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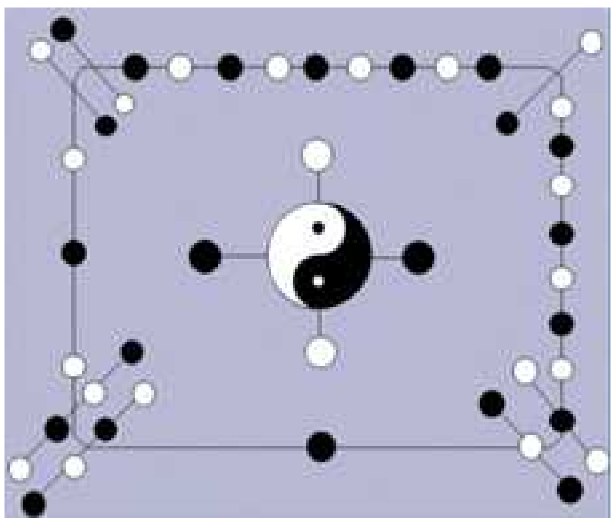

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All that we are is the result of what we have thought. The mind is everything. What we think, we become.

Buddha.

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Halfsubgroups

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Abstract: Let G be a group having a partially closed subset S such that S contains the identity element of G and each element in S has an inverse in S. Such subsets of G are called *halfsubgroups of* G. If a halfsubgroup S generates the group G, then S is called a *halfsubgroup generating the group* or *hsgg* in short. In this paper we prove some results on hsggs of a group. Order class of a group are special halfsubgroupoids. Elementary abelian groups are characterized as groups with maximum special halfsubgroupoids. Order class of a group with unity forms a typical halfsubgroup.

Key words: halfsubgroup, hsgg, order class of an element.

AMS(2000): 20Kxx, 20L05.

§1. Introduction

According to R.H.Bruck [2] a halfgroupoid is a partially closed set w.r.t. certain operation.

Definition 1.1 *Let $(G,*)$ be a group and S be a subset of G. Let $(S,*)$ be a halfgroupoid (partially closed subset) of G such that*

(i) $e \in S$, e is the identity element of G.

(ii) $a^{-1} \in S$, $\forall\, a \in S$.

Then $(S,*)$ is called a half subgroup of the group G.

Illustration 1.1 Every subgroup of a group G is also halfsubgroup of G but not vise-versa. For example, consider the multiplicative group $G = \{1, -1, i, -i\}$. Then $S = \{1, i, -i\}$ is a halfsubgroup of G which is not a subgroup.

Definition 1.2 *If for a group G there exists a halfsubgroup H without identity such that for all $x, y \in H$, $xy \in H$ whenever $y \neq x^{-1}$ then H is called a special halfsubgroup of G.*

Definition 1.3 *A halfsubgroup $(S,*)$ of a group $(G,*)$ is called a halfsubgroup generating the group (or hsgg in short) if it generates G.*

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[2]Supported by the University Grants Commission of India Under the Project No. 23-245/06.

It is easy to verify that the union of two hsgg of a group G is again a hsgg of G. In fact, union of any number of hsgg of G is also hsgg of G.

However, the intersection of two hsgg of a group G may not be a hsgg of G.

Theorem 1.1 *If S is a proper hsgg of a group G then $O(S) \geq 3$.*

Proof Let S be a proper hsgg of a group G. Then $S \neq \{e\}$. Let $a \in S, a \neq e$ then $S \neq \{e,a\}$, because if $S = \{e,a\}$ then $a = a^{-1}$ and S can not generate whole G, so S can not be a proper hsgg of G. Thus if $O(G) \leq 3$ then G can not have a proper hsgg. Now if $O(G) \geq 4$ then we can have a proper hsgg $S = \{e,a,b\}$ of G such that $a = a^{-1}$ and $b = b^{-1}$ or $a^{-1} = b$.

As a result there exists an hsgg S such that $O(S) = 3$. Hence we get the result. □

Remark If G is any cyclic group such that $G = (a)$, then $S = \{e, a, a^{-1}\}$ is a minimal hsgg of G.

Definition 1.4 *Let G be a group and S be an hsgg of G. The element $x(\neq e) \in S$ is called a redundant element of S if $S \setminus \{x\}$ is also an hsgg of G.*

An element of S which is not redundant is called an irredundant element.

Definition 1.5 *Let G be a group and S be a hsgg of G such that $a^2 \neq e, \forall a \in S$ and S has no redundant element. Then S is called pure hsgg of G.*

The following results follow trivially.

(1) *Every cyclic group of order ≥ 3 has at least one pure hsgg.*

(2) *A cyclic group of prime order p has $\frac{p-1}{2}$ number of distinct pure hsgg.*

We discuss some Abelian groups in terms of their pure hsgg.

Theorem 1.2 *Every group of prime order can be expressed as the union of its distinct pure hsgg. However, the converse is not true.*

Proof Every group of prime order p is cyclic. Hence the group has $\frac{p-1}{2}$ number of distinct pure hsgg. Each hsgg has two non-identity elements together with an identity element e common in all. Thus G has $2 \cdot \frac{p-1}{2} + 1 = p$ elements. Hence G is the union of all these distinct pure hsggs. □

Theorem 1.3 *If a group G can be written as the union of its distinct pure hsgg then G is a group of odd order.*

Proof It is easy to verify. □

Theorem 1.4 *An elementary Abelian $p-$group, $p > 3$ is a direct product of n cyclic groups each of which is a cyclic $p-$group which is the union of distinct pure hsggs.*

Proof By the definition of elementary Abelian $p-$group,

$$G = C_1 \times C_2 \times \cdots \times C_n$$

where $C_i's$ are cyclic $p-$groups of order p. Now each $C_i = \cup_{r=1}^{(p-1)/2} S_{i_r}$ where $1 \leq i \leq n$ and S_{i_r} are distinct pure hsgg representing each cyclic group C_i of order p given in the above decomposition. Thus G is n times the direct product of union of distinct pure hsgg. □

Theorem 1.5 *Let G be a finite Abelian group of order n. Let $G = C_1 \times C_2 \times \cdots \times C_k$ where each C_i is a cyclic group of order $p_i^{\alpha_i}$. That is $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where p_i are distinct primes and each $\alpha_i > 0$. Then*

$G = \prod_i \{\alpha_i \prod_1^{\alpha_i} \cup_{r=1}^{(p_i-1)/2} S_{i_r})\}$

where $i = 1, \cdots, k$.

Proof The proof follows Theorem 1.4. □

§2. Order Class

Definition 2.1 *Let G be a group. A subset O_α of G defined by*

$$O_a = \{b \in G : o(b) = o(a)\}$$

is called an order class of of a.

Definition 2.2 *Let G be a group. Let O_a be an order class of $a \in G$. Then the set of all xy such that $x, y \in O_a$ is called the closure of O_a.It is denoted by $\bar{O}_a$.*

Lemma 2.1 *If G is a finite group and $a \in G$, $a \neq e$ then*

(i) O_a is a halfgroupoid;

(ii) $a^{-1} \in O_a, \forall a \in O_a$.

Proof The proof follows by these definitions of halfgroupoid and order class. □

Notation: We use the notation Θ_a to denote order class of a with unity.

Definition 2.3 *If H is a halfsubgroup of a group G then $H \setminus \{e\}$ is called the halfsubgroupoid of G.*

Every group G has a unique maximum halfsubgroupoid $G \setminus \{e\}$ associated with it.

Definition 2.4 *Let G be a group. Then O_a is a halfsubgroupoid of G. It is called a special halfsubgroupoid of G.*

2.1 Groups with maximum special halfsubgroupoids

There exist groups which have only one order class other than $\{e\}$. For such groups closure of the order class of $a(\neq e)$ where $a \in G$, we give below a series of examples of such groups.

Example 2.1 Cyclic groups of prime order without unity such as $Z_5 \setminus \{e\}, Z_7 \setminus \{e\}, \ldots$ are the maximum special halfsubgroupoids.

Example 2.2 All groups with exponent p a prime are such groups.

Example 2.3 All elementary Abelian groups are such groups.

Example 2.4 Extra special groups of order 27 generated by three elements and of order 81 generated by 2 elements are such groups. This has been verified by using GAP ref[3]. The GAP Small Groups Library no. of these groups are [27,3] and [81,12]. These are polycyclic groups of order 27 and order 81 respectively.These are the only groups from the groups of order 100 which which have a single order class other than order class of $\{e\}$.

Example 2.5 The group $GL_3(F,p)$ for odd prime p is such a group.

Example 2.6 George Havas has constructed a biggest 5-group generated by 2 elements. It is of order 5^{34} with exponent 5.

Example 2.7 Dihedral groups of order D_{2p} are such groups.

2.2 Results

Theorem 2.1 *If G has a maximum special halfsubgroupoid then G is a p-group.*

Proof Let a group G has a maximum special halfsubgroupoid. Then every non-identity element of G has same order. If p divides order of G then there exists an element of order p in G. As a result all non-trivial elements of G are of order p. Thus, G is a p-group. □

Now we prove a theorem which gives the characterization of an elementary Abelian groups.

Theorem 2.2 *A group G is elementary Abelian if and only if G has a maximum special halfsubgroupoid.*

Proof Assume G is elementary Abelian, then every element of G is of same order p where p is a prime. Thus the collection of non-identity elements form an order class which is a maximum special halfsubgroupoid. Conversely, If G has a maximum special halfsubgroupoid then by Theorem 2.1 G is a p group and G has a maximum special halfsubgroupoid. Whence G is elementary Abelian. □

Theorem 2.3 *If G be a finite group, $a \in G$ then Θ_a is a halfsubgroup of G.*

Proof The Proof follows Lemma 2.1 and the definition of halfsubgroup of G. □

Definition 2.5 *A halfsubgroup S of a group G is normal in G if and only if $xSx^{-1} \in S, \forall x \in G$.*

Theorem 2.4 *If G is a finite group then Θ_a is a normal halfsubgroup of G.*

Proof If G is abelian then obviously Θ_a is normal in G. If G is non-abelian, then $o(a) = o(xax^{-1}), \forall a \in \Theta_a$. Therefore $xax^{-1} \in \Theta_a$. Hence Θ_a is normal in G. □

Theorem 2.5 *If G is a finite abelian group such that $O(G) = p_1 \cdot p_2 \cdots p_r$ for the primes $p_1, \cdots, p_r$ then G is the direct product of order classes with unity (i.e. halfsubgroup).*

Proof In the decomposition of G every Sylow p_i subgroup is an order class with unity which is also a halfsubgroup. Hence we get the result. □

Corollary 2.1 *Any finite abelian group is a direct product of some order classes with unity (halfsubgroup).*

Theorem 2.6 *Every finite group G is the union of halfsubgroups (namely order class with unity) Θ_a, $a \in G$ and $a \neq e$ which are normal in G such that $\cap_{a\in G}\Theta_a = \{e\}$.*

Proof The proof follows from Lemma 2.1 and Theorem 2.4. □

References

[1] Artin Micheal, *Algebra*, Prentice Hall of India Pvt. Ltd. M-9, 1994.

[2] Bruck R. H., *Survey of Binary Systems*, Springer Verlag, Berlin.

[3] *GAP,Computational Algebra System*, http://www.gap-system.org.

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Flexibility of Embeddings of a Halin Graph on the Projective Plane

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Abstract: A basic problem in graph embedding theory is to determine distinct embeddings of planar graphs on higher surfaces. Tutte's work on graph connectivity shows that wheels or wheel-like configurations plays a key role in 3-connected graphs. In this paper we investigate the flexibility of a Halin grap on N_1, the projective plane, and show that embeddings of a Halin graph on N_1 is determined by making either a twist or a 3-patchment of a vertex in a wheel. Further more, as applications, we give a correspondence between a Halin graph and its embeddings on the projective plane. Based on this, the numbers of some types of such embeddings are determined.

Key Words: Halin graph, embedding, face-width.

AMS(2000): 05C30, 05C45.

§1. Introduction

Throughout this paper we consider simple connected labeled graphs and their embeddings on surfaces. Terms and notations not defined may be found in [1,3] and [11].

A surface is a compact closed 2-manifold. An(A) orientable (non-orientable) surface of genus g is the sphere with g *handles* (or *crosscaps*) which is denoted by S_g (or N_g). A *map* M or *embedding* on $S_g (or N_g)$ is a graph drawn on the surface so that each vertex is a point on the surface, each edge {x,y}, $x \neq y$, is a simple open curve whose endpoints are x and y, each loop incident to a vertex x is a simple closed curve containing x, no edge contains a vertex to which it is not incident, and each connected region of the complement of the graph in the surface is homeomorphic to a disc and is called a *face*. It is clear that maps(or embeddings) here are combinatorial. A map or An embedding is called *strong* if the boundaries of all the facial walks are simply cycles. A curve (or circuit) C on a surface Σ is called *non-contractible* (or *essential*) if none of the regions of $\Sigma - C$ is homeomorphic to an open disc; otherwise it is called *contractible* (or *trivial*). Let T be a tree without subdivisions of edges and embedded in the plane with its one-valent vertices being $v_1, v_2, ..., v_m$ under the rotation of T. A *leaf* is an edge incident to a vertex of valence 1. If new edges $(v_i, v_{i+1}) (i = 1, 2, ..., m)$ are added to $E(T)$, the edge-set of T, then T together with the cycle $(v_1, v_2, ..., v_m)$ forms a planar map

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called *Halin graph*. This cycle is defined as *leaf-cycle* (i.e., the boundary of the outer face) and is denoted as ∂f_r. In convenience, we always let T to denote the tree which orients a Halin graph. It is clear that Halin graphs are generalized wheels on the plane. Tutte showed in his book[11] that a 3-connected graph are obtained from the wheels by a series of *edge* or *vertex splitting* operations. Further, Vitray[12] pointed out that wheels play a key role in embeddings of a 3-connected graph since in that case the *local structures* (i.e., neighbour of a vertex) may be viewed as a wheel.

A major subject about planar graphs is to determine all of their distinct embeddings on a non-planar surface. This theory has been developed and deepened by people such as R.Vitray[12], N.Robertson and R.Vitray[7], B.Mohar and N.Robertson[4,6], and C.Thomassen[8] etc. Recently, Mohar et al[5] showed the existence of upper bounds for the distinct embeddings of a 3-connected graph in general orientable surfaces. In this paper we investigate the embeddings of a Halin map on N_1 and show that strong embeddings of a Halin graph on N_1 is determined by making 3-patchments on inner vertices of a wheel and present a correspondence between a Halin graph and its (strong) embeddings in the projective plane. Based on this, the number of such embeddings is determined.

Let $\mathcal{H}, \mathcal{H}_p$ be the set of all the Halin graphs and their embeddings on the projective plane, respectively. Then we have the following result:

Theorem A. *For a map $M \in \mathcal{H}$ with $s(M)$ edges, there are*

$$\sum_{v \in V(M)} \binom{d(v)}{2} - s(M)$$

maps in $\mathcal{H}_p$ corresponding to M whose face-width are all 1.

Here, the concept of *face-width* of an embedding is defined in the next section. The readers may also see[12] for a reference. Based on Theorem 1 one may calculate the number of such embeddings on N_1. For instance, there are exactly 6 such embeddings of W_4 in N_1 as depicted in Fig.1.

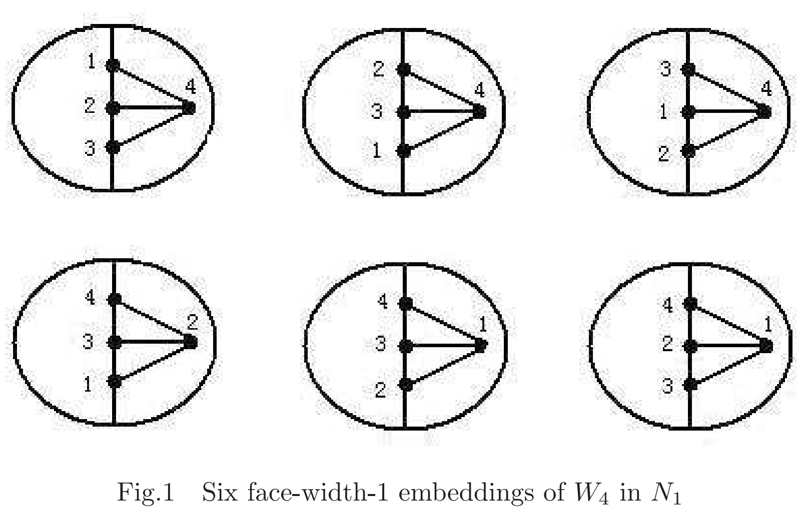

Fig.1 Six face-width-1 embeddings of W_4 in N_1

In the case of strong embeddings or maps (i.e., the boundary of each facial walk is a cycle), the following result shows that any strong embedding of a Halin graph in N_1 is in fact determined by a corresponding strong embedding of a wheel.

Theorem B. Let M be a Halin graph. Then its strong embeddings are determined by strong embeddings of wheels.

As applications of Theorem B, we have

Theorem C. *For a Halin graph G, there are*

$$\sum_{v\in V-\partial f_r}\left(2^{d(v)-1}-d(v)\right).$$

elements in $\mathcal{H}_p$ corresponding to G whose face-width are all 2.

Based on the formula presented in Theorem C, one may calculate the number of strong embeddings of a Halin graph on N_1. The following Fig.2 shows a Halin graph and its strong embeddings in N_1.

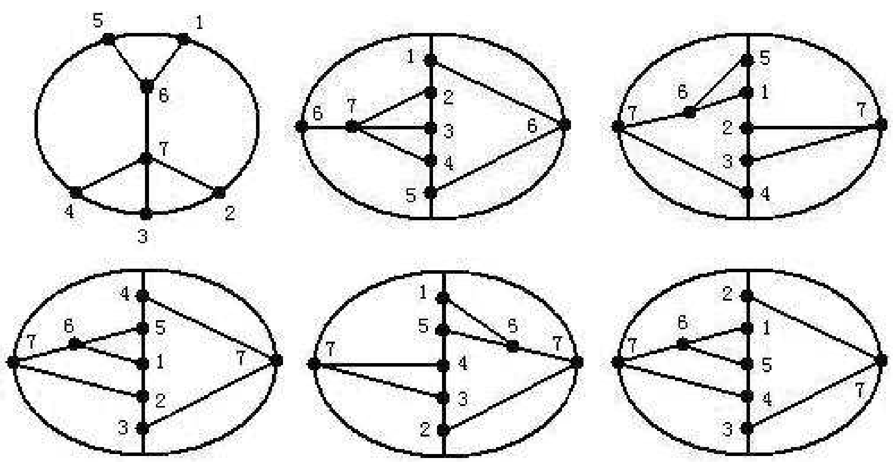

Fig.2 A Halin graph with five distinct strong embeddings in N_1

§2. Some Preliminary Works

In this section we shall give some lemmas on graph embeddings before proving of our main results.

Lemma 1 *A planar Halin graph is* 3*-connected and has at least two facial walks which are* 3*-gons.*

Proof. Let G be a planar Halin graph oriented by a tree, say $T(G)$. One may easily see that G is 3-connected. In fact, for any two distinct vertices x and y not on the leaf-cycle and with their valencies not less than 3, there are two paths in $T(G)$ connecting two leaves for each of them. Those four paths are pairwise inner disjoint. It is easy for one to see that those paths together with a pair of segments (which are determined by the four leaves) of the leaf-cycle form a pair of inner disjoint paths connecting x and y. If we consider the unique path from x to y in $T(G)$, then there are three inner disjoint paths joining x and y in G. Since the same property holds for other locations of x and y, G is 3-connected by *Menger's Theorem*. As for the existence of 3-gons, one may find at least two such triangles along the longest paths in $T(G)$. □

A fundamental result on topological graph theory by H.Whitney[13] states that any 3-connected graph has at most one planar embedding, i.e.,

Lemma 2 *There is only one way to embed a* 3*-connected planar graph in the plane.*

W.Tutte[10] obtained Whitney's uniqueness result from a combinatorial view of facial walks-*induced non-separating cycle* (for a reference, one may see[8]), i.e.,

Lemma 3 *Every facial walk of a* 3*-connected planar graph is an induced non-separating cycle.*

Later, C.Thomassen[9] generalized the above two results to *LEW-embeddings* (a concept by J.Hutchison[2]) in general surfaces and found that such embeddings share many properties with planar graphs.

Based on Lemmas 1, 2, and 3, we have the following

Lemma 4 *If a Halin graph is embedded in a non-planar surface* Σ*, then every facial walk of it (viewed as a planar map) is either a contractible cycle (hence also a facial walk) or a non-contractible cycle (or essential as some people called it) of* Σ*.*

When a planar graph G is embedded in a non-planar surface Σ, then some very important properties will appear. For instance, R.Vitray[12] found (late proved by N.Robertson et al[8] and C.Thomassen[9] independently) that the *face-width* $\rho_\Sigma(G)$ of G on Σ is at most 2, where $\rho_\Sigma(G)$ is defined as

$$\rho_\Sigma(G) = \min\{| C \cap V(G) | \, | C \text{ is an noncontractible curve of } \Sigma\}$$

In view of intuition, face-width is a measure of how densely a graph is embedded in a given surface. The above property says that every embedding, if possible, of a planar graph on non-planar surfaces is relatively sparse. A basic problem is how to determine the face-width of an embedding or to find that what operations performed on the graphs may not change its representativity. It is easy for one to check that the following result (which is depicted in Fig.3) presents such operations.

Lemma 5 *Let* G *be a graph embedded in a surface. Then the* $\triangle - Y$ *and* $\triangle - I$ *operations defined below do not change the face-width.*

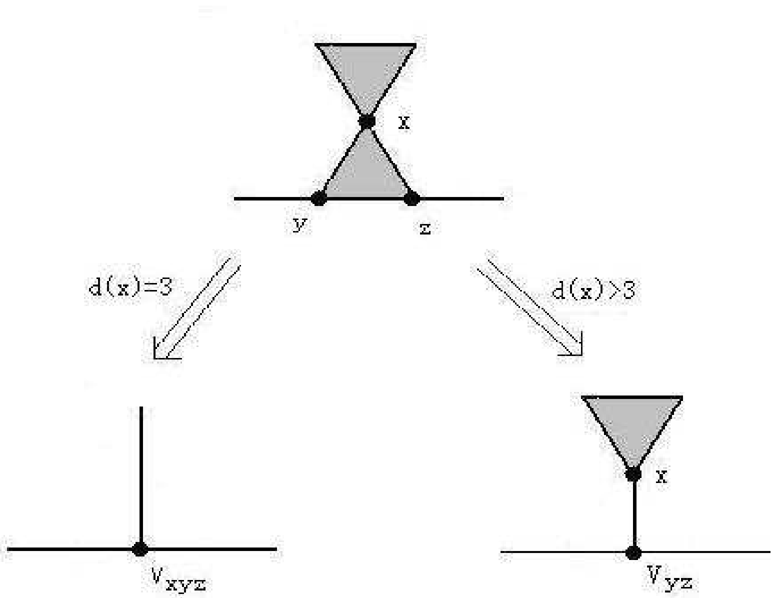

Fig.3 Popping a planar triangle into a vertex or an edge, $d(y) = d(z) = 3$

§3. Projective Planar Maps

In this section we shall prove Theorems A, B and C.

According to Lemma 3, the leaf-cycles of those in $\mathcal{H}_p$ are either facial walks or non-contractible cycles on N_1. Thus, $\mathcal{H}_p$ may be partitioned into two parts as

$$\mathcal{H}_p = \mathcal{H}_p(1) + \mathcal{H}_p(2),$$

where

$$\mathcal{H}_p(1) = \{M \mid \rho_{\mathrm{N}_1}(\mathrm{G(M)}) = 1\}; \quad (1)$$

$$\mathcal{H}_p(2) = \{M \mid \rho_{\mathrm{N}_1}(\mathrm{G(M)}) = 2\} \quad (2)$$

and $G(M)$ is the *underline graph* of $M(M)$.

Proof of Theorem A Let M be a map in $\mathcal{H}_p(1)$. Then it is determined by making a twist at a vertex of a Halin graph. On the other hand, by making a twist at each pair of corners around every vertex of a Halin graph G will induce an embedding of G on N_1. One may see that for each element in $\mathcal{H}_p(1)$ no more than one such twists are permitted since otherwise by reversing the specific twists (which will change a facial walk into two whose boundaries are simple cycles) we may see that a 3-connected planar Halin graph will have at least two distinct embeddings in the plane and hence contradicts Lemma 2 or 3. This completes the proof. □

We now concentrate on the structures of the maps in $\mathcal{H}_p(2)$.

Lemma 6 *Let M be a map in $\mathcal{H}_p(2)$. Then the leaf-cycle of M is non-contractible.*

Proof: It is easy to see its validity for smaller maps. Suppose it holds for those having fewer than n edges. Let $M \in \mathcal{H}_p(2)$ be a counter example with n edges. Then its leaf-cycle is contractible. Under this case we will show that its face-width is 1. By the definition of M, there exists a Halin graph G such that M is an embedding of it in N_1. Notice that both of the them share the same leaf-cycle (and consequently the same outer facial walk). By Lemmas 1 and 3, G has a 3-cycle, say (x, y, z), which is either a 3-gon or non-contractible in M. If (x, y, z) is non-contractible in M and the leaf-cycle is on the only one side of it, then we have $\rho_{N_1}(G(M)) = 1$ since at least two vertices of $\{x, y, z\}$ are on the leaf-cycle and trivalent and the two edges not on it are incident to them are on the same side of the 3-cycle. If (x, y, z) is non-contractible with edges of the leaf-cycle lying on the both sides of it, the leaf-cycle of M is not a simple cycle (i.e., containing vertices repeated more than twice), a contradiction as required. Next, we consider the case that the 3-cycle (x, y, z) is a 3-gon of M. In this case the face-width is 2 by performing operations in Lemma 5 and the Induction hypothesis says that the leaf-cycle is non-contractible. This contradiction completes the proof. □

Let M be a map in $\mathcal{H}_p(2)$. Then by Lemma 6 its leaf-cycle is non-contractible and all the *leaves* are distributed alternatively on the both sides of the leaf-cycle since otherwise we will have its face-width 1. Thus, leaves together with their 1-valent vertices are classified into two groups lying on the "both sides" of the leaf-cycles. One may see that this is not accurate since on N_1 every non-contractible cycle has only one side. But this description will not ruin our proofs. By a *foot* we mean a maximal group of leaves together with the 1-valent vertices which appear to the same side of the leaf-cycle consecutively. Further, we have

Lemma 7 *The feet on ∂f_r (the boundary of the leaf-cycle) must appear alternatively (i.e., there exists three feet B_1, B_2 and B_3 such that their appearing order is B_1, B_2, B_3, where B_1 and B_3 are on the same side of ∂f_r and B_2 on the other side (the right hand side of Fig.4 presents a case of this structure).*

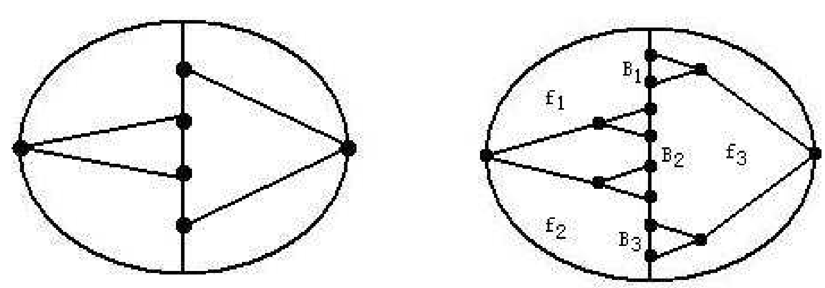

Fig.4 An embedding of W_5 which will induce
a strong embedding of a Halin map

Proof This follows from the fact that maps in $\mathcal{H}_p(2)$ have face-width 2. □

We say that a group of leaves will *induce a subtree* of T if there is a vertex v in T such that those leaves consist of all the leaves of a subtree of T rooted at v. By the definition of Halin graphs one may see the following

Lemma 8 *Let M be a map in $\mathcal{H}_p(2)$ which has a structure in Lemma 7. Then the foot B_2 will induce a subtree of T*

Proof Let the two ends of B_i be x_i and $y_i(1 \leq i \leq 3)$ and f be the face on the opposite of side of B_2. Let f_1 and f_2 be, respectively, the faces on the other side of the edges (y_1, x_2) and (y_2, x_3). Then one may see the following fact from $\rho_{N_1}(G(M)) = 2$ and the definition of Halin graphs.

Fact 1 $f_1 \neq f_2$.

Our next proofs are divided into two cases.

Case 1 $\partial f_1 \bigcap \partial f_2 \neq \emptyset$.

One may choose a vertex u on the common boundary of f_1 and f_2 such that the path from y_2 to u in T is shortest. Then by the 3-connectness of $G(M)$ and the definition of a Halin graph there is an unique path, say P, connecting u and a vertex v on ∂f. Choose v such that P is as short as possible. Then we have

Fact 2 $| V(P) | \leq 2$.

Since otherwise there will exist an internal vertex w on P. By the definition of a Halin map there is a path Q (in T) connecting w and a vertex w_1 on ∂f_r. It is clear that $w_1 \notin V(B_i)$. If we view B_i as a vertex v_{B_i} for $1 \leq i \leq 3$, then the set $\{v, w, w_1, v_{B_1}, v_{B_2}, v_{B_3}\}$ will guarantee the existence of a subgraph of $G(M)$ which is a subdivision of $K_{3,3}$, a Kuratowski graph.

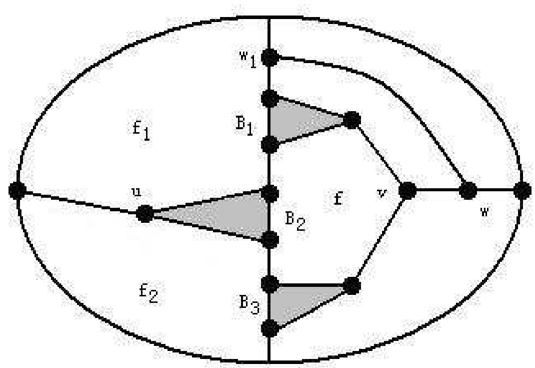

Fig.5

This is shown in Fig.5, contradicts to that Lemma 8.

Case 2 $\partial f_1 \bigcap \partial f_2 = \emptyset$.

Then there are two paths, say P and Q, from ∂f_1 and ∂f_2 to ∂f, respectively. We may choose P and Q such that they are from x_2 and y_2 to ∂f respectively and $V(P)\bigcap\partial f_{=}\{v\}$, $V(Q)\bigcap\partial f=\{u\}$. If $u\neq v$, then there would be a subgraph which

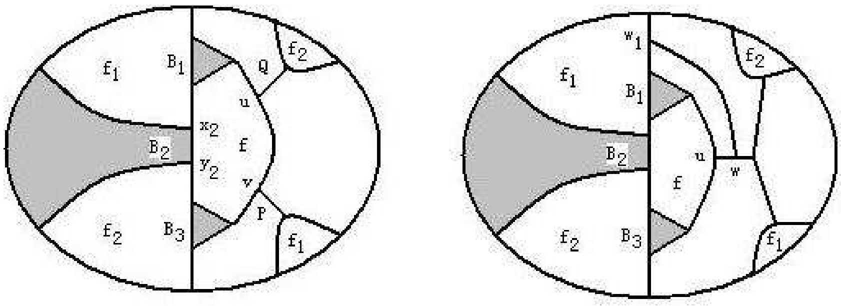

Fig.6

is a subdivision of $K_{3,3}$ (in fact it is induced by the set $\{u,v,x_2,y_2,v_{B_1},v_{B_3}\}$ as depicted in the left side of Fig.6). So, $u=v$. If $P\bigcap Q$ is a path with length≥ 1, then the length of $P\bigcap Q$ is 1. Since otherwise we may choose an internal vertex w (as we did previously) which may lead to a path from w to ∂f and hence will imply the existence of a non-planar subgraph of $G(M)$ (which is determined by the vertex-set $\{u,w,w_1,v_{B_1},v_{B_2},v_{B_3}\}$ as shown in the right side of Fig.6). This contradiction shows that $P\bigcap Q$ is a path with no more than two vertices. Combining all the possible situations in the two cases completes the proof. □.

Lemma 9 *Let M be an embedding of a Halin map with representativity 2. Let $B_i(1\leq i\leq 5)$ be five feet appearing alternatively on the two sides of ∂f_r. Then the two trees induced by B_2 and B_4 are rooted at the same vertex of T.*

Proof. Let f_i be the face on the opposite side of B_i and T_i be the subtree induced by B_i for $1\leq i\leq 5$. Then by Lemma 8 the tree T_4 (which corresponds to B_4) is rooted at some vertex u in ∂f_4. Let P be a path from u to B_5 along ∂f_4 and Q be a path from B_4 to u. Then the cycle $C=QuPx_5y_5$ is non-contractible. Similarly, choose Q_1 be a path from B_2 to a vertex v on ∂f_2 such that T_2 is rooted at v. Let P_1 be path from v to x_3 on ∂f_2. The cycle $C'=Q_1vP_1x_3y_2$ is also non-contractible. Notice that any pair of non-contractible cycles (curves) on N_1 will intersect at a vertex, we conclude that C and C' will intersect at a vertex w on the path P. If $u\neq v$, then as we have discussed before, there is a non-planar subgraph of $G(M)$. This contradiction shows that $u=v$. It follows from Lemma 8 that the vertex w is also on the boundary of f_2. This ends the proof. □

Proof of Theorem B Let M be a Halin Map and M' an embedding of it on N_1 with $\rho_{N_1}(G(M))=2$. Let B_i be the feet of M' and induces a subtree T_i for $1\leq i\leq s$. Then by Lemma 6 the leaf-cycle ∂f_r is non-contractible and all the feet are lying on the two sides of ∂f_r alternatively by Lemma 8. Lemmas 6-9 show that all the subtrees T_i are rooted at some vertex v of T. Let $d(v)=m$. For each $T_i(1\leq i\leq s)$, its root-vertex is v and edges incident to v is $e_{l_1},e_{l_2},...,e_{l_i}$. One may view T_i together B_i as a claw K_{1,l_i} whose edges are correspondent to $e_{l_1}, e_{l_2}, ..., e_{l_i}$. Then one may get a bigger claw $K_{1,m}$ which together with ∂f_r forms a wheel W_{m+1} whose underlying graph is planar, where $m = \sum_1^s l_i$. This procedure is shown in Fig.4 where the case of $m = 4$ is shown. It is clear that W_{m+1} is strongly embedded in N_1. Since this procedure is reversible, the theorem follows. □

Proof of Theorem C Let M be a Halin graph with T and ∂f as its orienting tree and leaf cycle. Then by Theorem B its strong embeddings are completely determined by performing 3-patchments on its inner vertices (i.e., those not on ∂f). So, taking an inner vertex, say $v \in V - \partial f$, and considering the number of ways of performing 3-patchments at v. Let $d(v) = m$. Then the corresponding 3-patchments is induced by those of W_{m+1}, the wheel with m spokes. So, we only need to restrain our procedure on the strong embeddings of W_{m+1} on N_1 . Notice that in the case of $m \geq 4$, there is only one leaf-cycle for W_{m+1} which is non-contractible. We can determine all the possible strong embeddings of W_{m+1} this way: We first draw the leaf-cycle $(1, 2, ..., m)$ into N_1 such that the leaf-cycle is non-contractible and then consider the ways of choosing alternating feet on $(1, 2, ..., m)$ as described in Lemma 7. It is clear that the number of alternating feet must be an odd number. Let the number of leaves in feet B_i is x_i. Then the total number of ways of choosing alternating feet is equal to the number of ways of assigning $2k + 1$ groups of consecutive vertices on a m-vertex-cycle. This correspondence is shown in Fig.7, where a 3-patchment on the center of a wheel will produce 7 alternating feet on the leaf-cycle.

