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Grassmannians in the Lattice Points of Dilations of the Standard Simplex

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Abstract: A remarkable connection between the cohomology ring $\mathrm{H}^*(\mathrm{Gr}(d, d+r), \mathbb{Z})$ of the Grasssmannian $\mathrm{Gr}(d, d+r)$ and the lattice points of the dilation $r\Delta_d$ of the standard d-simplex is investigated. The natural grading on the cohomology induces different gradings of the lattice points of $r\Delta_d$. This leads to different refinements of the Ehrhart polynomial $L_{\Delta_d}(r)$ of the standard d-simplex. We study two of these refinements which are defined by the weights $(1, 1, \cdots, 1)$ and $(1, 2, \cdots, d)$. One of the refinements interprets the Poincaré polynomial $\mathrm{P}(\mathrm{Gr}(d, d+r), z)$ as the counting of the lattice points which lie on the slicing hyperplanes of the dilation $r\Delta_d$. Therefore, on the combinatorial level the Poincaré polynomial of the Grassmannian $\mathrm{Gr}(d, d+r)$ is a refinement of the Ehrhart polynomial $L_{\Delta_d}(r)$ of the standard d-simplex Δ_d.

Key Words: Cohomology ring, Grassmannian, partition, lattice polytope, simplex.

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§1. Introduction

Consider the diagonal sequence D_d of natural numbers realized from Pascal triangle below:

D_1 : 1

D_2 : 1 1

D_3 : 1 2 1

D_4 : 1 3 3 1

D_5 : 1 4 6 4 1

D_6 : 1 5 10 10 5 1

$\cdots$ 1 6 15 20 15 6 1

$\ddots$ $\ddots$ $\ddots$ $\ddots$ $\ddots$ $\ddots$ $\ddots$

Pascal Triangle

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One of the combinatorial interpretations of the terms of the sequence $D_d := \binom{r+d}{d}_{r=0}$, $d \in \mathbb{N}$, has to do with the counting of the lattice points associated with the dilations $r\Delta_d$ of the standard d-simplex Δ_d. By the standard d-simplex Δ_d we mean the convex hull of the set $\{\underline{0}, e_1, \cdots, e_d\}$ where $e_i's$, $1 \leq i \leq d$ are the standard vectors in $\mathbb{R}^d$ and $\underline{0}$ is the origin. That is,

$$\Delta_d := \operatorname{conv}(\underline{0}, e_1, \dots, e_d) = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot e_i \geq 0, \ \sum_{i=1}^{d} \mathbf{x} \cdot e_i \leq 1\} \tag{1.1}$$

and the dilation $r\Delta_d$, is given by

$$r\Delta_d = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot e_i \geq 0, \ \sum_{i=1}^{d} \mathbf{x} \cdot e_i \leq r, \quad r \in \mathbb{N}\}. \tag{1.2}$$

Lattice points are the points whose coordinates are integers. Asking for the lattice points on $r\Delta_d$ is tantamount to counting the integer solutions for the inequality

$$\sum_{i=1}^{d} \mathbf{x} \cdot e_i \leq r. \tag{1.3}$$

The number of lattice points on any given lattice polytope is well known. This is central theme of Ehrhart polynomials, [3], [6], [10], [11] and [16]. In fact the number of the lattice points on $r\Delta_d$ is given by

$$|r\Delta_d \cap \mathbb{Z}^d_{\geq 0}| = \binom{r+d}{d} \tag{1.4}$$

and its generating function by

$$\mathrm{P}(r\Delta_d, z) = \sum_{r=0}^{\infty} A_r z^r = \frac{1}{(1-z)^{d+1}}, \text{ where } \ A_r = \binom{r+d}{d}. \tag{1.5}$$

On the other hand, Grassmannians are ubiquitous in nature and they constitute one of the best understood algebraic varieties. They admit algebraic, combinatorial and geometric structures which are very elegant. Their classical cohomology theory has taken the center stage in algebraic combinatorics in recent years, see [4], [5], [7] , [8], [9], [10] and [12]. It turns out that the lattice points on $r\Delta_d$ encode some vital information about the indexing partitions of the Schubert varieties contained in the Grassmannian $Gr(d, d+r)$. This sheds more light on the cohomology ring of the Grassmannian. It is well known that the multiplicative generators of the cohomology of the Grassmannian $\mathrm{Gr}(d, d+r)$ are given by the special Schubert cycles σ_λ, see [3]. These cycles are indexed by one-row partitions $\lambda = (k), 1 \leq k \leq r$ and they constitute the total Chern class of the quotient bundle $\mathcal{Q}$, that is,

$$c(\mathcal{Q}) = 1 + \sigma_{\square} + \sigma_{\square\square} + \cdots + \sigma_{\square\square\cdots\square_{1\times r}}.$$

We study the monomials identified with the semi standard tableaux of these one-row Young diagrams and realize a natural graded polynomial $T_r(t)$ called dilation polynomial. This is our first refinement of the Ehrhart polynomial $L_\Delta(r)$ of the standard d-simplex Δ_d. It comes with the natural weight $(1, 1, \cdots, 1)$. The second refinement is the the Poincaré polynomial $\mathbf{P}(Gr(d, d+r), z)$ of the Grassmannian $\mathrm{Gr}(d, d+r)$ interpreted as the slicing of $r\Delta_d$ with hyperplanes with respect to the weight $(1, 2, \ldots, d)$. It is interesting to note that the natural grading on the cohomology of the Grassmannian $\mathrm{Gr}(d, d+r)$ induces different gradings of the lattice points of the dilation $r\Delta_d$ which give various refinements of the Ehrhart polynomial $L_\Delta(r)$. The paper is a generalisation of the previous studies in [1] and [2]. In section 2, we introduce a technique of counting lattice points by grading with respect to the weight $\mathbf{a} = (1, 1, \ldots, 1)$. This is just the slicing of the dilation $r\Delta_d$ into parallel regular $(d-1)$-simplices. The The polynomial

$$T_r^{(1,\ldots,1)}(t) = \sum_{k=0}^{r} \binom{k+d-1}{d-1} t^k \tag{1.6}$$

refines the Ehrhart polynomial $L_\Delta(r)$. We give a generating function for such polynomials as r grows. This grading allows us to establish in Section 3, a bijection between the lattice points of the dilation $r\Delta_d$ and the semi standard tableaux of row Young diagrams indexing the special Schubert cycles of the Grassmanninan $\mathrm{Gr}(d, d+r)$. By using another weight $\mathbf{h} = (1, 2, \ldots, d)$ which gives a different slicing of the simplex, we construct a polynomial

$$P_{r\Delta_d}^{(1,2,\cdots,d)}(z) = \left[\binom{k+d}{d}\right]_z \quad \text{for } 0 \le k \le r \tag{1.7}$$

which is a z-binomial coefficient. This gives a bijection between the lattice points in $r\Delta_d$ and partitions fitting into an $r \times d$ rectangle, and establishes that the grading given here to a lattice point eventually identifies this polynomial with the Poincaré polynomial of the Grassmannian $\mathrm{Gr}(d, d+r)$.

§2. The Dilation Polynomial $T_{r\Delta_d}, r \ge 1$

We define the lexicographical order $<_{\text{lex}}$ on the set $r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$ of lattice points on $r\Delta_d$ as follows: Let $\mathbf{a} = (a_1, \cdots, a_d)$ and $\mathbf{b} = (b_1, \cdots, b_d)$ be any two lattice points in $r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$. We say $\mathbf{a} <_{\text{lex}} \mathbf{b}$ if, in the integer coordinate difference $\mathbf{a} - \mathbf{b} \in \mathbb{Z}^d$, the leftmost nonzero entry is negative. As noted earlier, the set $r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$ of lattice points on $r\Delta_d$ is the integer solution set of the inequality (1.3). It turns out that the upper bound r in (1.3) defines a relation on the lattice points of the solution set which brings about the disjoint subdivisions of the integer solution set.

