is four or five.*

In 1996, Kostochka [88] proved TCC for the largest degree of a graph being fewer than 6, as below.

Theorem 6.3 *The maximum five degree multigraph is at most 7-total colorable.*

Later, this result was improved to 6-total colorable, as shown below.

Theorem 6.4 *A multigraph having maximum degree four is at most 6-total colorable.*

Theorem 6.5 *A five-regular multigraph with perfect matching is at most seven-total colorable.*

In 1992, Seoud [128] established results for the Cartesian product of path graphs.

Theorem 6.6 *A graph* $(P_m \times P_n)$ *is of Type-I,* $m, n > 2$.

In 1999, Sanders, Daniel P., and Y. Zhao [125] proved that the TCC for the maximum degree of a planar graph is less than 8, as below.

Theorem 6.7 *If a planar graph has at the most degree seven, then it is nine-total colorable.*

In 2001, Bojarshinov [33] showed that TCC holds for an interval-graph. This is NP-hard problem with its polynomial time complexity as below.

Theorem 6.8 *An interval-graph* H *with odd maximum degree is of Type-I and its time complexity is* $O(|V(H)| + |E(H)| + (\Delta(H))^2)$.

Theorem 6.9 *An interval-graph* H *with even maximum degree is of Type-II and its time complexity is* $O(|V(H)| + |E(H)| + (\Delta(H))^2)$.

In 2003 and 2007, Campos and Mello [35,36], proved the results on a circulant graph that is a power of cycle graph, as below.

Theorem 6.10 *The power two of cycle graph* C_n *(circulant graph-*$C_n(1, 2)$*) is Type-I excluding* $n = 7$.

Theorem 6.11 *If* $C_n(1, 2, \cdots, k)$ *is a circulant graph where* $2 \leq k \leq \lfloor n/2 \rfloor$ *then it is type-II iff* k *is odd integer and* $k > \frac{(n-3)}{3}$.

In 2003, Hilton et al. [81], G. Li and L. Zhang [98] published results on total chromatic numbers of join graphs.

Theorem 6.12 *The graph* $H = H_1 + H_2$ *is a join graph, where* H_1 *and* H_2 *are bipartite graphs with maximum degree at most 2, then* H *is of Type-I if and only it if not isomorphic to* $K_{n,n}$ *or* K_4.

Theorem 6.13 *The graph* $H = H_1 + H_2$ *is a regular graph, where* H_1 *and* H_2 *are graphs having odd number of vertices, then* H *is of Type-II.*

Theorem 6.14 *The graph* $H = C_m + C_n$ *is a join graph. Then,* H *is of Type-II if and only* m *and* n *are odd integers with* $m = n$.

Theorem 6.15 *The graph* $H = K_{p,q} + C_n$ *is of Type-I for positive integers* n *and* $p > q$.

In 2005, Campos and Mello [37] proved some result on bipartite graph families as below.

Theorem 6.16 *A grid graph* $G_{m \times n}$, $m, n > 1$, *is of Type-I, a near ladder graph* B_k *is of Type-I and II for* k *is even and odd, respectively and a* k*-dimensional cube graph* Q_k *is of Type-I for integers* $k > 2$.

In 2008, Kowalik et al. [89] proved following result for maximum degree of planar graph is more than 8 as below.

Theorem 6.17 *If the largest degree of a planar graph is more than eight then it is Type-I.*

Khennoufa and Togni [92] discussed about fractional total coloring number for cubic circulant and four-regular graphs in 2008.

Theorem 6.18 *For a circulant H having n vertices and $(p, p, \cdots, p)$-stable for positive rational number p, the fractional total coloring number is $\leq (n/p)$.*

Theorem 6.19 *All cubic circulant graph $H = C_{2n}(1, n)$ with fractional $(p, p, 0)$-stable, then fractional total coloring number is $\leq (2n/p) + 1$.*

Theorem 6.20 *A four regular circulant graph $C_{5p}(1, k)$ is of Type-I for integers $p > 0$ and $(k - 2)$, $(k - 3)$ being multiple of 5, where $k < 3p/2$.*

Theorem 6.21 *A four regular circulant graph $C_{6p}(1, k)$ is of Type-I for integers $p > 3$ and $(k - 1)$, $(k - 2)$ being multiple of 3, where $k < 3p$.*

In 2010, Prnaver and Zmazek [113] results on direct product graph's total colorings.