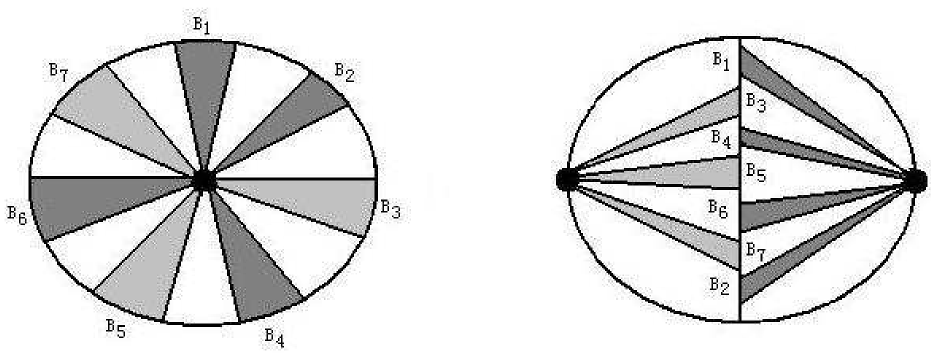

Fig.7 Generating a strong embedding in N_1 by performing a 3-patchment at the center of W_{m+1}

Let $f(m, k)$ be the number of ways of grouping k sets of consecutive vertices. Then it is clear that $f(m, k)$ satisfies the following recursive relation:

$$\begin{cases} f(m,k) = f(m-1,k-1) + f(m-1,k), & m \geq k \geq 2; \\ f(m,m) = f(0,0) = 1. \end{cases}$$

Since the combinatorial number $\binom{m}{k}$ also satisfies the above relation, we have that $f(m, k) = \binom{m}{k}$. Hence, Theorem C follows from the case of $f(m, 2k + 1)$. □

Final Remark By using the same procedure used in our proof of Theorem B, one may find that a Halin graph has no strong embeddings in orientable surface other than the sphere. This seems resulted from the fact that the face-distance (i.e., the shortest length of face-chain connecting two faces) is not greater than 2. With the increase of genera, the possibility of strong embedding is decrease. Hence, we think that the sphere and the projective plane are the only two possible surfaces on which a Halin graph may be strongly embedded.

References

[1] J.A.Bondy and U.S.R.Murty, *Graph Theory with Applications*, The Macmilan Press Ltd, 1976.

[2] J.P.Hutchinson, Automorphism Properties of Embedded Graphs, *J. Graph Theory*, 8(1984), 35-49.

[3] Yanpei Liu, *Embeddibility in Graphs*, Kluwer Academic Publishers, 1995.

[4] B.Mohar and N.Robertson, Planar Graphs on Non-Planar Surfaces, I, *J. Combin. Theory*, Ser.B 68 (1996), 87-111.

[5] B.Mohar and N.Robertson, Flexibility of polyhedral embeddings of a graphs in surfaces, *J. Combin. Theory*, Ser (B), 84, (2001), pp

[6] B.Mohar, N.Robertson, and R.P.Vitray, Planar Graphs on the Projective Plane, *Discrete Math.,* 149 (1996), 141-157.

[7] N.Robertson and R.Vitray, Representativity of Surface Embeddings, *Paths, Flows, and VLSI Layout,* B.Kort, L.Lovàsz, H.J.Prömel, and A.Schrijver. Eds., Springer- Verlag, Berlin, Heidelberg, (1999), 239–328.

[8] C.Thomassen and B.Toft, Non-separating Induced Cycles in Graphs, *J. Combin. Theory,* Ser. B 31 (1981), 199-224.

[9] C.Thomassen, Embeddings of Graphs with No Short Non-Contractible Cycles, *J. Combin. Theory,* Ser. B 48 (1990), 155-177.

[10] W.T.Tutte, How to Draw a Graph, *Proc. London Math. Soc.,* 13 (1963), 743–768.

[11] W.T.Tutte, *Graph Theory,* Addison-Wesley, Reading, MA.,1984.

[12] R.Vitray, Representativity and Flexibility of Drawings of Graphs on the Projective Plane, *Contemporary Math.*, 147 (1993), 341-347.

[13] H.Whitney, 2-Isomorphic Graphs, *Amer. J. Math.*, 55 (1933), 245-254.

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Curvature Equations on Combinatorial Manifolds with Applications to Theoretical Physics

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Abstract Curvature equations are very important in theoretical physics for describing various classical fields, particularly for gravitational field by Einstein. For applying Smarandache multi-spaces to *parallel universes*, the conception of combinatorial manifolds was introduced under a combinatorial speculation for mathematical sciences in [9], which are similar to manifolds in the local but different in the global. Similarly, we introduce curvatures on combinatorial manifolds and find their structural equations in this paper. These Einstein's equations for a gravitational field are established again by the choice of a combinatorial Riemannian manifold as its spacetime and some multi-space solutions for these new equations are also gotten by applying the *projective principle* on multi-spaces in this paper.

Key Words: curvature, combinatorial manifold, combinatorially Euclidean space, equations of gravitational field, multi-space solution.

AMS(2000): 51M15, 53B15, 53B40, 57N16, 83C05, 83F05.

§1. Introduction

As an efficiently mathematical tool used by Einstein in his general relativity, tensor analysis mainly dealt with transformations on manifolds had gotten considerable developments by both mathematicians and physicists in last century. Among all of these, much concerns were concentrated on an important tensor called curvature tensor for understanding the behavior of curved spaces. For example, the famous Einstein's gravitational field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}$$

are consisted of curvature tensors and energy-momentum tensors of the curved space.

Notice that all curved spaces considered in classical fields are homogenous. Achievements of physics had shown that the multiple behavior of the cosmos in last century, enables the model of parallel universe for the cosmos born([14]). Then *can we construct a new mathematical theory, or generalized manifolds usable for this multiple, non-homogenous physics appeared in 21st century*? The answer is YES in logic at least. That is the *Smarandache multi-space theory*, see [7] for details.

For applying Smarandache multi-spaces to *parallel universes*, combinatorial manifolds were

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introduced endowed with a topological or differential structure under a combinatorial speculation for mathematical sciences in [9], i.e. *mathematics can be reconstructed from or turned into combinatorization*([8]), which are similar to manifolds in the local but different in the global. Whence, geometries on combinatorial manifolds are nothing but these *Smarandache* geometries([12]-[13]).

Now we introduce the conception of combinatorial manifolds in the following. For an integer $s \geq 1$, let $n_1, n_2, \cdots, n_s$ be an integer sequence with $0 < n_1 < n_2 < \cdots < n_s$. Choose s open unit balls $B_1^{n_1}, B_2^{n_2}, \cdots, B_s^{n_s}$ with $\bigcap\limits_{i=1}^{s} B_i^{n_i} \neq \emptyset$ in $\mathbf{R}^n$, where $n = n_1 + n_2 + \cdots n_s$. A *unit open combinatorial ball of degree* s is a union

$$\widetilde{B}(n_1, n_2, \cdots, n_s) = \bigcup_{i=1}^{s} B_i^{n_i}.$$

A combinatorial manifold $\widetilde{M}$ is defined in the next.

Definition 1.1 *For a given integer sequence $n_1, n_2, \cdots, n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_m$, a combinatorial manifold $\widetilde{M}$ is a Hausdorff space such that for any point $p \in \widetilde{M}$, there is a local chart (U_p, φ_p) of p, i.e., an open neighborhood U_p of p in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \rightarrow \widetilde{B}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$ with $\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\}$ and $\bigcup\limits_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\}$, denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$ or $\widetilde{M}$ on the context and*

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$$

an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$. The maximum value of $s(p)$ and the dimension $\widehat{s}(p)$ of $\bigcap\limits_{i=1}^{s(p)} B_i^{n_i}$ are called the dimension and the intersectional dimensional of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ at the point p, denoted by $d(p)$ and $\widehat{d}(p)$, respectively.

A combinatorial manifold $\widetilde{M}$ is called *finite* if it is just combined by finite manifolds without one manifold contained in the union of others, is called *smooth* if it is finite endowed with a C^∞ differential structure. For a smoothly combinatorial manifold $\widetilde{M}$ and a point $p \in \widetilde{M}$, it has been shown in [9] that $\dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ and $\dim T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ with a basis

$$\{\frac{\partial}{\partial x^{hj}}|_p | 1 \leq j \leq \widehat{s}(p)\} \bigcup (\bigcup_{i=1}^{s(p)} \bigcup_{j=\widehat{s}(p)+1}^{n_i} \{\frac{\partial}{\partial x^{ij}}|_p \mid 1 \leq j \leq s\})$$

or

$$\{dx^{hj}|_p|\} 1 \leq j \leq \widehat{s}(p)\} \bigcup (\bigcup_{i=1}^{s(p)} \bigcup_{j=\widehat{s}(p)+1}^{n_i} \{dx^{ij}|_p \mid 1 \leq j \leq s\}$$

for a given integer $h, 1 \leq h \leq s(p)$.

Definition 1.2 *A connection $\widetilde{D}$ on a smoothly combinatorial manifold $\widetilde{M}$ is a mapping $\widetilde{D}$: $\mathscr{X}(\widetilde{M}) \times T^r_s\widetilde{M} \to T^r_s\widetilde{M}$ on tensors of $\widetilde{M}$ with $\widetilde{D}_X\tau = \widetilde{D}(X,\tau)$ such that for $\forall X, Y \in \mathscr{X}\widetilde{M}$, $\tau, \pi \in T^r_s(\widetilde{M})$,$\lambda \in \mathbf{R}$ and $f \in C^\infty(\widetilde{M})$,*

(1) $\widetilde{D}_{X+fY}\tau = \widetilde{D}_X\tau + f\widetilde{D}_Y\tau$; and $\widetilde{D}_X(\tau + \lambda\pi) = \widetilde{D}_X\tau + \lambda\widetilde{D}_X\pi$;

(2) $\widetilde{D}_X(\tau \otimes \pi) = \widetilde{D}_X\tau \otimes \pi + \sigma \otimes \widetilde{D}_X\pi$;

(3) *for any contraction C on $T^r_s(\widetilde{M})$, $\widetilde{D}_X(C(\tau)) = C(\widetilde{D}_X\tau)$.*

A combinatorially connection space is a 2-tuple $(\widetilde{M}, \widetilde{D})$ consisting of a smoothly combinatorial manifold $\widetilde{M}$ with a connection $\widetilde{D}$ and a torsion tensor $\widetilde{T} : \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$ on $(\widetilde{M}, \widetilde{D})$ is defined by $\widetilde{T}(X,Y) = \widetilde{D}_XY - \widetilde{D}_YX - [X,Y]$ for $\forall X, Y \in \mathscr{X}(\widetilde{M})$. If $\widetilde{T}|_U(X,Y) \equiv 0$ in a local chart $(U, [\varphi])$, then $\widetilde{D}$ is called torsion-free on $(U, [\varphi])$.

Similar to that of Riemannian geometry, metrics on a smoothly combinatorial manifold and the combinatorially Riemannian geometry are defined in next definition.

Definition 1.3 *Let $\widetilde{M}$ be a smoothly combinatorial manifold and $g \in A^2(\widetilde{M}) = \bigcup\limits_{p \in \widetilde{M}} T^0_2(p, \widetilde{M})$. If g is symmetrical and positive, then $\widetilde{M}$ is called a combinatorially Riemannian manifold, denoted by $(\widetilde{M}, g)$. In this case, if there is a connection $\widetilde{D}$ on $(\widetilde{M}, g)$ with equality following hold*

$$Z(g(X,Y)) = g(\widetilde{D}_Z, Y) + g(X, \widetilde{D}_ZY)$$

then $\widetilde{M}$ is called a combinatorially Riemannian geometry, denoted by $(\widetilde{M}, g, \widetilde{D})$.

It has been showed that there exists a unique connection $\widetilde{D}$ on $(\widetilde{M}, g)$ such that $(\widetilde{M}, g, \widetilde{D})$ is a combinatorially Riemannian geometry in [9].

We all known that curvature equations are very important in theoretical physics for describing various classical fields, particularly for gravitational field by Einstein. The main purpose of this paper is to establish curvature tensors with equations on combinatorial manifolds and apply them to describe the gravitational field. For this objective, we introduce the conception of curvatures on combinatorial manifolds and establish symmetrical relations for curvature tensors, particularly for combinatorially Riemannian manifolds in the next two sections. Structural equations of curvature tensors on combinatorial manifolds are also established. These generalized Einstein's equations of gravitational field on combinatorially Riemannian manifolds are constructed in Section 4. By applying the *projective principle* on multi-spaces, multi-space solutions for these new equations are gotten in Section 5.

Terminologies and notations used in this paper are standard and can be found in [1], [4] for those of manifolds [9] − [11] for combinatorial manifolds and [6] − [7] for graphs, respectively.

§2. Curvatures on Combinatorially Connection Spaces

As a first step for introducing curvatures on combinatorial manifolds, we define combinatorially curvature operators on smoothly combinatorial manifolds in the next.

Definition 2.1 *Let $(\widetilde{M},\widetilde{D})$ be a combinatorially connection space. For $\forall X,Y\in\mathscr{X}(\widetilde{M})$, a combinatorially curvature operator $\widetilde{\mathcal{R}}(X,Y):\mathscr{X}(\widetilde{M})\to\mathscr{X}(\widetilde{M})$ is defined by*

$$\widetilde{\mathcal{R}}(X,Y)Z=\widetilde{D}_X\widetilde{D}_YZ-\widetilde{D}_Y\widetilde{D}_XZ-\widetilde{D}_{[X,Y]}Z$$

for $\forall Z\in\mathscr{X}(\widetilde{M})$.

For a given combinatorially connection space $(\widetilde{M},\widetilde{D})$, we know properties following on combinatorially curvature operators similar to those of the Riemannian geometry.

Theorem 2.1 *Let $(\widetilde{M},\widetilde{D})$ be a combinatorially connection space. Then for $\forall X,Y,Z\in\mathscr{X}(\widetilde{M})$, $\forall f\in C^\infty(\widetilde{M})$,*

(1) $\widetilde{\mathcal{R}}(X,Y)=-\widetilde{\mathcal{R}}(Y,X)$;

(2) $\widetilde{\mathcal{R}}(fX,Y)=\widetilde{\mathcal{R}}(X,fY)=f\widetilde{\mathcal{R}}(X,Y)$;

(3) $\widetilde{\mathcal{R}}(X,Y)(fZ)=f\widetilde{\mathcal{R}}(X,Y)Z$.

Proof For $\forall X,Y,Z\in\mathscr{X}(\widetilde{M})$, we know that $\widetilde{\mathcal{R}}(X,Y)Z=-\widetilde{\mathcal{R}}(Y,X)Z$ by definition. Whence, $\widetilde{\mathcal{R}}(X,Y)=-\widetilde{\mathcal{R}}(Y,X)$.

Now since

$$\begin{aligned}\widetilde{\mathcal{R}}(fX,Y)Z &= \widetilde{D}_{fX}\widetilde{D}_YZ-\widetilde{D}_Y\widetilde{D}_{fX}Z-\widetilde{D}_{[fX,Y]}Z\\ &= f\widetilde{D}_X\widetilde{D}_YZ-\widetilde{D}_Y(f\widetilde{D}_XZ)-\widetilde{D}_{f[X,Y]-Y(f)X}Z\\ &= f\widetilde{D}_X\widetilde{D}_YZ-Y(f)\widetilde{D}_XZ-f\widetilde{D}_Y\widetilde{D}_XZ\\ &- f\widetilde{D}_{[X,Y]}Z+Y(f)\widetilde{D}_XZ\\ &= f\widetilde{\mathcal{R}}(X,Y)Z,\end{aligned}$$

we get that $\widetilde{\mathcal{R}}(fX,Y)=f\widetilde{\mathcal{R}}(X,Y)$. Applying the quality (1), we find that

$$\widetilde{\mathcal{R}}(X,fY)=-\widetilde{\mathcal{R}}(fY,X)=-f\widetilde{\mathcal{R}}(Y,X)=f\widetilde{\mathcal{R}}(X,Y).$$

This establishes (2). Now calculation shows that

$$\begin{aligned}\widetilde{\mathcal{R}}(X,Y)(fZ) &= \widetilde{D}_X\widetilde{D}_Y(fZ)-\widetilde{D}_Y\widetilde{D}_X(fZ)-\widetilde{D}_{[X,Y]}(fZ)\\ &= \widetilde{D}_X(Y(f)Z+f\widetilde{D}_YZ)-\widetilde{D}_Y(X(f)Z+f\widetilde{D}_XZ)\\ &- ([X,Y](f))Z-f\widetilde{D}_{[X,Y]}Z\\ &= X(Y(f))Z+Y(f)\widetilde{D}_XZ+X(f)\widetilde{D}_YZ\\ &+ f\widetilde{D}_X\widetilde{D}_YZ-Y(X(f))Z-X(f)\widetilde{D}_YZ-Y(f)\widetilde{D}_XZ\\ &- f\widetilde{D}_Y\widetilde{D}_XZ-([X,Y](f))Z-f\widetilde{D}_{[X,Y]}Z\\ &= f\widetilde{\mathcal{R}}(X,Y)Z.\end{aligned}$$

Whence, we know that

$$\widetilde{\mathcal{R}}(X,Y)(fZ) = f\widetilde{\mathcal{R}}(X,Y)Z.$$

□

Theorem 2.2 *Let $(\widetilde{M}, \widetilde{D})$ be a combinatorially connection space. If the torsion tensor $\widetilde{T} \equiv 0$ on $\widetilde{D}$, then the first and second Bianchi equalities following hold.*

$$\widetilde{\mathcal{R}}(X,Y)Z + \widetilde{\mathcal{R}}(Y,Z)X + \widetilde{\mathcal{R}}(Z,X)Y = 0$$

and

$$(\widetilde{D}_X\widetilde{R})(Y,Z)W + (\widetilde{D}_Y\widetilde{R})(Z,X)W + (\widetilde{D}_Z\widetilde{R})(X,Y)W = 0.$$

Proof Notice that $\widetilde{T} \equiv 0$ is equal to $\widetilde{D}_X Y - \widetilde{D}_Y X = [X,Y]$ for $\forall X, Y \in \mathscr{X}(\widetilde{M})$. Thereafter, we know that

$$\begin{aligned}
& \widetilde{\mathcal{R}}(X,Y)Z + \widetilde{\mathcal{R}}(Y,Z)X + \widetilde{\mathcal{R}}(Z,X)Y \\
= \ & \widetilde{D}_X\widetilde{D}_Y Z - \widetilde{D}_Y\widetilde{D}_X Z - \widetilde{D}_{[X,Y]}Z + \widetilde{D}_Y\widetilde{D}_Z X - \widetilde{D}_Z\widetilde{D}_Y X \\
- \ & \widetilde{D}_{[Y,Z]}X + \widetilde{D}_Z\widetilde{D}_X Y - \widetilde{D}_X\widetilde{D}_Z Y - \widetilde{D}_{[Z,X]}Y \\
= \ & \widetilde{D}_X(\widetilde{D}_Y Z - \widetilde{D}_Z Y) - \widetilde{D}_{[Y,Z]}X + \widetilde{D}_Y(\widetilde{D}_Z X - \widetilde{D}_X Z) \\
- \ & \widetilde{D}_{[Z,X]}Y + \widetilde{D}_Z(\widetilde{D}_X Y - \widetilde{D}_Y X) - \widetilde{D}_{[X,Y]}Z \\
= \ & \widetilde{D}_X[Y,Z] - \widetilde{D}_{[Y,Z]}X + \widetilde{D}_Y[Z,X] - \widetilde{D}_{[Z,X]}Y \\
+ \ & \widetilde{D}_Z[X,Y] - \widetilde{D}_{[X,Y]}Z \\
= \ & [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]].
\end{aligned}$$

By the Jacobi equality $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$, we get that

$$\widetilde{\mathcal{R}}(X,Y)Z + \widetilde{\mathcal{R}}(Y,Z)X + \widetilde{\mathcal{R}}(Z,X)Y = 0.$$

By definition, we know that

$$\begin{aligned}
& (\widetilde{D}_X\widetilde{R})(Y,Z)W = \\
& \widetilde{D}_X\widetilde{R}(Y,Z)W - \widetilde{R}(\widetilde{D}_X Y, Z)W - \widetilde{R}(Y, \widetilde{D}_X Z)W - \widetilde{R}(Y,Z)\widetilde{D}_X W \\
& = \widetilde{D}_X\widetilde{D}_Y\widetilde{D}_Z W - \widetilde{D}_X\widetilde{D}_Z\widetilde{D}_Y W - \widetilde{D}_X\widetilde{D}_{[Y,Z]}W - \widetilde{D}_{\widetilde{D}_X Y}\widetilde{D}_Z W \\
& + \widetilde{D}_Z\widetilde{D}_{\widetilde{D}_X Y}W + \widetilde{D}_{[\widetilde{D}_X Y, Z]}W - \widetilde{D}_Y\widetilde{D}_{\widetilde{D}_X Z}W + \widetilde{D}_{\widetilde{D}_X Z}\widetilde{D}_Y W \\
& + \widetilde{D}_{[Y,\widetilde{D}_X Z]}W - \widetilde{D}_Y\widetilde{D}_Z\widetilde{D}_X W + \widetilde{D}_Z\widetilde{D}_Y\widetilde{D}_X W + \widetilde{D}_{[Y,Z]}\widetilde{D}_X W.
\end{aligned}$$

Let

$$A^W(X,Y,Z) = \widetilde{D}_X\widetilde{D}_Y\widetilde{D}_Z W - \widetilde{D}_X\widetilde{D}_Z\widetilde{D}_Y W - \widetilde{D}_Y\widetilde{D}_Z\widetilde{D}_X W + \widetilde{D}_Z\widetilde{D}_Y\widetilde{D}_X W,$$

$$B^W(X,Y,Z) = -\widetilde{D}_X\widetilde{D}_{\widetilde{D}_Y Z}W + \widetilde{D}_X\widetilde{D}_{\widetilde{D}_Z Y}W + \widetilde{D}_Z\widetilde{D}_{\widetilde{D}_X Y}W - \widetilde{D}_Y\widetilde{D}_{\widetilde{D}_X Z}W,$$

$$C^W(X,Y,Z) = -\widetilde{D}_{\widetilde{D}_X Y}\widetilde{D}_Z W + \widetilde{D}_{\widetilde{D}_X Z}\widetilde{D}_Y W + \widetilde{D}_{\widetilde{D}_Y Z}\widetilde{D}_X W - \widetilde{D}_{\widetilde{D}_Z Y}\widetilde{D}_X W$$

and

$$D^W(X,Y,Z) = \widetilde{D}_{[\widetilde{D}_X Y,Z]}W - \widetilde{D}_{[\widetilde{D}_X Z,Y]}W.$$

Applying the equality $\widetilde{D}_X Y - \widetilde{D}_Y X = [X,Y]$, we find that

$$(\widetilde{D}_X \widetilde{R})(Y,Z)W = A^W(X,Y,Z) + B^W(X,Y,Z) + C^W(X,Y,Z) + D^W(X,Y,Z).$$

We can check immediately that

$$A^W(X,Y,Z) + A^W(Y,Z,X) + A^W(Z,X,Y) = 0,$$
$$B^W(X,Y,Z) + B^W(Y,Z,X) + B^W(Z,X,Y) = 0,$$
$$C^W(X,Y,Z) + C^W(Y,Z,X) + C^W(Z,X,Y) = 0$$

and

$$\begin{aligned}&D^W(X,Y,Z) + D^W(Y,Z,X) + D^W(Z,X,Y)\\ &= \widetilde{D}_{[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]}W = \widetilde{D}_0 W = 0\end{aligned}$$

by the Jacobi equality $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$. Therefore, we get finally that

$$\begin{aligned}&(\widetilde{D}_X \widetilde{R})(Y,Z)W + (\widetilde{D}_Y \widetilde{R})(Z,X)W + (\widetilde{D}_Z \widetilde{R})(X,Y)W\\ &= A^W(X,Y,Z) + B^W(X,Y,Z) + C^W(X,Y,Z) + D^W(X,Y,Z)\\ &+ A^W(Y,Z,X) + B^W(Y,Z,X) + C^W(Y,Z,X) + D^W(Y,Z,X)\\ &+ A^W(Z,X,Y) + B^W(Z,X,Y) + C^W(Z,X,Y) + D^W(Z,X,Y) = 0.\end{aligned}$$

This completes the proof. □

According to Theorem 2.1, the curvature operator $\widetilde{\mathcal{R}}(X,Y) : \mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$ is a tensor of type $(1,1)$. By applying this operator, we can define a curvature tensor in the next definition.

Definition 2.2 *Let $(\widetilde{M}, \widetilde{D})$ be a combinatorially connection space. For $\forall X,Y,Z \in \mathscr{X}(\widetilde{M})$, a linear multi-mapping $\widetilde{\mathcal{R}} : \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$ determined by*

$$\widetilde{\mathcal{R}}(Z,X,Y) = \widetilde{\mathcal{R}}(X,Y)Z$$

is said a curvature tensor of type $(1,3)$ on $(\widetilde{M}, \widetilde{D})$.

Let $(\widetilde{M}, \widetilde{D})$ be a combinatorially connection space and

$$\{e_{ij} | 1 \leq i \leq s(p), 1 \leq j \leq n_i \text{ and } e_{i_1 j} = e_{i_2 j} \text{ for } 1 \leq i_1, i_2 \leq s(p) \text{ if } 1 \leq j \leq \widehat{s}(p)\}$$

a local frame with a dual

$$\{\omega^{ij}|1 \le i \le s(p), 1 \le j \le n_i \text{ and } \omega^{i_1 j} = \omega^{i_2 j} \text{ for } 1 \le i_1, i_2 \le s(p) \text{ if } 1 \le j \le \widehat{s}(p)\},$$

abbreviated to $\{e_{ij}\}$ and $\{\omega^{ij}\}$ at a point $p \in \widetilde{M}$, where $\widetilde{M} = \widetilde{M}(n_1, n_2, \cdots, n_m)$. Then there exist smooth functions $\Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)} \in C^\infty(\widetilde{M})$ such that

$$\widetilde{D}_{e_{\kappa\lambda}} e_{\mu\nu} = \Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)} e_{\sigma\varsigma}$$

called connection coefficients in the local frame $\{e_{ij}\}$. Define

$$\omega^{\sigma\varsigma}_{\mu\nu} = \Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)} \omega^{\kappa\lambda}.$$

We get that

$$\widetilde{D} e_{\kappa\lambda} = \omega^{\sigma\varsigma}_{\mu\nu} e_{\sigma\varsigma}.$$

Theorem 2.3 *Let $(\widetilde{M}, \widetilde{D})$ be a combinatorially connection space and $\{e_{ij}\}$ a local frame with a dual $\{\omega^{ij}\}$ at a point $p \in \widetilde{M}$. Then*

$$\widetilde{d}\omega^{\mu\nu} - \omega^{\kappa\lambda} \wedge \omega^{\mu\nu}_{\kappa\lambda} = \frac{1}{2} \widetilde{T}^{\mu\nu}_{(\kappa\lambda)(\sigma\varsigma)} \omega^{\kappa\lambda} \wedge \omega^{\sigma\varsigma},$$

where $\widetilde{T}^{\mu\nu}_{(\kappa\lambda)(\sigma\varsigma)}$ is a component of the torsion tensor $\widetilde{T}$ in the frame $\{e_{ij}\}$, i.e., $\widetilde{T}^{\mu\nu}_{(\kappa\lambda)(\sigma\varsigma)} = \omega^{\mu\nu}(\widetilde{T}(e_{\kappa\lambda}, e_{\sigma\varsigma}))$ and

$$\widetilde{d}\omega^{\kappa\lambda}_{\mu\nu} - \omega^{\sigma\varsigma}_{\mu\nu} \wedge \omega^{\kappa\lambda}_{\sigma\varsigma} = \frac{1}{2} \widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)} \omega^{\sigma\varsigma} \wedge \omega^{\eta\theta}$$

with $\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)(\sigma\varsigma)} e_{\kappa\lambda} = \widetilde{R}(e_{\sigma\varsigma}, e_{\eta\theta}) e_{\mu\nu}$.

Proof By definition, for any given $e_{\sigma\varsigma}, e_{\eta\theta}$ we know that (see Theorem 3.6 in [9])

$$\begin{aligned}
(\widetilde{d}\omega^{\mu\nu} - \omega^{\kappa\lambda} \wedge \omega^{\mu\nu}_{\kappa\lambda})(e_{\sigma\varsigma}, e_{\eta\theta}) &= e_{\sigma\varsigma}(\omega^{\mu\nu}(e_{\eta\theta})) - e_{\eta\theta}(\omega^{\mu\nu}(e_{\sigma\varsigma})) - \omega^{\mu\nu}([e_{\sigma\varsigma}, e_{\eta\theta}]) \\
&- \omega^{\kappa\lambda}(e_{\sigma\varsigma})\omega^{\mu\nu}_{\kappa\lambda}(e_{\eta\theta}) + \omega^{\kappa\lambda}(e_{\eta\theta})\omega^{\mu\nu}_{\kappa\lambda}(e_{\sigma\varsigma}) \\
&= -\omega^{\mu\nu}_{\sigma\varsigma}(e_{\eta\theta}) + \omega^{\mu\nu}_{\eta\theta}(e_{\sigma\varsigma}) - \omega^{\mu\nu}([e_{\sigma\varsigma}, e_{\eta\theta}]) \\
&= -\Gamma^{\mu\nu}_{(\sigma\varsigma)(\eta\theta)} + \Gamma^{\mu\nu}_{(\eta\theta)(\sigma\varsigma)} - \omega^{\mu\nu}([e_{\sigma\varsigma}, e_{\eta\theta}]) \\
&= \omega^{\mu\nu}(\widetilde{D}_{e_{\sigma\varsigma}} e_{\eta\theta} - \widetilde{D}_{e_{\eta\theta}} e_{\sigma\varsigma} - [e_{\sigma\varsigma}, e_{\eta\theta}]) \\
&= \omega^{\mu\nu}(\widetilde{T}(e_{\sigma\varsigma}, e_{\eta\theta})) = \widetilde{T}^{\mu\nu}_{(\sigma\varsigma)(\eta\theta)}.
\end{aligned}$$

Whence,

$$\widetilde{d}\omega^{\mu\nu} - \omega^{\kappa\lambda} \wedge \omega^{\mu\nu}_{\kappa\lambda} = \frac{1}{2} \widetilde{T}^{\mu\nu}_{(\kappa\lambda)(\sigma\varsigma)} \omega^{\kappa\lambda} \wedge \omega^{\sigma\varsigma}.$$

Now since

$$
\begin{aligned}
&(\widetilde{d}\omega_{\mu\nu}^{\kappa\lambda} - \omega_{\mu\nu}^{\vartheta\iota} \wedge \omega_{\vartheta\iota}^{\kappa\lambda})(e_{\sigma\varsigma}, e_{\eta\theta}) \\
&= e_{\sigma\varsigma}(\omega_{\mu\nu}^{\kappa\lambda}(e_{\eta\theta})) - e_{\eta\theta}(\omega_{\mu\nu}^{\kappa\lambda}(e_{\sigma\varsigma})) - \omega_{\mu\nu}^{\kappa\lambda}([e_{\sigma\varsigma}, e_{\eta\theta}]) \\
&-\omega_{\mu\nu}^{\vartheta\iota}(e_{\sigma\varsigma})\omega_{\vartheta\iota}^{\kappa\lambda}(e_{\eta\theta}) + \omega_{\mu\nu}^{\vartheta\iota}(e_{\eta\theta})\omega_{\vartheta\iota}^{\kappa\lambda}(e_{\sigma\varsigma}) \\
&= e_{\sigma\varsigma}(\Gamma_{(\mu\nu)(\eta\theta)}^{\kappa\lambda}) - e_{\eta\theta}(\Gamma_{(\mu\nu)(\sigma\varsigma)}^{\kappa\lambda}) - \omega^{\vartheta\iota}([e_{\sigma\varsigma}, e_{\eta\theta}])\Gamma_{(\mu\nu)(\vartheta\iota)}^{\kappa\lambda} \\
&-\Gamma_{(\mu\nu)(\sigma\varsigma)}^{\vartheta\iota}\Gamma_{(\vartheta\iota)(\eta\theta)}^{\kappa\lambda} + \Gamma_{(\mu\nu)(\eta\theta)}^{\vartheta\iota}\Gamma_{(\vartheta\iota)(\sigma\varsigma)}^{\kappa\lambda}
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{R}(e_{\sigma\varsigma}, e_{\eta\theta})e_{\mu\nu} &= \widetilde{D}_{e_{\sigma\varsigma}}\widetilde{D}_{e_{\eta\theta}}e_{\mu\nu} - \widetilde{D}_{e_{\eta\theta}}\widetilde{D}_{e_{\sigma\varsigma}}e_{\mu\nu} - \widetilde{D}_{[e_{\sigma\varsigma}, e_{\eta\theta}]}e_{\mu\nu} \\
&= \widetilde{D}_{e_{\sigma\varsigma}}(\Gamma_{(\mu\nu)(\eta\theta)}^{\kappa\lambda}e_{\kappa\lambda}) - \widetilde{D}_{e_{\eta\theta}}(\Gamma_{(\mu\nu)(\sigma\varsigma)}^{\kappa\lambda}e_{\kappa\lambda}) - \omega^{\vartheta\iota}([e_{\sigma\varsigma}, e_{\eta\theta}])\Gamma_{(\mu\nu)(\vartheta\iota)}^{\kappa\lambda}e_{\kappa\lambda} \\
&= (e_{\sigma\varsigma}(\Gamma_{(\mu\nu)(\eta\theta)}^{\kappa\lambda}) - e_{\eta\theta}(\Gamma_{(\mu\nu)(\sigma\varsigma)}^{\kappa\lambda}) + \Gamma_{(\mu\nu)(\eta\theta)}^{\vartheta\iota}\Gamma_{(\vartheta\iota)(\sigma\varsigma)}^{\kappa\lambda} \\
&- \Gamma_{(\mu\nu)(\sigma\varsigma)}^{\vartheta\iota}\Gamma_{(\vartheta\iota)(\eta\theta)}^{\kappa\lambda} - \omega^{\vartheta\iota}([e_{\sigma\varsigma}, e_{\eta\theta}])\Gamma_{(\mu\nu)(\vartheta\iota)}^{\kappa\lambda})e_{\kappa\lambda} \\
&= (\widetilde{d}\omega_{\mu\nu}^{\kappa\lambda} - \omega_{\mu\nu}^{\vartheta\iota} \wedge \omega_{\vartheta\iota}^{\kappa\lambda})(e_{\sigma\varsigma}, e_{\eta\theta})e_{\kappa\lambda}.
\end{aligned}
$$

Therefore, we get that

$$(\widetilde{d}\omega_{\mu\nu}^{\kappa\lambda} - \omega_{\mu\nu}^{\vartheta\iota} \wedge \omega_{\vartheta\iota}^{\kappa\lambda})(e_{\sigma\varsigma}, e_{\eta\theta}) = \widetilde{R}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)}^{\kappa\lambda},$$

that is,

$$\widetilde{d}\omega_{\mu\nu}^{\kappa\lambda} - \omega_{\mu\nu}^{\sigma\varsigma} \wedge \omega_{\sigma\varsigma}^{\kappa\lambda} = \frac{1}{2}\widetilde{R}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)}^{\kappa\lambda}\omega^{\sigma\varsigma} \wedge \omega^{\eta\theta}.$$

□

Definition 2.3 *Let $(\widetilde{M}, \widetilde{D})$ be a combinatorially connection space. Differential 2-forms $\Omega^{\mu\nu} = \widetilde{d}\omega^{\mu\nu} - \omega^{\mu\nu} \wedge \omega_{\kappa\lambda}^{\mu\nu}$, $\Omega_{\mu\nu}^{\kappa\lambda} = \widetilde{d}\omega_{\mu\nu}^{\kappa\lambda} - \omega_{\mu\nu}^{\sigma\varsigma} \wedge \omega_{\sigma\varsigma}^{\kappa\lambda}$ and equations*

$$\widetilde{d}\omega^{\mu\nu} = \omega^{\kappa\lambda} \wedge \omega_{\kappa\lambda}^{\mu\nu} + \Omega^{\mu\nu}, \quad \widetilde{d}\omega_{\mu\nu}^{\kappa\lambda} = \omega_{\mu\nu}^{\sigma\varsigma} \wedge \omega_{\sigma\varsigma}^{\kappa\lambda} + \Omega_{\mu\nu}^{\kappa\lambda}$$

are called torsion forms, curvature forms and structural equations in a local frame $\{e_{ij}\}$ of $(\widetilde{M}, \widetilde{D})$, respectively.