Proposition 2.1 *Let $\mathbf{a}$ and $\mathbf{b}$ be two lattice points in $r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$ such that $\mathbf{a} <_{\text{lex}} \mathbf{b}$. The relation $\mathbf{a} \sim \mathbf{b}$ defined by $\sum_{i=1}^{d}(a_i - b_i) = 0$ is an equivalence relation.*

The relation partitions the set $r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$ into disjoint equivalence classes. Notice that the integer solution set is complete with respect to the bound r in the sense that the sum of integer coordinates of the lattice points in $r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$ takes all the values of the integers in the closed interval $[0, r]$. Completeness is one of the beautiful properties of the standard d-simplex not all the lattice polytopes enjoy this feature.

Corollary 2.2 *Any two lattice points in* $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ *belong to the same class if and only if they share the same sum of their respective integer coordinates.*

Corollary 2.3 $|r\Delta_d \cap \mathbb{Z}^d_{\geq 0}/\sim| = r + 1.$

Proof This follows corollary 2 and the fact that r$\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ is complete

$$r\Delta_d \cap \mathbb{Z}^d_{\geq 0}/\sim := \{X_k : \sum_{i=1}^{d} x_i = k,\ 0 \leq k \leq r, \forall x = (x_1, \cdots, x_d) \in X_k\} \tag{2.1}$$

and hence, $|r\Delta_d \cap \mathbb{Z}^d_{\geq 0}/\sim| = r + 1.$ □

Corollary 2.4 *The class of the origin* $\underline{0} \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ *is a singleton set.*

Proof The class of the origin denoted by X_0 is given by

$$X_0 = \{\underline{x} = (x_1, \cdots, x_d) \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0} : \sum_{i=1}^{d} x_i = 0\}. \tag{2.2}$$

Suppose that there is a lattice point $\mathbf{a}$ which belongs to X_0 such that $\mathbf{a}$ is not the origin. Since the origin $\underline{0}$ is $<_{\text{lex}}$ than every lattice point $\mathbf{a} \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$, so, $\underline{0} \sim \mathbf{a}$ implies that $\sum_{i=1}^{d}(0 - a_i) < 0$, This integer value is not in $[0, r]$, therefore, there is no lattice point $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ which is equivalent to the origin apart from itself hence $|X_0| = 1$. □

We now compute the size of each of the equivalence classes X_k such that $0 \leq k \leq r$.

Theorem 2.5 *Let* $\mathcal{A} = r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ *denote the set of lattice points on* $r\Delta_d$ *and let* $X_k \subset \mathcal{A}$ *be the collection of lattice points whose sum of their integer coordinates is* k *such that* $0 \leq k \leq r$. *Then* $|X_k| = \binom{k+d-1}{d-1}$.

Proof Notice that the chain of the following inclusions

$$\{(0, \cdots, 0)\} \subset \Delta_d \cap \mathbb{Z}^d_{\geq 0} \subset 2\Delta_d \cap \mathbb{Z}^d_{\geq 0} \cdots \subset r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$$

implies the following chain

$$\Delta_d \cap \mathbb{Z}^d_{>0} \subset 2\Delta_d \cap \mathbb{Z}^d_{>0} \subset \cdots \subset r\Delta_d \cap \mathbb{Z}^d_{>0}.$$

The subcollection X_k is given by

$$X_k = \{\underline{x} = (x_1, \cdots, x_d) \in \mathcal{A} : \quad \sum_{i=1}^{d} x_i = k, \quad 0 \leq k \leq r\},$$

$X_0 = \{(0, \cdots, 0)\}$ and so $|X_k| = 1$. Observe that

$$X_k = k\Delta_d \cap \mathbb{Z}^d_{\geq 0}/(k-1)\Delta_d \cap \mathbb{Z}^d_{\geq 0}, \quad 2 \leq k \leq r$$

In fact, $X_k's$ define a partition of the set $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ of the lattice points on $r\Delta_d$, that is,

$$\bigcap_{k=0}^{r} X_k = \emptyset, \qquad \bigcup_{k=0}^{r} X_k = \mathcal{A}$$

From Ehrhart theory, using (1.4),

$$|\Delta_d \cap \mathbb{Z}^d_{\geq 0}| = \binom{1+d}{d} = |X_0 \cup X_1|.$$

This implies that $|X_1| = d$. Similarly,

$$|2\Delta_d \cap \mathbb{Z}^d_{\geq 0}| = \binom{2+d}{d} = |X_0 \cup X_1 \cup X_2|,$$

which gives

$$|X_2| = \binom{2+d}{d} - d - 1 = \binom{1+d}{d-1}.$$

Continuing this way,

$$|X_k| = \binom{k+d}{d} - \sum_{j=1}^{k} \binom{k+d-j}{d} = \binom{k+d-1}{d-1}. \qquad \square$$

The disjoint union $\cup X_k$ of subcollections $X_k, \quad 0 \leq k \leq r$ of the set $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ of lattice points on $r\Delta_d$ defines a polynomial $T_r(t)$ of degree r in variable t given by

$$T_r(t) = \sum_{k=0}^{r} \binom{k+d-1}{d-1} t^k. \tag{2.3}$$

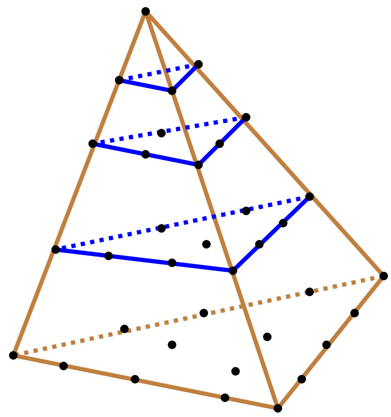

Figure 1 $T_4(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4$

We call $T_r(t)$ the dilation polynomial of degree r identified with the dilation $r\Delta_d$. This is precisely the slicing of $r\Delta_d$ with hyperplanes perpendicular to the direction $\mathbf{a} := (1, \cdots, 1)$ and enumerate all the lattice points in the different layers. That is,

$$\binom{k+d-1}{d-1} = \#\{v \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0} : v \cdot \mathbf{a} = k, 0 \leq k \leq r\}. \tag{2.4}$$

The dilation polynomial $T_4(t)$ for the 4th dilation of the standard 3-simplex is illustrated in Figure 1.

Remark 2.6 Dilation polynomials identified with $r\Delta_2$ and $r\Delta_3$ are called triangular and tetrahedral polynomials respectively.