Theorem 6.22 *The direct product of cycles C_m and C_n is 5-total colorable. Also, the direct product of cycle C_m and path P_n is 5-total colorable.*

In 2011, Campos et al. [40] published result for some snarks families graph's total colorings.

Theorem 6.23 *The total coloring number of infinite snarks families namely flower, Goldberg and twisted Goldberg snark is 4. Hence these graphs are of Type-I.*

In 2012, Campos et al. [41] published result of total-coloring of split indifference graph.

Theorem 6.24 *The total-coloring of split indifference graph with largest even and odd degree is Type-I and II respectively iff Hilton's condition satisfied.*

In 2013, Machadoa et al. [102] discussed TCC for chordless graph with its time complexity in polynomial time as below.

Theorem 6.25 *The chordless graph H having maximum degree is three and more is of Type-I and its time complexity is $O(|V(H)|^3|E(H)|)$.*

In 2015, Geetha and Somasundaram [65] published the total coloring for generalized sierpinski graph of hypergraph and cycle graph.

Theorem 6.26 *For $n > 1$, if a graph H is Type-I, so the Sierpinski graph $S(n, H)$ is Type-I.*

Theorem 6.27 *The Sierpinski graph $S(n, C_k)$ of cycle graph C_k is Type-I for positive integers $n > 1, k > 2$.*

Theorem 6.28 *The Sierpinski graph $S(n, Q_k)$ of hypercube graph Q_k is of Type-I for positive integer $n > 1$.*

In 3D topology, WK-recursive topology of graph G is constructed as l layers of 2D recursive topology of graph G. $K(l,n,G)$ for $l=1$ is snark family of graph G that is Sierpinski graph $S(n,G)$. For WK-recursive topology of complete graph, Geetha and Somasundaram [65] proved following result of total coloring.

Theorem 6.29 *The graph $K(l,n,K_k)$-WK-recursive topology of complete graph K_k is of Type-I for positive integers $n,k>1$ and $l>0$.*

Theorem 6.30 *The graph $K(l,n,C_k)$-WK-recursive topology of cycle C_k is of Type-I for positive integers $n>1, k>1$ and $l>0$.*

In 2016, Mohan et al. [100], discussed results of total coloring for compounded graph and rooted graph.

Theorem 6.31 *The compounded graph $H[G]$ for any two total colorable graphs H and G is of Type-I.*

Theorem 6.32 *A rooted graph $H \circ P_n$ for total colorable graph H and path P_n is Type-I.*

In 2017, Mohan, Geetha and Somasundaram [99], proved results of total colorings for corona product of two graphs.

Theorem 6.33 *For the path, cycle, complete and complete bipartite graphs, the corona product with any graph H is Type-I.*

In 2018, Geetha and Somasundaram [66] published the results on total coloring numbers of graph's product.

Theorem 6.34 *A graph $(K_n \times K_n)$ is Type-I for even positive integer n.*

Theorem 6.35 *A graph $(C_m \times C_n)$ is Type-I for positive integer $n \geq 3$ and m is multiple of $3,5$ and 8.*

In 2018, Golumbic [64] discussed total coloring of rooted path graph and its polynomial time complexity. He also gave algorithm by using greedy algorithm to find total coloring number.

Theorem 6.36 *A rooted path graph having even maximum degree is Type-I. Otherwise it is Type-II. Its time complexity is $O(|V(H)|+|E(H)|)$.*

In 2018, Vignesh [136] discussed total coloring numbers for double graph.

Theorem 6.37 *A double graph of Type-I graph is Type-I. Otherwise, it is Type-II.*

Theorem 6.38 *For two Type-I graph's deleted lexicon product is Type-I.*

Theorem 6.39 *The deleted lexicon product of any graph H with path P_m for $m>2$ is Type-I.*

Theorem 6.40 *Let K_n be a complete graph. Then, its line graph is Type-I.*

In 2020, Vignesh et al. [135] explained total coloring numbers of Cocktail Party, Core-Satellite, Shrikhande and Modular Product of Graphs.