By Theorem 2.3 and Definition 2.3, we get local forms for torsion tensor and curvature tensor in a local frame following.

Corollary 2.1 *Let $(\widetilde{M}, \widetilde{D})$ be a combinatorially connection space and $\{e_{ij}\}$ a local frame with a dual $\{\omega^{ij}\}$ at a point $p \in \widetilde{M}$. Then*

$$\widetilde{T} = \Omega^{\mu\nu} \otimes e_{\mu\nu} \quad \textit{and} \quad \widetilde{R} = \omega^{\mu\nu} \otimes e_{\kappa\lambda} \otimes \Omega_{\mu\nu}^{\kappa\lambda},$$

i.e., for $\forall X, Y \in \mathscr{X}(\widetilde{M})$,

$$\widetilde{T}(X, Y) = \Omega^{\mu\nu}(X, Y)e_{\mu\nu} \quad \textit{and} \quad \widetilde{R}(X, Y) = \Omega_{\mu\nu}^{\kappa\lambda}(X, Y)\omega^{\mu\nu} \otimes e_{\mu\nu}.$$

Theorem 2.4 *Let $(\widetilde{M},\widetilde{D})$ be a combinatorially connection space and $\{e_{ij}\}$ a local frame with a dual $\{\omega^{ij}\}$ at a point $p\in\widetilde{M}$. Then*

$$\widetilde{d}\Omega^{\mu\nu}=\omega^{\kappa\lambda}\wedge\Omega^{\mu\nu}_{\kappa\lambda}-\Omega^{\kappa\lambda}\wedge\omega^{\mu\nu}_{\kappa\lambda}\ \ and\ \ \widetilde{d}\Omega^{\kappa\lambda}_{\mu\nu}=\omega^{\sigma\varsigma}_{\mu\nu}\wedge\Omega^{\kappa\lambda}_{\sigma\varsigma}-\Omega^{\sigma\varsigma}_{\mu\nu}\wedge\omega^{\kappa\lambda}_{\sigma\varsigma}.$$

Proof Notice that $\widetilde{d}^2=0$. Differentiating the equality $\Omega^{\mu\nu}=\widetilde{d}\omega^{\mu\nu}-\omega^{\mu\nu}\wedge\omega^{\mu\nu}_{\kappa\lambda}$ on both sides, we get that

$$\begin{aligned}\widetilde{d}\Omega^{\mu\nu} &= -\widetilde{d}\omega^{\mu\nu}\wedge\omega^{\mu\nu}_{\kappa\lambda}+\omega^{\mu\nu}\wedge\widetilde{d}\omega^{\mu\nu}_{\kappa\lambda}\\ &= -(\Omega^{\kappa\lambda}+\omega^{\sigma\varsigma}\wedge\omega^{\kappa\lambda}_{\sigma\varsigma})\wedge\omega^{\mu\nu}_{\kappa\lambda}+\omega^{\kappa\lambda}\wedge(\Omega^{\mu\nu}_{\kappa\lambda}+\omega^{\sigma\varsigma}_{\kappa\lambda}\wedge\omega^{\mu\nu}_{\sigma\varsigma})\\ &= \omega^{\kappa\lambda}\wedge\Omega^{\mu\nu}_{\kappa\lambda}-\Omega^{\kappa\lambda}\wedge\omega^{\mu\nu}_{\kappa\lambda}.\end{aligned}$$

Similarly, differentiating the equality $\Omega^{\kappa\lambda}_{\mu\nu}=\widetilde{d}\omega^{\kappa\lambda}_{\mu\nu}-\omega^{\sigma\varsigma}_{\mu\nu}\wedge\omega^{\kappa\lambda}_{\sigma\varsigma}$ on both sides, we can also find that

$$\widetilde{d}\Omega^{\kappa\lambda}_{\mu\nu}=\omega^{\sigma\varsigma}_{\mu\nu}\wedge\Omega^{\kappa\lambda}_{\sigma\varsigma}-\Omega^{\sigma\varsigma}_{\mu\nu}\wedge\omega^{\kappa\lambda}_{\sigma\varsigma}.$$

□

Corollary 2.2 *Let (M,D) be an affine connection space and $\{e_i\}$ a local frame with a dual $\{\omega^i\}$ at a point $p\in M$. Then*

$$d\Omega^i=\omega^j\wedge\Omega^i_j-\Omega^j\wedge\omega^i_j\ \ and\ \ d\Omega^j_i=\omega^k_i\wedge\Omega^j_k-\Omega^k_i\wedge\Omega^j_k.$$

According to Theorems 2.1−2.4 there is a type $(1,3)$ tensor $\widetilde{\mathcal{R}}_p:T_p\widetilde{M}\times T_p\widetilde{M}\times T_p\widetilde{M}\rightarrow T_p\widetilde{M}$ determined by $\widetilde{\mathcal{R}}(w,u,v)=\widetilde{\mathcal{R}}(u,v)w$ for $\forall u,v,w\in T_p\widetilde{M}$ at each point $p\in\widetilde{M}$. Particularly, we get its a concrete local form in the standard basis $\{\frac{\partial}{\partial x^{\mu\nu}}\}$.

Theorem 2.5 *Let $(\widetilde{M},\widetilde{D})$ be a combinatorially connection space. Then for $\forall p\in\widetilde{M}$ with a local chart $(U_p;[\varphi_p])$,*

$$\widetilde{\mathcal{R}}=\widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}dx^{\sigma\varsigma}\otimes\frac{\partial}{\partial x^{\eta\theta}}\otimes dx^{\mu\nu}\otimes dx^{\kappa\lambda}$$

with

$$\widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}=\frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\mu\nu}}-\frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\kappa\lambda}}+\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\eta\theta}_{(\vartheta\iota)(\mu\nu)}-\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\eta\theta}_{(\vartheta\iota)(\kappa\lambda)})\frac{\partial}{\partial x^{\vartheta\iota}},$$

where, $\Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)}\in C^\infty(U_p)$ is determined by

$$\widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}}\frac{\partial}{\partial x^{\kappa\lambda}}=\Gamma^{\sigma\varsigma}_{(\kappa\lambda)(\mu\nu)}\frac{\partial}{\partial x^{\sigma\varsigma}}.$$

Proof We only need to prove that for integers $\mu,\nu,\kappa,\lambda,\sigma,\varsigma,\iota$ and θ,

$$\widetilde{\mathcal{R}}(\frac{\partial}{\partial x^{\mu\nu}},\frac{\partial}{\partial x^{\kappa\lambda}})\frac{\partial}{\partial x^{\sigma\varsigma}}=\widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}\frac{\partial}{\partial x^{\eta\theta}}$$

at the local chart $(U_p;[\varphi_p])$. In fact, by definition we get that

$$\begin{aligned}
&\widetilde{\mathcal{R}}(\frac{\partial}{\partial x^{\mu\nu}},\frac{\partial}{\partial x^{\kappa\lambda}})\frac{\partial}{\partial x^{\sigma\varsigma}}\\
&=\widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}}\widetilde{D}_{\frac{\partial}{\partial x^{\kappa\lambda}}}\frac{\partial}{\partial x^{\sigma\varsigma}}-\widetilde{D}_{\frac{\partial}{\partial x^{\kappa\lambda}}}\widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}}\frac{\partial}{\partial x^{\sigma\varsigma}}-\widetilde{D}_{[\frac{\partial}{\partial x^{\mu\nu}},\frac{\partial}{\partial x^{\kappa\lambda}}]}\frac{\partial}{\partial x^{\sigma\varsigma}}\\
&=\widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}}(\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}\frac{\partial}{\partial x^{\eta\theta}})-\widetilde{D}_{\frac{\partial}{\partial x^{\kappa\lambda}}}(\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}\frac{\partial}{\partial x^{\eta\theta}})\\
&=\frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\mu\nu}}\frac{\partial}{\partial x^{\eta\theta}}+\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}\widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}}\frac{\partial}{\partial x^{\eta\theta}}-\frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\kappa\lambda}}\frac{\partial}{\partial x^{\eta\theta}}-\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}\widetilde{D}_{\frac{\partial}{\partial x^{\kappa\lambda}}}\frac{\partial}{\partial x^{\eta\theta}}\\
&=(\frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\mu\nu}}-\frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\kappa\lambda}})\frac{\partial}{\partial x^{\eta\theta}}+\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\vartheta\iota}_{(\eta\theta)(\mu\nu)}\frac{\partial}{\partial x^{\vartheta\iota}}-\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\vartheta\iota}_{(\eta\theta)(\kappa\lambda)}\frac{\partial}{\partial x^{\vartheta\iota}}\\
&=(\frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\mu\nu}}-\frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\kappa\lambda}}+\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\eta\theta}_{(\vartheta\iota)(\mu\nu)}-\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\eta\theta}_{(\vartheta\iota)(\kappa\lambda)})\frac{\partial}{\partial x^{\vartheta\iota}}\\
&=\widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}\frac{\partial}{\partial x^{\eta\theta}}.
\end{aligned}$$

This completes the proof. □

For the curvature tensor $\widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}$, we can also get these *Bianchi identities* in the next result.

Theorem 2.6 *Let $(\widetilde{M},\widetilde{D})$ be a combinatorially connection space. Then for $\forall p\in\widetilde{M}$ with a local chart $(U_p,[\varphi_p])$, if $\widetilde{T}\equiv 0$, then*

$$\widetilde{R}^{\mu\nu}_{(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}+\widetilde{R}^{\mu\nu}_{(\sigma\varsigma)(\eta\theta)(\kappa\lambda)}+\widetilde{R}^{\mu\nu}_{(\eta\theta)(\kappa\lambda)(\sigma\varsigma)}=0$$

and

$$\widetilde{D}_{\vartheta\iota}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)}+\widetilde{D}_{\sigma\varsigma}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\eta\theta)(\vartheta\iota)}+\widetilde{D}_{\eta\theta}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\vartheta\iota)(\sigma\varsigma)}=0,$$

where,

$$\widetilde{D}_{\vartheta\iota}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)}=\widetilde{D}_{\frac{\partial}{\partial x^{\vartheta\iota}}}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)}.$$

Proof By definition of the curvature tensor $\widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}$, we know that

$$\begin{aligned}
&\widetilde{R}^{\mu\nu}_{(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}+\widetilde{R}^{\mu\nu}_{(\sigma\varsigma)(\eta\theta)(\kappa\lambda)}+\widetilde{R}^{\mu\nu}_{(\eta\theta)(\kappa\lambda)(\sigma\varsigma)}\\
&=\widetilde{R}(\frac{\partial}{\partial x^{\sigma\varsigma}},\frac{\partial}{\partial x^{\eta\theta}})\frac{\partial}{\partial x^{\kappa\lambda}}+\widetilde{R}(\frac{\partial}{\partial x^{\eta\theta}},\frac{\partial}{\partial x^{\kappa\lambda}})\frac{\partial}{\partial x^{\sigma\varsigma}}+\widetilde{R}(\frac{\partial}{\partial x^{\kappa\lambda}},\frac{\partial}{\partial x^{\sigma\varsigma}})\frac{\partial}{\partial x^{\eta\theta}}\\
&=0
\end{aligned}$$

with

$$X = \frac{\partial}{\partial x^{\sigma\varsigma}}, \quad Y = \frac{\partial}{\partial x^{\eta\theta}} \text{ and } Z = \frac{\partial}{\partial x^{\kappa\lambda}}. \quad \natural$$

in the first Bianchi equality and

$$\begin{aligned}
&\widetilde{D}_{\vartheta\iota}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)} + \widetilde{D}_{\sigma\varsigma}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\eta\theta)(\vartheta\iota)} + \widetilde{D}_{\eta\theta}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\vartheta\iota)(\sigma\varsigma)} \\
&= \widetilde{D}_{\vartheta\iota}\widetilde{R}(\frac{\partial}{\partial x^{\sigma\varsigma}}, \frac{\partial}{\partial x^{\eta\theta}})\frac{\partial}{\partial x^{\kappa\lambda}} + \widetilde{D}_{\sigma\varsigma}\widetilde{R}(\frac{\partial}{\partial x^{\eta\theta}}, \frac{\partial}{\partial x^{\vartheta\iota}})\frac{\partial}{\partial x^{\kappa\lambda}} + \widetilde{D}_{\eta\theta}\widetilde{R}(\frac{\partial}{\partial x^{\vartheta\iota}}, \frac{\partial}{\partial x^{\sigma\varsigma}})\frac{\partial}{\partial x^{\kappa\lambda}} \\
&= 0.
\end{aligned}$$

with

$$X = \frac{\partial}{\partial x^{\vartheta\iota}}, \ Y = \frac{\partial}{\partial x^{\sigma\varsigma}}, \ Z = \frac{\partial}{\partial x^{\eta\theta}}, \ W = \frac{\partial}{\partial x^{\kappa\lambda}}$$

in the second Bianchi equality of Theorem 2.2. □

§3. **Curvatures on Combinatorially Riemannian Manifolds**

Now we turn our attention to combinatorially Riemannian manifolds and characterize curvature tensors on combinatorial manifolds further.

Definition 3.1 *Let $(\widetilde{M}, g, \widetilde{D})$ be a combinatorially Riemannian manifold. A combinatorially Riemannian curvature tensor*

$$\widetilde{R} : \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \to C^\infty(\widetilde{M})$$

of type $(0,4)$ is defined by

$$\widetilde{R}(X, Y, Z, W) = g(\widetilde{R}(Z, W)X, Y)$$

for $\forall X, Y, Z, W \in \mathscr{X}(\widetilde{M})$.

Then we find symmetrical relations of $\widetilde{R}(X, Y, Z, W)$ following.

Theorem 3.1 *Let $\widetilde{R} : \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \to C^\infty(\widetilde{M})$ be a combinatorially Riemannian curvature tensor. Then for $\forall X, Y, Z, W \in \mathscr{X}(\widetilde{M})$,*

(1) $\widetilde{R}(X, Y, Z, W) + \widetilde{R}(Z, Y, W, X) + \widetilde{R}(W, Y, X, Z) = 0$.

(2) $\widetilde{R}(X, Y, Z, W) = -\widetilde{R}(Y, X, Z, W)$ *and* $\widetilde{R}(X, Y, Z, W) = -\widetilde{R}(X, Y, W, Z)$.

(3) $\widetilde{R}(X, Y, Z, W) = \widetilde{R}(Z, W, X, Y)$.

Proof For the equality (1), calculation shows that

$$\begin{aligned}
&\widetilde{R}(X, Y, Z, W) + \widetilde{R}(Z, Y, W, X) + \widetilde{R}(W, Y, X, Z) \\
&= g(\widetilde{R}(Z, W)X, Y) + g(\widetilde{R}(W, X)Z, Y) + g(\widetilde{R}(X, Z)W, Y) \\
&= g(\widetilde{R}(Z, W)X + \widetilde{R}(W, X)Z + \widetilde{R}(X, Z)W, Y) = 0
\end{aligned}$$

by definition and Theorem 2.1(4).

For (2), by definition and Theorem 2.1(1), we know that

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= g(\widetilde{R}(Z,W)X,Y) = g(-\widetilde{R}(W,Z)X,Y)\\ &= -g(\widetilde{R}(W,Z)X,Y) = -\widetilde{R}(X,Y,W,Z).\end{aligned}$$

Now since $\widetilde{D}$ is a combinatorially Riemannian connection, we know that ([9])

$$Z(g(X,Y)) = g(\widetilde{D}_Z X,Y) + g(X,\widetilde{D}_Z Y).$$

Therefore, we find that

$$\begin{aligned}g(\widetilde{D}_Z\widetilde{D}_W X,Y) &= Z(g(\widetilde{D}_W X,Y)) - g(\widetilde{D}_W X,\widetilde{D}_Z Y)\\ &= Z(W(g(X,Y))) - Z(g(X,\widetilde{D}_W Y))\\ &- W(g(X,\widetilde{D}_Z Y)) + g(X,\widetilde{D}_W\widetilde{D}_Z Y).\end{aligned}$$

Similarly, we have that

$$\begin{aligned}g(\widetilde{D}_W\widetilde{D}_Z X,Y) &= W(Z(g(X,Y))) - W(g(X,\widetilde{D}_Z Y))\\ &- Z(g(X,\widetilde{D}_W Y)) + g(X,\widetilde{D}_Z\widetilde{D}_W Y).\end{aligned}$$

Notice that

$$g(\widetilde{D}_{[Z,W]},Y) = [Z,W]g(X,Y) - g(X,\widetilde{D}_{[Z,W]}Y).$$

By definition, we get that

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= g(\widetilde{D}_Z\widetilde{D}_W X - \widetilde{D}_W\widetilde{D}_Z X - \widetilde{D}_{[Z,W]}X,Y)\\ &= g(\widetilde{D}_Z\widetilde{D}_W X,Y) - g(\widetilde{D}_W\widetilde{D}_Z X,Y) - g(\widetilde{D}_{[Z,W]}X,Y)\\ &= Z(W(g(X,Y))) - Z(g(X,\widetilde{D}_W Y)) - W(g(X,\widetilde{D}_Z Y))\\ &+ g(X,\widetilde{D}_W\widetilde{D}_Z Y) - W(Z(g(X,Y))) + W(g(X,\widetilde{D}_Z Y))\\ &+ Z(g(X,\widetilde{D}_W Y)) - g(X,\widetilde{D}_Z\widetilde{D}_W Y) - [Z,W]g(X,Y)\\ &- g(X,\widetilde{D}_{[Z,W]}Y)\\ &= Z(W(g(X,Y))) - W(Z(g(X,Y))) + g(X,\widetilde{D}_W\widetilde{D}_Z Y)\\ &- g(X,\widetilde{D}_Z\widetilde{D}_W Y) - [Z,W]g(X,Y) - g(X,\widetilde{D}_{[Z,W]}Y)\\ &= g(X,\widetilde{D}_W\widetilde{D}_Z Y - \widetilde{D}_Z\widetilde{D}_W Y + \widetilde{D}_{[Z,W]}Y)\\ &= -g(X,\widetilde{R}(Z,W)Y) = -\widetilde{R}(Y,X,Z,W).\end{aligned}$$

Applying the equality (1), we know that

$$\widetilde{R}(X,Y,Z,W)+\widetilde{R}(Z,Y,W,X)+\widetilde{R}(W,Y,X,Z)=0, \quad (3.1)$$

$$\widetilde{R}(Y,Z,W,X)+\widetilde{R}(W,Z,X,Y)+\widetilde{R}(X,Z,Y,W)=0. \quad (3.2)$$

Then $(3.1)+(3.2)$ shows that

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &+ \widetilde{R}(W,Y,X,Z)\\ &+ \widetilde{R}(W,Z,X,Y)+\widetilde{R}(X,Z,Y,W)=0\end{aligned}$$

by applying (2). We also know that

$$\begin{aligned}\widetilde{R}(W,Y,X,Z)-\widetilde{R}(X,Z,Y,W) &= -(\widetilde{R}(Z,Y,W,X)-\widetilde{R}(W,X,Z,Y))\\ &= \widetilde{R}(X,Y,Z,W)-\widetilde{R}(Z,W,X,Y).\end{aligned}$$

This enables us getting the equality (3)

$$\widetilde{R}(X,Y,Z,W)=\widetilde{R}(Z,W,X,Y).$$

□

Applying Theorems 2.2, 2.3 and 3.1, we also get the next result.

Theorem 3.2 *Let* $(\widetilde{M},g,\widetilde{D})$ *be a combinatorially Riemannian manifold and* $\Omega_{(\mu\nu)(\kappa\lambda)}=\Omega_{\mu\nu}^{\sigma\varsigma}g_{(\sigma\varsigma)(\kappa\lambda)}$. *Then*

(1) $\Omega_{(\mu\nu)(\kappa\lambda)}=\frac{1}{2}\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}\omega^{\sigma\varsigma}\wedge\omega^{\eta\theta}$;

(2) $\Omega_{(\mu\nu)(\kappa\lambda)}+\Omega_{(\kappa\lambda)(\mu\nu)}=0$;

(3) $\omega^{\mu\nu}\wedge\Omega_{(\mu\nu)(\kappa\lambda)}=0$;

(4) $\widetilde{d}\Omega_{(\mu\nu)(\kappa\lambda)}=\omega_{\mu\nu}^{\sigma\varsigma}\wedge\Omega_{(\sigma\varsigma)(\kappa\lambda)}-\omega_{\kappa\lambda}^{\sigma\varsigma}\wedge\Omega_{(\sigma\varsigma)(\mu\nu)}$.

Proof Notice that $\widetilde{T}\equiv 0$ in a combinatorially Riemannian manifold $(\widetilde{M},g,\widetilde{D})$. We find that

$$\Omega_{\mu\nu}^{\kappa\lambda}=\frac{1}{2}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)}\omega^{\sigma\varsigma}\wedge\omega^{\eta\theta}$$

by Theorem 2.2. By definition, we know that

$$\begin{aligned}\Omega_{(\mu\nu)(\kappa\lambda)} &= \Omega_{\mu\nu}^{\sigma\varsigma}g_{(\sigma\varsigma)(\kappa\lambda)}\\ &= \frac{1}{2}\widetilde{R}^{\sigma\varsigma}_{(\mu\nu)(\eta\theta)(\vartheta\iota)}g_{(\sigma\varsigma)(\kappa\lambda)}\omega^{\eta\theta}\wedge\omega^{\vartheta\iota}=\frac{1}{2}\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}\omega^{\sigma\varsigma}\wedge\omega^{\eta\theta}.\end{aligned}$$

Whence, we get the equality (1). For (2), applying Theorem 3.1(2), we find that

$$\Omega_{(\mu\nu)(\kappa\lambda)}+\Omega_{(\kappa\lambda)(\mu\nu)}=\frac{1}{2}(\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}+\widetilde{R}_{(\kappa\lambda)(\mu\nu)(\sigma\varsigma)(\eta\theta)})\omega^{\sigma\varsigma}\wedge\omega^{\eta\theta}=0.$$

By Corollary 2.1, a connection $\widetilde{D}$ is torsion-free only if $\Omega^{\mu\nu}\equiv 0$. This fact enables us to get these equalities (3) and (4) by Theorem 2.3. □

For any point $p\in\widetilde{M}$ with a local chart $(U_p,[\varphi_p])$, we can also find a local form of $\widetilde{R}$ in the next result.

Theorem 3.3 *Let $\widetilde{R}:\mathscr{X}(\widetilde{M})\times\mathscr{X}(\widetilde{M})\times\mathscr{X}(\widetilde{M})\times\mathscr{X}(\widetilde{M})\to C^\infty(\widetilde{M})$ be a combinatorially Riemannian curvature tensor. Then for $\forall p\in\widetilde{M}$ with a local chart $(U_p;[\varphi_p])$,*

$$\widetilde{R}=\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)}dx^{\sigma\varsigma}\otimes dx^{\eta\theta}\otimes dx^{\mu\nu}\otimes dx^{\kappa\lambda}$$

with

$$\begin{aligned}\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} &= \frac{1}{2}(\frac{\partial^2 g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}\partial x^{\eta\theta}}+\frac{\partial^2 g_{(\kappa\lambda)(\eta\theta)}}{\partial x^{\mu\nu\nu}\partial x^{\sigma\varsigma}}-\frac{\partial^2 g_{(\mu\nu)(\eta\theta)}}{\partial x^{\kappa\lambda}\partial x^{\sigma\varsigma}}-\frac{\partial^2 g_{(\kappa\lambda)(\sigma\varsigma)}}{\partial x^{\mu\nu}\partial x^{\eta\theta}})\\ &+ \Gamma^{\vartheta\iota}_{(\mu\nu)(\sigma\varsigma)}\Gamma^{\xi o}_{(\kappa\lambda)(\eta\theta)}g_{(\xi o)(\vartheta\iota)}-\Gamma^{\xi o}_{(\mu\nu)(\eta\theta)}\Gamma_{(\kappa\lambda)(\sigma\varsigma)^{\vartheta\iota}}g_{(\xi o)(\vartheta\iota)},\end{aligned}$$

where $g_{(\mu\nu)(\kappa\lambda)}=g(\frac{\partial}{\partial x^{\mu\nu}},\frac{\partial}{\partial x^{\kappa\lambda}})$.

Proof Notice that

$$\begin{aligned}\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} &= \widetilde{R}(\frac{\partial}{\partial x^{\sigma\varsigma}},\frac{\partial}{\partial x^{\eta\theta}},\frac{\partial}{\partial x^{\mu\nu}},\frac{\partial}{\partial x^{\kappa\lambda}})=\widetilde{R}(\frac{\partial}{\partial x^{\mu\nu}},\frac{\partial}{\partial x^{\kappa\lambda}},\frac{\partial}{\partial x^{\sigma\varsigma}},\frac{\partial}{\partial x^{\eta\theta}})\\ &= g(\widetilde{R}(\frac{\partial}{\partial x^{\sigma\varsigma}},\frac{\partial}{\partial x^{\eta\theta}})\frac{\partial}{\partial x^{\mu\nu}},\frac{\partial}{\partial x^{\kappa\lambda}})=\widetilde{R}^{\vartheta\iota}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)}g_{(\vartheta\iota)(\kappa\lambda)}\end{aligned}$$

By definition and Theorem 3.1(3). Now we have know that (eqn.(3.5) in [9])

$$\frac{\partial g_{(\mu\nu)(\kappa\lambda)}}{\partial x^{\sigma\varsigma}}=\Gamma^{\eta\theta}_{(\mu\nu)(\sigma\varsigma)}g_{(\eta\theta)(\kappa\lambda)}+\Gamma^{\eta\theta}_{(\kappa\lambda)(\sigma\varsigma)}g_{(\mu\nu)(\eta\theta)}.$$

Applying Theorem 2.4, we get that

$$\begin{aligned}
&\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)}\\
&=(\frac{\partial\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\mu\nu}}-\frac{\partial\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\kappa\lambda}}+\Gamma^{\xi o}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\vartheta\iota}_{(\xi o)(\mu\nu)}-\Gamma^{\xi o}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\vartheta\iota}_{(\xi o)(\kappa\lambda)})g_{(\vartheta\iota)(\eta\theta)}\\
&=\frac{\partial}{\partial x^{\mu\nu}}(\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}g_{(\vartheta\iota)(\eta\theta)})-\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}\frac{\partial g_{(\vartheta\iota)(\eta\theta)}}{\partial x^{\mu\nu}}-\frac{\partial}{\partial x^{\kappa\lambda}}(\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}g_{(\vartheta\iota)(\eta\theta)})\\
&+\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}\frac{\partial g_{(\vartheta\iota)(\eta\theta)}}{\partial x^{\kappa\lambda}}+\Gamma^{\xi o}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\vartheta\iota}_{(\xi o)(\mu\nu)}g_{(\vartheta\iota)(\kappa\lambda)}-\Gamma^{\xi o}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\vartheta\iota}_{(\xi o)(\kappa\lambda)}g_{(\vartheta\iota)(\eta\theta)}\\
&=\frac{\partial}{\partial x^{\mu\nu}}(\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}g_{(\vartheta\iota)(\eta\theta)})-\frac{\partial}{\partial x^{\kappa\lambda}}(\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}g_{(\vartheta\iota)(\eta\theta)})\\
&+\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}(\Gamma^{\xi o}_{(\vartheta\iota)(\kappa\lambda)}g_{(\xi o)(\eta\theta)}+\Gamma^{\xi o}_{(\eta\theta)(\kappa\lambda)}g_{(\vartheta\iota)(\xi o)})+\Gamma^{\xi o}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\vartheta\iota}_{(\xi o)(\mu\nu)}g_{(\vartheta\iota)(\kappa\lambda)}\\
&-\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}(\Gamma^{\xi o}_{(\vartheta\iota)(\mu\nu)}g_{(\xi o)(\eta\theta)}+\Gamma^{\xi o}_{(\eta\theta)(\mu\nu)}g_{(\vartheta\iota)(\xi o)})-\Gamma^{\xi o}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\vartheta\iota}_{(\xi o)(\kappa\lambda)})g_{(\vartheta\iota)(\eta\theta)}\\
&=\frac{1}{2}\frac{\partial}{\partial x^{\mu\nu}}(\frac{\partial g_{(\sigma\varsigma)(\eta\theta)}}{\partial x^{\kappa\lambda}}+\frac{\partial g_{(\kappa\lambda)(\eta\theta)}}{\partial x^{\sigma\varsigma}}-\frac{\partial g_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\eta\theta}})+\Gamma^{\xi o}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\vartheta\iota}_{(\xi o)(\mu\nu)}g_{(\vartheta\iota)(\kappa\lambda)}\\
&-\frac{1}{2}\frac{\partial}{\partial x^{\kappa\lambda}}(\frac{\partial g_{(\sigma\varsigma)(\eta\theta)}}{\partial x^{\mu\nu}}+\frac{\partial g_{(\mu\nu)(\eta\theta)}}{\partial x^{\sigma\varsigma}}-\frac{\partial g_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\eta\theta}})-\Gamma^{\xi o}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\vartheta\iota}_{(\xi o)(\kappa\lambda)})g_{(\vartheta\iota)(\eta\theta)}\\
&=\frac{1}{2}(\frac{\partial^2 g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}\partial x^{\eta\theta}}+\frac{\partial^2 g_{(\kappa\lambda)(\eta\theta)}}{\partial x^{\mu\nu}\partial x^{\sigma\varsigma}}-\frac{\partial^2 g_{(\mu\nu)(\eta\theta)}}{\partial x^{\kappa\lambda}\partial x^{\sigma\varsigma}}-\frac{\partial^2 g_{(\kappa\lambda)(\sigma\varsigma)}}{\partial x^{\mu\nu}\partial x^{\eta\theta}})\\
&+\Gamma^{\xi o}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\vartheta\iota}_{(\xi o)(\mu\nu)}g_{(\vartheta\iota)(\kappa\lambda)}-\Gamma^{\xi o}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\vartheta\iota}_{(\xi o)(\kappa\lambda)})g_{(\vartheta\iota)(\eta\theta)}.
\end{aligned}$$

This completes the proof. □

Combining Theorems 2.5, 3.1 and 3.3, we have the following consequence.

Corollary 3.1 *Let $\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}$ be a component of a combinatorially Riemannian curvature tensor $\widetilde{R}$ in a local chart $(U,[\varphi])$ of a combinatorially Riemannian manifold $(\widetilde{R},g,\widetilde{D})$. Then*

(1) $\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}=-\widetilde{R}_{(\kappa\lambda)(\mu\nu)(\sigma\varsigma)(\eta\theta)}=-\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\eta\theta)(\sigma\varsigma)}$;

(2) $\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}=\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)}$;

(3) $\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}+\widetilde{R}_{(\eta\theta)(\kappa\lambda)(\mu\nu)(\sigma\varsigma)}+\widetilde{R}_{(\sigma\varsigma)(\kappa\lambda)(\eta\theta)(\mu\nu)}=0$;

(4) $\widetilde{D}_{\vartheta\iota}\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}+\widetilde{D}_{\sigma\varsigma}\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\eta\theta)(\vartheta\iota)}+\widetilde{D}_{\eta\theta}\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\vartheta\iota)(\sigma\varsigma)}=0$.

§4. Einstein's Gravitational Equations on Combinatorial Manifolds

Application of results in last two sections enables us to establish these Einstein' gravitational filed equations on combinatorially Riemannian manifolds in this section and find their multi-space solutions in next section under a *projective principle* on the behavior of particles in multi-spaces.

Let $(\widetilde{M},g,\widetilde{D})$ be a combinatorially Riemannian manifold. A type $(0,2)$ tensor $\mathscr{E}:\mathscr{X}(\widetilde{M})\times\mathscr{X}(\widetilde{M})\rightarrow C^\infty(\widetilde{M})$ with

$$\mathscr{E}=\mathscr{E}_{(\mu\nu)(\kappa\lambda)}dx^{\mu\nu}\otimes dx^{\kappa\lambda}\quad(4.1)$$

is called an *energy-momentum tensor* if it satisfies the conservation laws $\widetilde{D}(\mathscr{E})=0$, i.e., for any indexes κ,λ, $1\leq\kappa\leq m$, $1\leq\lambda\leq n_\kappa$,

$$\frac{\partial \mathscr{E}_{(\mu\nu)(\kappa\lambda)}}{\partial x^{\kappa\lambda}} - \Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)}\mathscr{E}_{(\sigma\varsigma)(\kappa\lambda)} - \Gamma^{\sigma\varsigma}_{(\kappa\lambda)(\kappa\lambda)}\mathscr{E}_{(\mu\nu)(\sigma\varsigma)} = 0 \quad (4.2)$$

in a local chart $(U_p, [\varphi_p])$ for any point $p \in \widetilde{M}$. Define the *Ricci tensor* $\widetilde{R}_{(\mu\nu)(\kappa\lambda)}$, *Rocci scalar tensor* $\mathbf{R}$ and *Einstein tensor* $\mathscr{G}_{(\mu\nu)(\kappa\lambda)}$ respectively by

$$\widetilde{R}_{(\mu\nu)(\kappa\lambda)} = \widetilde{R}^{\sigma\varsigma}_{(\mu\nu)(\sigma\varsigma)(\kappa\lambda)}, \quad \mathbf{R} = g^{(\mu\nu)(\kappa\lambda)}\widetilde{R}_{(\mu\nu)(\kappa\lambda)} \quad (4.3)$$

and

$$\mathscr{G}_{(\mu\nu)(\kappa\lambda)} = \widetilde{R}_{(\mu\nu)(\kappa\lambda)} - \frac{1}{2}g_{(\mu\nu)(\kappa\lambda)}\mathbf{R}. \quad (4.4)$$

Then we get results following hold by Theorems 2.4, 2.5 and 3.1.

$$\widetilde{R}_{(\mu\nu)(\kappa\lambda)} = \widetilde{R}_{(\kappa\lambda)(\mu\nu)}, \quad (4.5)$$

$$\widetilde{R}_{(\mu\nu)(\kappa\lambda)} == \frac{\partial \Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)}}{\partial x^{\sigma\varsigma}} - \frac{\partial \Gamma^{\sigma\varsigma}_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}} + \Gamma^{\vartheta\iota}_{(\mu\nu)(\kappa\lambda)}\Gamma^{\sigma\varsigma}_{(\vartheta\iota)(\sigma\varsigma)} - \Gamma^{\vartheta\iota}_{(\mu\nu)(\sigma\varsigma)}\Gamma^{\sigma\varsigma}_{(\vartheta\iota)(\kappa\lambda)}. \quad (4.6)$$

and

$$\frac{\partial \mathscr{G}_{(\mu\nu)(\kappa\lambda)}}{\partial x^{\kappa\lambda}} - \Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)}\mathscr{G}_{(\sigma\varsigma)(\kappa\lambda)} - \Gamma^{\sigma\varsigma}_{(\kappa\lambda)(\kappa\lambda)}\mathscr{G}_{(\mu\nu)(\sigma\varsigma)} = 0. \quad (4.7)$$

i.e., $\widetilde{D}(\mathscr{G}) = 0$. *Einstein's* principle of general relativity says that *a law of physics should take a same form in any reference system*, which claims that a right form for a physics law should be presented by tensors in mathematics. For a multi-spacetime, we conclude that *Einstein's* principle of general relativity is still true, if we take the multi-spacetime being a combinatorially Riemannian manifold. Whence, a physics law should be also presented by tensor equations in the multi-spacetime case.