Theorem 2.7 *Let $\mathcal{M} = \{T_r(t)\}_{r=0}$ be the sequence of dilation polynomials of lattice points counting on $r\Delta_d$ for $r \geq 0$. Then its generating series $G(t,z) = \sum_{r=0} T_r(t)z^r$ is given by*

$$G(t,z) = \frac{z}{(1-z)(1-tz)^d}.$$

Proof Notice from the equation (2.3) that

$$T_r(t) = T_{r-1}(t) + \frac{(r+1)\cdots(r+d-1)}{(d-1)!}t^r \quad \text{and} \quad \sum_{r\geq 0} \frac{(r+1)\cdots(r+d-1)}{(d-1)!} z^r = \frac{1}{(1-z)^d}.$$

$$G(t,z) = \sum_{r\geq 0} T_r(t)z^r = \sum_{r\geq 0}\left[T_{r-1}(t) + \tfrac{(r+1)\cdots(r+d-2)}{(d-1)!}t^{r-1}\right]z^r.$$

$$G(t,z) = zG(t,z) + \sum_{r\geq 1}\left[\tfrac{(r+1)\cdots(r+d-1)}{(d-1)!}t^{r-1}\right]z^r,$$

and so

$$G(t,z) = \frac{z}{(1-z)(1-tz)^d}.$$

□

§3. The Cohomology ring of Grassmannian $Gr(d, d+r)$

Let V be an n-dimensional complex vector space. The set of all maximal chains of subspaces in V is called the flag variety $\mathcal{F}\ell_n(\mathbb{C})$ of dimension $\frac{n(n-1)}{2}$. That is,

$$\mathcal{F}\ell_n(\mathbb{C}) := \{V_\bullet := \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = V \ \text{ such that } \ \dim V_i = i\}.$$

The Grassmannian $Gr(d,n)$ is the spacial case of the flag variety being the set of all d-dimensional subspaces in V. Its dimension is $d(n-d)$. There is a projection

$$\pi : \mathcal{F}\ell_n(\mathbb{C}) \longrightarrow Gr(d,n)$$

from the full flag variety $\mathcal{F}\ell_n(\mathbb{C})$ to the Grassmannian $Gr(d,n)$ with $\pi^{-1}(X_\lambda(F_\bullet)) = X_{w(\lambda)}(F_\bullet)$, where $X_\lambda(F_\bullet)$ is a Schubert variety in the Grassmannian $Gr(d,n)$ defined as the closure of the Schubert cell $C_\lambda(F_\bullet)$ given by

$$C_\lambda(F_\bullet) = \{V_d \in Gr(d,n) : \dim V_d \cap F_{n+i-\lambda_i} = i,\ 1 \le i \le d\},$$

with respect to the fixed flag $F_\bullet$:

$$F_\bullet := \{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_n = V \ \text{ such that } \ \dim F_i = i$$

The partition λ is called fitted in the sense that it has at most length d and each part cannot exceed $n-d$. The permutation $w(\lambda)$ identified with the partition $\lambda = (\lambda_1, \cdots, \lambda_d)$ is given by

$$w_i = i + \lambda_{d+1-i},\ 1 \le i \le d \text{ and } w_j < w_{j+1},\ d+1 \le j \le n. \tag{3.1}$$

This permutation is called Grassmannian in that it has a unique descent by definition. Every such permutation has the code $c(w(\lambda))$ of the form $(w_1-1, w_2-2, \cdots, w_d-d, 0, \ldots, 0)$ which can be represented by $(m_1, m_2, \cdots, m_d)$ by disregarding the string of zeros at the right hand. It turns out that the partition λ indexing the Schubert variety X_λ can be recovered from this code as $\lambda = (m_{i_1}, m_{i_2}, \cdots, m_{i_d})$ where $m_{i_1} \ge m_{i_2} \ge \cdots \ge m_{i_d}$ and $m_{i_p} \ne 0,\ 1 \le i_p \le d$. Recall that for any permutation w in the symmetric group S_n, the code $c(w)$ of w is the sequence $(c_1(w), \cdots, c_n(w))$ where $c_i(w) = \mid \{j : 1 \le i < j \le n \text{ and } w(i) > w(j)\} \mid$. For instance the code $c(w)$ of the permutation $w = 315426 \in S_6$ is $(2,0,2,1,0,0)$. The string of zeros at the right hand may be discarded. Notice that $c_i(w) \le n-i$. The length $\ell(w)$ of w is $\#\{(i,j) : w(i) > w(j), 1 \le i < j \le n\}$, the number of inversions in w, that is, the sum of integer coordinates of the code of w. It is well known that the cohomology ring of the Grassmannian $Gr(d,n)$ is generated by the Schubert cycles σ_λ. These are Poincaré dual of the fundamental classes in the homology of Schubert varieties. The Grassmannian $\mathrm{Gr}(d,n)$ admits many important vector bundles, most importantly there is a universal short exact sequence: $0 \longrightarrow \mathcal{S} \longrightarrow \mathbb{C}^n \times \mathrm{Gr}(d,n) \longrightarrow \mathcal{Q} \longrightarrow 0$ of bundles on $Gr(d,n)$ which makes it easy to compute the Chern class $c(\mathcal{Q})$ of the quotient bundle $\mathcal{Q}$ on the Grassmannian $Gr(d,n)$. Recall that $\mathcal{Q}$ is a globally generated vector bundle of rank $r := n-d$ and all its global sections are from the trivial bundle $\mathbb{C}^{d+r} \times \mathrm{Gr}(d, d+r)$. The total Chern class is the sum over all the one-row partitions inside the rectangle $\square_{r\times d}$. That is,

$$c(\mathcal{Q}) = 1 + \sigma_{\square} + \sigma_{\square\square} + \cdots + \sigma_{\square\square\cdots\square_{1\times r}}. \tag{3.2}$$

It turns out that the set of all one-row Young diagrams indexing the multiplicative generators of the cohomology of the Grassmannian $\mathrm{Gr}(d, d+r)$ is deeply connected with the lattice points of $r\Delta_d$. Let $\mathcal{C}_{d,r}$ be the set of row Young diagrams with at most r boxes and adjoin the empty set ϕ. That is,

$$\mathcal{C}_{d,r} = \{\square_{1\times k} :\ 1 \le k \le r\} \cup \emptyset.$$

The filling of the boxes of the row Young diagrams in $\mathcal{C}_{d,r}$ using the numbers in $[d] := \{1, \cdots, d\}$ is semi standard, that is, the numbers weakly increase from the left to the right. We denote the collection of all such fillings by $\mathcal{C}^d_{d,r}$ and call it the d-filling set of the dilation $r\Delta_d$. For instance, the 3-filling set $\mathcal{C}^3_{3,3}$ associated the second dilation $3\Delta_3$ of the standard 3-simplex is the following collection

| . |, | 1 |, | 2 |, | 3 |, | 1 | 1 |, | 1 | 2 |, | 2 | 2 |, | 1 | 3 |, | 2 | 3 |,

| 3 | 3 |, | 1 | 1 | 1 |, | 1 | 1 | 2 |, | 1 | 2 | 2 |, | 2 | 2 | 2 |, | 1 | 1 | 3 |,

| 1 | 3 | 3 |, | 3 | 3 | 3 |, | 2 | 2 | 3 |, | 2 | 3 | 3 |, | 1 | 2 | 3 |.

These 20 semi standard Young tableaux can be organized in terms of their defining Young diagrams. It turns out that this arrangement can be expressed as a polynomial, given by $T_3(t) = 1 + 3t + 6t^2 + 10t^3$. This is the graded semi-standard polynomial of degree 3 illustrated in Figure 2.

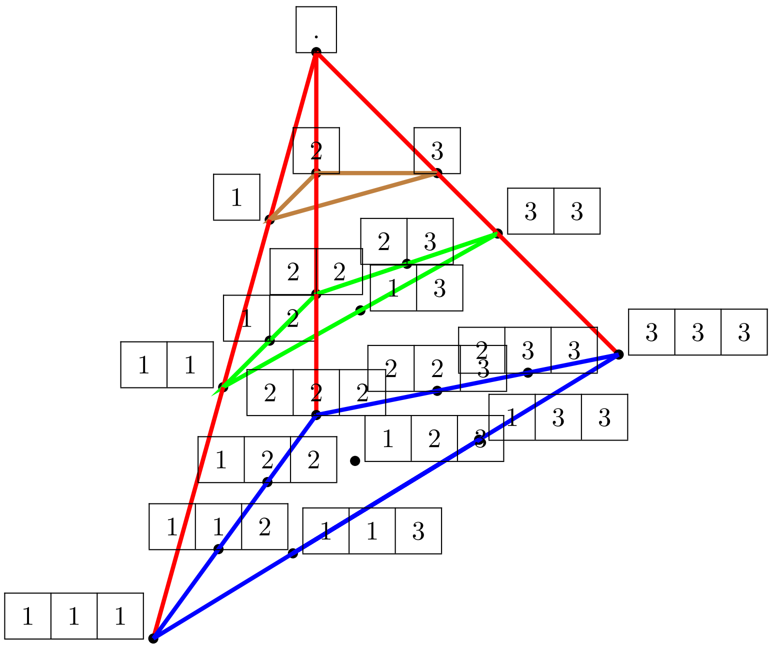

Figure 2. $T_3(t) = P_3(t) = 1 + 3t + 6t^2 + 10t^3$

Theorem 3.1 *(i) The size $L^d(r)$ of the d-filling set $\mathcal{C}^d_{d,r}$ is $\binom{r+d}{d}$ and the sequence $(L^d(r))_{r=0}^{\infty}$ of cardinalities as r grows is recorded by the generating function*