Theorem 6.41 *A core-satellite graph is total colorable and it is Type-I if the core and satellite cliques are of Type-I.*

Theorem 6.42 *A cocktail party graph of order n is Type-I for $n > 2$.*

Theorem 6.43 *The modular product of P_3 graph with Cycle C_n and path P_n are total colorable graph.*

Theorem 6.44 *The Shrikhande graph is Type-I.*

In 2021, Mauro et al. [109] proved 5-total coloring of four regular circulant graphs that are Type-I as follow.

Theorem 6.45 *A circulant graphs $C_{3kp}(1, p)$ is Type-I for integer $k > 0$, p is a multiple of 3. Also proved if k is even then $C_{3nk}(1, k)$ is Type-I and $C_{3n}(1, 3)$ is Type-I except that $C_{12}(1, 3)$ is Type-II.*

In 2022, Prajnanaswaroopa et al. [114] described result on total coloring of Caley graph.

Theorem 6.46 *A Caley graph is of Type-II.*

Theorem 6.47 *The TCC holds for odd and mock threshold graph.*

For the vertex, edge or total coloring of a graph G, there are open problems following.

Problem 6.1 *Let the complete graph K_n be decomposed into 3 subgraphs G_1, G_2, G_3 such that $\chi(G_1) = n_1$, $\chi'(G_2) = n_2$ and $\chi''(G_3) = n_3$ for integers $n_1, n_2, n_3 \geq 1$.*

(1) *Determine all possible subgraphs G_1, G_2 and G_3;*
(2) *Determine all possible integers n_1, n_2, n_3.*

Problem 6.2 *For any connected graph G, can it be decomposed into 3 subgraphs G_1, G_2, G_3 such that $\chi(G_1) = n_1$, $\chi'(G_2) = n_2$ and $\chi''(G_3) = n_3$ for integers $n_1, n_2, n_3 \geq 1$, particularly, with some special numbers such as $n_1 = 0, 1$, $n_2 = 0, 1$ or $n_3 = 0, 1$ or other integers?*

§7. Perfect Coloring

Definition 7.1 *A perfect coloring means proper coloring of all components of planar graph. If it needs least k colors, then it is called k-proper perfect coloring.*

In 2018, Bhapkar [15] introduced perfect coloring of graphs. The following results proved.

Theorem 7.1 *A star graph is perfectly $(n + 2)$-colorable.*

Theorem 7.2 *A rose graph is perfectly $(m + 2)$-colorable.*

Theorem 7.3 *A chain graph is perfectly 4-colorable.*

Theorem 7.4 *A tree with largest degree d is perfectly $(d + 2)$-colorable.*

Theorem 7.5 *A cycle C_n is perfectly 5-colorable if the integer n is multiple of 3. Otherwise, it is perfectly 6-colorable.*

In 2019, Archana Bhange [30] proved result of perfect coloring of corona product of cycle graph with cycle, path and null graphs as below.

Theorem 7.6 *A perfect coloring of corona product of cycle graph* C_n *with* C_m *is* $m+3$ *for* $n>4, m>4$.

Theorem 7.7 *A perfect coloring of corona product of cycle graph* C_n *with path graph* P_m *is* $m+3$ *for* $n>4, m>3$.

Theorem 7.8 *A perfect coloring of corona product of cycle graph* C_n *with null graph* N_m *is* $m+4$ *for* $n>4, m\geq 14$.

In 2020, Bhange [16] collaborated with Bhapkar to define upper and lower bound and kinds of perfect coloring. They also worked on some standard families and their Perfect coloring.

Theorem 7.19 *The bound for perfect coloring is* $\chi'' \leq \chi^p \leq \chi''+4$, *where* χ'' *is total coloring number.*

Theorem 7.10 *There are following kinds of perfect coloring of graphs.*

(1) *Kind 0 perfect coloring if* $\chi^p(H)=\chi''(H)$;
(2) *Kind 1 perfect coloring if* $\chi^p(H)=\chi''(H)+1$;
(3) *Kind 2 perfect coloring if* $\chi^p(H)=\chi''(H)+2$;
(4) *Kind 3 perfect coloring if* $\chi^p(H)=\chi''(H)+3$;
(5) *Kind 4 perfect coloring if* $\chi^p(H)=\chi''(H)+4$.

