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Group Connectivity
of 1-Edge Deletable IM-Extendable Graphs

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Abstract: A graph G is called a k-edges deletable IM-extendable graph, if $G-F$ is IM-extendable for every $F \subseteq E(G)$ with $|F|=k$. Denoted by $\wedge_g(G)$ the group connectivity of a graph G. In this paper, $\wedge_g(G)=3$ is gotten if G is a 4-regular claw-free 1-edge deletable IM-extendable graph.

Key Words: Graph, multi-group connectivity, group connectivity, induced matching.

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§1. Introduction and Lemmas

In 1950s, Tutte introduced the theory of nowhere-zero flows as a tool to investigate the coloring problem of maps, together with his most fascinating conjectures on nowhere-zero flows. These have been extended by Jaeger, Linial, Payan and Tarsil in 1992 to group connectivity, the generalized form of nowhere-zero flows. Let G be an undirected graph and $\widetilde{A}=(\bigcup\limits_{i=1}^{m} A_i;\{+_i, 1 \leq i \leq m\})$ be an Abelian multi-group. Let $\widetilde{A}^*$ denote the set of non-zero elements of $\widetilde{A}$. A function $b: V(G) \rightarrow \widetilde{A}$ is called an *$\widetilde{A}$-valued zero-sum function* of G if $\sum_{v \in V(G)} b(v)=0_{+_i}$, $1 \leq i \leq m$ in G. The set of all $\widetilde{A}$-valued zero-sum function on G is denoted by $Z(G,\widetilde{A})$. We define: $F(G,\widetilde{A})=\{f: E(G) \rightarrow \widetilde{A}\}$ and $F^*(G,\widetilde{A})=\{f: E(G) \rightarrow \widetilde{A}^*\}$. Let G^1 be an orientation of a graph G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let tail$(e)=u$ and head$(e)=v$. For a vertex $v \in V(G)$, let $E^-(v)=\{e \in E(G^1): v=tail(e)\}$, and $E^+(v)=\{e \in E(G^1): v=head(e)\}$. Given a function $f \in F(G,\widetilde{A})$, define $\partial f: V(G) \rightarrow A$ by $\partial f(v)=\sum\limits_{e \in E^+(v)} f(e)-\sum\limits_{e \in E^-(v)} f(e)$. A graph G is *$\widetilde{A}$-connected* if G has an orientation G^1 such that for every function $b \in Z(G,\widetilde{A})$, there is a function $f \in F^*(G^1,\widetilde{A})$ such that $b=\partial f$. Let $\langle\widetilde{A}\rangle$ be the family of graphs that are $\widetilde{A}$-connected. The *multi-group connectivity* of G is defined as: $\wedge_g(G)$=min{k | if $\widetilde{A}$ is an Abelian group with $|\widetilde{A}| \geq k$, then $G \in \langle\widetilde{A}\rangle$}. Particularly,

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if $m = 1$, i.e., $\widetilde{A} = (A, +)$ an Abelian group, such connectivity is called *group connectivity*.

Let v be a vertex of G, denote $N(v)$ *by* : $N(v) = \{u \in V(G) - v : uv \in E(G)\}$. Let u be a vertex of G, denote $N^2(u) = N(N(u))\backslash(N(u) \cup \{u\})$. Graph G is called *claw-free*, if it doesn't contain $k_{1,3}$ as an induced subgraph. Let C_n^k denote the graph with $V(C_n^k) = V(C_n)$, $E(C_n^k) = \{uv : u, v \in V(C_n)$ *and* $d_{c_n}(u, v) \leq k\}$, where $d_{c_n}(u, v)$ is a distance between u and v in C_n. Let G be a graph. A *triangle-path* in G is a sequence of distinct triangles $T_1\ T_2 \cdots T_m$ in G such that for $1 \leq i \leq m-1$, the following formula (*) holds:

$$|E(T_i) \cap E(T_{i+1})| = 1 \quad \textit{and} \quad E(T_i) \cap E(T_j) = \Omega \ \ \textit{if } j > i+1. \qquad (*)$$

Furthermore, if $m \geq 3$ and (*) holds for all i, $1 \leq i \leq m$, with the additionally taken mod m, then the sequence is called a *triangle-cycle*. The number m is the length of the triangle-path(triangle-cycle). A connected graph G is *triangularly connected* if for any distinct $e, e_1 \in E(G)$, which are not parallel, there is a triangle-path $T_1T_2 \cdots T_m$ such that $e \in E(T_1)$ and $e_1 \in E(T_m)$.