$$P(\mathcal{C}^d_{(d,r)}, z) = \frac{1}{(1-z)^{d+1}}.$$

(ii) More is true, there is a graded counting polynomial of the semi standard tableaux in $\mathcal{C}^d_{d,r}$ given by

$$P_r(t) = \sum_{k=0}^{r} \binom{k+d-1}{d-1} t^k$$

that is, a k-box row diagram gives $\binom{k+d-1}{d-1}$ semi standard Young tableaux. This has a generating function

$$G(t,z) = \frac{z}{(1-z)(1-tz)^d}.$$

Theorem 3.2 *There is a bijection $T \mapsto v(T)$ between the set $\mathcal{C}^d_{d,r}$ and the set $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ of the lattice points of the dilation $r\Delta_d$. Furthermore, the semi-standard polynomial $P_r(t)$ is precisely the dilation polynomial $T_r(t)$ identified with $r\Delta_d$.*

Proof To each semi standard tableau $T \in \mathcal{C}^d_{(d,r)}$ there exists a unique exponent vector $v(T) := (v(T)_1, \cdots, v(T)_d)$ in which the coordinate $v(T)_j$ is the number of appearances of j in T, $1 \leq j \leq d$. This is a bijection.

The number of semi standard fillings of each of the row diagram with shape $\lambda = (k), 0 \leq k \leq r$ using the elements of the set $\{1, \ldots, d\}$ has a well known closed formula. Notice that for a fixed point $\mathbf{a} = (1, \cdots, 1)$ the following identity holds

$$\prod_{1 \leq i < j \leq d} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \binom{k+d-1}{d-1} = \#\{v \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0} : v \cdot \mathbf{a} = k, 0 \leq k \leq r\}$$

Therefore, the semi-standard polynomial $P_r(t)$ can be viewed as the dilation polynomial $T_r(t)$. The bijection is a polynomial preserving map, see Figure 2. □

§4. Grassmannian Monomials

It is clear from the Theorem 3.1 that every standard tableau $T \in \mathcal{C}^d_{(d,r)}$ defines a monomial $\mathbf{t}^{v(T)}$ where $v(T) := (v(T)_1, \cdots, v(T)_d)$, that is,

$$\mathbf{t}^{v(T)} := \prod_{j=1}^{d} t_j^{\#\text{ times j appears in T}}, \quad \text{where } \ v(T) \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0}. \tag{4.1}$$

For instance, the monomial defined by $T = \boxed{1\,|\,1\,|\,2\,|\,3\,|\,3} \in \mathcal{C}^4_{(4,5)}$ is given by $\mathbf{t}^{\mathbf{a}} = t_1^2 t_2 t_3^2$ where $\mathbf{a} = (2,1,2,0)$. We call such monomials in $\mathcal{C}^d_{(d,r)}$ Grassmannian because they encode the data of indexing partitions of Schubert varieties in the Grassmannian $\mathrm{Gr}(d, d+r)$. We denote these monomials by W^r_d, that is,

$$W^r_d := \{t_1^{a_1} \cdots t_d^{a_d} : \sum_{i=1}^{d} a_i \leq r, \quad 0 \leq a_i \leq r\}$$

Proposition 4.1 *Let W^r_d and $W^{r'}_d$ be two Grassmannian monomial sets such that $r \leq r'$. Then $W^r_d \subseteq W^{r'}_d$.*

Proposition 4.2 *Every monomial $\mathbf{t}^{\mathbf{a}} \in \mathbb{Z}[t_1, \cdots, t_d]$ is Grassmannian.*

Proof It suffices to produce a Grassmannian set W^r_d containing $\mathbf{t}^{\mathbf{a}}$. By (4.1) there is a semi-standard tableau T which encodes the exponent vector $\mathbf{a}$ and this implies that there exists $r \in \mathbb{N}$ such that T is an element of the d- filling set $\mathcal{C}^d_{(d,r)}$, so $\mathbf{t}^{\mathbf{a}}$ belongs to the Grassmannian monomial set W^r_d. □

Corollary 4.3 *If $r = \sum_{i=1}^d a_i$, where a_i is an integer coordinate of $\mathbf{a}$ then the Grassmannian set W^r_d is the smallest set containing the monomial $\mathbf{t}^{\mathbf{a}}$.*

It is important to quickly point out that the sum $P_r(t_1, \ldots, t_d)$ of all the monomials in W^r_d, that is,

$$P_r(t_1, \cdots, t_d) = \sum_{T \in \mathcal{C}^d_{(d,r)}} \prod_{j=1}^d t_j^{\#\text{ times j appears in T}} \tag{4.2}$$

is deeply connected with a polynomial representation (V, ρ) of the general linear group $GL_d(\mathbb{C})$ where $V := \bigoplus_{k=0}^r \mathrm{Sym}^k(\mathbb{C}^d)$ is the space of the direct sum of homogeneous symmetric polynomials of degree k in d variables. Let $\mathbb{C}[X] := \mathbb{C}[x_{11}, x_{12} \cdots, x_{dd}]$ be the ring of polynomial functions on $d \times d$ matrices. There is an action of $G = GL_d(\mathbb{C})$ on $\mathbb{C}[X]$ by conjugation. The character of the polynomial representation (V, ρ) is the polynomial $\chi_\rho \in \mathbb{C}[X]$ given by the trace of the matrix $\rho(X)$. Recall that the character χ_ρ of every polynomial representation (V, ρ) lies in the invariant ring $\mathbb{C}[X]^G$. Interested reader can consult [15] and [17].

Theorem 4.4 *The character χ_V of $V := \bigoplus_{k=0}^r \mathrm{Sym}^k(\mathbb{C}^d)$ as a polynomial representation ρ of the general linear group $GL_d(\mathbb{C})$ is $P_r(t_1, \cdots, t_d)$, that is,*

$$\chi_V = \sum_{T \in \mathcal{C}^d_{(d,r)}} \prod_{j=1}^d t_j^{\#\text{ times j appears in T}}$$

The sum ranging over all the semi standard fillings of the row diagrams with at most r boxes.