Just as the establishing of *Einstein's* gravitational equations in the classical case, these equations should satisfy two conditions following.

(C1) *They should be* $(0, 2)$ *type tensor equations related to the energy-momentum tensor* $\mathscr{E}$ *linearly;*

(C2) *Their forms should be the same as in a classical gravitational field.*

By these two conditions, *Einstein's* gravitational equations in a multi-spacetime should be taken the following form

$$\mathscr{G} = c\mathscr{E}$$

with c a constant. Now since these equations should take the same form in the classical case, i.e.,

$$\mathscr{G}_{ij} = -8\pi G\mathscr{E}_{ij}$$

for $1 \le i, j \le n$ at a point p in a manifold of $\widetilde{M}$ not contained in the others. Whence, it must be $c = -8\pi G$ for c being a constant. This enables us finding these *Einstein's* gravitational equations in a multi-spacetime to be

$$\widetilde{\mathcal{R}}_{(\mu\nu)(\kappa\lambda)} - \frac{1}{2}\mathbf{R}g_{(\mu\nu)(\kappa\lambda)} = -8\pi G\mathscr{E}_{(\mu\nu)(\kappa\lambda)}. \quad (4.8)$$

Certainly, we can also add a cosmological term $\lambda g_{(\mu\nu)(\kappa\lambda)}$ in (4.8) and obtain these gravitational equations

$$\widetilde{\mathcal{R}}_{(\mu\nu)(\kappa\lambda)} - \frac{1}{2}\mathbf{R}g_{(\mu\nu)(\kappa\lambda)} + \lambda g_{(\mu\nu)(\kappa\lambda)} = -8\pi G\mathscr{E}_{(\mu\nu)(\kappa\lambda)}. \quad (4.9)$$

All of these equations (4.8) and (4.9) mean that there are multi-space solutions in classical *Einstein's* gravitational equations by a multi-spacetime view, which will be shown in the next section.

§5. Multi-Space Solutions of Einstein's Equations

For given integers $0 < n_1 < n_2 < \cdots < n_m, m \ge 1$, let $(\widetilde{M}, g, \widetilde{D})$ be a combinatorial Riemannian manifold with $\widetilde{M} = \widetilde{M}(n_1, n_2, \cdots, n_m)$ and $(U_p, [\varphi_p])$ a local chart for $p \in \widetilde{M}$. By definition, if $\varphi_p : U_p \to \bigcup\limits_{i=1}^{s(p)} B^{n_i(p)}$ and $\widehat{s}(p) = \dim(\bigcap\limits_{i=1}^{s(p)} B^{n_i(p)})$, then $[\varphi_p]$ is an $s(p) \times n_{s(p)}$ matrix shown following.

$$[\varphi_p] = \left[\begin{array}{cccccccc} \frac{x^{11}}{s(p)} & \cdots & \frac{x^{1\widehat{s}(p)}}{s(p)} & x^{1(\widehat{s}(p)+1)} & \cdots & x^{1n_1} & \cdots & 0 \\ \frac{x^{21}}{s(p)} & \cdots & \frac{x^{2\widehat{s}(p)}}{s(p)} & x^{2(\widehat{s}(p)+1)} & \cdots & x^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{x^{s(p)1}}{s(p)} & \cdots & \frac{x^{s(p)\widehat{s}(p)}}{s(p)} & x^{s(p)(\widehat{s}(p)+1)} & \cdots & \cdots & x^{s(p)n_{s(p)}-1} & x^{s(p)n_{s(p)}} \end{array}\right]$$

with $x^{is} = x^{js}$ for $1 \le i, j \le s(p), 1 \le s \le \widehat{s}(p)$.

For given non-negative integers $r, s, r + s \ge 1$, choose a type (r, s) tensor $\mathscr{F} \in T^r_s(\widetilde{M})$. Then how to get multi-space solutions of a tensor equation

$$\mathscr{F} = 0 \ ?$$

We need to apply the *projective principle* following.

[Projective Principle] *Let $(\widetilde{M}, g, \widetilde{D})$ be a combinatorial Riemannian manifold and $\mathscr{F} \in \left\langle T | T \in T^r_s(\widetilde{M})\right\rangle$ with a local form $\mathscr{F}_{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_s\nu_s)}\omega^{\mu_1\nu_1} \otimes \omega^{\mu_2\nu_2} \otimes \cdots \otimes \omega^{\mu_n\nu_n}$ in $(U_p, [\varphi_p])$. If*

$$\mathscr{F}_{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_n\nu_n)} = 0$$

for integers $1 \le \mu_i \le s(p), 1 \le \nu_i \le n_{\mu_i}$ with $1 \le i \le s$, then for any integer $\mu, 1 \le \mu \le s(p)$, there must be

$$\mathscr{F}_{(\mu\nu_1)(\mu\nu_2)\cdots(\mu\nu_s)} = 0$$

for integers ν_i, $1 \leq \nu_i \leq n_\mu$ *with* $1 \leq i \leq s$.

Now we solve these vacuum *Einstein's* gravitational equations

$$\widetilde{R}_{(\mu\nu)(\kappa\lambda)} - \frac{1}{2}g_{(\mu\nu)(\kappa\lambda)}\mathbf{R} = 0 \quad (5.1)$$

by the projective principle on a combinatorially Riemannian manifold $(\widetilde{M}, g, \widetilde{D})$. For a given point $p \in \widetilde{M}$, we get $s(p)$ tensor equations

$$\widetilde{R}_{(\mu\nu)(\mu\lambda)} - \frac{1}{2}g_{(\mu\nu)(\mu\lambda)}\mathbf{R} = 0, \ 1 \leq \mu \leq s(p) \quad (5.2)$$

as these usual vacuum *Einstein's* equations in classical gravitational field, where $1 \leq \nu, \lambda \leq n_\mu$. For line elements in $\widetilde{M}$, the next result is easily obtained.

Theorem 5.1 *If each line element* ds_μ *is uniquely determined by equations* (5.2), *Then* $\widetilde{ds}$ *is uniquely determined in* $\widetilde{M}$.

Proof For a given index μ, let

$$ds_\mu^2 = \sum_{i=1}^{n_\mu} a_{\mu i}^2 dx_{\mu i}^2.$$

Then we know that

$$\widetilde{ds}^2 = \sum_{i=1}^{\widehat{s}(p)} (\sum_{\mu=1}^{s(p)} a_{\mu i})^2 dx_{\mu i}^2 + \sum_{\mu=1}^{s(p)} \sum_{i=\widehat{s}(p)+1}^{n_\mu} a_{\mu i}^2 dx_{\mu i}^2.$$

Therefore, the line element $\widetilde{ds}$ is uniquely determined in $\widetilde{M}$ if $ds_{\mu i}$ is uniquely determined by (5.2). $\square$.

We consider a special case for these *Einstein's* gravitational equations (5.1), solutions of combinatorially Euclidean spaces $\widetilde{M} = \bigcup_{i=1}^{m} \mathbf{R}^{n_i}$ with a matrix ([11])

$$[\overline{x}] = \begin{bmatrix} x^{11} & \cdots & x^{1\widehat{m}} & x^{1(\widehat{m})+1)} & \cdots & x^{1n_1} & \cdots & 0 \\ x^{21} & \cdots & x^{2\widehat{m}} & x^{2(\widehat{m}+1)} & \cdots & x^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ x^{m1} & \cdots & x^{m\widehat{m}} & x^{m(\widehat{m}+1)} & \cdots & \cdots & x^{mn_m-1} & x^{mn_m} \end{bmatrix}$$

for any point $\overline{x} \in \widetilde{M}$, where $\widehat{m} = \dim(\bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i})$ is a constant for $\forall p \in \bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i}$ and $x^{il} = \frac{x^l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$. In this case, we have a unifying solution for these equations (5.1), i.e.,

$$\widetilde{ds}^2 = \sum_{i=1}^{\widehat{m}} (\sum_{\mu=1}^{m} a_{\mu i})^2 dx_{\mu i}^2 + \sum_{\mu=1}^{m} \sum_{i=\widehat{m}+1}^{n_\mu} a_{\mu i}^2 dx_{\mu i}^2$$

for each point $p \in \widetilde{M}$ by Theorem 5.1.

For usually undergoing, we consider the case of $n_\mu = 4$ for $1 \le \mu \le m$ since line elements have been found concretely in classical gravitational field in these cases. Now establish m spherical coordinate subframe $(t_\mu; r_\mu, \theta_\mu, \phi_\mu)$ with its originality at the center of the mass space. Then we have known its a spherically symmetric solution for the line element ds_μ with a given index μ by *Schwarzschild* (see also [3]) for (5.2) to be

$$ds_\mu^2 = (1 - \frac{r_{\mu s}}{r_\mu})c^2 dt_\mu^2 - (1 - \frac{r_{\mu s}}{r_\mu})^{-1} dr_\mu^2 - r_\mu^2 (d\theta_\mu^2 + \sin^2\theta_\mu d\phi_\mu^2).$$

for $1 \le \mu \le m$, where $r_{\mu s} = 2Gm_\mu/c^2$. Applying Theorem 5.1, the line element $\widetilde{ds}$ in $\widetilde{M}$ is

$$\widetilde{ds} = (\sum_{\mu=1}^{m} \sqrt{1 - \frac{r_{\mu s}}{r_\mu}})^2 c^2 dt^2 - \sum_{\mu=1}^{m} (1 - \frac{r_{\mu s}}{r_\mu})^{-1} dr_\mu^2 - \sum_{\mu=1}^{m} r_\mu^2 (d\theta_\mu^2 + \sin^2\theta_\mu d\phi_\mu^2)$$

if $\widehat{m} = 1$, $t_\mu = t$ for $1 \le \mu \le m$ and

$$\widetilde{ds} = (\sum_{\mu=1}^{m} \sqrt{1 - \frac{r_{\mu s}}{r_\mu}})^2 c^2 dt^2 - (\sum_{\mu=1}^{m} \sqrt{(1 - \frac{r_{\mu s}}{r_\mu})^{-1}})^2 dr^2 - \sum_{\mu=1}^{m} r_\mu^2 (d\theta_\mu^2 + \sin^2\theta_\mu d\phi_\mu^2)$$

if $\widehat{m} = 2$, $t_\mu = t, r_\mu = r$ for $1 \le \mu \le m$ and

$$\widetilde{ds} = (\sum_{\mu=1}^{m} \sqrt{1 - \frac{r_{\mu s}}{r_\mu}})^2 c^2 dt^2 - (\sum_{\mu=1}^{m} \sqrt{(1 - \frac{r_{\mu s}}{r_\mu})^{-1}})^2 dr^2 - m^2 r^2 d\theta^2 - \sum_{\mu=1}^{m} r_\mu^2 \sin^2\theta_\mu d\phi_\mu^2$$

if $\widehat{m} = 3$, $t_\mu = t, r_\mu = r, \theta_\mu = \theta$ for $1 \le \mu \le m$ and

$$\widetilde{ds} = (\sum_{\mu=1}^{m} \sqrt{1 - \frac{r_{\mu s}}{r_\mu}})^2 c^2 dt^2 - (\sum_{\mu=1}^{m} \sqrt{(1 - \frac{r_{\mu s}}{r_\mu})^{-1}})^2 dr^2 - m^2 r^2 d\theta^2 - m^2 r^2 \sin^2\theta d\phi^2$$

if $\widehat{m} = 4$, $t_\mu = t, r_\mu = r, \theta_\mu = \theta$ and $\phi_\mu = \phi$ for $1 \le \mu \le m$.

For another interesting case, let $\widehat{m} = 3, r_\mu = r, \theta_\mu = \theta, \phi_\mu = \phi$ and

$$d\Omega^2(r, \theta, \phi) = (1 - \frac{r_s}{r})^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2).$$

Then we can choose a multi-time system $\{t_1, t_2, \cdots, t_m\}$ to get a cosmic model of $m, m \ge 2$ combinatorially $\mathbf{R}^4$ spaces with line elements

$$ds_1^2 = -c^2 dt_1^2 + a^2(t_1) d\Omega^2(r, \theta, \phi),$$

$$ds_2^2 = -c^2 dt_2^2 + a^2(t_2) d\Omega^2(r, \theta, \phi),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$

$$ds_m^2 = -c^2 dt_m^2 + a^2(t_m)d\Omega^2(r,\theta,\phi).$$

In this case, the line element $\widetilde{ds}$ is

$$\widetilde{ds} = \sum_{\mu=1}^{m}(1-\frac{r_{\mu s}}{r_\mu})c^2dt_\mu^2 - (\sum_{\mu=1}^{m}\sqrt{(1-\frac{r_{\mu s}}{r_\mu})^{-1}})^2dr^2 - m^2r^2d\theta^2 - m^2r^2\sin^2\theta d\phi^2.$$

As a by-product for our universe $\mathbf{R}^3$, these formulas mean that these beings with time notion different from human being will recognize differently the structure of our universe if these beings are intellectual enough for the structure of the universe.

References

[1] R.Abraham and J.E.Marsden, *Foundation of Mechanics*(2nd edition), Addison-Wesley, Reading, Mass, 1978.

[2] G.Bassini and S.Capozziello, Multi-spaces and many worlds from conservation laws, *Progress in Physics*, Vol.4(2006), 65-72.

[3] M.Carmeli, *Classical Fields–General Relativity and Gauge Theory*, World Scientific, 2001.

[4] W.H.Chern and X.X.Li, *Introduction to Riemannian Geometry*, Peking University Press, 2002.

[5] L.Kuciuk and M.Antholy, An Introduction to Smarandache Geometries, *JP Journal of Geometry and Topology*, 5(1), 2005,77-81.

[6] L.F.Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005.

[7] L.F.Mao, *Smarandache multi-space theory*, Hexis, Phoenix, AZ2006.

[8] L.F.Mao, Combinatorial speculation and the combinatorial conjecture for mathematics, *Selected Papers on Mathematical Combinatorics*(I), 1-22, World Academic Union, 2006.

[9] L.F.Mao, Geometrical theory on combinatorial manifolds, *JP J.Geometry and Topology*, Vol.7, No.1(2007),65-114.

[10] L.F.Mao, An introduction to Smarandache multi-spaces and mathematical combinatorics, *Scientia Magna*, Vol.3, No.1(2007), 54-80.

[11] L.F.Mao, A combinatorially generalized Stokes' theorem on integrations, *International J.Mathematical Combinatorics*, Vol.1, No.1(2007), 1-26.

[12] F.Smarandache, Mixed noneuclidean geometries, *eprint arXiv: math.GM/0010119*, 10/2000.

[13] F.Smarandache, A unifying field in logic: neutrosophic field, *Multi-value Logic*, Vol.8, No.3(2002), 385-438.

[14] M.Tegmark, Parallel Universes, in *Science and Ultimate Reality: From Quantum to Cosmos*, ed. by J.D.Barrow, P.C.W.Davies and C.L.Harper, Cambridge University Press, 2003.

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A Pair of Smarandachely Isotopic Quasigroups and Loops of the Same Variety

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Abstract: The isotopic invariance or universality of types and varieties of quasigroups and loops described by one or more equivalent identities has been of interest to researchers in loop theory in the recent past. A variety of quasigroups(loops) that are not universal have been found to be isotopic invariant relative to a special type of isotopism or the other. Presently, there are two outstanding open problems on universality of loops: semi-automorphic inverse property loops(1999) and Osborn loops(2005). Smarandache isotopism(S-isotopism) was originally introduced by Vasantha Kandasamy in 2002. But in this work, the concept is restructured in order to make it more explorable. As a result of this, the theory of Smarandache isotopy inherits the open problems as highlighted above for isotopy. In this paper, the question: *Under what type of S-isotopism will a pair of S-quasigroups(S-loops) form any variety?* is answered by presenting a pair of specially Smarandachely isotopic quasigroups(loops) that both belong to the same variety of S-quasigroups(S-loops). This is important because pairs of specially Smarandachely isotopic S-quasigroups(e.g Smarandache cross inverse property quasigroups) that are of the same variety are useful for applications, for example, to cryptography.

Key words: Smarandache holomorph, S-isotopism, variety of S-quasigroups (S-loops).

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§1. Introduction

1.1 Isotopy theory of quasigroups and loops

The isotopic invariance of types and varieties of quasigroups and loops described by one or more equivalent identities, especially those that fall in the class of Bol-Moufang type loops as first named by Fenyves [16]-[17] in the 1960s and later on in this 21^{st} century by Phillips and Vojtěchovský [35], [36] and [39] have been of interest to researchers in loop theory in the recent past. Among such is Etta Falconer's Ph.D [14] and her paper [15] which investigated isotopy invariants in quasigroups. Loops such as Bol loops, Moufang loops, central loops and extra loops are the most popular loops of Bol-Moufang type whose isotopic invariance have been

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considered. For more on loops and their properties, readers should check [34], [8], [11], [13], [18] and [40].

Bol-Moufang type of quasigroups(loops) are not the only quasigroups(loops) that are isomorphic invariant and whose universality have been considered. Some others are flexible loops, F-quasigroups, totally symmetric quasigroups(TSQ), distributive quasigroups, weak inverse property loops(WIPLs), cross inverse property loops(CIPLs), semi-automorphic inverse property loops(SAIPLs) and inverse property loops(IPLs). As shown in Bruck [34], a left(right) inverse property loop is universal if and only if it is a left(right) Bol loop, so an IPL is universal if and only if it is a Moufang loop. Jaíyéọlá [20] investigated the universality of central loops. Recently, Michael Kinyon et. al. in [25]-[27] solved the Belousov problem concerning the universality of F-quasigroup which has been open since 1967. The universality of WIPLs and CIPLs have been addressed by OSborn [32] and Artzy [2] respectively while the universality of elasticity (flexibility) was studied by Syrbu [39]. In 1970, Basarab [4] later continued the work of J. M. Osborn of 1961 on universal WIPLs by studying isotopes of WIPLs that are also WIPLs after he had studied a class of WIPLs ([3]) in 1967. The universality of SAIPLs is still an open problem to be solved as stated by Michael Kinyon during the LOOPs'99 conference. After the consideration of universal AIPLs by Karklinsh and Klin [24], Basarab [6] obtained a sufficient condition for which a universal AIPL is a G-loop. Although Basarab in [5], [7] considered universal Osborn loops but the universality of Osborn loops was raised as an open problem by Michael Kinyon in [28]. Up to the present moment, this problem is still open.

Interestingly, Adeniran [1] and Robinson [37], Oyebo and Adeniran [33], Chiboka and Solarin [12], Bruck [9], Bruck and Paige [10], Robinson [38], Huthnance [19] and Adeniran [1] have respectively studied the holomorphs of Bol loops, central loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops, Osborn loops and Bruck loops. Huthnance showed that if $(L,\cdot)$ is a loop with holomorph $(H,\circ)$, $(L,\cdot)$ is a WIPL if and only if $(H,\circ)$ is a WIPL in [19]. The holomorphs of an AIPL and a CIPL are yet to be studied.

1.2 Isotopy theory of Smarandache quasigroups and loops

The study of Smarandache loops was initiated by W.B. Vasantha Kandasamy in 2002. In her book [40], she defined a Smarandache loop(S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. In that book, she introduced over 75 Smarandache concepts on loops. In [41], she introduced Smarandachely left (right) alternative loops, S-Bol loops, S-Moufang loops, and S-Bruck loops. Similarly, in Jaíyéọlá [21], these conceptions Smarandachely inverse property loops (IPL), Smarandachely weak inverse property loops (WIPL), G-loops, Smarandachely conjugacy closed loops (CC-loop), Smarandachely central loops, extra loops, Smarandachely A-loops, Smarandachely K-loops, Smarandachely Bruck loops, Smarandachely Kikkawa loops, Smarandachely Burn loops and homogeneous loops were also introduced and studied relative to the holomorphs of loops. It is particularly established that a loop is a Smarandache loop if and only if its holomorph is a Smarandache loop. This statement was also shown to be true for some weak Smarandache loops(inverse property, weak inverse property) but false for others(conjugacy closed, Bol, central, extra, Burn, A-, homogeneous) except if their holomorphs are nuclear or central. The study of Smarandache quasigroups was carried out in Jaíyéọlá [22] after the introduction in Muktibodh [30]-[31]. In Jaíyéọlá [23], the universality of some Smarandache loops of Bol-Moufang types was studied and some necessary and sufficient conditions for their universality were established.

In this paper, the question: *Under what type of S-isotopism will a pair of S-quasigroups(S-loops) form any variety?* is answered by presenting a pair of specially Smarandachely isotopic quasigroups (loops), abbreviated to *S-isotopic quasigroups (loops)* that both belong to the same variety of S-quasigroups(S-loops). This fact is important because pairs of specially S-isotopic quasigroups, e.g Smarandache cross inverse property quasigroups that are of the same variety are useful for applications, for example, to cryptography.

§2. Definitions and Notations

Definition 2.1 *Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x\cdot y\in L$, for $\forall\ x,y\in L$, $(L,\cdot)$ is called a groupoid. If the equation system $a\cdot x=b$ and $y\cdot a=b$ have a unique solution x and y for a given $a,b\in L$, then $(L,\cdot)$ is called a quasigroup. Furthermore, if there exists a unique element $e\in L$ called the identity element such that $\forall\ x\in L$, $x\cdot e=e\cdot x=x$, $(L,\cdot)$ is called a loop.*

If there exists at least a non-empty and non-trivial subset M of a groupoid (quasigroup or semigroup or loop) L such that $(M,\cdot)$ is a non-trivial subsemigroup (subgroup or subgroup or subgroup) of $(L,\cdot)$, then L is called a S-groupoid, or S-quasigroup, or S-semigroup, or S-loop with S-subsemigroup, or S-subgroup, or S-subgroup, or S-subgroup M.

A quasigroup (loop) is called a Smarandachely certain quasigroup (loop) if it has at least a non-trivial subquasigroup (subloop) with the certain property and the later is referred to as the Smarandachely certain subquasigroup (subloop). For example, a loop is called a Smarandachely Bol-loop if it has at least a non-trivial subloop that is a Bol-loop and the later is referred to as the Smarandachely Bol-subloop. By an initial S-quasigroup L with an initial S-subquasigroup L', we mean that L and L' are purely quasigroups, i.e., they do not obey a certain property (not of any variety).

Let $(G,\cdot)$ be a quasigroup(loop). The bijections $L_x:G\to G$ and $R_x:G\to G$ defined by $yL_x=x\cdot y$or $yR_x=y\cdot x$ for $\forall\ x,y\in G$ is called a left (right) translation of G.

The set $SYM(L,\cdot)=SYM(L)$ of all bijections in a groupoid $(L,\cdot)$ forms a group called the permutation(symmetric) group of the groupoid $(L,\cdot)$. If L is a S-groupoid with a S-subsemigroup H, then the set $SSYM(L,\cdot)=SSYM(L)$ of all bijections A in L such that $A\ :\ H\to H$ forms a group called the Smarandachely permutation (symmetric) group of the S-groupoid. In fact, $SSYM(L)\leq SYM(L)$.

Definition 2.2 *If $(L,\cdot)$ and $(G,\circ)$ are two distinct groupoids, then the triple $(U,V,W):(L,\cdot)\to(G,\circ)$ such that $U,V,W:L\to G$ are bijections is called an isotopism if and only if*

$$xU\circ yV=(x\cdot y)W,\ \ for\ \forall\ x,y\in L.$$

So we call L and G groupoid isotopes.

If $U=V=W$, then U is called an isomorphism, hence we write $(L,\cdot)\cong(G,\circ)$.

Now, if $(L,\cdot)$ and $(G,\circ)$ are S-groupoids with S-subsemigroups L' and G' respectively such that $A\ :\ L'\to G'$, where $A\in\{U,V,W\}$, then the isotopism $(U,V,W):(L,\cdot)\to(G,\circ)$ is called a Smarandache isotopism(S-isotopism).

Thus, if $U=V=W$, then U is called a Smarandache isomorphism, hence we write $(L,\cdot)\succsim(G,\circ)$.

If $(L,\cdot)=(G,\circ)$, then the triple $\alpha=(U,V,W)$ of bijections on $(L,\cdot)$ is called an autotopism of the groupoid(quasigroup, loop) $(L,\cdot)$. Such triples form a group $AUT(L,\cdot)$ called the autotopism group of $(L,\cdot)$. Furthermore, if $U=V=W$, then U is called an automorphism of the groupoid(quasigroup, loop) $(L,\cdot)$. Such bijections form a group $AUM(L,\cdot)$ called the automorphism group of $(L,\cdot)$.

Similarly, if $(L,\cdot)$ is an S-groupoid with S-subsemigroup L' such that $A\in\{U,V,W\}$ is a Smarandache permutation, then the autotopism (U,V,W) is called a Smarandache autotopism (S-autotopism) and they form a group $SAUT(L,\cdot)$ which will be called the Smarandache autotopism group of $(L,\cdot)$. Observe that $SAUT(L,\cdot)\le AUT(L,\cdot)$.

Discussions To be more precise about the notion of S-isotopism in Definition 2.2, the following explanations are given. For a given S-groupoid, the S-subsemigroup is arbitrary. But in the proofs, we make use of one arbitrary S-subsemigroup for an S-groupoid at a time for our arguments. Now, if $(L,\cdot)$ and $(G,\circ)$ are S-isotopic groupoids with arbitrary S-subsemigroups L' and G' respectively under the triple (U,V,W). In case the S-subsemigroup L' of the S-groupoid L is replaced with another S-groupoid L'' of L(i.e a situation where by L has at least two S-subsemigroups), then under the same S-isotopism (U,V,W), the S-groupoid isotope G has a second S-subsemigroups G''. Hence, when studying the S-isotopism (U,V,W), it will be for the system

$$\{(L,\cdot),(L',\cdot)\}\to\{(G,\circ),(G',\circ)\}\text{ or }\{(L,\cdot),(L'',\cdot)\}\to\{(G,\circ),(G'',\circ)\}$$

and not

$$\{(L,\cdot),(L',\cdot)\}\to\{(G,\circ),(G'',\circ)\}\text{ or }\{(L,\cdot),(L'',\cdot)\}\to\{(G,\circ),(G',\circ)\}.$$

This is because $|L'|=|G'|$ and $|L''|=|G''|$ since $(L')A=G'$ and $(L'')A=G''$ for all $A\in\{U,V,W\}$ while it is not compulsory that $|L'|=|G''|$ and $|L''|=|G'|$. It is very easy to see that from the definition the component transformations U,V,W of isotopy after restricting them to the S-subsemigroup or S-subgroup L' are bijections. Let $x_1,x_2\in L'$, then $x_1A=x_2A$ implies that $x_1=x_2$ because $x_1,x_2\in L'$ implies $x_1,x_2\in L$, hence $x_1A=x_2A$ in L implies $x_1=x_2$. The mappings $A\ :L'\to G'$ and $A\ :L-L'\to G-G'$ are bijections because $A\ :L\to G$ is a bijection. Our explanations above are illustrated with the following examples.

Example 2.1 The systems $(L,\cdot)$ and $(L,*)$, with the multiplication shown in tables below are S-quasigroups with S-subgroups $(L',\cdot)$ and $(L'',*)$ respectively, where $L=\{0,1,2,3,4\}$, $L'=\{0,1\}$ and $L''=\{1,2\}$. Here, $(L,\cdot)$ is taken from Example 2.2 in [31]. The triple (U,V,W) such that

$$U=\begin{pmatrix}0&1&2&3&4\\1&2&3&4&0\end{pmatrix},\ V=\begin{pmatrix}0&1&2&3&4\\1&2&4&0&3\end{pmatrix}\text{ and }W=\begin{pmatrix}0&1&2&3&4\\1&2&0&4&3\end{pmatrix}$$

are permutations on L, is an S-isotopism of $(L, \cdot)$ onto $(L, *)$. Notice that $A(L') = L''$ for all $A \in \{U, V, W\}$ and $U, V, W : L' \to L''$ are all bijcetions.

$\cdot$	0	1	2	3	4
0	0	1	3	4	2
1	1	0	2	3	4
2	3	4	1	2	0
3	4	2	0	1	3
4	2	3	4	0	1

$*$	0	1	2	3	4
0	1	0	4	2	3
1	3	1	2	0	4
2	4	2	1	3	0
3	0	4	3	1	2
4	2	3	0	4	1

Example 2.2 According Example 4.2.2 in [43], the system $(\mathbb{Z}_6, \times_6)$ i.e the set $L = \mathbb{Z}_6$ under multiplication modulo 6 is an S-semigroup with S-subgroups $(L', \times_6)$ and $(L'', \times_6)$, where $L' = \{2, 4\}$ and $L'' = \{1, 5\}$. This can be deduced from its multiplication table below. The triple (U, V, W) such that

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 & 0 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 5 & 4 & 2 & 3 \end{pmatrix}$$

are permutations on L, is an S-isotopism of $(\mathbb{Z}_6, \times_6)$ unto an S-semigroup $(\mathbb{Z}_6, *)$ with S-subgroups $(L''', *)$ and $(L'''', *)$, where $L''' = \{2, 5\}$ and $L'''' = \{0, 3\}$ as shown in the second table below. Notice that $A(L') = L'''$ and $A(L'') = L''''$ for all $A \in \{U, V, W\}$ and $U, V, W : L' \to L'''$ and $U, V, W : L'' \to L''''$ are all bijections.

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	1	1	4	4	1
2	5	1	5	2	1	2
3	3	1	5	0	4	2
4	1	1	1	1	1	1
5	2	1	2	5	1	5

From Example 2.1 and Example 2.2, it is very clear that the study of S-isotopy of two S-groupoids, or S-quasigroups, or S-semigroups, or S-loops is independent of the S-subsemigroup or S-subgroup that is in consideration. All results in this paper are true for any given S-subsemigroups or S-subgroups of two S-isotopic groupoids, or S-quasigroups, or S-semigroups, or S-loops. More examples of S-isotopic groupoids can be constructed by using S-groupoids in [42].

Remark 2.1 Taking careful look at Definition 2.2 and comparing it with Definition 4.4.1 in [40], it will be observed that the author did not allow the component bijections U,V and W

in (U, V, W) to act on the whole S-loop L but only on the S-subloop(S-subgroup) L'. We feel this is necessary to adjust here so that the set $L - L'$ is not out of the study. Apart from this, our adjustment here will allow the study of Smarandache isotopy to be explorable. Therefore, the S-isotopism and S-isomorphism here are clearly special types of relations(isotopism and isomorphism) on the whole domain into the whole co-domain but those of Vasantha Kandasamy [40] only take care of the structure of the elements in the S-subloop and not the S-loop.

For each loop $(L, \cdot)$ with automorphism group $AUM(L, \cdot)$, there corresponds another loop. Let the set $H = (L, \cdot) \times AUM(L, \cdot)$. If we define '$\circ$' on H such that $(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y)$ for all $(\alpha, x), (\beta, y) \in H$, then $H(L, \cdot) = (H, \circ)$ is a loop as shown in Bruck [9] and is called the Holomorph of $(L, \cdot)$. Let $(L, \cdot)$ be an S-quasigroup(S-loop) with S-subgroup $(L', \cdot)$. Define the Smarandache automorphism of L to be the set $SAUM(L) = SAUM(L, \cdot) = \{\alpha \in AUM(L) | \alpha : L' \to L'\}$. It is easy to see that $SAUM(L) \leq AUM(L)$. $SAUM(L)$ will be called a *Smarandachely automorphism group(SAG) of L*. Now, let $H_S = (L, \cdot) \times SAUM(L, \cdot)$. If we define '$\circ$' on H_S such that $(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y)$ for all $(\alpha, x), (\beta, y) \in H_S$, then $H_S(L, \cdot) = (H_S, \circ)$ is a S-quasigroup(S-loop) with S-subgroup $(H', \circ)$ where $H' = L' \times SAUM(L)$ and thus will be called a *Smarandache Holomorph(SH) of* $(L, \cdot)$.

§3. Main Results

Theorem 3.1 *Let $U = (L, \oplus)$ and $V = (L, \otimes)$ be initial S-quasigroups such that $SAUM(U)$ and $SAUM(V)$ are conjugates in $SSYM(L)$ i.e., there exists a $\psi \in SSYM(L)$ such that for any $\gamma \in SAUM(V)$, $\gamma = \psi^{-1}\alpha\psi$ where $\alpha \in SAUM(U)$. Then, $H_S(U) \succsim H_S(V)$ if and only if $x\delta \otimes y\gamma = (x\beta \oplus y)\delta$ for $\forall\ x, y \in L$, $\beta \in SAUM(U)$ and some $\delta, \gamma \in SAUM(V)$. Hence,*

(1) *$\gamma \in SAUM(U)$ if and only if $(I, \gamma, \delta) \in SAUT(V)$.*

(2) *if U is a initial S-loop, then,*
$(a)\mathcal{L}_{e\delta} \in SAUM(V)$;
$(b)\beta \in SAUM(V)$ if and only if $\mathcal{R}_{e\gamma} \in SAUM(V)$,
where e is the identity element in U and $\mathcal{L}_x$, $\mathcal{R}_x$ are respectively the left and right translations mappings of $x \in V$.