Let G be a connected graph, $V(G)$ and $E(G)$ denote its sets of vertices and edges, respectively. For $S \subseteq V(G)$, let $E(S) = \{uv \in E(G), u, v \in S\}$. For $M \subseteq E(G)$, let $V(M) = \{u \in V(G)$: there is $v \in V(G)$ such that $uv \in M\}$. A set of edges $M \subseteq E(G)$ is called a *matching* of G if they are independent in G, and no two of them share a common end vertex. A matching is called *perfect* if it covers all vertices of G. A matching M is called induced matching if $E(V(M)) = M$. G is called *induced matching extendable* if every induced matching M of G is contained in a perfect matching of G. For simplicity, induced matching extendable will often be abbreviated as *IM-extendable*.

Notations undefined in this paper will follow [1]. In this paper, we give some properties of the 4-regular claw-free 1-edge deletable IM-extendable and prove that its group connectivity is 3.

Lemma 1.1.([1]) *A graph G has a perfect matching if and only if for every $S \subseteq V(G)$, $o(G-S) \leq |S|$, Where $o(H)$ is the number of odd components of H.*

For group connectivity, some conclusions have reached. For example, complete graphs, complete bipartite graphs and triangularly connected graphs etc in [2,3,4,5,6]. For more results about IM-extendable graphs, one can see references [7,8,9,10].

A *k-circuit* is a circuit of k vertices. A *wheel* W_k is the graph obtained from a k-circuit by adding a new vertex, called the center of the wheel, which is joined to every vertex of the k-circuit. W_k is an odd(even) wheel if k is odd(even). For a technical reason, a single edge is regarded as 1-circuit, and thus W_1 is a triangle, called the trivial wheel.

Lemma 1.2([6]) (1)$W_{2n} \in \langle Z_3 \rangle$.
(2) *Let $G \cong W_{2n+1}$, $b \in Z(G, Z_3)$. Then there exists a (Z_3, b)-NZF $f \in F^*(G, Z_3)$ if and only if $b \neq 0$.*

Lemma 1.3([4]) *Let G be a connected graph with n vertices and m edges. Then $\wedge_g(G)$=2 if and only if $n = 1$ (and so G has m loops).*

Lemma 1.4([3]) *Let $H \leq G$ be Z_k-connected. If G/H is Z_k-connected, then so is G.*

§2. Main Results

Lemma 2.1 *C_6^2 is 4-regular claw-free 1-edge deletable connected IM-extendable graph.*

Proof Obviously, C_6^2 is 4-regular and claw-free. The following we will prove C_6^2 is 1-edge deletable IM-extendable graph. Supposing the vertices of C_6^2 denoted by v_i, $1 \leq i \leq 6$, along a clockwise. Supposing M is an induced matching of an induced graph $G[N(u)]$. Since $G[N(u)]$ has four vertices, so $|M| \leq 2$. If $|M| = 2$, u is an isolated vertex of $G - V(M)$, this conflict with that G is 1-edge deletable IM-extendable graph, thus, $|M| \leq 1$. Since $E(C_6^2) = E_1 \cup E_2$ where $E_1 = E(C_6)$, $E_2 = E \setminus E_1$, the following discussions are divided into two cases.

Case 1 Deleting one edge in E_1. Without loss of generality, suppose deleting edge v_1v_2. From the structure of C_6^2, if M is an induced matching of C_6^2, then $|M| < 2$. If let $M = v_2v_3$ be an induced matching of $G - \{v_1v_2\}$, it extended to a perfect matching $\{v_2v_3, v_4v_5, v_6v_1\}$. Otherwise, let $M = \{v_2v_4\}$ be an induced matching of $G - \{v_1v_2\}$, it extended to a perfect matching $\{v_2v_4, v_3v_5, v_6v_1\}$.