Proof Let $t_1, \cdots, t_d$ be eigenvalues of a generic $d \times d$ matrix X. The map $\mathbb{C}[X]^G \longrightarrow \mathbb{C}[t_1, t_2, \cdots, t_d]^{S_n}$ defined by$f \mapsto f(\mathrm{diag}(t_1, \ldots, t_d))$ is an isomorphism. Set $\lambda = (k)$ since $k's$ define the rows diagrams with at most r boxes, so the image of the character $f_\rho(X)$ is

$$\sum_{k=0}^r \frac{\det(t_i^{\lambda_i + d - j})_{1 \le i,j \le d}}{\det(t_i^{d-j})_{1 \le i,j \le d}}.$$ □

Corollary 4.5 *The dimension of the vector space $V := \bigoplus_{k=0}^r \mathrm{Sym}^k(\mathbb{C}^d)$ is $\chi_V(1, 1, \cdots, 1) := |r\Delta_d \cap \mathbb{Z}_{\ge 0}|$, the number of lattice points of the dilation $r\Delta_d$.*

Proof The Grassmannian set W^r_d spans the vector space $V := \bigoplus_{k=0}^r \mathrm{Sym}^k(\mathbb{C}^d)$. □

Now to every monomial $\mathbf{t}^{\mathbf{a}} \in \mathbb{Z}[t_1, \cdots, t_d]$ we associate a weight $w_{\mathbf{a}}$ defined by

$$w_{\mathbf{a}} = \sum_{k=1}^d k a_k. \tag{4.3}$$

It turns out that $w_{\mathbf{a}}$ admits two important partitions $\lambda, \lambda^* \vdash w_{\mathbf{a}}$ which can be identified with the monomial $\mathbf{t}^{\mathbf{a}}$. These partitions, λ and λ^* are called α-partition and β-partition respectively. A partition $\lambda \vdash w_{\mathbf{a}}$ is said to be the α-partition of the monomial $t_1^{a_1} \cdots t_d^{a_d}$ if the number of parts of size i in λ is a_i, $1 \leq i \leq d$. The length $\ell(\lambda)$ of α-partition is $a_1 + \cdots + a_d$. The β partition $\lambda^* = (\lambda_1^*, \cdots, \lambda_d^*)$ of $w_{\mathbf{a}}$ is such that $\lambda_k^* = \sum_{i \geq k}^d a_i$, $1 \leq k \leq d$ and its length is d. For instance, the α-partition associated with the monomial $t_1^3 t_2^2 t_3^3 t_4^2 \in \mathbb{Z}[t_1, t_2, t_3, t_4]$ is (4,4,3,3,3,2,2,1,1,1) while its β partition i λ^* is $(10, 7, 5, 2)$. In fact, α and β partitions identified with the monomial $\mathbf{t}^{\mathbf{a}}$ can be realized in terms of the sum of the entries of the $d \times d$ upper triangular matrix $\mathrm{M}_{\mathbf{a}}$ associated with the exponent vector $\mathrm{a} = (a_1, \cdots, a_d)$ of the monomial, that is,

$$\mathrm{M}_{\mathbf{a}} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_d \\ & a_2 & a_3 & \cdots & a_d \\ & & a_3 & \cdots & a_d \\ & & & & \vdots \\ & & & & a_d \end{bmatrix} \tag{4.4}$$

The sum of the entries in the column k divided by k is the number of parts of size k in the α- partition λ of $w_{\mathbf{a}}$. The β partition $\lambda^* = (\lambda_1^* \cdots \lambda_d^*)$ of $w_{\mathbf{a}}$ is such that λ_k^* is the sum of the entries in the row k where $1 \leq k \leq d$. For instance, the 4×4 matrix $\mathrm{M}_{\mathbf{a}}$ corresponding to the monomial $t_1^3 t_2^2 t_3^3 t_4^2 \in \mathbb{Z}[t_1, t_2, t_3, t_4]$ is

$$\mathrm{M}_{\mathbf{a}} = \begin{bmatrix} 3 & 2 & 3 & 2 \\ & 2 & 3 & 2 \\ & & 3 & 2 \\ & & & 2 \end{bmatrix}$$

so the α-partition λ and the β-partition λ^* identified with the matrix $\mathrm{M}_{\mathbf{a}}$ are $1^3 2^2 3^3 4^2$ and $(10, 7, 5, 2)$ respectively.

Proposition 4.6 *Let λ be the α-partition of $w_{\mathbf{a}}$ associated with the monomial $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_d^{a_d} \in \mathbb{Z}[t_1, \cdots, t_d]$. Then its corresponding β-partition λ^* is the transpose of λ and vice versa.*

Proof Let $\lambda = (\lambda_1, \cdots, \lambda_{a_1 + \cdots + a_d})$ and $\lambda^* = (\lambda_1^*, \cdots, \lambda_d^*)$. It is obvious that these partitions satisfy the following identity

$$\sum_{k=1}^{a_1 + \cdots + a_d} (2k-1)\lambda_k = \sum_{k=1}^{d} \lambda_k^{*2}. \qquad \square$$

It would be interesting to characterize and study all the monomials for which α-partition and β-partition coincide. This amounts to the characterization of all self conjugate partitions. Recall that for all $n \in \mathbb{N}$ such that $n > 2$ there is a bijection between the set of self conjugate partitions of n and the set of all distinct odd parts partitions of n. For instance, a square free monomial of the form $t_1t_2\cdots t_d$ admits the stair case partition $(d, d-1, \cdots, 1)$, this is deeply connected with the distribution of triangular numbers in the set $\mathbb{N}$ of natural numbers. We give a few other examples of such monomials.

Example 4.7 Some monomials following for which α and β-partitions coincide:

(*i*) All monomials of the form $t_{\frac{d}{2}}^{\frac{d}{2}} t_d^{\frac{d}{2}} \in \mathbb{Z}[t_1, t_2, \cdots, t_d]$ for even d;

(*ii*) All monomials of the form $t_1 t^{d-2} t_d \in \mathbb{Z}[t_1, t_2, \cdots, t_d]$;

(*iii*) All monomials of the form $t_1^{d-1} t_d \in \mathbb{Z}[t_1, t_2, \cdots, t_d]$;

(*iv*) All monomials of the form $t_{d-2} t_{d-1} t_d^{d-2} \in \mathbb{Z}[t_1, t_2, \cdots, t_d]$.

Lemma 4.8 *Let $\lambda^* = (\lambda_1, \cdots, \lambda_d)$ be the β-partition identified with the monomial $t_1^{a_1} \cdots t_d^{a_d} \in \mathbb{Z}[t_1, t_2, \cdots, t_d]$. Then the exponent vector $(a_1, \cdots, a_d)$ is equivalent to $(\lambda_1-\lambda_2, \lambda_2-\lambda_3, \cdots, \lambda_{d-1}-\lambda_d, \lambda_d)$.*

Proof It follows from the construction of the β partition λ^* from the exponent vector $(a_1, \cdots, a_d)$. □

Theorem 4.9 *Let $\mathbf{t^a} \in W_d^r$ be a Grassmannian monomial associated with exponent vector $\mathbf{a} \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$. If a partition λ^* is the β-partition identified with $\mathbf{t^a}$ then the length $\ell(w(\lambda^*))$ of the Grassmannian permutation $w(\lambda^*)$ is the weight $w_{\mathbf{a}}$.*

Proof The code $c(w(\lambda^*))$ of Grassmannian permutation $w(\lambda^*)$ is of the form $(m_1, m_2, \cdots, m_d, 0, 0, \cdots, 0)$. The rearrangement of $m_1, m_2, \cdots, m_d$ in weakly decreasing order yields the fitted partition $\lambda^* = (\lambda_1^*, \cdots, \lambda_d^*)$. The sum of entries of the code $c(w) = (c_1(w), c_2(w), \cdots, c_n(w))$ of any permutation w is the length $\ell(w)$ of the partition, since each entry $c_i(w)$ is the number of inversions associated to the value w_i in the position i. Hence the length $\ell(w(\lambda^*))$ of $w(\lambda^*)$ is the size $|\lambda^*|$ of λ^*. Next we show that the weight $w_{\mathbf{a}}$ of the exponent vector $\mathbf{a} = (a_1, \cdots, a_d)$ of the Grassmannian monomial $\mathbf{t^a} = t_1^{a_1} \cdots t_d^{a_d}$ is $|\lambda^*|$. From Lemma 3.12 $a_i = \lambda_i^* - \lambda_{i+1}^*, \ \ 1 \leq i \leq d-1, \ \ a_d = \lambda_d^*$. Therefore, the weight $w_{\mathbf{a}} = \sum_{i=1}^{d-1} i(\lambda_i^* - \lambda_{i+1}^*) + d\lambda_d^* = |\lambda^*|$. □