Theorem 7.11 *There are no graph with*

$$\chi^p(H)=\chi''(H)+5.$$

Theorem 7.12 *The diamond graph is kind 0.*

Theorem 7.13 *The null graph, trees, friendship graph, ladder rung graph are kind 1.*

Theorem 7.14 *A prism and circular ladder graph are kind 2.*

Theorem 7.15 *A ladder graph is kind 3.*

Theorem 7.16 *A helm graph is kind 4.*

In 2022, Archana Bhange and Bhapkar [31] worked on perfect coloring of corona product of Fan graphs with sunlet graph, tadpole graph and proved following results.

Theorem 7.17 *A perfect coloring of corona product of sunlet graph and fan graph is* $\Delta+1$.

Theorem 7.18 *Perfect coloring of corona product of Tadpole graph and Fan graph is* $\Delta+1$.

We observed that the concept of vertex coloring was extended to edge and face coloring as well as the combination of these elements. After this many researchers in the field of graph theory have defined various types of graph coloring by enforcing some different conditions while coloring graphs. Now, let we discuss some other special types of coloring and their results.

§8. Strong Edge Coloring of Graph

Definition 8.1 *A properly-edge colored graph fulfills the condition $C(u_1) \neq C(u_2)$ for every edge u_1u_2 with color set $C(u)$ and $C(v)$ is known as strong edge coloring of graph.*

In 1997, Burris [26] introduced this coloring notion. It is also named vertex distinguishing proper edge coloring. It has the following properties

(1) Adjacent edges have the different color;

(2) If two vertices u_1 and u_2 are neighbors, their color sets are separate. i.e. $C(u_1) \neq C(u_2)$.

Notice that a color set $C(u)$ for a vertex u means set of colors of all edges incident at a vertex u after proper coloring of edge and the strong edge chromatic number symbolized by $\chi'_s(H)$ means minimum colors needed for this coloring. Burris, Schelp [26] validated the following result for cycle graph, bipartite, and complete bipartite.

Theorem 8.1 *For a cycle C_n,*

(1) $\chi'_s(C_n) = 5$, *for $n = 5$;*

(2) $\chi'_s(C_n) = 3$, *for n is multiple of 3;*

(3) $\chi'_s(C_n) = 4$, *else.*

Theorem 8.2 *For a complete bipartite graph $K_{m,n}, 1 \leq m \leq n$,*

(1) $\chi'_s(K_{m,n}) = (n+1)$, *if $m < n$ and*

(2) $\chi'_s(K_{m,n}) = (n+2)$, *if $m = n > 1$.*

Theorem 8.3 *For a complete graph $K_n, n \geq 3, \chi'_s(K_n)$ is n and $n+1$ for n is odd and even, respectively.*

Theorem 8.4 *For a star graph $K_{1,n}, n \geq 3$, $\chi'_s(K_{1,n}) = n$.*

For a graph H with n_k at least k colors as there are vertices having degree k. Thus lower bound is $\chi'_s(H) \geq max\{(k!n_k)^{(1/k)} + (k-1)/2 : for\ 1 \leq k \leq d\}$. We can improve this lower bound upto additive 1 as below.

Theorem 8.5 *If graph H has largest degree is d and for smallest integer j such that ${}^jC_k \geq n_k$ for $1 \leq k \leq d$. Then strong edge coloring $\chi'_s(H) = j$ or $j+1$.*

The upper bond for strong edge coloring was proved as below.

Theorem 8.6 *If a graph H is strong edge coloring and $n_i^{1/i} = max\{n_j^{1/j}$: for $j = 1$ to maximum degree d of graph $H\}$. Then $\chi'_s(H) \leq (\Delta+1)(2n_i^{1/i}+5)$. This is upper bound.*

In 1997, Bazgan et al. [27] verified following result on strong edge coloring.

Theorem 8.7 *If H is any graph with n vertices, consists of no more than one isolated vertex and no isolated edges, then $\chi'_s(H) \leq n+1$.*

In 2002, Zhongfu Zhang et al. [152] published following results.