Case2 Deleting one edge in E_2. Without loss of generality, let $M = \{v_2v_3\}$ be an induced matching of $G - \{v_1v_3\}$, it extended to a perfect matching $\{v_2v_3, v_4v_5, v_6v_1\}$. Otherwise, let $M = \{v_3v_5\}$ be an induced matching of $G - \{v_1v_3\}$, it extended to a perfect matching $\{v_3v_5, v_2v_4, v_6v_1\}$. So C_6^2 is a 4-regular claw-free 1-edge deletable connected IM-extendable graph. □

Lemma 2.2 *Let G be a 4-regular claw-free 1-edge deletable connected IM-extendable graph, then $G = C_6^2$.*

Proof For a given $u \in V(G)$, since G is 4-regular, let $N(u) = \{x_1, x_2, x_3, x_4\}$, $N^2(u) = N(N(u)) \setminus (N(u) \cup \{u\})$, $f(u) = |E(N(u))|$, $g(N(u)) = |N^2(u)|$.

Since G is 4-regular and claw-free, we have $2 \leq f(u) \leq 6$. If $f(u) = 6$, G is isomorphic to K_5, there is no perfect matching in K_5 which is a contradiction with 1-edge deletable IM-extendable graph. If $f(u) = 5$, $G[N(u)]$ is isomorphic to $K_4 - e$. Without loss of generality, let $e = x_2x_4$ and $y_1 \in N(x_2) \setminus (\{u\} \cup N(u))$. Since G is 4-regular, there exists $z \in N(y_1) \setminus N(u)$. Let $M = \{y_1z, ux_1\}$ be an induced matching of $G - \{x_2x_3\}$. It could not extend to a perfect matching of $G - \{x_2x_3\}$ which is a contradiction. Next we discuss: $2 \leq f(u) \leq 4$. There are three cases according to the value of $f(u)$.

Case 1 $f(u) = 2$.

Let x_1, x_2, x_3, x_4 be neighbor vertices of u, we have three different subcases: Subcase(a) $x_1x_2, x_2x_3 \in E(G)$; Subcase(b) $x_1x_2, x_3x_4 \in E(G)$; Subcase(c) $x_1x_2, x_1x_3 \in E(G)$.

For subcases (a), the induced subgraph of G by vertices u, x_1, x_3, x_4 consists of an induced subgraph $k_{1,3}$ which contradict with the assumption. For subcase (b), $M = \{x_1x_2, x_3x_4\}$ is an induced matching of $G - \{ux_2\}$, however, u could not included in vertex set of any perfect matching of $G - \{ux_2\}$ which contradict with the assumption. For subcase (c), the induced subgraph of G by vertices u, x_2, x_3, x_4 consists of $k_{1,3}$, which contradicts with the assumption. Therefore $f(u) \neq 2$.

Case 2 $f(u) = 3$.

$G[N(u)]$ is isomorphic to P_4 or $K_3 \cup K_1$ or $K_{1,3}$, where P_n is the path with n vertices.

If $G[N(u)]$ is isomorphic to P_4. Supposing $x_1x_2, x_2x_3, x_3x_4 \in E(G)$, then $M = \{x_1x_2, x_3x_4\}$ is an induced matching of $G - \{x_2x_3\}$, but it does not extended to a perfect matching of $G - \{x_2x_3\}$.

If $G[N(u)]$ is isomorphic to $k_{1,3}$, obviously G consists of $k_{1,3}$ as its induced subgraph, contradiction.

If $G[N(u)]$ is isomorphic to $K_3 \cup K_1$, supposing $x_1x_2, x_2x_3, x_1x_3 \in E(G)$, then $3 \leq g(N(u)) \leq 6$, there are four subcases according to the value of $g(N(u))$.