Corollary 4.10 *Every β-partition λ^* identified with each of the monomials $\mathbf{t^a} \in W_d^r$ fits into the $r \times d$ rectangle $\square_{r\times d}$.*

Proof It is sufficient to establish that the parts of λ^* cannot exceed r and the length $\ell(\lambda^*)$ of λ^* is d. Notice that the exponent vector $\mathbf{a}$ is a lattice point of $r\Delta_d$ and by definition $a_1 + \cdots + a_d \leq r$. Therefore each part λ_k^* of λ^* is at most r and length $\ell(\lambda^*)$ is d by the definition of λ^*. □

Corollary 4.11 *The set of β-partitions λ^* identified with monomials in W_d^r index the Schubert varieties in the Grassmannian $Gr(d, d+r)$, giving a bijection between lattice points in $r\Delta_d$ and partitions fitting into an $r \times d$ rectangle.*

The weight $w_{\mathbf{a}}$ defined in the equation (3.2) gives another refinement $P^h_{r\Delta_d}(z)$ of the Ehrhart polynomial of $r\Delta_d$ with respect to a fixed point $h = (1, 2, \cdots d)$.

$$P^h_{r\Delta_d}(z) = \sum_{m=0}^{dr} A_m z^m. \tag{4.5}$$

where $A_m = \#\{\mathbf{a} \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0} : \mathbf{a} \cdot h = m,\ 0 \leq m \leq dr\}$, that is, the number of exponent vectors $\mathbf{a}$ which share the weight m. We call $P^h_{r\Delta_d}(z)$ the weighted polynomial associated with the dilation $r\Delta_d$.

Lemma 4.12 *The polynomial $P^h_{r\Delta_d}(z) = \sum_{m=0}^{dr} A_m z^m$ specializes at $z = 1$ to the Ehrhart polynomial $L_{\Delta_d}(r)$.*

Remark 4.13 Notice that A_m is precisely the number of lattice points in the intersection of the dilation $r\Delta_d$ with the hyperplane H_m perpendicular to the direction $\mathbf{h} := (1, 2, \cdots, d)$. It is also interesting to note that the grading given here to a lattice point eventually identifies the weighted polynomial $P^h_{r\Delta_d}(z)$ with the Poincaré polynomial of the Grassmannian $\mathrm{Gr}(d, d + r)$.

Theorem 4.14 *Let $P^h_{r\Delta_2}(z)$ be the weighted polynomial of the lattice points of the dilation $r\Delta_d$. Then the Poincaré polynomial $P(Gr(d, d + r), t)$ of the Grassmannian $Gr(d, d + r)$ coincides with the weighted polynomial $P^h_{r\Delta_d}(z)$.*

Proof It is well known from the Borel presentation of the cohomology ring $H^*(Gr(d, d + r), \mathbb{Z})$ of the Grassmannian $\mathrm{Gr}(d, d+r)$ that the Poincaré polynomial $\mathrm{P}(\mathrm{Gr}(d, d+r), t)$ is given by the following Gaussian polynomial

$$\frac{(1-t)(1-t^2)\cdots(1-t^{d+r})}{(1-t)\cdots(1-t^d)(1-t)\cdots(1-t^r)}.$$

This is combinatorially simplified as

$$\sum_{\lambda \subseteq \square_{d\times r}} t^{|\lambda|}$$

where $|\lambda|$ is the number of boxes in the Young diagram of shape λ. The size $|\lambda|$ coincides with the length $\ell(w(\lambda))$ (the number of inversions) of the Grassmannian permutation $w(\lambda)$ identified with λ in the equation (3.1). Notice that $|\lambda| \leq dr$, therefore, It follows from the Theorem 4.9 that $|\lambda|$ is the weight $w_{\mathbf{a}}$ of the monomial $t^{\mathbf{a}} \in W^r_d$, $a \in r\Delta \cap \mathbb{Z}^d_{\geq 0}$, therefore, $\sum_{\lambda \subseteq \square_{2\times r}} t^{|\lambda|}$ is precisely the polynomial $\sum_{m=0}^{dr} A_m z^m$. □

Question 4.15 *Does the set $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ encode some data about the degree and the Hilbert polynomial of $Gr(d, d + r)$?*

The goal of this paper is the general study of some combinatorial geometry of the lattice points $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ associated with $r\Delta_d$. That is, we evoke some geometric information about these lattice points. In particular, we answer the following questions:

(*i*) *What kind of geometric information can be extracted from these integral solutions to (1.4)?*

(ii) What kind of combinatorial object parameterizes this solution set?

(iii) Is there an interesting polynomial P_r which keeps track of the integral points? In other words, Is there a generic polynomial of degree r whose exponents of its monomials with nonzero coefficients satisfy (1.3)*?*

Theorem 4.16 *Every linear polynomial function of the form $y = ax + 1$ such that $a \in \mathbb{N}$ is the fundamental polynomial of a certain standard d-simplex whose dimension is the slope of the polynomial function.*

Proof Consider the family $\mathcal{G}$ of'Cartesian graphs of all linear functions of the form $y = ax+1$ such that $a \in \mathbb{N}$. It is obvious that these graphs are parametrized by the x-intercepts since they all share the same y-intercept $(0, 1)$. Consider the sequence $\mathcal{E} = (-\frac{1}{a})_{a=1}$ of x-intercepts. $\mathcal{E}$ is strictly monotone decreasing and lies in the interval $[-1, 0)$. There is a bijection $a \mapsto -\frac{1}{a}$, between the sequence $\mathcal{K} = (\Delta_a)_{a=1}$ of standard a-simplices and the sequence $\mathcal{E}$ of x-intercepts of $\mathcal{G}$. As $\mathcal{K}$ diverges, $\mathcal{E}$ converges. □

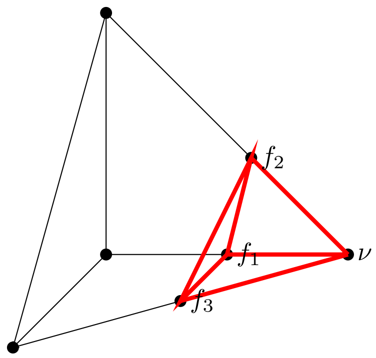

Figure 3. 3-Simplex

The sum of all the monomials in W_d^r is called the symbolic polynomial corresponding to the d-filling set $\mathcal{C}^d_{(d,r)}$. That is,

$$P_r(t_1, \cdots, t_d) = \sum_{T \in \mathcal{C}^d_{(d,r)}} \mathbf{t}^{wt(T)}. \tag{4.6}$$

For every lattice point $\mathbf{a} \in r\Delta_2 \cap \mathbb{Z}^2_{\geq 0}$, there is a corresponding monomial $\mathbf{t}^{\mathbf{a}}$ in the polynomial ring $\mathbb{Z}[t_1, t_2]$ given by $\mathbf{t}^{\mathbf{a}} := t_1^{a_1} t_2^{a_2}$. We call these monomials Grassmannian and denote their collection by W_2^r,that is,

$$W_2^r = \{t_1^{a_1} t_2^{a_2} \in \mathbb{Z}[t_1, t_2] : (a_1, a_2) \in r\Delta_2 \cap \mathbb{Z}^2_{\geq 0}\}. \tag{4.7}$$

To every monomial $\mathbf{t}^{\mathbf{a}} \in W_2^r$ we associate a weight $w_{\mathbf{a}}$ defined by

$$w_{\mathbf{a}} = \sum_{k=1}^{d} k a_k. \tag{4.8}$$

It turns out that $w_{\mathbf{a}}$ admits two important partitions $\lambda, \lambda^* \vdash w_{\mathbf{a}}$ which can be identified with the monomial $\mathbf{t}^{\mathbf{a}}$. These partitions, λ and λ^* are called α-partition and β-partition respectively. A partition $\lambda \vdash w_{\mathbf{a}}$ is said to be the α-partition of the monomial $t_1^{a_1}t_2^{a_2} \in W_2^r$ if the number of parts of size i in λ is a_i, $1 \leq i \leq 2$, while the β partition $\lambda^* = (\lambda_1^*, \lambda_2^*)$ of $w_{\mathbf{a}}$ is such that $\lambda_k^* = \sum_{i\geq k}^{2} a_i$, $1 \leq k \leq 2$. This is not exclusively for only Grassmannian monomial, it is true for all monomials. For instance, given a monomial $t_1^2t_2^3$. The corresponding *alpha*-partition λ and β-partition λ^* are (2,2,2,1,1) and (5,3) respectively.