Theorem 8.8 *A graph H is formed with n connected components H_i then its strong edge coloring is $\chi'_s(H) = max\{\chi'_s(H_i) : \ for \ all \ i\}$.*

Theorem 8.9 *If T is tree graph with 3 and more vertices, then $\chi'_s(T) = d$ for two maximum degree vertices are not neighbors, otherwise $\chi'_s(T) = d + 1$.*

Theorem 8.10 *If two highest degree d vertices are neighbors in any graph, then $\chi'_s \geq d + 1$.*

Theorem 8.11 *A graph with highest degree d vertices are not neighbors and two neighbor vertices of different degree, then $\chi'_s = d$.*

In 2007 Balister et al. [32] proved following results for strong edge-colorings.

Theorem 8.12 *If H be any connected graph with 6 and more vertices, then $\chi'_s(H) \leq \Delta + 2$.*

Theorem 8.13 *If a graph H has non-isolated edges and the largest degree is 3, then $\chi'_s(H) \leq 6$.*

Theorem 8.14 *If a bipartite graph B has no isolated edge, then $\chi'_s(B) \leq \Delta + 2$.*

Theorem 8.15 *If a graph H is non-isolated and it is k-chromatic, then $\chi'_s(H) \leq \Delta + O(log\ k)$.*

In 2010, Wang et al. [142] proved strong edge coloring for maximum degree of graph more than 4 with the condition on maximum average degree (mad) as follows.

Theorem 8.16 *If a connected graph H has highest degree Δ with mad(H), then*

(1) *If $\Delta = 3$; $mad(H) < 7/2$, then $\chi'_s(H) \leq \Delta + 1$;*
(2) *If $\Delta \geq 3$; $mad(H) < 3$, then $\chi'_s(H) \leq \Delta + 2$;*
(3) *If $\Delta \geq 4$; $mad(H) < 5/2$, then $\chi'_s(H) \leq \Delta + 1$;*
(4) *If $\Delta \geq 5$; $mad(H) < 5/2$, then $\chi'_s(H) \leq \Delta + 1$ iff graph H has adjacent vertices of highest degree.*

In 2013, Hocquard, Montassier [78] generalized these results and proven the below results for the largest degree $\Delta \geq 5$ with a condition on mad.

Theorem 8.17 *For every graph H with $\Delta \geq 5$, $mad(H) < 3 - (2/\Delta)$, then $\chi'_s(H) \leq \Delta + 1$.*

In 2021, Borut et al [97] proved strong edge coloring of regular graph.

Theorem 8.18 *If a graph H is r-regular, then strong-edge-chromatic number $\chi'_s(H)$ is equal to $(2r - 1)$ if and only if it covers the Kneser graph $K(2r - 1,\ r - 1)$.*

Theorem 8.19 *A cubic graph is* 5 *strong-edge-chromatic iff it covers Petersen graph.*

§9. Vertex Distinguishing Total Coloring

Definition 9.1 *There is an additional constraint joined in total coloring, if the color sets of any two neighboring vertices should be different, such a coloring is known as AVD total coloring.*

In 2005, Zhang et al. [153] introduced this coloring type after adding one more restriction in the definition of total coloring. It has the following properties.

(1) Adjacent-vertices colored differently;

(2) Adjacent-edges colored differently;

(3) An edge with its end vertices are colored differently;

(4) For each two neighbor vertices u_1, u_2 of a graph H, both vertices color sets are different. That means $C(u_1) \neq C(u_2)$.

The AVD-total-chromatic number symbolized by $\chi_{at}(H)$. It is the least colors needed for AVD-total-coloring of a graph. Many researchers provided results on AVD-Total Coloring. The lower and upper bounds were discussed by Zhang et al. [153] in 2005.

Theorem 9.1(Lower bound) *If two maximum degree vertices are adjacent in a simple graph H, then $\chi_{at}(H) \geq \Delta + 2$; otherwise, $\chi_{at}(H) \geq \Delta + 1$.*

Theorem 9.2 *If graph H is simple and connected with minimum order 2, then $\chi_{at}(H) \leq \Delta+3$.*

Thus, by the AVD-total-coloring conjecture we conclude that $\Delta + 1 \leq \chi_{at}(H) \leq \Delta + 3$. This is true for various graph families such as the graphs with $\Delta = 3$, bipartite graphs, complete graphs.

In 2007, Wang [143] and in 2008, Chen [42] separately verified result of AVD-total-coloring for maximum degree three graphs as below.