Subcase 1. If $g(N(u)) = 3$, let $N^2(u) = \{y_1, y_2, y_3\}$ then $y_ix_4 \in E(G)$ $(i = 1, 2, 3)$. Since G is claw-free and 4-regular, then $y_1y_2, y_1y_3, y_2y_3 \in E(G)$ and each vertex of x_i is adjacent to only one vertices of y_j (where $i, j \in \{1, 2, 3\}$ are distinct). Without loss of generality, let x_1, x_2, x_3 are adjacent to y_1, y_2, y_3 respectively. Let $M = \{x_1x_3, x_4y_2\}$ be an induced matching of $G - \{ux_2\}$. It does not extended to a perfect matching of $G - \{ux_2\}$ which is a contradiction.

Subcase 2. If $g(N(u)) = 4$, let $y_i \in N^2(u), i = 1, 2, 3, 4$. Since $d(x_4) = 4$, there is three of $y_i (1 \leq i \leq 4)$ adjacent to x_4, without loss of generality, supposing $y_ix_4 \in E(G)$ $(i = 2, 3, 4)$. Because G is claw-free, one have $y_2y_3, y_3y_4, y_2y_4 \in E(G)$. Obviously y_1 is adjacent to at least one vertex of x_i (i=1,2,3).

If y_1 is adjacent to each of x_i (i=1,2,3), that is $x_1y_1, x_2y_1, x_3y_1 \in E(G)$. If $y_1y_4 \notin E(G)$, let $M = \{x_1y_1, x_4y_4\}$ be an induced matching of $G - \{x_2x_3\}$ which could not extended to a perfect matching of $G - \{x_2x_3\}$. If $y_1y_4 \in E(G)$, let $M = \{x_1y_1, x_4y_3\}$ be an induced matching of $G - \{x_2x_3\}$ which could not extended to a perfect matching of $G - \{x_2x_3\}$.

If y_1 is adjacent to two vertices of x_i (i=1,2,3), without loss of generality, let $x_1y_1, x_2y_1 \in E(G)$ and $x_3y_1 \notin E(G)$, x_3 is adjacent to one of y_i (which $i \in \{2, 3, 4\}$). If $x_3y_3 \notin E(G)$, let $M = \{x_2x_3, x_4y_3\}$ be an induced matching of $G - \{y_2y_4\}$ which could not extended to a perfect matching of $G - \{y_2y_4\}$. If $x_3y_3 \in E(G)$, let $M = \{x_2x_3, x_4y_4\}$ be an induced matching of $G - \{ux_1\}$ which could not extended to a perfect matching of $G - \{ux_1\}$.

If y_1 is adjacent to only one of x_i (i=1,2,3), without loss of generality, let $x_1y_1 \in E(G)$. Supposing $x_2y_2, x_3y_3 \in E(G)$, let $M = \{x_1x_2, x_4y_3\}$ be an induced matching of $G - \{ux_3\}$ which could not extended to a perfect matching of $G - \{ux_3\}$.

Subcase 3. If $g(N(u)) = 5$, let $N^2(u) = \{y_1, y_2, y_3, y_4, y_5\}$ and supposing $y_3x_4, y_4x_4, y_5x_4 \in E(G)$. Because G is claw-free, $y_3y_4, y_4y_5, y_5y_3 \in E(G)$. Since no more than two vertices of x_i $(i = 1, 2, 3)$ is adjacent to y_j $(j = 1, 2)$, without loss of generality, let $x_1y_1, x_2y_2 \in E(G)$.

If x_3 is adjacent to y_1 or y_2, supposing $x_3y_1 \in E(G)$. Let $M = \{x_1x_3, x_4y_3\}$ be an induced matching of $G - \{ux_2\}$. It does not extended to a perfect matching of $G - \{ux_2\}$ which is a contradiction.

If $x_3y_i \notin E(G)$ (i=1,2). Without loss of generality, supposing $x_3y_3 \in E(G)$. Let $M = \{x_1x_3, x_4y_5\}$ be an induced matching of $G - \{ux_2\}$ which could not extended to a perfect matching of $G - \{ux_2\}$.