Corollary 4.17 *The triangular polynomial $T_r(t) = \sum_{c=0}^{r}(c+1)t^c$ specialises at $t = 1$ to the Ehrhart polynomial $\binom{r+2}{2}$.*

We now give a combinatorial construction of a certain discrete object $\mathcal{C}_{d,r}^d$ identified with the lattice points of $r\Delta_d$ which we call the 3-filling set of the dilation. It describes certain fillings of a row Young diagram with the numbers from the set $[d] := \{1, \cdots, d\}$.

Theorem 4.18 The size $L^d(r)$ of the d-filling set $\mathcal{C}_{d,r}^d$ associated with the lattice points of the r^{th} dilation $r\Delta_d$ of the standard d-simplex is $\binom{r+d}{d}$. Moreover, the sequence $(L^d(r))_{r=0}^{\infty}$ as r grows is recorded by the generating function

$$P(\mathcal{C}_{(d,r)}^d, z) = \frac{1}{(1-z)^{d+1}}$$

Proof The size is given by $\sum_{k=0}^{r}\prod_{1\leq i<j\leq d}\frac{\lambda_i-\lambda_j+j-i}{j-i}$ since these are the semi standard fillings of the Young diagrams of shapes $\lambda = (k)$ $0 \leq k \leq r$ using the numbers from the set $\{1, \cdots, d\}$ and hence $\binom{r+d}{d}$. The sequence $(L^d(r))_{r=0}^{\infty}$ is given by triangular numbers which is well known. It is obvious that the generating series is in the coefficient of the polynomial $\binom{r+d}{d}$, that is, the general term of the sequence. Therefore, it is given by

$$\sum_{r\geq 0}\binom{r+d}{d}z^r = \frac{1}{(1-z)^{d+1}}. \qquad \square$$

Corollary 4.19 *There is a bijection between the set $\mathcal{C}_{d,r}^d$ of semi standard fillings of the row Young diagrams with at most r boxes using the numbers from $[d]$ and the set $r\Delta_d \cap \mathbb{Z}_{\geq 0}^d$ of the lattice points in the r^{th} dilation of the standard d-simplex.*

This bijection can be clearly understood in the language of monomials. This is the subject of discussion in what follows.

The symbolic polynomial $P_r(t_1, \cdots, t_d)$ is deeply connected with a polynomial representation (V, ρ) of the general linear group $GL_d(\mathbb{C})$ where $V := \bigoplus_{k=0}^{r}\operatorname{Sym}^k(\mathbb{C}^d)$. The space of homogeneous symmetric polynomials of degree k in d variables is denoted by $\operatorname{Sym}^k(\mathbb{C}^d)$. Let $\mathbb{C}[X] := \mathbb{C}[x_{11}, x_{12}\cdots, x_{dd}]$ be the ring of polynomial functions on $d \times d$ matrices. There is an action of $G = GL_d(\mathbb{C})$ on $\mathbb{C}[X]$ by conjugation. The character of a polynomial representation (V, ρ) is the polynomial $\chi_\rho \in \mathbb{C}[X]$ given by the trace of the matrix $\rho(X)$. Recall that the character χ_ρ of every polynomial representation (V, ρ) lies in the invariant ring $\mathbb{C}[X]^G$.

Theorem 4.20 *The character χ_V of $V := \bigoplus_{k=0}^{r} \operatorname{Sym}^k(\mathbb{C}^d)$ as a polynomial representation ρ of the general linear group $GL_d(\mathbb{C})$ is the symbolic polynomial*

$$\chi_V = \sum_{T\in\mathcal{C}^d_{(d,r)}} \prod_{j=1}^{d} t_j^{\#\text{ times j appears in T}}$$

The sum ranging over all the semi standard fillings of the row diagrams with at most r boxes.

Proof Let $t_1, \cdots, t_d$ be eigenvalues of a generic $d\times d$ matrix X. The map $\mathbb{C}[X]^G \longrightarrow \mathbb{C}[t_1, t_2, \cdots, t_d]^{\mathcal{S}_n}$ defined by $f \mapsto f(\operatorname{diag}(t_1, \cdots, t_d))$ is an isomorphism. Set $\lambda = (k)$ since $k's$ define the rows diagrams with at most r boxes, so the image of the character $f_\rho(X)$ is

$$\sum_{k=0}^{r} \frac{\det(t_i^{\lambda_i+d-j})_{1\le i,j\le d}}{\det(t_i^{d-j})_{1\le i,j\le d}}.$$

□

Corollary 4.21 *The dimension of the vector space $V := \bigoplus_{k=0}^{r} \operatorname{Sym}^k(\mathbb{C}^d)$ given by the value of $\chi_V(1, 1, \cdots, 1)$ is the number of lattice points of the dilation $r\Delta_d$.*

Proof The number of semi standard tableaux $T \in \mathcal{C}^d_{(d,r)}$ defined by the set of k-box row diagrams with at most r boxes. That is,

$$\sum_{k=0}^{r} \prod_{1\le i<j\le d} \frac{\lambda_i - \lambda_j + j - i}{j-i}$$

where $\lambda = (k)$ $0 \le k \le r$. This is precisely the number of monomials which constitute the character χ_V of each of these has coefficient 1. The value of $\chi_V(1, 1, \cdots, 1)$ is $\binom{r+d}{d}$. □

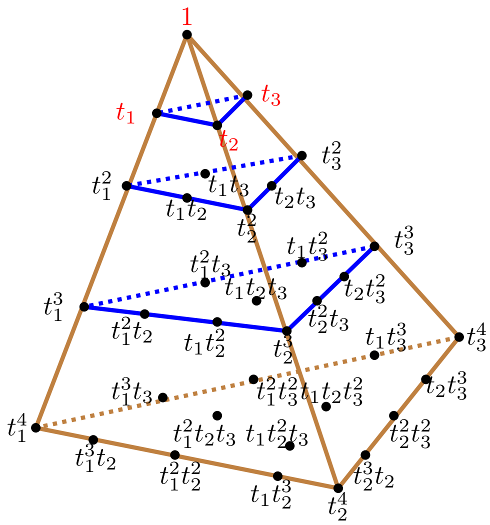

Figure 4. The monomial basis elements of W_3^4

The elements of W_d^r which span the vector space $V := \bigoplus_{k=0}^{r} \operatorname{Sym}^k(\mathbb{C}^d)$ encode the indexing partitions of the Schubert cycles of the cohomology ring of the Grassmanian $Gr(d, d + r)$ and therefore they are called Grassmannian monomials. This will dominate the discussion in what follows but we shall first describe in general how a monomial encodes information about partitions in the next section. Recall that partition λ of $n \in \mathbb{N}$ denoted $\lambda \vdash n$ is a list $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The length k of the partition λ is denoted by $\ell(\lambda)$ and each λ_i is called a part of the partition λ. Associated to every partition $\lambda \vdash n$ is its conjugate partition, $\lambda^t = (\lambda^t_1, \cdots, \lambda^t_m)$, which is also a partition of n where λ^t_i counts the parts of λ which are at least i. For example, the conjugate λ^t of the partition $\lambda = (4, 4, 3, 3, 3, 2, 2, 1, 1, 1)$ is given by $\lambda^t = (10, 7, 5, 2)$. A partition is said to be self conjugate if it coincides with its conjugate.