Theorem 9.3 *If the largest degree of graph H is at most three, then $\chi_{at}(H) \leq \Delta + 3$.*

In 2012, Huang [45] proved following theorem.

Theorem 9.4 *If the largest degree of a simple graph H is more than 2, then $\chi_{at}(H) \leq 2\Delta$.*

An algorithmic procedure described for four regular graph's AVD-total-coloring by Papaioannou and Raftopoulou [112] in 2014.

Theorem 9.5 *$\chi_{at}(H) \leq \Delta + 3$ for any four regular graphs having its maximum degree is Δ.*

In 2017 Yang [150] proved the result for planar graph.

Theorem 9.6 *If the largest degree of a planar graph is more than 10, then $\chi_{at}(H) \leq \Delta + 2$.*

In 2019, Wang [141] and Hu [77] independently demonstrated result for planar graph having extreme degree nine.

Theorem 9.7 *If the largest degree of a planar graph H is more than 8, then $\chi_{at}(H) \leq \Delta + 3$.*

In 2020, Yulin Chang et al. [43] revised this result for maximum degree is more than 7.

Theorem 9.8 *If the largest degree of a planar graph H is more than 7, then $\chi_{at}(H) \leq \Delta + 3$.*

§10. Acyclic Coloring

Definition 10.1 *If every two-chromatic subgraph is acyclic after graph's vertex coloring, then it is known as acyclic coloring. In other words, each cycle in a graph uses minimum three* *colors for proper vertex coloring. The least colors needed for such coloring is known as acyclic chromatic number.*

In 1973, Grunbaum [60] started work on acyclic coloring and proved result for graph with largest degree is 3 as below.

Theorem 10.1 *For any graph H with $\Delta = 3$, $A(H) \leq 4$.*

In 1979, Burstein [12] verified result for largest 4 degree graph.

Theorem 10.2 *Any graph H with $\Delta = 4$ is acyclic 5-coloring.*

In 1979, Borodin [11] proved result of acyclic coloring for planar graph.

Theorem 10.3 *Any planar graph H is at most acyclic 5-coloring.*

In 2011, Varagani et al. [138] verified result for largest 6 degree graph.

Theorem 10.4 *Any graph H with $\Delta = 6$ is at most acyclic 12-coloring.*

In 2011, Kostochka and Stocker [4] proved result for graph with maximum degree is 5.

Theorem 10.5 *Any graph H with $\Delta = 5$ is at most acyclic 7-coloring.*

§11. **List Coloring**

Definition 11.1 *List vertex coloring means proper vertex coloring of graph with color every vertex from available list of color only. The minimum colors necessary for this coloring is called List vertex chromatic number or vertex-choosability.*

Definition 11.2 *If we color edges of graph from an available list of colors for each edge, then it is called List edge-coloring. Thus choose a color for every edge from a list of colors only. The minimum colors necessary for this coloring is called List edge chromatic number or edge-choosability.*

Definition 11.13 *The total coloring from available list of colors for each vertex and edge is termed as list total-coloring.*

This type of coloring was introduced for the first time by Erdos et al. [53] in 1980.

Theorem 11.1 *For any graph, its lower bound of list vertex coloring is chromatic number.*

Theorem 11.2 *For any graph, the lower bound of list edge coloring is chromatic index of a graph.*

In 1976 Vizing [137] proved result for list edge coloring.

Theorem 11.3 *The vertex-choosability of every graph is at most $\Delta + 1$.*

Borodin [25] proved above result is true for planar graphs for $\Delta = 8$ in 1990. Woodall et al. [24] verified list chromatic index is Δ for planar graph for maximum 11 degree. In 1994, Borodin [23] proved the result for list coloring.

Theorem 11.4 *In a planar graph, if largest degree is more than 9; then total coloring $\leq$ list total coloring $\leq$ maximum degree $+2$.*

In 1995, Woodall [148] proved the result of list total coloring on planar graph.

Theorem 11.5 *A planar graph with largest degree is more than 5 and also girth is more than 5; then total coloring $=$ list total coloring $=$ maximum degree $+1$. Thus it is of Type-I graph.*

Above result proved for maximum degree 7 and its girth more than 3 by Borodin et al. [24] in 1997. Hou [74] in 2006 proved above result if there is no 4-cycle in graph.