Subcase 4. If $g(N(u)) = 6$, without loss of generality, supposing $y_ix_4 \in E(G)$ $(i = 4, 5, 6)$.

Because each vertex of x_i (i=1,2,3) has degree 3 in $N(u)$, each of x_i $(i = 1, 2, 3)$ is adjacent to only one of y_j $(j = 1, 2, 3)$, without loss of generality, let $x_iy_i \in E(G)$ $(i = 1, 2, 3)$. Let $M = \{x_1x_3, x_4y_4\}$ be an induced matching of $G - \{ux_2\}$ which could not extended to a perfect matching of $G - \{ux_2\}$. So $f(u) \neq 3$.

Case 3 If $f(u) = 4$, $G[N(u)]$ is isomorphic to C_4 or $K_{1,3} + e$.

If $G[N(u)]$ is isomorphic to C_4. Since there are only four edges between $N(u)$ and $N^2(u)$, then $1 \leq g(N(u)) \leq 4$.

(3.1) If $g(N(u)) = 1$. Supposing $v \in N^2(u)$, there exists $x_iv \in E(G)$ $(i = 1, 2, 3, 4)$, it is isomorphic to C_6^2.

(3.2) If $g(N(u)) = 2$. Supposing $y_1, y_2 \in N^2(u)$. There are two subcases. Subcases 1, there are two vertices adjacent to y_1, two vertices adjacent to y_2. Subcases 2, there are three vertices adjacent to y_1, only one vertex adjacent to y_2.

Subcase 1. Supposing $x_1y_1, x_2y_1, x_3y_2, x_4y_2 \in E(G)$, there exists z, satisfying $y_1z \in E(G)$. Let $M = \{x_3x_4, y_1z\}$ be an induced matching of $G - \{x_1x_2\}$ which could not extended to a perfect matching of $G - \{x_1x_2\}$.

Subcase 2. Supposing $x_1y_1, x_2y_1, x_3y_1, x_4y_2 \in E(G)$. However, x_4, x_3, x_1, y_2 induced a $K_{1,3}$ which is a contradiction.

(3.3) If $g(N(u)) = 3$. Supposing $y_1, y_2, y_3 \in N^2(u)$. Obviously, only two vertices of x_i (i=1,2,3,4) are adjacent to one of y_i (i=1,2,3). Without loss of generality, let $x_iy_i, x_4y_3 \in E(G)$ $(i = 1, 2, 3)$. x_2, x_1, y_2, x_3 induced a $K_{1,3}$ which is a contradiction.

(3.4) If $g(N(u)) = 4$. Supposing $y_1, y_2, y_3, y_4 \in N^2(u)$. Without loss of generality, let $x_iy_i \in E(G)$ $(i = 1, 2, 3, 4)$. x_2, x_1, y_1, x_4 induced a $K_{1,3}$ subgraph. If $G[N(u)]$ is isomorphic to C_4, it does not extended to a perfect matching. If $G[N(u)]$ is isomorphic to $K_{1,3} + e$, supposing $x_1x_2, x_1x_3, x_1x_4, x_3x_4 \in E(G)$, since there are only 4 edges between $N(u)$ and $N^2(u)$, one have $2 \leq g(N(u)) \leq 4$. There are three subcases according to the value of $g(N(u))$.

Subcase 1. If $g(N(u)) = 2$. Supposing $N^2(u) = \{y_1, y_2\}$, then $x_2y_1, x_2y_2 \in E(G)$. Because G is claw-free, $y_1y_2 \in E(G)$. If both of x_3, x_4 are adjacent to y_1, let $M = \{ux_1, y_1y_2\}$ be an induced matching of $G - \{x_3x_4\}$ which could not extended to a perfect matching of $G - \{x_3x_4\}$. If $x_3y_1, x_4y_2 \in E(G)$, let $M = \{ux_1, y_1y_2\}$ be an induced matching of $G - \{x_3x_4\}$ which could not extended to a perfect matching of $G - \{x_3x_4\}$.