(3) *if $\delta = I$, then $|SAUM(U)| = |SAUM(V)| = 3$ and so $SAUM(U)$ and $SAUM(V)$ are Boolean groups.*

(4) *if $\gamma = I$, then $|SAUM(U)| = |SAUM(V)| = 1$.*

Proof Let $H_S(L, \oplus) = (H_S, \circ)$ and $H_S(L, \otimes) = (H_S, \odot)$. $H_S(U) \succsim H_S(V)$ if and only if there exists a bijection $\phi\ :\ H_S(U) \to H_S(V)$ such that $[(\alpha, x) \circ (\beta, y)]\phi = (\alpha, x)\phi \odot (\beta, y)\phi$ and $(H', \oplus) \overset{\phi}{\cong} (H'', \otimes)$, where $H' = L' \times SAUM(U)$ and $H'' = L'' \times SAUM(V)$, $(L', \oplus)$ and $(L'', \otimes)$ are initial S-subquasigroups of U and V. Define $(\alpha, x)\phi = (\psi^{-1}\alpha\psi, x\psi^{-1}\alpha\psi)\ \forall\ (\alpha, x) \in (H_S, \circ)$ where $\psi \in SSYM(L)$. Then we find that

$H_S(U) \cong H_S(V) \Leftrightarrow (\alpha\beta, x\beta \oplus y)\phi = (\psi^{-1}\alpha\psi, x\psi^{-1}\alpha\psi) \odot (\psi^{-1}\beta\psi, y\psi^{-1}\beta\psi) \Leftrightarrow (\psi^{-1}\alpha\beta\psi, (x\beta \oplus y)\psi^{-1}\alpha\beta\psi) = (\psi^{-1}\alpha\beta\psi, x\psi^{-1}\alpha\beta\psi \otimes y\psi^{-1}\beta\psi) \Leftrightarrow (x\beta \oplus y)\psi^{-1}\alpha\beta\psi = x\psi^{-1}\alpha\beta\psi \otimes y\psi^{-1}\beta\psi \Leftrightarrow x\delta \otimes y\gamma = (x\beta \oplus y)\delta$ where $\delta = \psi^{-1}\alpha\beta\psi$, $\gamma = \psi^{-1}\beta\psi$.

Notice that $\gamma\mathcal{L}_{x\delta} = L_{x\beta}\delta$ and $\delta\mathcal{R}_{y\gamma} = \beta R_y\delta\ \forall\ x, y \in L$. So, when U is an S-loop, $\gamma\mathcal{L}_{e\delta} = \delta$ and $\delta\mathcal{R}_{e\gamma} = \beta\delta$. These can easily be used to prove the remaining part of this theorem. □

Theorem 3.2 *Let $\mathfrak{F}$ be any class of variety of S-quasigroups(loops). Let $U = (L, \oplus)$ and $V = (L, \otimes)$ be initial S-quasigroups(S-loops) that are S-isotopic under the triple of the form $(\delta^{-1}\beta, \gamma^{-1}, \delta^{-1})$ for all $\beta \in SAUM(U)$ and some $\delta, \gamma \in SAUM(V)$ such that their SAGs are non-trivial and are conjugates in $SSYM(L)$ i.e there exists a $\psi \in SSYM(L)$ such that for any $\gamma \in SAUM(V)$, $\gamma = \psi^{-1}\alpha\psi$ where $\alpha \in SAUM(U)$. Then, $U \in \mathfrak{F}$ if and only if $V \in \mathfrak{F}$.*

Proof By Theorem 3.1, we have known that $H_S(U) \cong H_S(V)$. Let $U \in \mathfrak{F}$, then since $H(U)$ has an initial S-subquasigroup(S-subloop) that is isomorphic to U and that initial S-subquasigroup(S-subloop) is isomorphic to an S-subquasigroup(S-subloop) of $H(V)$ which is isomorphic to V, $V \in \mathfrak{F}$. The proof for the converse is similar. □

References

[1] J. O. Adeniran, On holomorphic theory of a class of left Bol loops, *Al.I.Cuza*, Vol.51, 1(2005), 23-28.

[2] R. Artzy, Crossed inverse and related loops, *Trans. Amer. Math. Soc.*, Vol. 91, 3(1959), 480-492.

[3] A. S. Basarab, A class of WIP-loops, *Mat. Issled*, Vol.2, 2(1967), 3-24.

[4] A. S. Basarab, Isotopy of WIP loops, *Mat. Issled*, Vol.5, 2(1970), 3-12.

[5] A. S. Basarab, The Osborn loop, *Studies in the theory of quasigroups and loops*, 193 (1973), Shtiintsa, Kishinev, 12-18.

[6] A. S. Basarab and A. I. Belioglo, Universal automorphic inverse G-loops, *Quasigroups and loops, Mat. Issled*, Vol. 71(1979), 3-7.

[7] A. S. Basarab and A. I. Belioglo, UAI Osborn loops, *Quasigroups and loops, Mat. Issled*, Vol.51(1979), 8-16.

[8] R. H. Bruck, *A survey of binary systems*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1966, 185pp.

[9] R. H. Bruck , Contributions to the theory of loops, *Trans. Amer. Math. Soc.*, Vol.55(1944), 245-354.

[10] R. H. Bruck and L. J. Paige , Loops whose inner mappings are automorphisms, *The Annuals of Mathematics*, Vol.63, 2(1956), 308-323.

[11] O. Chein, H. O. Pflugfelder and J. D. H. Smith, *Quasigroups and loops : Theory and applications*, Heldermann Verlag, 1990, 568pp.

[12] V. O. Chiboka and A. R. T. Solarin, Holomorphs of conjugacy closed loops, *Scientific Annals of Al.I.Cuza. Univ.*, Vol.37, 3(1991), 277-284.

[13] J. Déne and A. D. Keedwell, *Latin squares and their applications*, the English University press Lts, 1974, 549pp.

[14] E. Falconer, *Quasigroup identities invariant under isotopy*, Ph.D thesis, Emory University, 1969.

[15] E. Falconer, Isotopy invariants in quasigroups, *Trans. Amer. Math. Soc.*, Vol.151, 2(1970), 511-526.

[16] F. Fenyves , Extra loops I, *Publ. Math. Debrecen*, Vol.15(1968), 235-238.

[17] F. Fenyves, Extra loops II, *Publ. Math. Debrecen*, Vol.16(1969), 187-192.

[18] E. G. Goodaire, E. Jespers and C. P. Milies , *Alternative loop rings*, NHMS(184), Elsevier, 1996, 387pp.

[19] E. D. Huthnance Jr., *A theory of generalised Moufang loops*, Ph.D. thesis, Georgia Institute of Technology, 1968.

20 T. G. Jaíyéọlá, *An isotopic study of properties of central loops*, M.Sc. dissertation, University of Agriculture, Abeokuta, 2005.

[21] T. G. Jaíyéọlá, An holomorphic study of the Smarandache concept in loops, *Scientia Magna Journal*, Vol.2, 1 (2006), 1-8.

[22] T. G. Jaíyéọlá , Parastrophic invariance of Smarandache quasigroups, *Scientia Magna Journal*, Vol.2, 3(2006), 48-53.

[23] T. G. Jaíyéọlá, On the universality of some Smarandache loops of Bol- Moufang type, *Scientia Magna Journal*, Vol.2, 4(2006), 45-48.

[24] B. B. Karklinsh and V. B. Klin, Universal automorphic inverse property loops, *Collections: sets and quasigroups*, Shtintsa, 1976.

[25] T. Kepka, M. K. Kinyon, J. D. Phillips, The structure of F-quasigroups, communicated for publication.

[26] T. Kepka, M. K. Kinyon, J. D. Phillips, F-quasigroups and generalised modules, communicated for publication.

[27] T. Kepka, M. K. Kinyon, J. D. Phillips, F-quasigroups isotopic to groups, communicated for publication.

[28] M. K. Kinyon, A survey of Osborn loops, *Milehigh conference on loops, quasigroups and non-associative systems*, University of Denver, Denver, Colorado, 2005.

[29] M. K. Kinyon, J. D. Phillips and P. Vojtěchovský , Loops of Bol-Moufang type with a subgroup of index two, *Bul. Acad. Stiinte Repub. Mold. Mat.*, Vol.49, 3(2005), 71-87.

[30] A. S. Muktibodh , Smarandache quasigroups rings, *Scientia Magna Journal*, Vol.1, 2(2005), 139-144.

[31] A. S. Muktibodh , Smarandache Quasigroups, *Scientia Magna Journal*, Vol.2, 1(2006), 13-19.

[32] J. M. Osborn, Loops with the weak inverse property, *Pac. J. Math.*, Vol.10(1961), 295-304.

[33] Y. T. Oyebo and O. J. Adeniran, On the holomorph of central loops, Pre-print.

[34] H. O. Pflugfelder, *Quasigroups and loops : Introduction*, Sigma series in Pure Math. 7, Heldermann Verlag, Berlin, 1990, 147pp.

[35] J. D. Phillips and P. Vojtěchovský, The varieties of loops of Bol-Moufang type, *Alg. Univer.*, Vol.54, 3(2005), 259-383.

[36] J. D. Phillips and P. Vojtěchovský, The varieties of quasigroups of Bol-Moufang type : An equational approach, *J. Alg.*, Vol.293(2005), 17-33.

[37] D. A. Robinson, *Bol loops*, Ph. D thesis, University of Wisconsin, Madison, Wisconsin, 1964.

[38] D. A. Robinson, Holomorphic theory of extra loops, *Publ. Math. Debrecen*, Vol.18(1971), 59-64.

[39] P. N. Syrbu, On loops with universal elasticity, *Quasigroups and Related Systems*, Vol.3(1996), 41-54.

[40] W. B. Vasantha Kandasamy, *Smarandache loops*, Department of Mathematics, Indian Institute of Technology, Madras, India, 2002, 128pp.

[41] W. B. Vasantha Kandasamy, Smarandache Loops, *Smarandache Notions Journal*, Vol.13(2002), 252-258.

[42] W. B. Vasantha Kandasamy, *Groupoids and Smarandache Groupoids*, American Research Press Rehoboth, 2002, 114pp.

[43] W. B. Vasantha Kandasamy, *Smarandache Semigroups*, American Research Press Rehoboth, 2002, 94pp.

[44] W. B. Vasantha Kandasamy, *Smarandache Semirings, Semifields, And Semivector Spaces*, American Research Press Rehoboth, 2002, 121pp.

[45] W. B. Vasantha Kandasamy, *Linear Algebra And Smarandache Linear Algebra*, American Research Press, 2003, 174pp.

[46] W. B. Vasantha Kandasamy, *Bialgebraic Structures And Smarandache Bialgebraic Structures*, American Research Press Rehoboth, 2003, 271pp.

[47] W. B. Vasantha Kandasamy and F. Smarandache, *N-Algebraic Structures And Smarandache N-Algebraic Structures*, Hexis Phoenix, Arizona, 2005, 174pp.

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A Revision to Gödel's Incompleteness Theorem by Neutrosophy

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Abstract: According to Smarandache's neutrosophy, the Gödel's incompleteness theorem contains the truth, the falsehood, and the indeterminacy of a statement under consideration. It is shown in this paper that the proof of Gödel's incompleteness theorem is faulty, because all possible situations are not considered (such as the situation where from some axioms wrong results can be deducted, for example, from the axiom of choice the paradox of the doubling ball theorem can be deducted; and many kinds of indeterminate situations, for example, a proposition can be proved in 9999 cases, and only in 1 case it can be neither proved, nor disproved). With all possible situations being considered with Smarandache's neutrosophy, the Gödel's Incompleteness theorem is revised into the incompleteness axiom: Any proposition in any formal mathematical axiom system will represent, respectively, the truth (T), the falsehood (F), and the indeterminacy (I) of the statement under consideration, where T, I, F are standard or non-standard real subsets of $]^-0, 1^+[$. Considering all possible situations, any possible paradox is no longer a paradox. Finally several famous paradoxes in history, as well as the so-called unified theory, ultimate theory, $\cdots$, etc. are discussed.

Key words: Smarandache's Neutrosophy, Gödel's Incompleteness theorem, Incompleteness axiom, paradox, unified theory.

The most celebrated results of Gödel are as follows.

Gödel's First Incompleteness Theorem: *Any adequate axiomatizable theory is incomplete.*

Gödel's Second Incompleteness Theorem: *In any consistent axiomatizable theory which can encode sequences of numbers, the consistency of the system is not provable in the system.*

In literature, the Gödel's incompleteness theorem is usually stated by *any formal mathematical axiom system is incomplete, because it always has one proposition that can neither be proved, nor disproved.*

Gödel's incompleteness theorem is a significant result in the history of mathematical logic, and has greatly influenced to mathematics, physics and philosophy among others. But, any

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theory cannot be the ultimate truth. Accompanying with the science development, new theories will replace the old ones. That is also for the Gödel's incompleteness theorem. This paper will revise the Gödel's Incompleteness theorem into the incompleteness axiom with the Smarandache's neutrosophy.

§1. An Introduction to Smarandache's Neutrosophy

Neutrosophy is proposed by F.Smarandache in 1995. *Neutrosophy* is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea $\langle A \rangle$ together with its opposite or negation $\langle Anti - A \rangle$ and the spectrum of *neutralities* $\langle Neut - A \rangle$, i.e., notions or ideas located between the two extremes, supporting neither $\langle A \rangle$ nor $\langle Anti - A \rangle$). The $\langle Neut - A \rangle$ and $\langle Anti - A \rangle$ ideas together are referred to as $\langle Non - A \rangle$.

Neutrosophy is the base of neutrosophic logic, neutrosophic set, neutrosophic probability and statistics used in engineering applications, especially for software and information fusion, medicine, military, cybernetics and physics, etc..

Neutrosophic Logic is a general framework for unification of existent logics, such as the fuzzy logic, especially intuitionistic fuzzy logic, paraconsistent logic, intuitionistic logic,$\cdots$, etc.. The main idea of Neutrosophic Logic (NL) is to characterize each logical statement in a 3D Neutrosophic Space, where each dimension of the space represents respectively the truth (T), the falsehood (F), and the indeterminacy (I) of the statement under consideration, where T, I, F are standard or non-standard real subsets of $]^{-}0, 1^{+}[$ without necessarily connection between them.

More information on Neutrosophy may be found in references [1-3].

§2. Some Errors in the Proof of Gödel's Incompleteness Theorem

It has been pointed out some errors in the proofs of Gödel's first and second incompleteness theorems in the reference [4]. This paper will again show that the proof of Gödel's incompleteness theorems contain some errors, but from other point of view. It will be shown that in the proof of Gödel's incompleteness theorem, all possible situations are not considered.

First, in the proof, the following situation is not considered: *wrong results can be deduced from some axioms*. For example, from the axiom of choice a paradox, the doubling ball theorem, can be deduced, which says that a ball of volume 1 can be decomposed into pieces and reassembled into two balls both of volume 1. It follows that in certain cases, the proof of Gödel's incompleteness theorem may be faulty.

Second, in the proof of Gödel's incompleteness theorem, only four situations are considered, that is, one proposition can be proved to be true, cannot be proved to be true, can be proved to be false, cannot be proved to be false and their combinations such as one proposition can neither be proved to be true nor be proved to be false. But those are not all possible situations. In fact, there may be many kinds of indeterminate situations, including it can be proved to be true in some cases and cannot be proved to be true in other cases; it can be proved to be false in some cases and cannot be proved to be false in other cases; it can be proved to be true in some cases and can be proved to be false in other cases; it cannot be proved to be true in some cases and cannot be proved to be false in other cases; it can be proved to be true in some cases and can neither be proved to be true, nor be proved to be false in other cases; and so on.

Because so many situations are not considered, we may say that the proof of Gödel's incompleteness theorem is faulty, at least, is not one with all sided considerations.

In order to better understand each case, we consider an extreme situation where one proposition as shown in Gödel's incompleteness theorem can neither be proved, nor disproved. It may be assumed that this proposition can be proved in 9999 cases, only in 1 case it can neither be proved, nor disproved. We will see whether or not this situation has been considered in the proof of Gödel's incompleteness theorem.

Some people may argue that, this situation is equivalent to that of a proposition can neither be proved, nor disproved. But the difference lies in the distinction between the part and the whole. If one case may represent the whole situation, many important theories cannot be applied. For example the general theory of relativity involves singular points; the law of universal gravitation does not allow the case where the distance r is equal to zero. Accordingly, whether or not one may say that the general theory of relativity and the law of universal gravitation cannot be applied as a whole? Similarly, the situation also cannot be considered as the one that can be proved. But, this problem may be easily solved with the neutrosophic method.

Moreover, if we apply the Gödel's incompleteness theorem to itself, we may obtain the following possibility: *in one of all formal mathematical axiom systems, the Gödel's incompleteness theorem can neither be proved, nor disproved.*

If all possible situations can be considered, the Gödel's incompleteness theorem can be improved in principle. But, with our boundless universe being ever changing and being extremely complex, it is impossible considering all possible situations. As far as considering all possible situations is concerned, the Smarandache's neutrosophy is a quite useful way, and possibly the best. Therefore this paper proposes to revise the Gödel's incompleteness theorem into the incomplete axiom with Smarandache's neutrosophy.

§3. The Incompleteness Axiom

Considering all possible situations with Smarandache's neutrosophy, one may revise the Gödel's Incompleteness theorem into the incompleteness axiom following.

Any proposition in any formal mathematical axiom system will represent the truth (T), the falsehood (F), and the indeterminacy (I) of the statement under consideration, where T, I, F are standard or non-standard real subsets of $]^{-}0, 1^{+}[$*, respectively.*

§4. Several Famous Paradoxes in History

The proof of Gödel's incompleteness theorem has a close relation with some paradoxes. However, after considering all possible situations, any paradox may no longer be a paradox.

Now we discuss several famous paradoxes in history.

Example 1. *The Barber paradox, one of Russell's paradoxes.*

Consider all men in a small town as members of a set. Now imagine that a barber puts up a sign in his shop that reads *I shave all those men, and only those men, who do not shave themselves.* Obviously, we may divide the set of men in this town into two subsets, those who shave themselves, and those who are shaved by the barber. To which subset does the barber himself belong? The barber cannot belong to the first subset, because if he shaves himself, he will not be shaved by the barber, or by himself; he cannot not belong to the second subset as well, because if he is really shaved by the barber, or by himself, he will not be shaved by the barber.

Now we will see from where comes the contradiction.

The contradiction comes from the fact that the barber's rule does not take all possible situations into consideration.

First, we should divide the set of men in this town into three subsets, those who shave themselves, those who are shaved by the barber, and those who neither shave themselves, nor are shaved by the barber. This contradiction can be avoided by the neutrosophy as follows. If the barber belongs to the third subset, no contradiction will appear. For this purpose, the barber should declare himself that he will be the third kind of person, and from now on, he will not be shaved by anyone; otherwise, if the barber's mother is not a barber, he can be shaved by his mother.

Second, the barber cannot shave all men in this town. For example, the barber cannot shave those who refuse to be shaved by the barber. Therefore, if the barber is the one who cannot shave himself and "who refuse to be shaved by the barber" , no contradiction will occur.

There also exist indeterminate situations to avoid the contradiction. The barber may say: *If I meet men from another universe, I will shave myself, otherwise I will not shave myself.*

Example 2. *Liar's paradox, another Russell's paradox.*

Epimenides was a Cretan who said that *all Cretans are liars*. Is this statement true or false? If this statement is true, he (a Cretan) is a liar, therefore, this statement is false; if this statement is false, that means that he is not a liar, this statement will be true. Therefore, we always come across a contradiction.

Now we will see from where comes the contradiction.

First, here the term "liar" should be defined. Considering all possible situations, a "liar" can be one of the following categories: those whose statements are all lies; those whose statements are partly lies, and partly truths; those whose statements are partly lies, partly truths and sometimes it is not possible to judge whether they are truths or lies. For the sake of convenience, at this movement we do not consider the situation where it is not possible to judge whether the statements are true or false.

Next, the first kind of liar is impossible, i.e., a Cretan could not be a liar whose statements are all lies. This conclusion can not be reached by deduction, instead, it is obtained through experience and general knowledge. With the situation where a liar's statements are partly truths, and partly lies, Epimenides' statement *all Cretans are liars*, will not cause any contradiction. According to the definitions of liar of the second category and the fact that Epimenides' statements could not be all lies, this particular statement of Epimenides' can be true and with his other statements being possibly lies, Epimenides may still be a liar.

This contradiction can be avoided by the neutrosophy as follows.

For this statement of *all Cretans are liars*, besides true or false, we should consider the situation where it is not possible to judge whether the statement is true or false. According to this situation, this *Russell's paradox* can be avoided.

Example 3. *Dialogue paradox*.

Considering the following dialogue between two persons A and B.

A: *what B says is true.*

B: *what A says is false.*

If the statement of A is true, it follows that the statement of B is true, that is, the statement what A says is false is true, which implies that the statement of A must be false. We come to a contradiction.

On the other hand, if the statement of A is false, it follows that the statement of B must be false, that is, the statement what A says is false is false, which implies that the statement of A must be true. We also come to a contradiction.

So the statement of A could neither be true nor false.

Now we will see that how to solve this contradiction.

It should be noted that, this dialogue poses a serious problem. If A speaks first, before B says anything, how can A know whether or not what B says is true? Otherwise, if B speaks first, B would not know whether what A says is true or false. If A and B speak at the same time, they would not know whether the other's statement is true or false.

For solving this problem, we must define the meaning of *lie*. In general situations a *lie* may be defined as follows:

with the knowledge of the facts of cases, a statement does not show with the facts.

But in order to consider all possible situations, especially those in this dialogue, another definition of lie must be given. For the situation when one does not know the facts of the case, and one makes a statement irresponsibly, can this statement be defined as a lie? There exist two possibilities: *it is a lie, and it is not a lie*. For either possibility, the contradiction can be avoided.

Consider the first possibility, i.e., it is a lie.

If A speaks first, before B makes his statement, it follows that A does not know the facts of the case, and makes the statement irresponsibly, it is a lie. Therefore the statement of A is false. B certainly also knows this point, therefore B's statement: what A says is false is a truth.

Whereas, if B speaks first before A makes his statement, it follows that B does not know the facts of the case, and makes the statement irresponsibly, it is a lie. Therefore the statement of B is false. A certainly also knows this point, therefore A's statement: *what B says is true* is false.

If A and B speak at the same time, it follows that A and B do not know the facts of the case, and make their statements irresponsibly, these statements are all lies. Therefore, the statements of A and B are all false.

Similarly, consider the second possibility, i.e., it is not a lie, the contradiction can be also avoided.

If we do not consider all the above situations, what can we do? With a lie detector! The results of the lie detector can be used to judge whose statement is true, whose statement is false.

§5. On the So-Called Unified Theory, Ultimate Theory and So on

Since Einstein proposed the theory of relativity, the so-called unified theory, ultimate theory and so on have made their appearance.

Not long ago, some scholars pointed out that if the physics really has the unified theory, ultimate theory or theory of everything, the mathematical structure of this theory also is composed by the finite axioms and their deductions. According to the Gödel's incompleteness theorem, there inevitably exists a proposition that cannot be derived by these finite axioms and their deductions. If there is a mathematical proposition that cannot be proved, there must be some physical phenomena that cannot be forecasted. So far all the physical theories are both inconsistent, and incomplete. Thus, the ultimate theory derived by the finite mathematical principles is impossible to be created.

The above discussion is based on the Gödel's incompleteness theorem. With Smarandache's neutrosophy and the incompleteness axiom, the above discussion should be revised.

For example, the proposition *this theory is the ultimate theory* should represent respective the truth (T), the falsehood (F) and the indeterminacy (I) of the statement under consideration, where T, I, F are standard or non-standard real subsets of $]^-0, 1^+[$.

Now we discuss the proposition *Newton's law of gravity is the ultimate theory of gravitation* (Proposition A).

According to the Gödel's incompleteness theorem, the ultimate theory is impossible, therefore, the above proposition is 0% true, 0% indeterminate, and 100% false. It may be written as $(0, 0, 1)$.

While according to the incomplete axiom, we may say that the Proposition A is 16.7% true, 33.3% indeterminate, and 50% false. It may be written as $(0.167, 0.333, 0.500)$. The reason for this sentence is on the following.

Consider the containing relation between the ultimate theory of gravitation and Newton's law of gravity. According to the incompleteness axiom, the proposition the ultimate theory of gravitation contains Newton's law of gravity (Proposition B) should represent respective the truth (T), the falsehood (F) and the indeterminacy (I). For the sake of convenience, we may assume that $T = I = F = 33.3\%$.

If the Proposition B is equivalent to the Proposition A , the Proposition A also is 33.3% true, 33.3% indeterminate, and 33.3% false. But they are not equivalent. Therefore we have to see how the ultimate theory of gravitation contains Newton's law of gravity. As is known, to establish the field equation of the general theory of relativity, one has to do a series of mathematical reasoning according to the principle of general covariance and so on, with Newton's law of gravity as the final basis. Suppose that the ultimate theory of gravitation is similar to the general theory of relativity, it depends upon some principle and Newton's law of gravity. Again this principle and Newton's law of gravity are equally important, they all have the same share of truthfulness, namely 16.7% (one half of 33.3%), but the 16.7% shared by this principle may be added to 33.3% for falsehood. Therefore, the Proposition A is 16.7% true, 33.3% indeterminate, and 50% false. It may be written as $(0.167, 0.333, 0.500)$.

This conclusion indicates that Newton's law of universal gravitation will continue to occupy a proper position in the future gravitational theory.

References

[1] F.Smarandache, *A Unifying Field in Logics: Neutrosophic Logic-Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics*(third edition), Xiquan Publishing House, Phoenix, 2003.

[2] F.Smarandache, V. Christianto, Fu Yuhua, R. Khrapko and J. Hutchison, *Unfolding the Labyrinth: Open Problems in Mathematics, Physics, Astrophysics, and Other Areas of Science*, Hexis, Phoenix, 2006

[3] D. Rabounski, F. Smarandache and L.Borissova, *Neutrosophic Methods in General Relativity*, Hexis, 2005

[4] Zhou Xunwei, *Mutually-Inversistic Mathematical Logic*, Science Publishing Press, Beijing, 2004.

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On the Basis Number and the Minimum Cycle Bases of the Wreath Product of Two Wheels

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Abstract: A construction of a minimum cycle bases for the wreath product of two wheels is presented. Moreover, the basis numbers for the wreath product of the same classes are investigated.

Key Words: Cycle space, basis number, minimum cycle basis, wreath product.

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§1. Introduction.

Cycle bases of a cycle space have a variety of applications which go back at least as far as Kirchoff's treatise on electrical network [20]. The required bases have been used to give rise to a better understanding and interpretations of the geometric properties of a given graph when MacLane [21] made a connection between the planarity of a graph G and the number of occurrence of edges of G in elements of cycle bases. Recently, the minimum cycle bases are employed in sciences and engineering; for examples, in structural flexibility analysis [19], in chemical structure and in retrieval systems [7] and [9].

In this paper, we investigate the basis number for the wreath product of two wheels and we construct minimum cycle bases for same; also, we give their total length and the length of the longest cycles.

§2. Definitions and Preliminaries

Recall that for a given simple graph $G = (V(G), E(G))$ the set $\mathcal{E}$ of all subsets of $E(G)$ forms an $|E(G)|$-dimensional vector space over Z_2 with vector addition $X \oplus Y = (X\backslash Y) \cup (Y \backslash X)$ and scalar multiplication $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for all $X, Y \in \mathcal{E}$. The *cycle space*, $\mathcal{C}(G)$, of a graph G is the vector subspace of $(\mathcal{E}, \oplus, \cdot)$ spanned by the cycles of G. Note that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that the dimension of the cycle space is the *cyclomatic number* or the *first Betti number* $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + r$ where r is the number of components (see [8]).

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A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is *cycle basis* of G. A cycle basis $\mathcal{B}$ of G is called a d-fold if each edge of G occurs in at most d of the cycles in $\mathcal{B}$. *The basis number,* $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a *d-fold* basis. A *required basis* of $\mathcal{C}(G)$ is a $b(G)$-fold basis. The *length* $l(\mathcal{B})$ *of a cycle basis* $\mathcal{B}$ is the sum of the lengths of its elements: $l(\mathcal{B}) = \sum_{C\in\mathcal{B}} |C|$. $\lambda(G)$ is defined to be the minimum length of the longest element in an arbitrary cycle basis of G. A *minimum cycle basis* (MCB) is a cycle basis with minimum length. Since the cycle space $\mathcal{C}(G)$ is a matroid in which an element C has weight $|C|$, the greedy algorithm can be used to extract a MCB (see [24]). Chickering, Geiger and Heckerman [6], showed that $\lambda(G)$ is the length of the longest element in a MCB.

Horton [12] presents a polynomial time algorithm that finds a minimum cycle basis in any graph, but the algorithm approach can lead us to miss deeper connections between the structures of graphs and their cycle bases. Therefore, some authors have directly constructed minimum cycle bases and determined the basis number for certain classes of graphs (see [3], [22] and [23]).

Recently, the study of minimum cycle bases and basis numbers of graph products have attracted many authors: Imrich and Stadler [14], Ali and Marougi [2] and Jaradat [16] have each constructed minimum cycle bases and given upper bounds on the basis number of the Cartesian and strong products. Also, Alsardary and Wojciechowski [4] gave an upper bound on the basis number of the Cartesian products of complete graphs. Hammack [10] constructed a minimum cycle basis of the direct product of two bipartite graphs and Jaradat [15] gave an upper bound on the basis number of the same. Most recently, Hammack [11] presented a minimum cycle basis of the direct product of two complete graphs of order greater than 2. Jaradat [16] and Jaradat and Al-Qeyyam [5] investigated basis numbers and constructed minimum cycle bases for certain classes of graphs.

For completeness, we recall the following definitions: Let G and H be two graphs. Then

(1) the Cartesian product $G\Box H$ is the graph whose vertex set is the Cartesian product $V(G)\times V(H)$ and whose edge set is $E(G\Box H) = \{(u_1,v_1)(u_2,v_2)|u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2\}$.

(2) the lexicographic product $G_1[G_2]$ is the graph with vertex set $V(G)\times V(H)$ and edge set $E(G[H]) = \{(u_1,u_2)(v_1,v_2)|u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_1v_1 \in E(G)\}$ and the wreath product $G \ltimes H$ is the graph with vertex set $V(G)\times V(H)$ and edge set $E(G\rho H) = \{(u_1,v_1)(u_2,v_2)|u_1 = u_2$ and $v_1v_2 \in H$, or $u_1u_2 \in G$ and there is $\alpha\in$Aut(H) such that $\alpha(v_1) = v_2\}$ (see [1] and [13]).

The following results will be used frequently in the sequel.

Theorem 2.1(MacLane [21]) *A graph G is planar if and only if $b(G)\le 2$.*

Lemma 2.2 (Jaradat, et al. [18]) *Let A, B be sets of cycles of a graph G, and suppose that both A and B are linearly independent, and that $E(A)\cap E(B)$ induces a forest in G (we allow the possibility that $E(A)\cap E(B)=\varnothing$). Then $A\cup B$ is linearly independent.*

In this paper, we continue the study initiated in [5] and [17] by investigating the basis number for the wreath products of two wheels W_n and W_m. Moreover, we construct a minimum cycle basis and we give the total lengths and the lengths of longest cycles of the minimum cycle bases of the same.

In the rest of this paper, we let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set of W_n (the star S_n), with $d_{W_n}(u_1) = n-1$ ($d_{S_n}(u_1) = n-1$), and $\{v_1, v_2, \ldots, v_m\}$ be the vertex set W_m (the star S_m), with $d_{W_m}(v_1) = m-1$ ($d_{S_m}(u_1) = m-1$). Wherever they appear a, b, c and l stand for vertices and abc, lab are paths of order 3. Also, $f_B(e)$ stands for the number of elements of B containing the edge e, and $E(B) = \cup_{C\in B} E(C)$ where $B \subseteq \mathcal{C}(G)$.

§3. The Basis Number of $W_n \rho W_m$

In this section, we investigate the basis number of the wreath product of two wheels. Throughout this work we use the notations $\mathcal{V}_{ab}^{(k)}$ and $\mathcal{U}_{lab}^{(k)}$ which were introduced by Jaradat [17] and Al-Qeyyam and Jaradat [5]: For each $k = 1, 2, \ldots, m$,

$$\begin{aligned}\mathcal{V}_{ab}^{(k)} &= \left\{\mathcal{V}_{ab}^{(k,j)} = (b, v_k)(a, v_j)(a, v_{j+1})(b, v_k) \mid 2 \le j \le m-1\right\},\\ \mathcal{U}_{lab}^{(k)} &= \{(l, v_k)(a, v_k)(b, v_k)(l, v_k)\},\end{aligned}$$

and

$$\mathcal{H}_{ab} = \{(a, v_j)(b, v_i)(a, v_{j+1})(b, v_{i+1})(a, v_j) \mid 2 \le i, j \le m-1\}.$$

Note that $\mathcal{H}_{ab}$ is Schemeichel's 4-fold basis of $\mathcal{C}(ab[N_{m-1}])$ (see Theorem 2.4 in [22]). Moreover, (1) if $e = (a, v_2)(b, v_m)$ or $e = (a, v_m)(b, v_2)$ or $e = (a, v_2)(b, v_2)$ or $e = (a, v_m)(b, v_m)$, then $f_{\mathcal{H}_{ab}}(e) = 1$; (2) if $e = (a, v_2)(b, v_l)$ or $(a, v_j)(b, v_2)$ or $(a, v_m)(b, v_l)$ or $(a, v_j)(b, v_m)$, then $f_{\mathcal{H}_{ab}}(e) \le 2$; and (3) If $e \in E(ab[N_{m-1}])$ and is not of the above forms, then $f_{\mathcal{H}_{ab}}(e) \le 4$.

The following result of Jaradat [17] will be needed in the sequel.