In 2006 and 2007, Hou, Liu, Cai [74,75] proved the result below.

Theorem 11.6 *If a planar graph has largest degree is more than five and does not contains 4-8 length cycles; then List total coloring $=$ total coloring $=$ maximum degree $+1$. Thus it is of Type-I graph.*

The following are some equivalent results.

Theorem 11.7 *A planar graph with largest degree is more than eight, then list total coloring $=$ total coloring $=$ maximum degree $+1$, if the graph satisfies below conditions:*

(1) *There is no intersecting 3-cycle. (Wu-Wang [149] proved in 2008);*

(2) *Does not contain 5-cycle or 6-cycle. (Ma, Wu, Yu [101] proved in 2009).*

In 2006, Borowiecki et al. [22] discussed list coloring of product graphs.

Theorem 11.8 *The vertex-choosability of product of two graphs P and Q has tight upper bound, $\chi_l(P \times Q) \leq min\{\chi_l(P) + col(Q), \chi_l(Q) + col(P)\} - 1$.*

In 2006, Hou, Liu, Cai [74,75] proved the result of List edge coloring for graph without 14-cycle.

Theorem 11.9 *The edge-choosability of graph is Δ, if largest degree is more than 3 and without 14-cycle.*

In 2008, Wu-Wang [149] verified result for the maximum degree was more than 8 for planar graphs.

Theorem 11.10 *For a more than 8 largest degree planar graph, the List total-coloring $=$ total coloring $=$ maximum degree $+1$, if there is no 3 cycle.*

Certainly, all the previous colorings can be determined on a graph G, which enables us to generalize Problem 6.2 as follows.

Problem 11.1 *How to decompose a connected graph G, particularly, the complete graph K_n of order n into subgraphs $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8$, i.e., $G = \bigcup_{i=1}^{8} G_i$ such that $\chi(G_1) = n_1$,*

$\chi'(G_2) = n_2$, $\chi''(G_3) = n_3$, $\chi^p(G_4) = n_4$, $\chi'_s(G_5) = n_5$ $\chi_{at}(G_6) = n_6$, $\chi_l(G_7) = n_7$ and $A(G_8) = n_8$ for chosen integers $n_i, 1 \leq i \leq 8$?

Clearly, Problem 11.1 is Problems 6.1 and 6.2 if $n_4 = n_5 = n_6 = n_7 = n_8 = 0$.

§12. π-Coloring and Incident Vertex π-Coloring

Bhapkar and Thakare [155, describe the π-coloring idea, which is based on properly coloring graph components with distinctive color patterns.

Definition 12.1(π-Coloring, [155]) *Let $H = (V, E)$ be a simple connected graph where V is vertex set, E is edge set, and $X = \{X_1, X_2, \cdots, X_r\}$ is a collection of distinct subsets of elements of graph H having some common properties. If there exists a function $f : X \longrightarrow P(C)$, where C is a set of colors and $P(C)$ is its power set, such that $f(X_p) \neq f(X_q)$, for all $p \neq q$ with some conditions, then it is called π-coloring of graph H. The least number of colors of set C is called π-chromatic number of a graph H corresponding to function $f(X)$. It is denoted by $\pi_f(H)$ or $\pi(H)$.*

Assigning distinct colors to each incidence vertex on the edges in set X, which is the collection of all incident vertices pairs of each edge in the graph is known as incident vertex π-coloring and it is defined as below [155].

Definition 12.2(Incident Vertex π-Coloring) *Let $H = (V, E)$ be a simple connected graph where V is vertex set, E is edge set, and $H = \{H_1, H_2, \cdots, H_r\}$, where $H_i = \{e_i = (u, v)|$ for all $u, v \in E\}$, that is a collection of order pair incident vertices of every single edge e in $E(H)$. Define a function $f : X \longrightarrow P(C)$, where C is set of colors and $P(C)$ is its power set, such that $f(X_i) \neq f(X_j)$, for all $i \neq j$, then it is called incident vertex π-coloring (IVPI) of graph H. The least number of colors of set C called Incident Vertex π chromatic number of graph H corresponding to function $f(X)$, and it is represented by $IV\Pi_f(H)$ or $IVPI(H)$.*

Bhapkar and Thakare [155] discussed the incident vertex π coloring of graphs namely star graph, double star graph, complete graph, wheel graph, fan graph, double fan graph and complete bipartite graph.