Subcase 2. If $g(N(u)) = 3$, supposing $N^2(u) = \{y_1, y_2, y_3\}$. Without loss of generality, let $x_2y_1, x_2y_2 \in E(G)$. Because G is claw-free, $y_1y_2 \in E(G)$. If both of x_3, x_4 are adjacent to y_3, let $M = \{y_1y_2, x_3x_4\}$ be an induced matching of $G - \{x_1x_3\}$. However, not all vertices of x_1, x_2, u could be included in a perfect matching vertices of $G - \{x_1x_3\}$ which is a contradiction. If $x_3y_3, x_4y_2 \in E(G)$, let $M = \{x_1x_3, y_1y_2\}$ be an induced matching of $G - \{x_1x_2\}$. However, not all vertices of x_2, x_4, u could be included in a vertex set of perfect matching of $G - \{x_1x_1\}$ which is a contradiction.

Subcase 3. If $g(N(u)) = 4$, Supposing $N^2(u) = \{y_1, y_2, y_3, y_4\}$. Without loss of generality, let $x_2y_1, x_2y_2, x_3y_3, x_4y_4 \in E(G)$. Because G is claw-free, $y_1y_2 \in E(G)$. Let $M = \{x_1x_4, y_1y_2 \in E(G)\}$ be an induced matching of $G - \{ux_2\}$ which could not extended to a perfect matching of $G - \{ux_2\}$. The lemma 2.2 is proved. □

Lemma 2.3 *If* $n \geq 5$, *then* C_n^2 *is* Z_3*-connected.*

Proof By the definition of C_n^k, for $n \geq 5$, there exists a subgraph of C_n^2 isomorphic to W_{2k}. By lemma 1.2, W_{2k} is Z_3-connected. Since C_n^2 is triangularly connected, by contracting W_{2k} in C_n^2 and Lemma 1.4, we have C_n^2 is Z_3-connected. □

Theorem 2.4 *The group connectivity of* 4*-regular claw-free* 1*-edge deletable* IM*-extendable graph is* 3.

Proof By applying lemma 2.1 and lemma2.2, 4-regular claw-free 1-edge deletable IM-extendable graph is C_6^2. From Lemma 2.3, the group connectivity of 4-regular claw-free 1-edge deletable IM-extendable graph is not more than 3. By lemma 1.3 we conclude the group connectivity of 4-regular claw-free 1-edge deletable IM-extendable graph is more than 2. Therefore, the group connectivity of 4-regular claw-free 1-edge deletable IM-extendable graph is 3. This theorem is proved. □

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier, New York, 1976.

[2] Jing-jing Chen, Elaine Eschen and Hong-Jian Lai, Group connectivity of certain graphs, *Ars Combinatoria*, 141-158(2008).

[3] M. DeVos, R. Xu and G. Yu, Nowhere-zero Z_3-flows through Z_3-connectivity, *Discrete Math.* 306(2006), 26-30.

[4] Genghua Fan, Hongjian Lai, Rui Xu, Cun-Quan Zhang, Chuixiang Zhou, Nowhere-zero 3-flows in triangularly connected graphs, *Journal of Combin. Theory*, Series B98(2008) 1325-1336.

[5] F. Jaeger, N. Linial, C. Payan and M. Tarsi, Group connectivity of graphs-a nonhomogeneous analogue of nowhere-zero flow properties, *J.Combin.Theory* Ser.B 56(1992) 165-182.

[6] H.-J. Lai, Group connectivity of 3-edge-connected chordal graphs, *Graphs and Combinatorics*, 16(2000), 165-176.

[7] Y. Liu, J.-J. Yuan and S.-Y. Wang, Degree conditions of IM-extendable graphs, *Appl. Math. JCU*, 15B 1 (2000) 1-6.

[8] Q. Wang and J.-J. Yuan, 4-regular claw-free IM-extendable graphs, *Discrete Math.*, 294 (2005), 303-309.

[9] Q. Wang and J.-J. Yuan, Degree sum conditions of induced matching extendable graphs, *Zhengzhou Univ.* 32 (1) (2000), 19-21.

[10] J.-J. Yuan, Induced matching extendable graph, *J. Graph Theory*, 28 (1998) 203-313.