Lemma 3.1 **([17])** *$(\cup_{k=2}^{m} \mathcal{V}_{ab}^{(k)}) \cup (\mathcal{V}_{ba}^{(l)})$ is linearly independent for any $2 \le l \le m$.*

Let

$$\mathcal{D}_{lab} = \mathcal{U}_{lab}^{(m)} \cup \mathcal{H}_{ab} \cup \mathcal{V}_{ba}^{(2)} \cup \mathcal{V}_{ab}^{(2)} \cup \mathcal{U}_{lab}^{(1)}.$$

Lemma 3.2 *$\mathcal{D}_{lab}$ is linearly independent.*

Proof By Schmeichel's Theorems and Lemma 3.1, each of $\mathcal{H}_{ab}, \mathcal{V}_{ba}^{(2)}$ and $\mathcal{V}_{ab}^{(2)}$ is linearly independent. Since $E(\mathcal{U}_{lab}^{(m)}) \cap E(\mathcal{H}_{ab}) = \{(a, v_m)(b, v_m)\}$ which is an edge, $\mathcal{U}_{lab}^{(m)} \cup \mathcal{H}_{ab}$, is linearly independent by Lemma 2.2. By specializing $l = 2$ in Lemma 3.1, we have that $\mathcal{V}_{ba}^{(2)} \cup \mathcal{V}_{ab}^{(2)}$ is linearly independent. Since $E(\mathcal{V}_{ba}^{(2)}) \cup E(\mathcal{V}_{ab}^{(2)}) - \{(a, v_j)(a, v_{j+1}), (b, v_j)(b, v_{j+1}) : 2 \le j \le m-1\}$ is a tree and since any linear combinations of cycles is a cycle or an edge disjoint union of cycles, any linear combination of cycles of $\mathcal{V}_{ba}^{(2)} \cup \mathcal{V}_{ab}^{(2)}$ must contain an edge of the form $(a, v_j)(a, v_{j+1})$

or $(b, v_j)(b, v_{j+1})$ which is not in any cycle of $\mathcal{U}_{lab}^{(m)} \cup \mathcal{H}_{ab}$. Thus, $\mathcal{U}_{lab}^{(m)} \cup \mathcal{H}_{ab} \cup \mathcal{V}_{ba}^{(2)} \cup \mathcal{V}_{ab}^{(2)}$ is linearly independent. Note that $E(\mathcal{U}_{lab}^{(1)}) \cap E(\mathcal{U}_{lab}^{(m)} \cup \mathcal{H}_{ab} \cup \mathcal{V}_{ba}^{(2)} \cup \mathcal{V}_{ab}^{(2)}) = \varnothing$. Therefore, $\mathcal{D}_{lab}$ is linearly independent. □

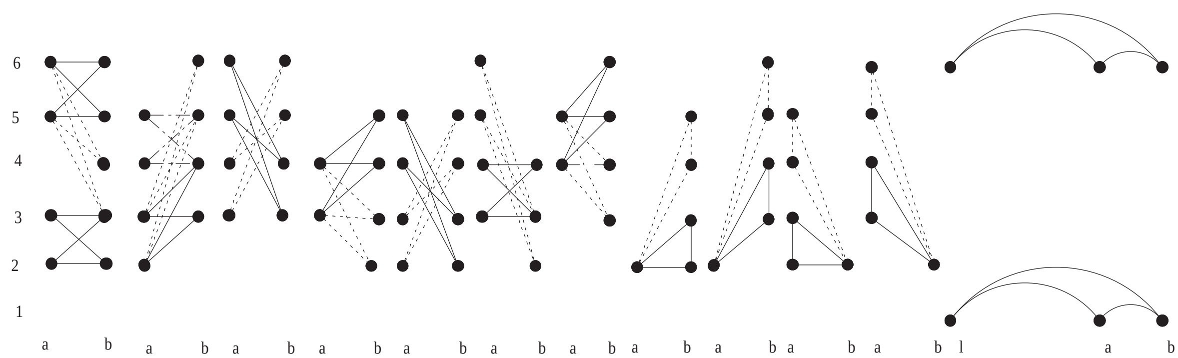

Fig.1 Cycles of $\mathcal{D}_{lab}$ for $m = 6$.

Remark 3.3 Let $e \in E(lab\rho W_m)$. From the definitions of $\mathcal{D}_{lab}$ and by the aid of Figure 2, one can easily see the following:

(1) *If* $e = (a, v_1)(b, v_1)$ *or* $(l, v_1)(a, v_1)$ *or* $(l, v_1)(b, v_1)$ *or* $(l, v_m)(a, v_m)$ *or* $(l, v_m)(b, v_m)$, *then* $f_{\mathcal{D}_{lab}}(e) = 1$.

(2) *If* $e = (a, v_j)(a, v_{j+1})$ *or* $(b, v_j)(b, v_{j+1}), 2 \geq j \geq m-1$, *then* $f_{\mathcal{D}_{lab}}(e) = 1$.

(3) *If* $e = (a, v_2)(b, v_2)$, *then* $f_{\mathcal{D}_{lab}}(e) = 3$.

(4) *If* $e = (a, v_j)(b, v_m)$ *or* $(a, v_m)(b, v_j), 2 \geq j \geq m$, *then* $f_{\mathcal{D}_{lab}}(e) = 2$.

(5) *If* $e = (a, v_j)(b, v_k), 2 \geq j, k \geq m$ *which is not as in (1)-(4), then* $f_{\mathcal{D}_{lab}}(e) \leq 4$.

(6) *If* $e \in E(lab\rho W_m)$ *which is not as in any of (1)-(6), then* $f_{\mathcal{D}_{lab}}(e) = 0$.

The graph $W_n\rho W_m$ is decomposable into $(S_n\rho W_m) \cup C_{n-1}[N_{m-1}] \cup \{(u_j, v_1)(u_{j+1}, v_1) \mid 2 \leq j \ \leq n-1\} \cup \{(u_n, v_1)(u_2, v_1)\}$ where $C_{n-1} = u_2u_3 \ldots u_nu_2$, and N_{m-1} is the null graph with vertex set $V(N_{m-1}) = \{v_2, v_3, \ldots, v_m\}$. Thus, $|E(W_n\rho W_m)| = |E(S_n\rho W_m)| + (n-1)(m-1)^2 + (n-1) = |E(S_n\rho W_m)| + (n-1)(m^2 - 2m + 2)$. Hence,

$$\dim \mathcal{C}(W_n\rho W_m) = \dim \mathcal{C}(S_n\rho W_m) + (n-1)(m^2 - 2m + 2).$$

By Theorem 3.3.2 of [15], we have that

$$\dim \mathcal{C}(S_n\rho W_m) = m^2(n-1) - nm + 2m - 1.$$

Therefore,

$$\dim \mathcal{C}(W_n\rho W_m) = (n-1)(2m^2 - 3m + 2) + (m-1).$$

Lemma 3.4 *$\mathcal{D} = \cup_{i=2}^{n} \mathcal{D}_{u_1 u_i u_{i+1}}$ is linearly independent where $\mathcal{D}_{u_1 u_n u_{n+1}} = \mathcal{D}_{u_1 u_n u_2}$.*

Proof We use the mathematical induction on n. If $n = 2$, then $\mathcal{D} = \mathcal{D}_{u_1 u_2 u_3}$ which is linearly independent by Lemma 3.2. Assume that $n > 2$ and it is true for less than n. Note that $\mathcal{D} = \mathcal{D}_{u_1 u_n u_2} \cup \left(\cup_{i=2}^{n-1} \mathcal{D}_{u_1 u_i u_{i+1}}\right)$. By Lemma 3.2 and the inductive step, each of $\mathcal{D}_{u_1 u_n u_2}$ and $\cup_{i=2}^{n-1} \mathcal{D}_{u_1 u_i u_{i+1}}$ is linearly independent. Note that

$$\begin{aligned} E(\mathcal{D}_{u_1 u_n u_2}) \cap E(\cup_{i=2}^{n-1} \mathcal{D}_{u_1 u_i u_{i+1}}) &= \{(u_1, v_1)(u_n, v_1), (u_1, v_1)(u_2, v_1), (u_1, v_m)(u_n, v_m), \\ &\quad (u_1, v_m)(u_2, v_m)\} \cup \{(u_n, v_j)(u_n, v_{j+1}), (u_2, v_j)(u_2, \\ &\quad v_{j+1}) \mid 2 \leq j \leq m-1\} \end{aligned}$$

which is an edge set of a forest. Thus, by Lemma 2.2, $\mathcal{D}$ is linearly independent. □

The following set of cycles which were introduced in [17] and [5] will be needed in the coming results:

$$\mathcal{G}_{ab} = \left\{\mathcal{G}_{ab}^{(j)} = (a, v_1)(a, v_j)(b, v_2)(a, v_{j+1})(a, v_1) \mid 2 \leq j \leq m-1\right\},$$

$$\mathcal{W}_{cab} = \{(c, v_1)(c, v_2)(a, v_2)(b, v_m)(b, v_1)(a, v_1)(c, v_1)\},$$

$$\mathcal{E}_{cab} = \left\{\mathcal{E}_{cab}^{(j)} = (c, v_2)(a, v_j)(b, v_m)(a, v_{j+1})(c, v_2) \mid 2 \leq j \leq m-1\right\},$$

$$\mathcal{P}_{a} = \left\{\mathcal{P}_{a}^{(j)} = (a, v_1)(a, v_j)(a, v_{j+1})(a, v_1) \mid 2 \leq j \leq m-1\right\},$$

$$\mathcal{S}_{ab} = \{(a, v_1)(a, v_2)(b, v_2)(b, v_1)(a, v_1)\},$$

and

$$\mathcal{I}_{a} = \{(a, v_2)(a, v_3) \ldots (a, v_m)(a, v_2)\}.$$

Let

$$\mathcal{F}_{ab} = \mathcal{H}_{ab} \cup \mathcal{G}_{ab} \cup \mathcal{G}_{ba} \cup \mathcal{S}_{ab}$$

and

$$\mathcal{F}_{cab} = \mathcal{E}_{cab} \cup \mathcal{H}_{ca} \cup \mathcal{G}_{ca} \cup \mathcal{W}_{cab}$$

Theorem 3.5 ([5]) *For any star S_n with $n \geq 2$ and wheel W_m with $m \geq 5$, we have that $\mathcal{B}(S_n \rho W_m) = (\cup_{i=2}^{n-1} \mathcal{F}_{u_{i+1} u_1 u_i}) \cup \mathcal{F}_{u_1 u_2} \cup (\cup_{i=1}^{n} \mathcal{P}_{u_i}) \cup (\cup_{i=1}^{n} \mathcal{I}_{u_i})$ is a 4-fold basis of $\mathcal{C}(S_n \rho W_m)$.*

Theorem 3.6 *For any two wheels W_n and W_m with $n \geq 4$ and $m \geq 5$, $b(W_n \rho W_m) \leq 4$.*

Proof Define $\mathcal{B}(W_n \rho W_m) = \mathcal{B}(S_n \rho W_m) \cup \mathcal{D}$ where $\mathcal{B}(S_n \rho W_m)$ is as in Theorem 3.5. By Theorem 3.5 and Lemma 3.4, each of $\mathcal{B}(S_n \rho W_m)$ and $\mathcal{D}$ is linearly independent. Note that,

$$E(\mathcal{B}(S_n \rho W_m)) \cap E(\mathcal{D}) = E(S_n \Box \{v_1, v_m\}) \cup E(V(C_{n-1}) \Box P_{m-1})$$

which is an edge set of a forest where $C_{n-1} = u_2u_3 \dots u_nu_2$ and $P_{m-1} = v_2v_3 \dots v_m$. Therefore, by Lemma 2.2, $\mathcal{B}(W_n\rho W_m)$ is linearly independent. Now,

$$|\mathcal{V}^{(2)}_{ba}| = (m-2) \text{ and } |\mathcal{H}_{ab}| = (m-2)^2 \tag{3}$$

and so

$$\begin{aligned} |\mathcal{D}_{u_1u_iu_{i+1}}| &= |\mathcal{D}_{lab}| = |\mathcal{U}^{(3)}_{lab}| + |\mathcal{H}_{ab}| + |\mathcal{V}^{(2)}_{ba}| + |\mathcal{V}^{(2)}_{ab}| + |\mathcal{U}^{(1)}_{lab}| \\ &= 1 + (m-2)^2 + (m-2) + (m-2) + 1 \\ &= (m-2)^2 + 2(m-2) + 2. \end{aligned} \tag{4}$$

By equation (3),

$$\begin{aligned} |\mathcal{D}| &= \sum_{i=2}^{n} |\mathcal{D}_{u_1u_iu_{i+1}}| \\ &= (n-1)((m-2)^2 + 2(m-2) + 2). \end{aligned}$$

Thus,

$$\begin{aligned} |\mathcal{B}(W_n\rho W_m)| &= |\mathcal{B}(S_n\rho W_m)| + |\mathcal{D}| \\ &= m^2(n-1) - nm + 2m - 1 + (n-1)((m-2)^2 + 2(m-2) + 2) \\ &= (n-1)(2m^2 - 3m + 2) + (m-1) \\ &= \dim \mathcal{C}(W_n\rho W_m) \end{aligned}$$

where the last equality followed from (1). Thus $\mathcal{B}(W_n\rho W_m)$ is a basis for $\mathcal{C}(W_n\rho W_m)$. Now,we show that $b(W_n\rho W_m) \leq 4$, for all $n \geq 4$, $m \geq 5$. Let $e \in E(W_n\rho W_m)$. Then we consider the following:

Case a $e \in E(W_n\rho W_m) - E(S_n\Box\{v_1, v_m\}) \cup E(V(C_{n-1})\Box P_{m-1})$ where C_{n-1} and P_{m-1} are as defined above. Then we have the following:

(1) $e = (u_i, v_j)(u_{i+1}, v_k)$ or $(u_i, v_1)(u_{i+1}, v_1)$ with $i \leq n-1$ and $2 \leq j, k \leq m$. Then e occurs only in cycles of $\mathcal{D}_{u_1u_iu_{i+1}}$. And so, by Remark 3.3, $f_{\mathcal{B}(W_n\rho W_m)}(e) = f_{\mathcal{D}_{u_1u_iu_{i+1}}}(e) \leq 4$.

(2) $e = (u_2, v_j)(u_n, v_k)$ or $(u_2, v_1)(u_n, v_1)$ with $2 \leq j, k \leq m$. Then e occurs only in cycles of $\mathcal{D}_{u_1u_nu_2}$. And so, by Remark 3.3, $f_{\mathcal{B}(W_n\rho W_m)}(e) = f_{\mathcal{D}_{u_1u_nu_2}}(e) \leq 4$.

(3) e is not as in (1) or (2). Then e occurs only in cycles of $\mathcal{B}(S_n\rho W_m)$ and so, by Theorem 3.5, $f_{\mathcal{B}(W_n\rho W_m)}(e) \leq f_{\mathcal{B}(S_n\rho W_m)}(e) \leq 4$.

Case b $e \in E(S_n\Box\{v_1, v_m\}) \cup E(V(C_{n-1})\Box P_{m-1})$. Then we have the following:

(1) $e \in E(u_i\Box P_{m-1})$ with $2 \leq i \leq n$. Then e occurs only in $\mathcal{D}_{u_1u_{i-1}u_i}, \mathcal{D}_{u_1u_iu_{i+1}}$ and $\mathcal{B}(S_n\rho W_m)$. Thus, by Remark 3.3 and Theorem 3.5, $f_{\mathcal{B}(W_n\rho W_m)}(e) = f_{\mathcal{D}_{u_1u_{i-1}u_i}}(e) + f_{\mathcal{D}_{u_1u_iu_{i+1}}} + f_{\mathcal{B}(S_n\rho W_m)} \leq 1 + 1 + 2$.

(2) $e = (u_1, v_1)(u_2, v_1)$ or $(u_1, v_m)(u_2, v_m)$. Then e occurs only in cycles of $\mathcal{D}_{u_1u_2u_3}, \mathcal{D}_{u_1u_3u_4}$ and $\mathcal{B}(S_n\rho W_m)$. And so, by Remark 3.3 and Theorem 3.5, $f_{\mathcal{B}(W_n\rho W_m)}(e) = f_{\mathcal{D}_{u_1u_2u_3}}(e) + f_{\mathcal{D}_{u_1u_3u_4}} + f_{\mathcal{B}(S_n\rho W_m)} \leq 1 + 1 + 2$.

(3) $e = (u_1, v_1)(u_i, v_1)$ or $(u_1, v_m)(u_i, v_m)$. Then e occurs only in cycles of $\mathcal{D}_{u_1u_{i-1}u_i}, \mathcal{D}_{u_1u_iu_{i+1}}$ and $\mathcal{B}(S_n\rho W_m)$. And so, by Remark 3.3 and Theorem 3.5, $f_{\mathcal{B}(W_n\rho W_m)}(e) = f_{\mathcal{D}_{u_1u_2u_3}}(e) + f_{\mathcal{D}_{u_1u_3u_4}} + f_{\mathcal{B}(S_n\rho W_m)} \leq 1 + 1 + 2$. □

Corollary 3.7 *For any $n \geq 4$ and $m \geq 6$, we have $3 \leq b(W_n\rho S_m) \leq 4$.*

Proof By Theorem 3.6, it is enough to show that $b(W_n\rho S_m) \geq 3$. Since $S_n\rho S_m$ is a subgraph of $W_n\rho W_m$ and $b(S_n\rho S_m) = 4$ (Theorem 3.2.5 of [17]), $b(W_n\rho S_m) \geq 3$ by MacLane Theorem. □

§4. The Minimum Cycle Basis of $W_n\rho W_m$

In this section, we construct a minimum cycle basis of the wreath product of two wheels. Let

$$\mathcal{X}^*_{lab} = (\cup_{k=2}^m \mathcal{V}_{ab}^{(k)}) \cup \mathcal{V}_{ba}^{(m)} \cup \mathcal{U}_{lab}^{(1)} \cup \mathcal{U}_{lab}^{(m)}$$

Lemma 4.1 *$\mathcal{X}^*_{lab}$ is linearly independent.*

Proof $(\cup_{k=2}^m \mathcal{V}_{ab}^{(k)}) \cup (\mathcal{V}_{ba}^{(m)})$ is a linearly independent set by Lemma 3.1. Since $E((\cup_{k=2}^m \mathcal{V}_{ab}^{(k)}) \cup \mathcal{V}_{ba}^{(m)}) \cap E(\mathcal{U}_{lab}^{(1)}) = \varnothing$, $(\cup_{k=2}^m \mathcal{V}_{ab}^{(k)}) \cup (\mathcal{V}_{ba}^{(m)}) \cup \mathcal{U}_{lab}^{(1)}$ is linearly independent by Lemma 2.2. Similarly, since $E((\cup_{k=2}^m \mathcal{V}_{ab}^{(k)}) \cup (\mathcal{V}_{ba}^{(m)}) \cup \mathcal{U}_{lab}^{(1)}) \cap E(\mathcal{U}_{lab}^{(3)}) = \{(a, v_m)(b, v_m)\}$ which is an edge, we have $\mathcal{X}^*_{lab}$ is linearly independent. □

Lemma 4.2 *$\left(\cup_{i=2}^{n-1} \mathcal{X}^*_{u_1u_iu_{i+1}}\right) \cup \mathcal{X}^*_{u_1u_nu_2}$ is linearly independent.*

Proof We prove that $\cup_{i=2}^{n-1} \mathcal{X}^*_{u_1u_iu_{i+1}}$ is linearly independent using the mathematical induction on n. If $n = 3$, then $\cup_{i=2}^{2} \mathcal{X}^*_{u_1u_iu_{i+1}} = \mathcal{X}^*_{u_1u_2u_3}$ which is linearly independent by Lemma 4.1. Assume that $n \geq 4$ and it is true for less than $n-1$. Note that $\cup_{i=2}^{n-1} \mathcal{X}^*_{u_1u_iu_{i+1}} = (\cup_{i=2}^{n-2} \mathcal{X}^*_{u_1u_iu_{i+1}}) \cup \mathcal{X}^*_{u_{n-1}u_n}$. Since

$$\begin{aligned} E(\cup_{i=2}^{n-2} \mathcal{X}^*_{u_1u_iu_{i+1}}) \cap E(\mathcal{X}^*_{u_{n-1}u_n}) &= \{(u_1, v_1)(u_{n-1}, v_1), (u_1, v_m)(u_{n-1}, v_m)\} \\ &\quad \cup \{(u_{n-1}, v_j)(u_{n-1}, v_{j+1}) \mid 2 \leq j \leq m-1\} \end{aligned}$$

which is an edge set of a forest, $\cup_{i=2}^{n-1} \mathcal{X}^*_{u_1u_iu_{i+1}}$ is linearly independent by Lemma 2.2. Similarly, Since

$$\begin{aligned} E(\cup_{i=2}^{n-1} \mathcal{X}^*_{u_1u_iu_{i+1}}) \cap E(\mathcal{X}^*_{u_1u_nu_2}) &= \{(u_1, v_1)(u_n, v_1), (u_1, v_m)(u_n, v_m), (u_1, v_1)(u_2, v_1), \\ &\quad (u_1, v_m)(u_2,\ v_m)\} \\ &\quad \cup \{(u_n, v_j)(u_n, v_{j+1}), (u_2, v_j)(u_2, v_{j+1}) \mid 2 \leq j \leq m-1\} \end{aligned}$$

which is an edge set of a forest, $\left(\cup_{i=2}^{n-1}\mathcal{X}^*_{u_1u_iu_{i+1}}\right)\cup\mathcal{X}^*_{u_1u_nu_2}$ is linearly independent. □

Throughout the following results, $B_{a\Box W_m}$ stands for the cycle basis of $a\Box W_m$ which consists of 3-cycles.

Lemma 4.3 *$\mathcal{B}^*(W_n\rho W_m)=\mathcal{B}^*(S_n\rho W_m)\cup(\cup_{i=2}^{n-1}\mathcal{X}^*_{u_1u_iu_{i+1}})\cup\mathcal{X}^*_{u_1u_nu_2}$ is a cycle basis of $\mathcal{C}(W_n\rho W_m)$ where $\mathcal{B}^*(S_n\rho W_m)=(\cup_{i=2}^{n}\cup_{j=2}^{m}\mathcal{V}^{(j)}_{u_1u_i})\cup(\cup_{i=2}^{n}\mathcal{V}^{(m)}_{u_iu_1})\cup(\cup_{i=1}^{n}B_{u_i\Box W_m})\cup(\cup_{i=2}^{n}\mathcal{S}_{u_1u_i})$.*

Proof $\mathcal{B}^*(S_n\rho W_m)$ is linearly independent by Lemma 4.3.2 of [5]. Since $E(\mathcal{B}^*(S_n\rho W_m))\cap E(\left(\cup_{i=2}^{n}\mathcal{X}^*_{u_1u_iu_{i+1}}\right)\cup\mathcal{X}^*_{u_1u_nu_2})=E(S_n\Box\{v_1,v_m\})\cup E(V(P_{n-1})\Box P_{m-1})$, which is an edge set of a forest, as a result $\mathcal{B}^*(W_n\rho W_m)$ is linearly independent by Lemma 2.2 where $P_{n-1}=u_2u_3\cdots u_n$ and $P_{m-1}=v_2v_3\ldots v_m$. Now,

$$\begin{aligned}|\mathcal{X}^*_{u_1u_iu_{i+1}}| &= |\mathcal{X}^*_{lab}|\\ &= \sum_{k=2}^{m}|V^{(k)}_{ab}|+|V^{(m)}_{ab}|+2\\ &= \sum_{k=2}^{m}(m-2)+(m-2)+2\\ &= m(m-2)+2.\end{aligned}$$

Thus,

$$\begin{aligned}|\mathcal{B}^*(W_n\rho W_m)| &= |\mathcal{B}^*(S_n\rho W_m)|+|(\cup_{i=2}^{n}\mathcal{X}^*_{u_1u_iu_{i+1}})|\\ &= m^2(n-1)-mn+2m-1+\sum_{i=2}^{n}(m(m-2)+2)\\ &= m^2(n-1)-mn+2m-1+(n-1)(m(m-2)+2)\\ &= (n-1)(2m^2-3m+2)+(m-1)\\ &= \dim\mathcal{C}(W_n\rho W_m).\end{aligned}$$

Therefore, $\mathcal{B}^*(W_n\rho W_m)$ is a cycle basis for $W_n\rho W_m$. □

Theorem 4.4 *$\mathcal{B}^*(W_n\rho W_m)$ is minimum cycle basis of $\mathcal{C}(S_n\rho W_m)$ for each $n,m\geq 5$.*

Proof Let $P^*=\cup_{i=1}^{n}B_{u_i\Box W_m}$. Since $B_{u_i\Box W_m}$ is a basis for $\mathcal{C}(u_i\Box W_m)$ for each $1\leq i\leq n$ and since $E(u_i\Box W_m)\cap E(u_i\Box W_m)=\varnothing$ for any $i\neq j$, we have P^* is a cycle basis for the subgraph $\cup_{i=1}^{n}(u_i\Box W_m)$. Let $Q^*=\mathcal{B}^*(W_n\rho W_m)-(P^*\cup(\cup_{i=2}^{n}\mathcal{S}_{u_1u_i}))$ and $(W_n\rho W_m)^-=(W_n\rho W_m)-\cup_{i=1}^{n}(E((u_i\Box S_m)\cup\{(u_i,v_2)(u_i,v_m)\}))$. Note that $(W_n\rho W_m)^-$ consists of two components with $V((W_n\rho W_m)^-)=V(W_n\rho W_m)$. Also,

$$\begin{aligned}|E((W_n\rho W_m)^-)| &= |E(W_n\rho W_m)|-\sum_{i=1}^{n}(|E(u_i\Box S_m)+1)\\ &= |E(W_n\rho W_m)|-nm.\end{aligned}$$

Thus,

$$\begin{aligned}\dim\mathcal{C}((W_n\rho W_m)^-) &= |E(W_n\rho W_m)|-nm-mn+2\\ &= \dim\mathcal{C}(W_n\rho W_m)-mn+1\end{aligned}$$

Now,

$$|B_{a\Box W_m}| = m-1$$

Hence,

$$\begin{aligned}|Q^*| &= |\mathcal{B}^*(W_n\rho W_m)| - |P^*| - |\cup_{i=2}^{n} \mathcal{S}_{u_1u_i}| \\ &= \dim \mathcal{C}(W_n\rho W_m) - n(m-1) - (n-1) \\ &= \dim \mathcal{C}(W_n\rho W_m) - mn + 1 \\ &= \dim \mathcal{C}((W_n\rho W_m)^-).\end{aligned}$$

Therefore, Q^* is a basis for $(W_n\rho W_m)^-$. Now, we show that $L = \mathcal{B}^*(W_n\rho W_m) - (\cup_{i=2}^{n}\mathcal{S}_{u_1u_i})$ is the largest linearly independent subset of $W_n\rho W_m$ containing L and consisting of 3-cycles. Suppose that $\{C\} \cup L$ is linearly independent where C is a 3-cycle of $W_n\rho W_m$. Then we have the following three cases:

Case 1: $E(C) \subseteq E(\cup_{i=1}^{n} u_i\Box W_m)$. Then $C \in P^*$ because the cycles of P^* is the only 3-cycles of $\cup_{i=1}^{n}(u_i\Box W_m)$. This is a contradiction.

Case 2: $E(C) \subseteq E((W_n\rho W_m)^-)$. Then C can be written as a linear combination of Q^* because Q^* is a basis for $(W_n\rho W_m)^-$. This is a contradiction.

Case 3: $E(C)$ neither a subset of $E(\cup_{i=1}^{n} u_i\Box W_m)$ nor of $E((W_n\rho W_m)^-)$. Thus, C contains at least one edge which does not belong to $\cup_{i=1}^{n} u_i\Box W_m$ and at least one edge which does not belong to $(W_n\rho W_m)^-$. Note that

$$E((W_n\rho W_m)^-) \cap E(\cup_{i=1}^{n} u_i\Box W_m) = \cup_{i=1}^{n} (u_i\Box v_2v_3\ldots v_m).$$

Thus, C must contains at least one edge of $(\cup_{i=1}^{n} u_i\Box W_m) - (\cup_{i=1}^{n} u_i\Box v_2v_3\ldots v_m)$ and at least one edge of $(W_n\rho W_m)^- - (\cup_{i=1}^{n} u_i\Box v_2v_3\ldots v_m)$. To this end, we have two subcases:

Subcase 3a: $(u_i,v_2)(u_i,v_m) \in E(C)$ for some i. Then $C = (u_i,v_2)(u_i,v_m)(u_k\ ,v_s)(u_i,v_2)$ where $u_iu_k \in E(W_n)$ and $2 \le s \le m$. Thus, C can be written as a linear combination of 3-cycle as follows:

$$\begin{aligned}C \;=\; &\left(\oplus_{j=2}^{m-1}(u_i,v_j)(u_i,v_{j+1})(u_i,v_1)(u_i,v_j)\right) \oplus (u_i,v_2)(u_i,v_m)(u_i,v_1)(u_i,v_2) \\ &\oplus_{j=2}^{m-1}(u_i,v_j)(u_i,v_{j+1})(u_k,v_s)(u_i,v_j).\end{aligned}$$

Note that each of $(u_i,v_j)(u_i,v_{j+1})(u_i,v_1)(u_i,v_j)$ and $(u_i,v_2)(u_i,v_m)(u_i,v_1)(u_i,\ v_2)$ belongs to P^*. Also, $(u_i,v_j)(u_i,v_{j+1})(u_k,v_s)(u_i,v_2)$ is a linear combinations of $(\cup_{l=2}^{m}\mathcal{V}_{u_iu_k}^{(l)}) \cup (\mathcal{V}_{u_iu_k}^{(m)})$ because $(u_i,v_j)(u_i,v_{j+1})(u_k,v_s)(u_i,v_2) \subseteq u_iu_k[v_2v_3\ldots v_m]$ and $(\cup_{l=2}^{m}\mathcal{V}_{u_iu_k}^{(l)}) \cup (\mathcal{V}_{u_iu_k}^{(m)})$ is a basis for $u_iu_k[v_2v_3\ldots v_m]$. Thus, C is a linear combinations of L. That is a contradiction.

Subcase 3b: $(u_i,v_2)(u_i,v_m) \notin E(C)$ for each i. Then C contains at least one edge of $\cup_{i=1}^{n}E(u_i\Box S_m)$ and one edge of $(W_n\rho W_m)^-$. Therefore, by the construction of $W_n\rho W_m$, C must contains at least two edges of $\cup_{i=1}^{n} (u_i\Box W_m)$ and two other edges of $(W_n\rho W_m)^-$. This is a contradiction.

Since the cycle space is a matroid and each cycle of $\cup_{i=2}^{n}\mathcal{S}_{u_1u_i}$ is of length 4. Then $\mathcal{B}^*(W_n\rho W_m)$ is a minimum cycle basis for $W_n\rho W_m$. ■

Corollary 3.5 $l(W_n\rho W_m) = 3((n-1)(2m^2-3m+1)+(m-1))+4(n-1)$, *and* $\lambda(W_n\rho W_m) = 4$.

References

[1] M. Anderson and M. Lipman, The wreath product of graphs, Graphs and Applications (Boulder, Colo., 1982), 23-39, Wiley-Intersci. Publ., Wiley, New York, 1985.

[2] A. A. Ali and G. T. Marougi, The basis number of cartesian product of some graphs, *The J. of the Indian Math. Soc.* 58 (1992), 123-134.

[3] S. Alsardary and A. A. Ali, The basis number of some special non-planar graphs, *Czechoslovak Math. J.* 53 (2003), no. 2, 225–240.

[4] A. Alsardary and J. Wojciechowski, The basis number of the powers of the complete graph, *Discrete Math.* 188 (1998), no. 1-3, 13-25.

[5] K. M. Al-Qeyyam and M. M. M. Jaradat, On the basis number and the minimum cycle bases of the wreath product of some graphs II, *JCMCC* (to appear).

[6] D. M. Chickering, D. Geiger and D. HecKerman, On finding a cycle basis with a shortest maximal cycle, *Inform. Processing Let.*, 54(1994), 55-58.

[7] L. O. Chua and L. Chen, On optimally sparse cycles and coboundary basis for a linear graph, *IEEE Trans. Circuit Theory*, 20 (1973), 54-76.

[8] R. Diestel, *Graph Theory*, Graduate Texts in Mathematics, 173, Springer-Verlag, New York, 1997.

[9] G. M. Downs, V.J. Gillet, J.D. Holliday and M.F. Lynch, Review of ring perception algorithms for chemical graphs, *J. Chem. Inf. Comput. Sci.*, 29 (1989), 172-187.

[10] R. Hammack, Minimum cycle bases of direct products of bipartite graphs, *Australas. J. Comb.* 36 (2006), 213-221.

[11] R. Hammack, Minimum cycle bases of direct products of complete graphs. *Inform. Process. Lett.* 102 (2007), no. 5, 214–218.

[12] J. D. Horton, A polynomial-time algorithm to find the shortest cycle basis of a graph, *SIAM J. Comput.* 16 (1987), 359-366.

[13] W. Imrich and S. Klavžar, Product Graphs: Structure and Recognition, Wiley, New York, 2000.

[14] W. Imrich and P. Stadler, Minimum cycle bases of product graphs, *Australas. J. Combin.* 26 (2002), 233-244.

[15] M. M. M. Jaradat, On the basis number of the direct product of graphs, *Australas. J. Combin.* 27 (2003), 293-306.

[16] M. M. M. Jaradat, An upper bound of the basis number of the strong product of graphs, *Discussion Mathematicea Graph Theory* 25 (2005), no. 3, 391-406.

[17] M. M. M. Jaradat, On the basis number and the minimum cycle bases of the wreath product of some graphs I, *Discussiones Mathematicae Graph Theory* 26 (2006), 113-134.

[18] M. M. M. Jaradat, M.Y. Alzoubi and E.A. Rawashdeh, The basis number of the Lexicographic product of different ladders, *SUT Journal of Mathematics* 40 (2004), no. 2, 91-101.

[19] A. Kaveh, Structural Mechanics, Graph and Matrix Methods. Research Studies Press, Exeter, UK, 1992.

[20] G. Kirchoff, Uber die Auflösung der Gleichungen, auf welche man bei Untersuchung der linearen Vertheilung galvanischer Ströme gefürt wird, *Annalen der Physik und Chemie* 72 (1847) no.12, 497-508.

[21] S.MacLane, A combinatorial condition for planar graphs, *Fundamenta Math.* 28 (1937), 22-32.

[22] E. F. Schmeichel, The basis number of a graph, *J. Combin. Theory Ser. B* 30 (1981), no. 2, 123-129.

[23] P. Stadler, Minimum cycle bases of Halin graphs, *J. graph Theory,* 43 (2003), no. 2, 150-155.

[24] D. J. A. Welsh, Kruskal's theorem for matroids, *Proc. Cambridge Phil, Soc.*, 64 (1968), 3-4.

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Theory of Relativity on the Finsler Spacetime

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Abstract: Einstein's theory of special relativity and the principle of causality imply that the speed of any moving object cannot exceed that of light in a vacuum (c). Nevertheless, there exist various proposals for observing faster-than-c propagation of light pulses, using anomalous dispersion near an absorption line, nonlinear and linear gain lines, or tunnelling barriers. However, in all previous experimental demonstrations, the light pulses experienced either very large absorption or severe reshaping, resulting in controversies over the interpretation. Recently, L.J.Wang, A.Kuzmich and A.Dogariu use gain-assisted linear anomalous dispersion to demonstrate superluminal light propagation in atomic caesium gas. The group velocity of a laser pulse in this region exceeds c and can even become negative, while the shape of the pulse is preserved. The textbooks say nothing can travel faster than light, not even light itself. New experiments show that this is no longer true, raising questions about the maximum speed at which we can send information. On the other hand, the light speed reduction to 17 meters per second in an ultracold atomic gas. This shows that the light speed could taken on voluntariness numerical value, This paper shows that if ones think of the possibility of the existence of the superluminal-speeds (the speeds faster than that of light) and redescribe the special theory of relativity following Einstein's way, it could be supposed that the physical spacetime is a Finsler spacetime, characterized by the metric

$$ds^4 = g_{ijkl}dx^i dx^j dx^k dx^l.$$

If so, a new spacetime transformation could be found by invariant ds^4 and the theory of relativity is discussed on this transformation. It is possible that the Finsler spacetime $F(x, y)$ may be endowed with a catastrophic nature. Based on the different properties between the $ds^2$and ds^4, it is discussed that the flat spacetime will also have the catastrophe nature on the Finsler metric ds^4. The spacetime transformations and the physical quantities will suddenly change at the catastrophe set of the spacetime, the light cone. It will be supposed that only the dual velocities of the superluminal-speeds could be observed. If so, a particle with the superluminal-speeds $v > c$ could be regarded as its anti-particle with the dual velocity $v_1 = c^2/v < c$. On the other hand, it could be assumed that the horizon of the field of the general relativity is also a catastrophic set. If so, a particle with the superluminal-speeds could be projected near the horizon of these fields, and the particle will move on the spacelike curves. It is very interesting that, in the Schwarzschild fields, the theoretical calculation for

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the spacelike curves should be in agreement with the data of the superluminal expansion of extragalactic radio sources observed year after year.(see Cao,1992b)
The catastrophe of spacetime has some deep cosmological means. According to the some interested subjects in the process of evolution of the universe the catastrophe nature of the Finsler spacetime and its cosmological implications are discussed. It is shown that the nature of the universal evolution could be attributed to the geometric features of the Finsler spacetime (see Cao,1993).