Theorem 12.1 *The incident vertex π chromatic number of $K_{1,n}$ is $n + 1$.*

Theorem 12.2 *The incident vertex π chromatic number of $K_{1,n,n}$ is $n + 1$.*

Theorem 12.3 *The incident vertex π chromatic number of a complete graph is n.*

Theorem 12.4 *The incident vertex π chromatic number of wheel graph W_{n+1} is $n + 1$.*

Theorem 12.5 *The incident vertex π chromatic number of fan graph $F_{1,n}$ is $\Delta + 1$.*

Theorem 12.6 *The incident vertex π chromatic number of double fan graph $F_{2,n}$ is $\Delta + 2$.*

Theorem 12.7 *The incident vertex π chromatic number of complete bipartite graph $B_{m,n}$ is $m + n$.*

§13. Applications of Graphs Coloring

The graph theory has a wide-ranging applications because it deals with real-world problems and their solutions (for more details, see Narsingh Deo [48], Roberts [119], and Berge [8]). Certainly, the graph theory is used in mathematics to solve problems involving linear systems such as signal flow problems. The Markov process is one of the most important methods in statistics and probability theory for solving problems in various areas such as statistical information, analysis of various computer programs, control theory, problems in genetics and inventory theory. As a result, the graph theory is used to solve Markov processes. In chemistry, the graph theory is used to represent and match the chemical structure of molecules. By using graph enumeration techniques, we can identify or characterize new chemical composites. Designing computer programs and analyzing them are the two most crucial aspects of computer engineering. In computer programming, the graph theory is used for running time estimation and storage requirements, identifying errors, segmenting and flow of a program, and creating a stochastic model for a program. It is also used for programme optimization, automatic flow charts, data structure as graph, and determining the equivalence and validity of various programmes by transforming their diagraph into canonical form.

There are numerous real-world applications for graph coloring (see [119] for the latest study), so it has received renewed interest in recent years. We can solve lots of real-world problems in sciences by using the graph coloring concept, which includes computer network problems, artificial intelligence problems, machine learning problems from computer engineering, electrical circuit problems from electrical engineering, and communication network problems from electronics and communication engineering. One of the most well-known of these applications is frequency allocation. Each radio transmitter in a radio transmitter network has its own set of operating frequencies. Once two adjacent transmitters utilize the same frequency, they can cause interference. The frequency bands assigned to these transmitter pairs should have been distinct in the simplest model. The aim is to reduce the overall frequency number used. The graph coloring solves this problem. Vertices are emitters in this case, an edge is added in among pair of emitters (vertices) that may interfere. As a result, the frequencies match the colors allocated to the vertices, and nearby vertices must have different colors. The required frequency separation in larger designs may be greater for closer transmitter pairs or for number of frequency bands assigned to the same transmitter; and hence, the goal is to generally reduce the variation between the lowest and highest frequency used (allocation range). It was widely assumed that assigned frequencies should be distinct and regularly spaced points on the spectrum (see Hale [79]). As a result, colors are commonly considered as numbers. Later from Hale's paper, two frequency assignment models are developed, one T-coloring and another is channel assignment. After this, Tesman [131] developed list T-coloring model for frequencies assignment which is with the restriction of frequencies available (using the concept of list coloring) for the transmitter.

In 2018, Bhapkar [29], explained how to generate security key with the help of perfect weighted planar graph. This paper also describes algorithm for public key and secret key generation.

In 1920, the virus graphs and its use were explained by Bhapkar et al. [28] in details. There are four types of virus graphs, i.e., the virus graph type I to IV, where types I and II are not death-defying but types III and IV are extremely harmful for human beings. In this paper, they also discussed about the importance of graph modeling during pandemic conditions and its rate spreading. To control the spreading of COVID-19, the cut set concept is used as an isolation of people.

In 2022, Ghorpade and Bhapkar [67, 68] worked together on brain MRI separation and used the cut-set concept to find the exact infected area that helps for medical treatment. They discussed brain MRI segmentation by cut and watershed model.

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