§5. The α and β Partitions of Monomial $t_1^{a_1} \cdots t_d^{a_d}$

Let $\mathbb{Z}[\mathbf{t}] := \mathbb{Z}[t_1, \cdots, t_d]$ be the polynomial ring over $\mathbb{Z}$ in the variables $t_1, \cdots, t_d$. We recall that by associating a monomial $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_d^{a_d}$ with its d-tuple exponent vector $\mathbf{a} = (a_1, \cdots, a_d) \in \mathbb{Z}^d_{\geq 0}$, a bijection between monomials in $\mathbb{Z}[t_1, \cdots, t_d]$ and exponent vectors in $\mathbb{Z}^d_{\geq 0}$ is realized.

We now construct the weighted polynomial Γ_{P_r} parameterized by the exponent vectors of the monomials the symbolic polynomial $P_r(t_1, \ldots, t_d)$ identified with the d-filling set $\mathcal{C}^d_{(d,r)}$. Recall that this is the character χ_V of the vector space $V := \bigoplus_{k=0}^r \mathrm{Sym}^k(\mathbb{C}^d)$ as a polynomial representation of the general linear group $GL_d(\mathbb{C})$ and notice that these exponent vectors are precisely $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$. Let the weight $w_{\mathbf{a}}$ defined in 5.1 be identified with each of the vectors. The identification realizes the weighted polynomial Ω_{P_r}

$$\Omega_{P_r}(z) = \sum_{m=0}^{dr} A_m z^m, \tag{5.1}$$

where A_m is number of exponent vectors of the monomials of $P_r(t_1, \cdots, t_d$ which share the same weight m. For instance, the weighted polynomial Ω_{P_3} parameterized by the exponent vectors of the symbolic polynomial $P_3(t_1, t_2, t_3)$ corresponding to $\mathcal{C}^3_{(3,3)}$ is given by

$$\Gamma_{P_3}(z) = 1 + z + 2z^2 + 3z^3 + 3z^4 + 3z^5 + 3z^6 + 2z^7 + z^8 + z^9.$$

This combinatorially defined polynomial from the lattice points of the dilation $r\Delta_d$ of the standard d-simplex Δ_d has an interesting interpretation in the cohomology of the Grassmannian $Gr(d, d + r)$.

The projection π induces a monomorphism π^* at the level of cohomology.

$$\pi^* : H^*(Gr(d, n), \mathbb{Z}) \longrightarrow H^*(\mathcal{F}\ell_n(\mathbb{C}), \mathbb{Z})$$

which takes cycle σ_λ to the cycle $\sigma_{w(\lambda)}$. The cohomology ring of the Grassmannian $Gr(d, n)$ is generated by the Schubert cycles σ_λ. These are Poincaré dual of the fundamental classes in the homology of Schubert varieties. Denote by Γ, the $\mathbb{Q}$-algebra of homogeneous symmetric functions in n variables $x_1, x_2, \cdots, x_n$. It well known that Γ is generated by Schur polynomials

s_λ among others, see [5], [8], [9] and [14]. By specializing $x_i = 0$ for $d+1 \le i \le n$, let Γ_d be the space of homogenous symmetric polynomials in variables $x_1, \cdots, x_d$, so Γ_d has the following presentation

$$\Gamma_d \cong \Gamma/\langle s_\lambda : \lambda \subsetneq \square_{d\times n-d}\rangle$$

The cohomology ring $H^*(Gr(d,n),\mathbb{Z})$ of the Grassmannian $Gr(d,n)$ by Borel presentation is given by

$$H^*(Gr(d,n),\mathbb{Z}) \cong \Gamma/\langle s_\lambda : \lambda \subsetneq \square_{d\times n-d}\rangle.$$

The interested readers may consult the following references [2], [4],[6],[7] and [12].

Recall that the Poincaré polynomial $P(X,t)$ associated with a given n-dimensional real manifold X is defined as

$$P(X,t) = \sum_{i=0}^{n} b_i(X)t^i$$

where $b_i(X) = \dim_{\mathbb{R}} H^i(X,\mathbb{R})$ is the i-th Betti number of X. This polynomial carries a lot of information about the topological and geometric invariants of X. It is well known that the cohomology ring $H^*(Gr(d,d+r),\mathbb{Z})$ has a polynomial description, that is,

$$H^*(Gr(d,d+r),\mathbb{Z}) \cong \mathbb{Z}[e'_1,\cdots,e'_d,e''_1,\ldots,e''_r]/\langle e_1,\cdots,e_{d+r}\rangle$$

where e'_i and e''_i are the i-th elementary symmetric functions in $x_1,\cdots,x_d$ and $x_{d+1},\cdots,x_{d+r}$ respectively and each x_i is the Chern class for the canonical bundle, so the Poncaré polynomial $P(Gr(d,d+r),t)$ is the following Gaussian polynomial

$$\frac{(1-t)(1-t^2)\cdots(1-t^{d+r})}{(1-t)\cdots(1-t^d)(1-t)\cdots(1-t^r)}$$

Theorem 5.1 *Let $P_w(z)$ be the weighted polynomial of the lattice points of the dilation $r\Delta_d$ of the standard d-simplex. Then the Poincaré polynomial $P(Gr(d,d+r),t)$ of the Grassmannian $Gr(d,d+r)$ coincides with the weighted polynomial $P_w(z)$.*

Example 5.2 The lattice points $(0,0,0),(1,0,0),(0,1,0),(2,0,0),(1,1,0),(0,0,1),(1,0,1)(0,1,1),$ $(0,0,2)(0,2,0),(3,0,0),(2,0,1),(1,2,0),(0,3,0),(0,2,1),(0,1,2),(0,0,3),(1,0,2),(2,0,1),(1,1,1)$ of $3\Delta_3$ graded by weights give the polynomial

$$1+t+3t^2+3t^3+3t^4+3t^5+3t^6+3t^7+t^8+t^9,$$

which is the Poincaré polynomial of the Grassmannian $Gr(3,6)$ so $3\Delta_3\cap\mathbb{Z}^3_{\ge 0}$ encodes the Young poset of $Gr(3,6)$ shown in Figure 5.

Corollary 5.3 *Let λ^* be the β-partition identified with the monomial $\mathbf{t}^{\mathbf{a}} \in W^r_d$ then the length $\ell(w(\lambda^*))$ of the Grassmannian permutation $w(\lambda^*)$ is the weight $w_{\mathbf{a}}$ of the exponent vector $\mathbf{a} \in r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$.*

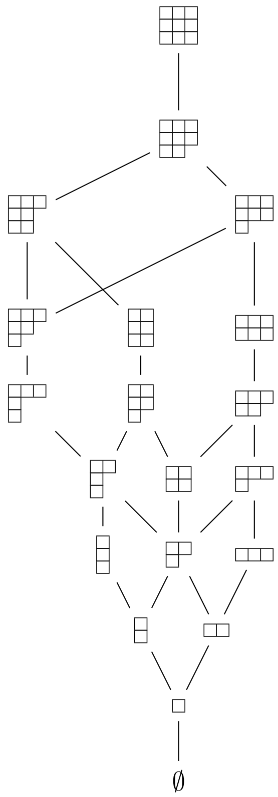

Figure 5

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