Key words: Spacetime, catastrophe, Finsler metric, Finsler spacetime, speed faster than light.

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It is known that in his first paper on the special theory of relativity: "On the electrodynamics of moving bodies", Einstein clearly states (cf. Einstein, 1923) that 'Velocities greater than that of light have, no possibility of existence.' But he neglected to point out the applicable range of Lorentz transformation. In fact, his whole description must be based on velocities smaller than that of light which we call subluminal-speed. So, the special theory of relativity cannot negate that real motion at a speed greater than the speed of light in vacuum which we call superluminal-speed could exist. In this paper, it is shown that if we think of the possibility of existence of the superluminal-speed and redescribe the special theory of relativity following Einstein's way, a new theory would be founded on the Finsler spacetime. The new theory would retain all meaning of the special theory of relativity when matters move with subluminal-speed and would give new content when matters move with superluminal-speed. If we assume that the superluminal-speed will accord with the spacelike curves in the general theory of relativity, calculations indicate that the superluminal expansion of extragalactic radio sources exactly corresponds with the spacelike curves of the Schwarzschild geometry.

Our discussion is still based on the principle of relativity and on the principle of constancy of the velocity of light which have been defined by Einstein as follows:

(1)The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of coordinates in uniform translatory motion (see Einstein, 1923;p.41).

(2)Any ray of light moves in the 'stationary' system of coordinates with the determined velocity c, whether the ray be emitted by stationary or by a moving body.

Note that these two postulates do not impose any constraint on the relative speed v of the two inertial observers.

§1 The General Theory of the Transformation of Spacetime

1.1 Definition of simultaneity and temporal order

In his description about definition of simultaneity, Einstein stated: "Let us take a system of coordinates in which the equations of Newtonian mechanics hold good", $\cdots$, "Let a ray of light start at the 'A time' t_A from A towards B, let it at the B time' t_B be reflected at B in the direction of A, and arrive again at A at the 'A time' t'_A." In accordance with definition, the two clocks synchronize if (see Einstein, 1923; p.40)

$$t_B - t_A = t'_A - t_B. \qquad (1.1)$$

"In agreement with experience we further assume the quantity

$$\frac{2AB}{t_B - t_A} = c, \qquad (1.2)$$

to be a universal constant - the velocity of light in empty space."

"It is essential to have time defined by means of stationary clocks in the stationary system, and the time now defined being appropriate to the stationary system we call it 'the time of the stationary system'." In this way, Einstein finished his definition of simultaneity. But he did not consider the applicable condition of this definition, still less the temporal order and as it appears to me these discussions are essential too. Let us continue these discussions following Einstein's way.

First and foremost, let us assume if the point B is moving with velocity v relative to the point A, in agreement with experience we must use the following equations instead of Equation:

$$\frac{2AB}{t_A - t_B} = \begin{cases} c - v, & when \quad B \quad is \quad leaving \quad A \quad (a) \\ c + v, & when \quad B \quad is \quad approaching \quad A \quad (b) \end{cases} \qquad (1.3)$$

Obviously, Equation (1.3a) is not always applicable, it must require v<c, but Equation (1.3b) is always applicable-i.e., for $v < c$ and $v > c$ Einstein's whole discussion is based on the following formulae:

$$t_B - t_A = \frac{r_{AB}}{c - v} \text{ and } t'_A - t_B = \frac{r_{AB}}{c + v}. \qquad (1.4)$$

It must require $v < c$, because $t_B \ \ t_A$ must be larger than zero. Particularly, in order to get the Lorentz transformation, Einstein was based on the following formula (see Einstein, 1923; p.44)

$$\frac{1}{2}[\tau(0,0,0,t) + \tau(0,0,0,t + \tfrac{x'}{c-v} + \tfrac{x'}{c+v})] = \tau(x', 0, 0, t + \tfrac{x'}{c-v}), \qquad (1.5)$$

where $\frac{x'}{c-v}$ is just $t_B \ \ t_A$, so must require $v < c$, i.e., B must be the motion with the subluminal-speed. Then the Lorentz transformation only could be applied to the motion with subluminal-speed. It could not presage anything about the motion with the superluminal-speed, i.e., the special theory of relativity could not negate that the superluminal-speed would exist.

In order for our discussion to be applied to the motion with the superluminal-speed, we will only use Equation (1.3b), i.e., let the point B approach A. Now, let another ray of light (it must be distinguished from the first) start at the 'A time' t_{A1} from A towards B (when B will be at a new place B_1) let it at the 'B time' t_{B1} be reflected at B in the direction of A, and arrive again at A at the 'A time' t_{A1}.

According to the principle of relativity and the principle of the constancy of the velocity of light, we obtain the following formulas:

$$\frac{1}{2}(t'_A - t_A) = t_B - t_A = \frac{AB}{c+v}, \quad (1.6)$$

$$\frac{1}{2}(t'_{A1} - t_{A1}) = t_{A1} - t_{B1} = \frac{AB_1}{c+v}, \quad (1.7)$$

$$AB - AB_1 = v(t_{A1} - t_A). \quad (1.8)$$

Let

$$\Delta t_A = t_{A1} - t_A, \Delta t_B = t_{B1} - t_B \quad \text{and} \quad \Delta t'_A = t'_{A1} - t'_A, \quad (1.9)$$

where $\Delta t_A, \Delta t_B$, and $\Delta t'_A$ represent the temporal intervals of the emission from A, the reflection from B, and arrival at A for two rays of light, respectively. The symbols of the temporal intervals describe the temporal orders. When Δt >0 it will be called the forward order and when Δt <0, the backward order.

From Equations (1.6)-(1.9) we can get

$$\Delta t_B = \frac{c}{c+v}\Delta t_A, \quad (1.10)$$

and

$$\Delta t'_A = \frac{c-v}{c+v}\Delta t_A. \quad (1.11)$$

Then we assume that, if $\Delta t_A > 0$, i.e., two rays of light were emitted from A, successively we must have Δt_B >0 i.e., for the observer at system A these two rays of light were reflected by the forward order from B. But

$$\Delta t'_A \geq 0, \text{ if and only if } v \leq c$$

and

$$\Delta t'_A < 0, \text{ if and only if } v > c.$$

It means that for the observer at system A these two rays of light arrived at A by the forward order only when the point B moves with subluminal-speed, and by the backward order only when with superluminal-speed. In other words, the temporal order is not always constant. It is constant only when v<c, and it is not constant when v>c.

Usually, one thinks that this is a backward flow of time. In fact, it is only a procedure of time in the system B with the superluminal-speed which gives the observer in the ‘stationary system’ A an inverse appearance of the procedure of the time. It is an inevitable outcome when the velocity of the moving body is faster than the transmission velocity of the signal. This outcome will be called the relativity of the temporal order. It is a new nature of the time when the moving body attains the supeluminal-speed. It is known that it is not spacetime that impresses its form on things, but the things and their physical laws that determine spacetime. So, the superluminal-speed need not be negated by the character of the spacetime of the special theory of relativity, but will represent the new nature of the spacetime, the relativity of the temporal order.

1.2 The temporal order and the chain of causation

In order to explain the disparity between the backward flow of time and the relativity of the temporal order, we will use spacetime figure (as Fig.1-1)

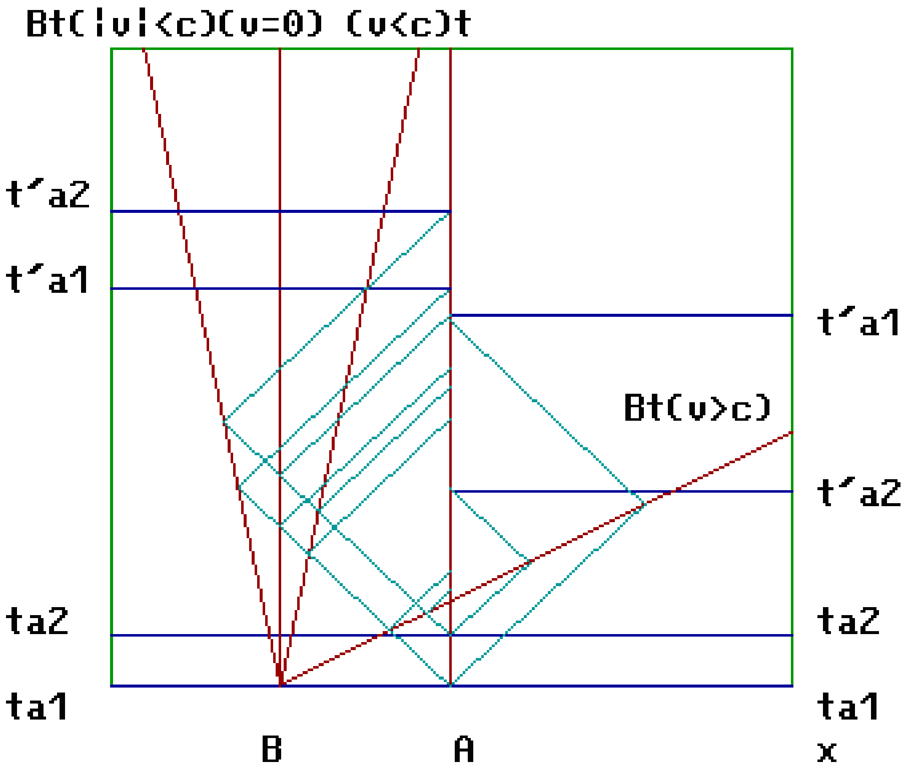

Fig.1-1. The spacetime figure

and take following definitions.

(1)The chain of the event, t_{A0},t_{A1},...,t_{Ai},.... The ith ray of light will be started at t_{Ai} and $\Delta t_{Ai} = t_{A(i+1)}$ t_{Ai} >0 It may or may not be chain of causality.

(2)The chains of the transference of the light t_{A0}, t_{B0}, t'_{A0}; t_{A1}, t_{B1}, t'_{A1};.... Every chain t_{Ai}, t_{Bi}, t'_{Ai} must be a chain of causality -i.e.

$$\frac{1}{2}(t'_{Ai} - t_{Ai}) = t_{Bi} - t_{Ai} = t'_{Ai} - t_{Bi} > 0. \qquad (1.12)$$

If they take a negative sign it will be the backward flow of time and will violate the principle of causality.

(3)The chains of the motion are the rays of the light, which will be reflected at B, but it will have different features when B moves with different velocity. Let us assume that:

(a) v >0 when B is approaching A;
(b) v <0 when B is leaving A;
(c) c >0 when the ray of light from A backwards B;
(d) c <0 when the ray of light from A towards B.

So, if v=0, we must have c <0. Then

$$t_{A(i+1)} - t_{Ai} = t_{B(i+1)} - t_{Bi} = t'_{A(i+1)} - t'_{Ai}. \qquad (1.13)$$

If $v < c$, we must have c <0 and when v >0,

$$t_{A(i+1)} - t_{Ai} > t_{B(i+1)} - t_{Bi} > t'_{A(i+1)} - t'_{Ai} > 0. \qquad (1.14)$$

But when $v < 0$,

$$0 < t_{A(i+1)} - t_{Ai} < t_{B(i+1)} - t_{Bi} < t'_{A(i+1)} - t'_{Ai}. \qquad (1.15)$$

Last of all, if $v > c$, must have v >0; and when c <0,

$$t_{A(i+1)} - t_{Ai} > t_{B(i+1)} - t_{Bi} > |t'_{A(i+1)} t'_{Ai}| > 0. \qquad (1.16)$$

But

$$t'_{A(i+1)} - t'_{Ai} < 0. \qquad (1.17)$$

When c >0,

$$0 < t_{A(i+1)} - t_{Ai} < |t_{B(i+1)} - t_{Bi}| < |t'_{A(i+1)} - t'_{Ai}| \qquad (1.18)$$

and

$$t_{B(i+1)} - t_{Bi} < 0 \quad \textit{and} \quad t'_{A(i+1)} - t_{Ai} < 0. \qquad (1.19)$$

These are rigid relations of causality.

4.The chains of the observation t'_{A0},t'_{A1},...,t'_{Ai},... and t_{B0},t_{B1},...,t_{Bi},... are not chains of causality. The relativity of temporal order is just that they could be a positive when $v < c$ or a negative when $v > c$ and the vector v and c have the same direction.

In (1.4) when $v > c$, $t_B \;\; t_A$ <0 it does not mean that velocities greater than that of light have no possibility of existence but only that the ray of light cannot catch up with the body with superluminal-speed.

1.3 Theory of the transformation of coordinates

From equations (1.10) and (1.11) we can get

$$\Delta t_B = \frac{c}{c+v} \Delta t_A \qquad (1.20)$$

and

$$\Delta t_B = \frac{c}{c-v}\Delta t'_A. \quad quad \qquad (1.21)$$

It has been pointed out that Δt_A and $\Delta t'_A$ are measurable by observer of the system A, but Δt_B is unmeasurable. Accordingly, the observer must conjecture Δt_B from Δt_A or $\Delta t'_A$. In form, Δt_B in Equation (1.20) and Δt_B in (1.21) are different. If we can find a transformation of coordinates it will satisfy following equation:

$$\Delta\tau^2 = \Delta t_A \cdot \Delta t'_A \qquad (1.22)$$

and, according to Equations (1.10) and (1.11), could get

$$\Delta\tau^2 = \begin{cases} >0, & iff \quad v<c, \\ =0, & iff \quad v=c, \\ <0, & iff \quad v>c. \end{cases} \qquad (1.23)$$

Then, we get

$$\Delta t_B^2 = \frac{c^2}{c^2-v^2}\Delta\tau^2$$

or

$$dt^2 = \frac{c^2}{c^2-v^2}d\tau^2. \qquad (1.24)$$

Let $ds^2 = c^2 d\tau^2$. We get

$$ds^2 = c^2 d\tau^2 = (c^2\ v^2)dt^2. \qquad (1.25)$$

So

$$ds^2 = \begin{cases} >0, & v<c \quad timelike, \\ =0, & v=c \quad lightlike, \\ <0, & v>c \quad spacelike. \end{cases} \qquad (1.26)$$

What merits special attention is that $ds^2 = (c^2 - v^2)dt^2$ and $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ are not identical. Usually, the special theory of relativity does not recognize their difference because motion with subluminal-speed does not involve the relative change of temporal orders, so the symbol of ds^2 remains unchanged when the inertial system changes.

Now let

$$ds^2 = ds_v^2 + ds_0^2, \qquad (1.27)$$

where

$$ds_v^2 = (c^2 - v^2)dt^2, \qquad (1.28)$$

$$ds_0^2 = dx^2 + dy^2 + dz^2, \qquad (1.29)$$

then

$$ds^2 = \begin{cases} +ds_v^2 + ds_0^2, & v < c, \\ -ds_v^2 + ds_0^2, & v > c. \end{cases} \tag{1.30}$$

Between any two inertial systems

$$ds_v^2 + ds_0^2 = \begin{cases} +ds_v^2 + ds_0^2, & v < c, \\ -ds_v^2 + ds_0^2, & v > c. \end{cases} \tag{1.31}$$

According to classical mechanics, we can determine the state of a system with n degrees of freedom at time t by measuring the 2n position and momentum coordinates $q^i(t)$, $p_i(t)$, i=1,2,...,n. These quantities are commutative each other, i.e., $q^i(t) \quad p_j(t) = p_j \ (t) q^i(t)$. But, in quantum mechanics the situation is entirely different. The operators Q_{op} and P_{op} corresponding to the classical observable position vector q and momentum vector p. These operators are non-commutative each other, i.e.,

$$QP \neq PQ.$$

So, ones doubt whether the quantum mechanics is not a good theory at first. But, ones discover that the non-commutability of operators is closely related to the uncertainty principle, it is just an essential distinction between the classical and quantum mechanics.

So, I doubt that whether the non-positive definite metrics ds^2 is just the best essential nature in the relativity theory? But, it was cast aside in Einstein's theory. Now, we could assume that

$$ds^4 = ds_v^4 + ds_0^4. \tag{1.32}$$

In general, we could let

$$ds^4 = g_{ijkl} dx^i dx^j dx^k dx^l, \qquad i, j, k, l = 0, 1, 2, 3. \tag{1.33}$$

Equations (1.32) and (1.33) which are defined as a Finsler metric are the base of the spacetime transformations. From the physical point of view this means that a new symmetry between the timelike and the spacelike could exist.

In his memoir of 1854, Riemann discusses various possibilities by means of which an n-dimensional manifold may be endowed with a metric, and pays particular attention to a metric defined by the positive square root of positive definite quadratic differential form. Thus the foundations of Riemannian geometry are laid; nevertheless, it is also suggested that the positive fourth root of a fourth-order differential form might serve as metric function (see Rund, 1959; Introduction X).

In his book of 1977, Wolfgang Rindler stated: "Whenever the squared differential distance $d\sigma^2$ is given by a homogeneous quadratic differential form in the surface coordinates, as in (7.10), we say that $d\sigma^2$ is a Riemannian metric, and that the corresponding surface is Riemannian. It is, of course, not a foregone conclusion that all metrics must be of this form: one could define, for example, a non-Riemannian metric $d\sigma^2 = \sqrt{dx^4 + dy^4}$ for some two-dimensional space, and investigate the resulting geometry.(Such more general metrics give rise to 'Finsler' geometry.)" (see W. Rindler,1997).

§2 The Special Theory of Relativity on the Finsler Spacetime ds^4

2.1 Spacetime transformation group on the Finsler metric ds^4

If $v = v_x$, then, between any two inertial systems we have

$$\begin{aligned} &c^4dt^4 + dx^4 - 2c^2dt^2dx^2 + dy^4 + dz^4 + 2dy^2dz^2 \\ &\quad = c^4dt'^4 + dx'^4 - 2c^2dt'^2 + dy'^4 + dz'^4 + 2dy'^2dz'^2 \end{aligned} \tag{2.1}$$

From (2.1) we could get transformations

$$t = \frac{t' + \frac{v}{c^2}x'}{\sqrt[4]{1 - 2\beta^2 + \beta^4}}, \qquad x = \frac{x' + vt'}{\sqrt[4]{1-2\beta^2+\beta^4}}, \qquad y = y', \qquad z = z'. \tag{2.2}$$

These transformations are called spacetime transformations. All spacetime transformations form into a group, called the spacetime transformation group (The Lorentz transformations group is only subgroup of the spacetime transformation group). The inverse transformations are of the form

$$\pm t' = \frac{t - \beta\frac{x}{c}}{\sqrt[4]{1 - 2\beta^2 + \beta^4}}, \qquad \pm x' = \frac{x - vt}{\sqrt[4]{1-2\beta^2+\beta^4}}, \qquad y' = y, \qquad z' = z, \tag{2.3}$$

where $\beta = \frac{v}{c}$. We could also use dual velocity $v_1 = \frac{c^2}{v}$ to represent the spacetime transformations. In fact, the transformations (2.2) can be rewritten as

$$t = \frac{\beta_1 t' + \frac{x'}{c}}{\sqrt[4]{1 - 2\beta_1^2 + \beta_1^4}}, \qquad x = \frac{\beta_1 x' + ct'}{\sqrt[4]{1-2\beta_1^2+\beta_1^4}}, \qquad y = y', \qquad z = z'. \tag{2.4}$$

Their inverse transformations are of the form

$$\pm t' = \frac{\beta_1 t - \frac{x}{c}}{\sqrt[4]{1 - 2\beta_1^2 + \beta_1^4}}, \qquad \pm x' = \frac{\beta_1 x - ct}{\sqrt[4]{1-2\beta_1^2+\beta_1^4}}, \qquad y' = y, \qquad z' = z. \tag{2.5}$$

where $\beta_1 = \frac{v_1}{c} = \frac{c}{v} = \frac{1}{\beta}$.

It is very interesting that all spacetime transformations are applicable to both the subluminal-speed (i.e.,$\beta<1$ or $\beta_1 >1$) and the superluminal-speed (i.e.,$\beta>1$ or $\beta_1 <1$). Whether the velocity is superluminal- or subluminal-speed, it is characterized by minus or plus sign of their inverse transformations, respectively.

Lastly, all spacetime transformations have the same singularity as the Lorentz transformation when the $\beta = \beta_1 = 1$.

2.2 Kinematics on the ds^4 invariant

We shall now consider the question of the measurement of length and time increment. In order to find out the length of a moving body, we must simultaneously plot the coordinates of its ends in a fixed system. From Equation (2.2) and (2.4), an expression for the length of a moving scale $\Delta x'$ measured by a fixed observer follows as

$$\pm\Delta x' = \Delta x \sqrt[4]{1 - 2\beta^2 + \beta^4}, \qquad (2.6)$$

and

$$\pm\Delta x' = c\Delta t \sqrt[4]{1 - 2\beta_1^2 + \beta_1^4}, \qquad (2.7)$$

Einstein stated: "For $v = c$ all moving objects - viewed from the 'stationary' system - shrivel up into plain figures. For velocities greater than that of light our deliberations become meaningless." However, formula (2.6) can applied to the case for velocities greater than that of light. Fig.2.1 give the relation between the length of a moving scale L and the velocity.

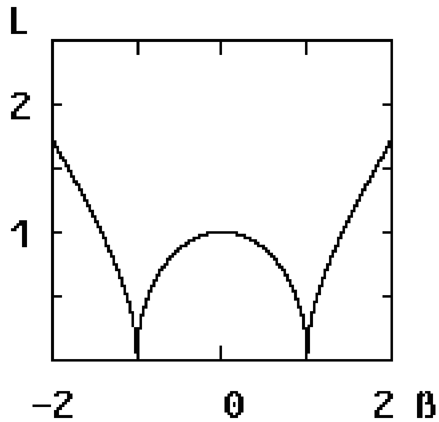

Fig.2.1. L-β curve

Let Δt be the time increment when the clock is at rest with respect to the stationary system, and $\Delta\tau$ be the time increment when the clock is at rest with respect to the moving system. Then

$$\pm\Delta\tau = \Delta t \sqrt[4]{1 - 2\beta^2 + \beta^4} \qquad (2.8)$$

and

$$\pm\Delta\tau = \frac{\Delta x}{c} \sqrt[4]{1 - 2\beta_1^2 + \beta_1^4}, \qquad (2.9)$$

Differentiating (2.3) or (2.5) and dividing dx' by dt' we obtain

$$\frac{dx'}{dt'} = v_x' = \frac{dx/dt - v}{1 - v/c^2 \, dx/dt} = \frac{v_x - v}{1 - vv_x/c^2}, \qquad (2.10)$$

Noting that $dy' = dy$, $dz' = dz$, we have a transformation of the velocity components perpendicular to v:

$$\frac{dy'}{dt'} = v'_y = \frac{v_y \sqrt[4]{1-2\beta^2+\beta^4}}{1-vv_x/c^2}, \frac{dz'}{dt'} = v'_z = \frac{v_z \sqrt[4]{1-2\beta^2+\beta^4}}{1-vv_x/c^2}, \quad (2.11)$$

where

$$v^2 = v_x^2 + v_y^2 + v_z^2. \quad (2.12)$$

From Equation (2.8), we could see that the composition of velocities have four physical implications: i.e.,

(1)A subluminal-speed and another subluminal-speed will be a subluminal-speed.

(2)A superluminal-speed and a subluminal-speed will be a superluminal-speed.

(3)The composition of two superluminal-speeds is a subluminal-speed.

(4)The composition of light-speed with any other speed (subluminal-,light-, or superluminal-speed) still is the light-speed.

There are the essential nature of the spacetime transformation group. The usual Lorentz transformation is a only subgroup of the spacetime transformation group.

It is necessary to point out that if $1 - vv_x/c^2 = 0$, i.e.,

$$v_x = v/c^2, \quad (2.13)$$

then $v_x \to \infty$. It implies that if two velocities are dual to each other and in opposite directions, then their composition velocity is an infinitely great velocity. We guess that it may well become an effective way to make an appraisal of a particle with the superluminal-speed.

2.3 Dynamics on the ds^4 invariant

The Lagrangian for a free particle with mass m is

$$L = -mc^2 \sqrt[4]{1-2\beta^2+\beta^4}, \quad (2.14)$$

The momentum energy, and mass of motion of the particle are of the forms:

$$p = \frac{mv}{\sqrt[4]{1-2\beta^2+\beta^4}}, E = \frac{mc^2}{\sqrt[4]{1-2\beta^2+\beta^4}}, M = \frac{m}{\sqrt[4]{1-2\beta^2+\beta^4}}. \quad (2.15)$$

Those could also be represented by dual velocity v_1:

$$p(v) = \frac{mv}{\sqrt[4]{1-2\beta^2+\beta^4}} = \frac{mc}{\sqrt[4]{1-2\beta_1^2+\beta_1^4}} = \frac{1}{c}E(v_1), \quad (2.16)$$

$$E(v) = \frac{mc^2}{\sqrt[4]{1-2\beta^2+\beta^4}} = \frac{mv_1c}{\sqrt[4]{1-2\beta_1^2+\beta_1^4}} = cp(v_1), \quad (2.17)$$

$$M(v) = \frac{m}{\sqrt[4]{1-2\beta^2+\beta^4}} = \frac{\beta_1 m}{\sqrt[4]{1-2\beta_1^2+\beta_1^4}} = \beta_1 M(v_1). \quad (2.18)$$

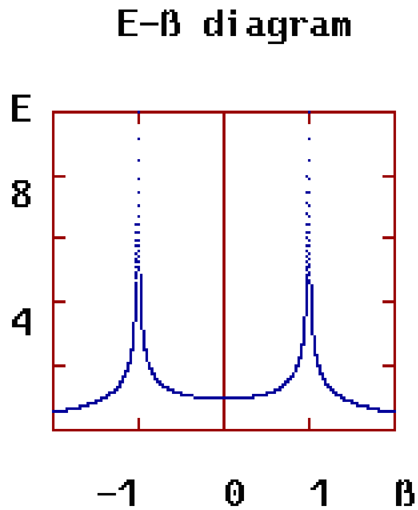

Fig.2.2. E-β diagram

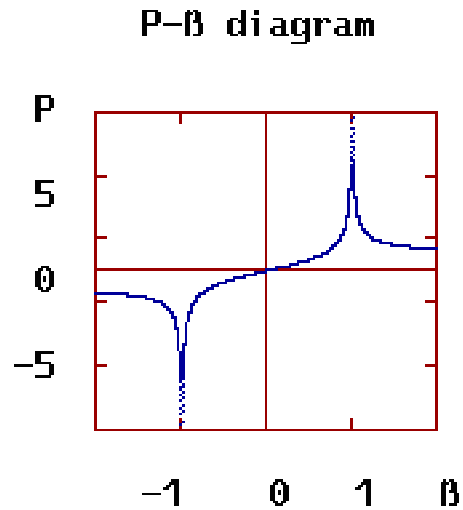

Fig. 2.3. p-β diagram

Einstein stated: "Thus, when $v = c$, E becomes infinite, velocities greater than that of light have - as in our previous results - no possibility of existence." But, formula (2.7) can also applied to the case for velocities greater than that of light. Fig.2.2 give the relation between the energy of a moving particle and its velocity, and Fig.2.3 give the relation between the momentum of a moving particle and its velocity.

It is very interesting that the momentum (or energy) in the v's representation will change into the energy (or momentum) in the v_1's representation. From (2.15) (or (2.16) and (2.17)), we could get the following relation between the momentum and energy of a free material particle:

$$p(v) = \frac{v}{c^2}E(v) \qquad or \qquad p(v_1) = \frac{v_1}{c^2}E(v_1), \qquad (2.19)$$

where the relation (2.19) keeps up the same form as the special theory of relativity. But a new invariant will be obtained as

$$E^4 + c^4p^4 - 2c^2p^2E^2 = m^4c^8. \qquad (2.20)$$

The relation (2.20) is correct for both of the v's and the v_1's representations. It is a new relation on the ds^4 invariant.

2.4 A charged particle in an electromagnetic field on the Finsler spacetine ds^4

Let us now turn to the equations of motion for a charged particle in an electromagnetic field, A,Φ, E_e andH_e. Their Lagrangian is

$$L = -mc^2\sqrt[4]{1 - 2\beta^2 + \beta^4} + \frac{e}{c}Av - e\Phi. \qquad (2.21)$$

The derivative $\partial L/\partial v$ is the generalized momentum of the particle. We denote it by p_e

$$p_e = mv\sqrt[4]{1 - 2\beta^2 + \beta^4} + \frac{e}{c}A = p + \frac{e}{c}A. \qquad (2.22)$$

where p denotes momentum in the absence of a field.

From the Lagrangian we could find the Hamiltonian function for a particle in a field from the general formula

$$H = mc^2 \sqrt[4]{1-2\beta^2+\beta^4} + e\Phi. \qquad (2.23)$$

However, the Hamiltonian must be expressed not in terms of the velocity, but rather in terms of the generalized momentum of the particle. From equations (2.2) and (2.3), we can get the relation

$$[(\frac{H-e\Phi}{c})^2 - (p-\frac{e}{c}A)^2]^2 = m^4c^4. \qquad (2.24)$$

Now we write the Hamilton-Jacobi equation for a particle in an electromagnetic field in the Finsler spacetime. It is obtained by replacing, in the equation for the Hamiltonian, P by $\partial S/\partial r$, and H by $-\partial S/\partial t$. Thus we get from (2.24)

$$[(\nabla S - \frac{e}{c}A)^2 - \frac{1}{c^2}(\frac{\partial S}{\partial t} + e\Phi)^2]^2 - m^4c^4 = 0. \qquad (2.25)$$

Now we consider the equation of motion of a charge in an electromagnetic field. It could be written by Lagrangian (2.21) as

$$\frac{d}{dt}\frac{mv}{\sqrt[4]{1-2\beta^2+\beta^4}} = eE_e + \frac{e}{c}v \times H_e. \qquad (2.26)$$

where

$$E_e = -\frac{1}{c}\frac{\partial A}{\partial t} - grad\Phi, \qquad H_e = curl A. \qquad (2.27)$$

It is easy to check the $dE_e = vdP$, i.e.,

$$v\frac{d}{dt}\frac{mv}{\sqrt[4]{1-2\beta^2+\beta^4}} = mc^2\frac{d}{dt}\frac{1}{\sqrt[4]{1-2\beta^2+\beta^4}}. \qquad (2.28)$$

Then from (2.26) we have

$$\frac{dE}{dt} = eE_e v. \qquad (2.29)$$

Integrate (2.29) and get

$$\frac{mc^2}{\sqrt[4]{1-2\beta^2+\beta^4}} - \frac{mc^2}{\sqrt[4]{1-2\beta_0^2+\beta_0^4}} = eU. \qquad (2.30)$$

where

$$\beta_0 = \frac{v_0}{c}, \qquad U = \int_{r_0}^{r} E_e dr. \qquad (2.31)$$

From (2.26) and (2.29), if we write it in terms of components, it is easy to obtain the spacetime transformation equations for the field components, and we could obtain the field transformation equation

$$\begin{cases} H'_x = H_x, & E'_x = E_x, \\ H'_y = \frac{H_y+\beta E_z}{\sqrt[4]{1-2\beta^2+\beta^4}}, & E'_y = \frac{E_y-\beta H_z}{\sqrt[4]{1-2\beta^2+\beta^4}}, \\ H'_z = \frac{H_z-\beta E_y}{\sqrt[4]{1-2\beta^2+\beta^4}}, & E'_z = \frac{E_z+\beta H_y}{\sqrt[4]{1-2\beta^2+\beta^4}}. \end{cases} \tag{2.32}$$

We could also use dual velocity v_1 to represent the field transformation equation

$$\begin{cases} H'_x = H_x, & E'_x = E_x, \\ H'_y = \frac{\beta_1 H_y+E_z}{\sqrt[4]{1-2\beta_1^2+\beta_1^4}}, & E'_y = \frac{\beta_1 E_y-H_z}{\sqrt[4]{1-2\beta_1^2+\beta_1^4}}, \\ H'_z = \frac{\beta_1 H_z-E_y}{\sqrt[4]{1-2\beta_1^2+\beta_1^4}}, & E'_z = \frac{\beta_1 E_z+H_y}{\sqrt[4]{1-2\beta_1^2+\beta_1^4}}. \end{cases} \tag{2.33}$$

An invariant will be obtained as
$H_e^4 + E_e^4 - 2H_e^2E_e^2$ =constant,
of new nature for the electromagnetic field in Finsler spacetime.

§3 The Catastrophe of the Spacetime and Its Physical Meaning

3.1 Catastrophe of the spacetime on the Finsler metric ds^4

The functions $y = x^2$ and $y = x^4$ are topologically equivalent in the theory of the singularities of differentiable maps (see Arnold et al.,1985). But the germ $y = x^2$ is topologically (and even differentially) stable at zero. the germ $y = x^4$ is differentially (and even topologically) unstable at zero. So, there is a great difference between the theories of relativity on the ds^2 and the ds^4.

On the other hand, a great many of the most interesting macroscopic phenomena in nature involve discontinuities. The Newtonian theory and Einstein's relativity theory only consider smooth, continuous processes. The catastrophe theory, however, provides a universal method for the study of all jump transitions, discontinuities and sudden qualitative changes. The catastrophe theory is a program. The object of this program is to determine the change in the solutions to families of equations when the parameters that appear in these equations change.

In general, a small change in parameter values only has a small quantitative effect on the solutions of these equations. However, under certain conditions a small change in the value of some parameters has a very large quantitative effect on the solutions of these equations. Large quantitative changes in solutions describe qualitative changes in the behaviour of the system modeled.

Catastrophe theory is, therefore, concerned with determining the parameter values at which there occur qualitative changes in solutions of families of equations described by parameters.

The double-cusp is the simplest non-simple in the sense of Arnold (see Arnold et al.,1985), but the double-cusp is unimodal.

The double-cusp is compact, in the sense that the sets f≤constant are compact. In Arnold's notation, the double-cusp belongs to the family X9 and in that family there are three real types of germ, according as to whether the germ has 0,2, or 4 real roots. For example representatives of the three types are: type $1x^4 + y^4$, type $2x^4$ y^4, type $3x^4 + y^4$ $2\delta x^2y^2$,respectively, and only the type 1 is compact.

Compact germs play an important role in application (see Zeeman, 1977), because any perturbation of a compact germ has a minimum; therefore if minima represent the stable equilibria of some system, then for each point of the unfolding space there exists a stable state of the system.

3.2 Catastrophe of the spacetime on the Finsler metric ds^4

In accordance with the Finsler metric ds^4 of the spacetime, we could

$$f(T,X,Y,Z) = T^4 + X^4 + Y^4 + Z^4 - 2T^2X^2 + 2Y^2Z^2, \tag{3.1}$$

here T=ct. Equation (3.1) that describes the behaviour of the spacetime is a smooth function.

As the catastrophe theory, first we must find the critical points of this function. Let $f = 0$, and $f' = 0$, here $f' = \partial f/\partial s$,$s = T, X, Y, Z$. i.e.,

$$\begin{aligned}
&f = T^4 + X^4 + Y^4 + Z^4 - 2T^2X^2 + 2Y^2Z^2 = 0,\\
&f'_T = \partial f/\partial T = 4T(T^2 - X^2) = 0,\\
&f'_X = \partial f/\partial X = 4X(X^2 - T^2) = 0,\\
&f'_Y = \partial f/\partial Y = 4Y(Y^2 + Z^2) = 0,\\
&f'_Z = \partial f/\partial Z = 4Z(Z^2 + Y^2) = 0.
\end{aligned}$$

So, the critical point are

$$X = \pm T, \qquad T = X = Y = Z = 0.$$

Then, we form the stability matrix $(\partial^2 f/\partial x^i \partial x^j)$. It is of the form

$$H(T,X,Y,Z) = \begin{bmatrix} 12T^2 - 4x^2 & -8Tx & 0 & 0 \\ -8Tx & 12x^2 - 4T^2 & 0 & 0 \\ 0 & 0 & 12y^2 + 4z^2 & 8yz \\ 0 & 0 & 8yz & 12z^2 + 4y^2 \end{bmatrix}.$$

Obviously, for the submatrix

$$H(Y,Z) = \begin{pmatrix} 12y^2 + 4z^2 & 8yz \\ 8yz & 12z^2 + 4y^2 \end{pmatrix},$$

its determinant does not vanish, unless Y=Z=0.

With the Thom theorem (splitting lemma), we could get

$$f_M(Y,Z) = Y^4 + Z^4 + 2Y^2Z^2, \tag{3.2}$$

$$f_{NM}(T,X) = T^4 + X^4 - 2T^2X^2, \tag{3.3}$$

where f_M Morse function, can be reduced to the Morse canonical form

$$M_0^2 = Y^2 + Z^2,$$

and f_{NM}, non-Morse function, is a degenerate form of the double-cusp catastrophe (see Zeeman, 1977). For another submatrix of $H(T, X, Y, Z)$

$$H(T,X) = \begin{vmatrix} 12T^2 - 4X^2 & -8TX \\ -8XT & 12X^2 - 4T^2 \end{vmatrix} = -48(T^4 + X^4 - 2T^2X^2).$$

So, the spacetime submanifold $M(T, X)$ will be divided into four parts by the different values of the $H(T, X)$:

$$\begin{array}{lll} H(T,X) \neq 0 & T^2 - X^2 < 0 & \textit{spacelike} \quad \textit{state} \\ (\textit{material} \quad \textit{states}) & T^2 - X^2 > 0 & \textit{timelike} \quad \textit{state} \\ H(T,X) = 0 & T = \pm X & \textit{lightlike} \quad \textit{state} \\ (\textit{singularities}) & T = X = 0 & \textit{the} \quad \textit{origin} \quad (\textit{indeterminate}). \end{array} \tag{3.4}$$

It means that the light cone is just a catastrophe set on the spacetime manifold, and both the timelike state and spacelike state are possible states of moving particles.

So, from the point of view of the catastrophe theory, the light cone is just a set of degenerate critical points on the spacetime manifold. The spacetime is structurally unstable at the light cone. It means that a lightlike state could change suddenly into a timelike state and a spacelike state. Also, a timelike state and a spacelike state could change suddenly into a lightlike state. It very much resembles the fact that two photons with sufficient energy could change suddenly into a pair of a particle and an anti-particle and contrarily, a pair of a particle and an antiparticle could annihilate and change into two photons.

According to the nature of catastrophe of the spacetime, the spacetime transformations (2.2) could be resolved into two parts at the light cone:

$$t = \frac{t' + \frac{\beta}{c}x'}{\sqrt{1-\beta^2}}, x = \frac{x' + vt'}{\sqrt{1-\beta^2}}, y = y', z = z'; \quad \beta = \frac{v}{c} < 1. \tag{3.5}$$

and

$$t = \frac{t' + \frac{\beta}{c}x'}{\sqrt{\beta^2 - 1}}, x = \frac{x' + vt'}{\sqrt{\beta^2 - 1}}, y = y', z = z'; \quad \beta = \frac{v}{c} > 1. \tag{3.6}$$

In the same way, the transformation (2.4) could also be resolved into two parts at the light cone:

$$t = \frac{\beta_1 t' + \frac{1}{c}x'}{\sqrt{\beta_1^2 - 1}}, x = \frac{\beta_1 x' + ct'}{\sqrt{\beta_1^2 - 1}}, y = y', z = z'; \quad \beta_1 = \frac{v_1}{c} > 1. \tag{3.7}$$

and

$$t = \frac{\beta_1 t' + \frac{1}{c}x'}{\sqrt{1-\beta_1^2}}, x = \frac{\beta_1 x' + ct'}{\sqrt{1-\beta_1^2}}, y = y', z = z'; \quad \beta_1 = \frac{v_1}{c} < 1. \qquad (3.8)$$

It is very interesting that transformations (3.5) and (3.7) have two major features: Firstly, they keep the same sign between the ds^2 and the ds'^2;i.e.,

$$ds_v^2 = ds'^2_v. \qquad (3.9)$$

Secondly, their inverse transformations are of the form

$$t' = \frac{t - \frac{\beta}{c}x}{\sqrt{1-\beta^2}}, x' = \frac{x - vt}{\sqrt{1-\beta^2}}, y' = y, z' = z; \quad \beta < 1. \qquad (3.10)$$

and

$$t' = \frac{\beta_1 t - \frac{1}{c}x}{\sqrt{\beta_1^2 - 1}}, x' = \frac{\beta_1 x - ct}{\sqrt{\beta_1^2 - 1}}, y' = y, z' = z; \quad \beta_1 > 1. \qquad (3.11)$$

These transformations keep the same sign between x, t and x', t'. So, they will be called the timelike transformations and (3.5) will be called the timelike representation of the timelike transformation (TRTT),and (3.7) the spacelike representation of timelike transformation (SRTT).

In the same manner, transformations (3.6) and (3.8) have two common major features, too. Firstly, they will change the sign between ds^2 and ds'^2,i.e.,

$$-ds_v^2 = ds'^2_v. \qquad (3.12)$$

Secondly, their inverse transformations are of the form

$$-t' = \frac{t - \frac{\beta}{c}x}{\sqrt{\beta^2 = 1}}, -x' = \frac{x - vt}{\sqrt{\beta^2 - 1}}, y' = y, z' = z; \quad \beta > 1. \qquad (3.13)$$

and

$$-t' = \frac{\beta_1 t - \frac{1}{c}x}{\sqrt{1-\beta_1^2}}, -x' = \frac{\beta_1 x - ct}{\sqrt{1-\beta_1^2}}, y' = y, z' = z; \quad \beta_1 < 1. \qquad (3.14)$$

These transformations will change the sign between x, t andx', t'. They will be called the spacelike transformations and (3.6) will be called the spacelike representation of spacelike transformation (SRST); and (3.8) the timelike representation of spacelike transformation (TRST).

Now, we have had four types of form of the spacetime transformation under ds^2:

Type I. TRTT, (3.5), it is just the Lorentz transformation;

Type II. SRTT, (3.7), it is the spacelike representation of the Lorentz transformation with the dual velocity $v_1 = c^2/v$, it is larger than the velocity of light;

Type III. SRST, (3.6), it is just the superluminal Lorentz transformation (see Recami, 1986 and Sen Gupta, 1973);

Type IV. TRST, (3.8), it is the timelike representation of the superluminal Lorentz transformation with the dual velocity $v_1 = c^2/v$, but it is less than the velocity of light.

3.3 The catastrophe of physical quantities on the Finsler metric ds^4

Firstly, we shall consider the question of the catastrophe of the measurement of length and time increment. According to the nature of catastrophe of spacetime, the expression for the length of a moving scale $\Delta x'$ measured by a fixed observer (2.6)-(2.9) could be resolved into two parts,

$$\Delta x' = \Delta x\sqrt{1-\beta^2}, \quad \beta < 1. \tag{3.15}$$

$$-\Delta x' = \Delta x\sqrt{\beta^2-1}, \quad \beta > 1. \tag{3.16}$$

$$-\Delta x' = c\Delta t\sqrt{1-\beta_1^2}, \quad \beta_1 < 1. \tag{3.17}$$

$$\Delta x' = c\Delta t\sqrt{\beta_1^2-1}, \quad \beta_1 > 1. \tag{3.18}$$

The expression for the time increment $\Delta\tau$ of the clock at rest with respect to the moving system could be resolved into two parts at the light cone:

$$\Delta\tau = \Delta t\sqrt{1-\beta^2}, \quad \beta < 1, \tag{3.19}$$

$$-\Delta\tau = \Delta t\sqrt{\beta^2-1}, \quad \beta > 1. \tag{3.20}$$

$$-\Delta\tau = \frac{\Delta x}{c}\sqrt{1-\beta_1^2}, \quad \beta_1 < 1, \tag{3.21}$$

$$\Delta\tau = \frac{\Delta x}{c}\sqrt{\beta_1^2-1}, \quad \beta_1 > 1; \tag{3.22}$$

It is very interesting that the $\Delta x'$, (or Δx) will exchange with Δt (or $\Delta\tau$) in the expressions (3.17)-(3.18) and (3.21)-(3.22).

If we let (see the formula (3.20))

$$f(E,P) = E^4 + c^4P^4 - 2c^2E^2P^2 \tag{3.23}$$

as the catastrophe theory, we could find a catastrophe set

$$E = \pm P \tag{3.24}$$

and we could have four types of the representation for the momentum, the energy, and the mass of a moving particle with the rest mass m:

Type I. TRTT

$$p^T(v) = \frac{mv}{\sqrt{1-\beta^2}}, E^T(v) = \frac{mc^2}{\sqrt{1-\beta^2}}, M^T(v) = \frac{m}{\sqrt{1-\beta^2}}; \quad \beta < 1. \tag{3.25}$$

Type II. SRTT

$$p^S\{v_1\} = \frac{mv_1}{\sqrt{\beta_1^2-1}}, E^S(v_1) = \frac{mc^2}{\sqrt{\beta_1^2-1}}, M^S(v_1) = \frac{m}{\sqrt{\beta_1^2-1}}; \quad \beta_1 > 1. \tag{3.26}$$

Type III. SRST

$$p^S\{v\} = \frac{-mv}{\sqrt{\beta^2-1}}, E^S(v) = \frac{-mc^2}{\sqrt{\beta^2-1}}, M^S(v) = \frac{-m}{\sqrt{\beta^2-1}}; \quad \beta > 1. \tag{3.27}$$

Type IV. TRST

$$p^S\{v_1\} = \frac{-mv_1}{\sqrt{1-\beta_1^2}}, E^S(v_1) = \frac{-mc^2}{\sqrt{1-\beta_1^2}}, M^S(v_1) = \frac{-m}{\sqrt{1-\beta_1^2}}; \quad \beta_1 < 1. \tag{3.28}$$

The transformations between type I (or type II) and type III (or type IV) have the forms

$$p^T(v) = \frac{mv}{\sqrt{1-\beta^2}} = \frac{mc}{\sqrt{\beta_1^2-1}} = \frac{1}{c}E^T(v_1), \tag{3.29}$$

$$E^T(v) = \frac{mc^2}{\sqrt{1-\beta^2}} = \frac{mv_1c}{\sqrt{\beta_1^2-1}} = cp^T(v_1), \tag{3.30}$$

$$M^T(v) = \frac{m}{\sqrt{1-\beta^2}} = \frac{\beta_1 m}{\sqrt{\beta_1^2-1}} = \beta_1 M^T(v_1) \tag{3.31}$$

and

$$p^S(v) = \frac{-mv}{\sqrt{\beta^2-1}} = \frac{-mc}{\sqrt{1-\beta_1^2}} = \frac{1}{c}E^S(v_1), \tag{3.32}$$

$$E^S(v) = \frac{-mc^2}{\sqrt{\beta^2-1}} = \frac{-mv_1c}{\sqrt{1-\beta_1^2}} = cp^S(v_1), \tag{3.33}$$

$$M^S(v) = \frac{-m}{\sqrt{\beta^2 - 1}} = \frac{-\beta_1 m}{\sqrt{1-\beta_1^2}} = \beta_1 M^S(v_1). \tag{3.34}$$

With these forms above, we could get that when β=β_1=1,

$$cP(c) = E(c) = mc^2 \quad and \quad M(c) = m. \tag{3.35}$$

Note that although all through Einstein's relativistic physics there occur indications that mass and energy are equivalent according to the formula

$$E = mc^2.$$

But it is only an Einstein's hypothesis.

It is very interesting that from type I and type IV we could get

$$E^2 - c^2p^2 = m^2c^4, \quad v < c \quad \text{and} \quad v_1 < c \quad (i.e., v > c) \tag{3.36}$$

and from type II and type III

$$E^2 - c^2p^2 = -m^2c^4, \quad v > c \quad \text{and} \quad v_1 > c \quad (i.e., v < c) \tag{3.37}$$

Here, we have forgotten the indices for the types in Equations (3.35) to (3.37). If we let the $H^2(E,P) = E^2 \quad c^2P^2$, then we could get

$$f(H, mc) = H^4 - (mc^2)^4. \tag{3.38}$$

It is a type II of the double-cusp catastrophe, we could also get (3.36) and (3.37) from it.

3.4 The catastrophe a charged particle in an electromagnetic field on the Finsler spacetime ds^4

The Hamilton-Jacobi equation for a particle in an electromagnetic field in the Finsler spacetime, formula (2.25) is a type II of the double-cusp catastrophe. We could get that

$$c^2(\nabla S - \frac{e}{c}A)^2 - (\frac{\partial S}{\partial t} + c\Phi)^2 + m^2c^4 = 0 \tag{3.39}$$

for type I and type IV of the spacetime transformation.

$$c^2(\nabla S - \frac{e}{c}A)^2 - (\frac{\partial S}{\partial t} + c\Phi)^2 - m^2c^4 = 0 \tag{3.40}$$

for type II and type III of the spacetime transformation.

Now, we consider the catastrophe change of the equation of a charge in an electromagnetic field. By equation (2.26), we could get

$$\frac{d}{dt}\frac{mv}{\sqrt{1-\beta^2}} = eE_e + \frac{e}{c}v \times H_e, \qquad v < c \tag{3.41}$$

and

$$-\frac{d}{dt}\frac{mv}{\sqrt{\beta^2-1}} = eE_e + \frac{e}{c}v \times H_e, \qquad v > c \ . \tag{3.42}$$

If we integrate (3.41) and (3.42), then

$$\frac{mc^2}{\sqrt{1-\beta^2}} - \frac{mc^2}{\sqrt{1-\beta_0^2}} = eU, \qquad v_0 < c \tag{3.43}$$

and

$$\frac{mc^2}{\sqrt{\beta_0^2-1}} - \frac{mc^2}{\sqrt{\beta^2-1}} = eU, \qquad v_0 > c \ . \tag{3.44}$$

So, the velocity v has

$$v = c\sqrt{1-\left(\frac{eU}{mc}+1/\sqrt{1-\beta_0^2}\right)^{-2}} < c, \qquad iff \quad v_0 < c, \tag{3.45}$$

and

$$v = c\sqrt{1+\left(\frac{eU}{mc}-1/\sqrt{\beta_0^2-1}\right)^{-2}} > c, \qquad iff \quad v_0 > c \ . \tag{3.46}$$

The expressions (3.45) and (3.46) mean that if $v_0 < c$, then for the charged particle always $v < c$; and if $v_0 > c$, then $v > c$. The velocity of light will be a bilateral limit: i.e., it is both of the maximum for the subluminal-speeds and the minimum for the superluminal-speeds.

If we let

$$f(H_e, E_e) = H_e^4 + E_e^4 - 2H_e^2E_e^2, \tag{3.47}$$

we will get that the catastrophe set is

$$H_e = \pm E_e \tag{3.48}$$

and could obtain the spacetime transformation equations for the electromagnetic field components(by (2.31) and (2.32)):

Type I. TRTT

$$\begin{cases} H_x' = H_x, & E_x' = E_x, \\ H_y' = \frac{H_y+\beta E_z}{\sqrt{1-\beta^2}}, & E_y' = \frac{E_y-\beta H_z}{\sqrt{1-\beta^2}}, \\ H_z' = \frac{H_z-\beta E_y}{\sqrt{1-\beta^2}}, & E_z' = \frac{E_z+\beta H_y}{\sqrt{1-\beta^2}}. \end{cases} \tag{3.49}$$

Type II. SRTT

$$\begin{cases} H_x' = H_x, & E_x' = E_x, \\ H_y' = \frac{\beta_1 H_y+E_z}{\sqrt{\beta_1^2-1}}, & E_y' = \frac{\beta_1 E_y-H_z}{\sqrt{\beta_1^2-1}}, \\ H_z' = \frac{\beta_1 H_z-E_y}{\sqrt{\beta_1^2-1}}, & E_z' = \frac{\beta_1 E_z+H_y}{\sqrt{\beta_1^2-1}}. \end{cases} \tag{3.50}$$

Type III. SRST

$$\begin{cases} H_x' = H_x, & E_x' = E_x, \\ -H_y' = \frac{H_y+\beta E_z}{\sqrt{\beta^2-1}}, & -E_y' = \frac{E_y-\beta H_z}{\sqrt{\beta^2-1}}, \\ -H_z' = \frac{H_z-\beta E_y}{\sqrt{\beta^2-1}}, & -E_z' = \frac{E_z+\beta H_y}{\sqrt{\beta^2-1}}. \end{cases} \tag{3.51}$$

Type IV. TRST

$$\begin{cases} H_x' = H_x, & E_x' = E_x, \\ -H_y' = \frac{\beta_1 H_y+E_z}{\sqrt{1-\beta_1^2}}, & -E_y' = \frac{\beta_1 E_y-H_z}{\sqrt{1-\beta_1^2}}, \\ -H_z' = \frac{\beta_1 H_z-E_y}{\sqrt{1-\beta_1^2}}, & -E_z' = \frac{\beta_1 E_z+H_y}{\sqrt{1-\beta_1^2}}. \end{cases} \tag{3.52}$$

3.5 The interchange of the forces between the attraction and the rejection

Usually, because of the equivalence of energy and mass in the relativity theory, ones believe that an object has due to its motion will add to its mass. In other words, it will make it harder to increase its speed. This effect is only really significant for objects moving at speeds close to the speed of light. So, only light, or other waves that have no intrinsic mass, can move at the speed of light.

The mass is the measure of the gravitational and inertial properties of matter. Once thought to be conceivably different, gravitational mass and inertial mass have recently been shown to be the same to one part in 10^{11}.

Inertial mass is defined through Newton's second law, F=ma, in which m is mass of body. F is the force action upon it, and a is the acceleration of the body induced by the force. If two bodies are acted upon by the same force (as in the idealized case of connection with a massless spring), their instantaneous accelerations will be in inverse ratio to their masses.

Now, we need discuss the problem of defining mass m in terms of the force and acceleration. This, however, implies that force has already been independently defined, which is by no means the case.

3.5.1 Electromagnetic mass and electromagnetic force

It is well known that the mass of the electron is about 2000 times smaller than that of the hydrogen atom. Hence the idea occurs that the electron has, perhaps, no "ordinary" mass at all, but is nothing other than an "atom of electricity", and that its mass is entirely electromagnetic in origin. Then, the theory found strong support in refined observations of cathode rays and of the β-rays of radioactive substances, which are also ejected electrons. If magnetic action on these rays allows us to determine the ratio of the charge to the mass, $\frac{e}{m_{el}}$, and also their velocity v, and that at first a definite value for $\frac{e}{m_{el}}$ was obtained, which was independent of v if $v << c$. But, on proceeding to higher velocities, a decrease of $\frac{e}{m_{el}}$ was found. This effect was particularly clear and could be measured quantitatively in the case of the β-rays of radium, which are only slightly slower than light. The assumption that an electric charge should depend on the velocity is incompatible with the ideas of the electron theory. But, that the mass should depend on the velocity was certainly to be expected if the mass was to be electromagnetic in origin. To arrive at a quantitative theory, it is true, definite assumptions had to be made about the form of the electron and the distribution of the charge on it. M. Abraham (1903) regarded the electron as a rigid sphere, with a charge distributed on the one hand, uniformly over the interior, or, on the other, over the surface, and he showed that both assumptions lead to the same dependence of the electromagnetic mass on the velocity, namely, to an increase of mass with increasing velocity. The faster the electron travels, the more the electromagnetic field resists a further increase of velocity. The increase of m_{el} explains the observed decrease of $\frac{e}{m_{el}}$, and Abraham's theory agrees quantitatively very well with the results of measurement of Kaufmann (1901) if it is assumed that there is no "ordinary" mass present. But, the electromagnetic force $F = e[E + \frac{1}{c}(v \times H)]$ was believed to be a constant and be independent of the velocity v.

Note that if we support that the mass m is independent of the velocity v, but the electromagnetic force $F = e[E + \frac{1}{c}(v \times H)]$ is dependent of the velocity v, it will be incompatible with neither the ideas of the electron theory nor the results of measurement of Kaufmann. One further matter needs attention: the E and$H$ occurring in the formula for the force F are supposed to refer to that system in which the electron is momentarily at rest.

3.5.2 The mass and the force in the Einstein's special relativity

In the Einstein's special relativity, Lorentz's formula for the dependency of mass on velocity has a much more general significance than is the electromagnetic mass apparent. It must hold for every kind of mass, no matter whether it is of electrodynamics origin or not.

Experiments by Kaufmann (1901) and others who have deflected cathode rays by electric and magnetic fields have shown very accurately that the mass of electrons grows with velocity according to Lorentz's formula (**??**). On the other hand, these measurements can no longer be regarded as a confirmation of the assumption that all mass is of electromagnetic origin. For Einstein's theory of relativity shows that mass as such, regardless of its origin, must depend on velocity in the way described by Lorentz's formula.

Up to now, if we support that all kinds of the mass, m, are independent of the velocity v, but all forces are dependent of the velocity v, it will be incompatible with neither the ideas of the physical theory nor the results of measurement of physics. Could make some mew measurements of physics (or some observations of astrophysics) to support this viewed from another standpoint.

3.5.3 The interchange of the forces between the attraction and the rejection

Let us return to the Newton's second law, $F{=}ma$, we can see that the product of mass and acceleration is a quantity antisymmetric with respect to the two interaction particles B and C. We shall now make the hypothesis that the value of this quantity in any given case depends on the relative position of the particles and sometimes on their relative velocities as well as the time. We express this functional dependence by introducing a vector function $F_{BC}(r, \dot{r}, t)$, where r is the position vector of B with respect to C and $\dot{r}$ is the relative velocity. We then write

$$m_B a_{BC} = F_{BC}. \qquad (3.53)$$

and define the function F_{BC} as the force acting on the particle B due to the particle C. It is worth while to stress the significance of the definition of force presented here. It will be noted that no merely anthropomorphic notion of push of pull is involved. Eq.(3.53) states that the product of mass and acceleration, usually known as the *kinetic reaction*, is equal to the *force*.

Now, if we explain the experiments by Kaufmann (1901) with here point of view, then, we could say that the electromagnetic force $F = e[E + \frac{1}{c}(v \times H)]$ is a function dependent of the velocity v, $F = F(v)$.

From the above mentioned, the relativity theory provides for an increase of apparent inertial mass with increasing velocity according to the formula

$$m = \frac{m_0}{\sqrt{1-\beta^2}}$$

could be understood equivalently as a decrease of the effective force of the fields with increasing relativistic velocity between the source of the field and the moving body according to the formula

$$F_{eff} = F\sqrt{1-\beta^2}.$$

Further, the negative apparent inertial mass could be understood equivalently as the effective forces of the fields have occurred the interchange between the attraction and the rejection according to the formula.

$$F_{eff} = -F\sqrt{\beta^2 - 1}.$$

3.5.4. The character velocity and effective forces for a forces

Up to now, one common essential feature for forces is neglected that the character velocities for forces. Ones commonly believe that if the resistance on the wagon with precisely the same force with which the horse pulls forward on the wagon then the wagon will keep the right line moving with a constant velocity. However, we could ask that if the resistance on the wagon is zero force then will the wagon be continue accelerated by the horse? How high velocity could be got by the wagon? It is very easy understood that the maximum velocity of the wagon, v_{max}, will be the fastest running velocity of the horse, v_{fst}. The velocity v_{fst} is just the character velocity, v_c, for the pulling force of the horse. When the velocity of the wagon is zero velocity, the pulling force of the horse to the wagon has the largest effective value $F_{eff} = F$. We assume that a decrease of the effective force with increasing velocity of the wagon, and $F_{eff} = 0$ if and only if $\beta = \frac{v_w}{v_c} = 1$. If $\beta = \frac{v_w}{v_c} > 1$ then $F_{eff}= -kF$. It means that when the velocity of the wagonv_w is larger the character velocity v_c, not that the horse pulls the wagon, but that the wagon pushes the horse.

If the interactions of the fields traverse empty space with the velocity of light, c, then the velocity of light is just the character velocity for all kinds of the interactions of the fields. We guess that the principle of the constancy of the velocity of light is just a superficial phenomenon of the character of the interactions of the fields.

3.5.5. One possible experiment for distinguish between moving mass and effective force

The Newtonian law of universal gravitation assumes that, two bodies attract each other with a force that is proportional to the mass of each body and is inversely proportional to the square of their distance apart:

$$F = G\frac{m_1 m_2}{r^2}. \tag{3.54}$$

According as Einstein's special relativity, if the body$_1$ is moving with constant speed v with respect to the body$_2$, then the mass of the body$_1$ will become with respect to the body$_2$ that

$$M_1 = \frac{m_1}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{3.55}$$

According to the principle of equivalence the body's gravitational mass equal to its inertia mass. So, the force of gravitational interaction between the two bodies will be

$$F_{M.M.} = G\frac{m_1 m_2}{r^2\sqrt{1-\frac{v^2}{c^3}}}. \tag{3.56}$$

But, according as the theory of the effective force, the force of gravitational interaction between the two bodies will be

$$F_{E.F.} = G\frac{m_1 m_2}{r^2}\sqrt{1-\frac{v^2}{c^2}}. \tag{3.57}$$

We hope that could design some new experiments to discover this deviation.

3.6 Decay of particles

On the Einstein's special relativity theory, consider the spontaneous decay of a body of mass M into two parts with masses m_1 and m_2. The law of conservation of energy in the decay, applied in the system of reference in which the body is at rest, gives

$$M = E_{10} + E_{20}, \tag{3.58}$$

where E_{10} and E_{20} are the energies of the emerging particles. Since $E_{10} > m_1$ and $E_{20} > m_2$, the equality (120) can be satisfied only if M¿m_1+m_2, i.e. a body can disintegrate spontaneously into parts the sum of whose masses is less than the mass of the body. On the other hand, if M¡$m_1 + m_2$, the body is stable (with respect to the particular decay) and does not decay spontaneously. To cause the decay in this case, we would have to supply to the body from outside an amount of energy at least equal to its "binding energy" $(m_1 + m_2 - M)$.

Usually, ones believe that momentum as well as energy must be conserved in the decay process. Since the initial momentum of the body was zero, the sum of the momenta of the emerging particles must be zero: $\mathbf{p}_{10}$+$\mathbf{p}_{20}$=0 in the special relativity theory. Consequently $p_{10}^2 = p_{20}^2$, or

$$E_{10}^2 - m_1^2 = E_{20}^2 - m_2^2. \tag{3.59}$$

The two equations (3.58) and (3.59) uniquely determine the energies of the emerging particles

$$E_{10} = \frac{M^2 + m_1^2 - m_2^2}{2M}, \quad E_{20} = \frac{M^2 - m_1^2 + m_2^2}{2M}. \tag{3.60}$$

In a certain sense the inverse of this problem is the calculation of the total energy M of two colliding particles in the system of reference in which their total momentum is zero. (This is abbreviated as the "system of the center of inertia" or the "C-system".) The computation of this quantity gives a criterion for the possible occurrence of various inelastic collision processes, accompanied by a change in state of the colliding particles, or the "creation" of new particles. A process of this type can occur only if the sum of the masses of the "reaction products" does not exceed M.

Suppose that in the initial reference system (the "laboratory" system) a particle with mass m_1 and energy E_1 collides with a particle of mass m_2 which is at rest. The total energy of the two particles is

$$E = E_1 + E_2 = E_1 + m_2,$$

and their total momentum is $\boldsymbol{p}$=$\boldsymbol{p}_1$+$\boldsymbol{p}_2$ =$\boldsymbol{p}_1$. Considering the two particles together as a single composite system, we find the velocity of its motion as a whole from (2.19):

$$V = \frac{p}{E} = \frac{p_1}{E_1 + m_2}. \tag{3.61}$$

This quantity is the velocity of the C-system with respect to the laboratory system (the L-system).

However, in determining the mass M, there is no need to transform from one reference frame to the other. Instead we can make direct use of formula (3.36), which is applicable to the composite system just as it is to each particle individually. We thus have

$$M^2 = E^2 - p^2 = (E_1 + m_2)^2 - (E_1^2 - m_1^2),$$

from which

$$M^2 = m_1^2 + m_2^2 + 2m_2E_1. \qquad (3.62)$$

§4 Conclusions

From the discussion in this paper, we could get the following conclusions:

(1) The special theory of relativity cannot negate the possibility of the existence of superluminal-speed.

(2) The essential nature of the superluminal-speed is the relativity of the temporal order. If one does not know how to distinguish the temporal orders, a particle moving with superluminal-speed could be taken for one moving with a subluminal-speed of some unusual nature.

(3) The specal theory of relativity could be discussed in the Finsler spacetime. The spacetime transformation on the Finsler metric ds^4 contains a new symmetry between the timelike and spacelike.

(4) Some new invariants describe the catastrophe nature of the Finsler spacetime ds^4. They obey the double-cusp catastrophe. The timelike state cannot change smoothly into the spacelike state for a motion particle. But a lightlike state could change suddenly into a timelike state and spacelike state. Also, a timelike state and a spacelike state could change suddenly into a lightlike state.

(5) The length x will exchange the position with the time increment t between v's representation and v_1's representation. The momentum (or energy) in the timelike (or spacelike) representation will be transformed into the energy (or momentum) in the spacelike (or timelike) representation.

(6) The difference between the subluminal- and superluminal-speed would be described as follows: a particle with the subluminal-speed has positive momentum, energy, and moving mass, and a particle with the superluminal-speed has negative ones.

(7) Usually, it is believed that Tachyons have a spacelike energy-momentum four-vector so that

$$E^2 < c^2P^2.$$

Hence, the square of the rest mass m defined by

$$m^2c^4 = E^2 - c^2P^2 < 0$$

requires the 'rest mass' to be imaginary' (see Hawking and Ellis, 1973).

As has been said in this paper, from the expressions (3.25)-(3.28) it is clear that, no matter whether a particle is moving with a subluminal- or superluminal-speed, in the timelike representation it will obey Equation (3.36), but, in the spacelike representation it will obey

Equation (3.37). So, for a particle with superluminal-speed its mass $M(v)$ (energy $E(v)$, and momentum $P(v)$) is negative rather than imaginary. As expression (3.28)

$$E^S(v_1) = -mc^2$$

when $\beta \to 0$.

So the particle with superluminal-speed, in the timelike representation, will remain a negative 'rest-mass'. We shall write:

$$E = \begin{cases} +mc^2 & for \quad sublu\min al - speed, \qquad i.e., \quad v < c(\;\; or \;\; v_1 > c), \\ -mc^2 & for \quad \sup erlu\min al - speed, \quad i.e., v > c(\;\; or \;\; v_1 < c). \end{cases}$$

It was just analyzed by Dirac for the anti-particle. So, we guess that a particle with the superluminal-speed$v > c$ could be regarded as its anti-particle with the dual velocity $v_1 = c^2/v < c$.

References

[1] Arnold, V.I.,*Catastrophe Theory*, Springer-Verlag, Berlin, 1986.

[2] Asanov,G.S., *Finsler Geometry, Relativity and Gauge Theories*, D.Reidel Publ.Co., Dordrecht, Holland, 1985.

[3] Shenglin Cao :1988,Astrophys.Space Sci.145,299.

[4] Shenglin Cao:1990,Astrophys.Space Sci.174,165.

[5] Shenglin Cao:1992,Astrophys.Space Sci.190,303.

[6] Shenglin Cao:1992b,Astrophys.Space Sci.193,123.

[7] Shenglin Cao:1993,Astrophys.Space Sci.208,191.

[8] Einstein, A.: 1923, in A.Sommerfeld(ed.), *The Principle of Relativity*, Dover ,New York.

[9] Hawking, S.W. and Ellis,G.F.R.: 1973, *The Large-Scale Structure of Space-Time*, Cambridge University Press. Cambridge.

[10] Recami, E.: 1986,Nuovo Cimento 9, 1.

[11] Rindler,W.:1977, *Essential Relativity*, Springer-Verlag,Berlin.

[12] Rund,H.:1959, *Differential Geometry of Finsler Spaces*, Springer-Verlag,Berlin.

[13] Sen Gupta N.D.:1973,Indian J. Physics 47,385.

[14] Zeeman,E.C.:1977, *Catastrophe Theory*, Selected Papers 1972 1977, Addison-Wesley.

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On the Number of Graceful Trees

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Abstract: Applying a relation of graceful trees with permutations, we enumerate non-equivalent graceful trees and get a closed formula for such number in this paper.

Key Words: Graceful tree, labeling, permutation.

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§1. Introduction

For a simple graph $G = (V(G), E(G))$, a vertex *labeling* of G is a mapping $\theta : V(G) \to \mathbf{Z}$ of non-negative integers that induces for each edge xy a label depending on $\theta(x)$ and $\theta(y)$. A labeling is called a *graceful labeling* of a graph G if it satisfying three conditions following:

(i) $\forall u, v \in V(G)$, if $u \neq v$, then $\theta(u) \neq \theta(v)$;

(ii) $max\{\theta(v)|v \in V(G)\} = |E(G)|$;

(iii) For $\forall e = xy \in E(G)$, let $\theta(e) = |\theta(x) - \theta(y)|$. Then $\forall e_1, e_2 \in E(G)$, if $e_1 \neq e_2$, then $\theta(e_1) \neq \theta(e_2)$.

Many research works on graph labeling can be found in the reference [2], particularly, graceful graphs. Gracefulness of some graph families can be also seen in references $[4] - [10]$. In this paper, we concentrate on the enumeration problem of graceful trees with given order.

Let $K_n = (V, E)$ be a complete graph with n vertices $v_1, v_2, \cdots, v_n$. All edges of K_n can be denoted by $e_{ij} = v_iv_j$, where $i, j \in N = \{1, 2, \cdots, n\}), (i \neq j$. We denote the vertex labeling of v_i by $\theta(v_i)$, and label it with $\theta(v_i) = i$. Then all edges labeling are respective $\theta(v_nv_1) = n-1$, $\theta(v_nv_2) = n-2$, $\theta(v_{n-1}v_1) = n-2, \cdots, \theta(v_nv_{n-1}) = 1$, $\theta(v_{n-1}v_{n-2}) = 1, \cdots, \theta(v_2v_1) = 1$. Obviously, all edge labels $\theta(v_iv_j)$ make up $(n-1)!$ graceful graphs. Certainly, these graceful graphs include disconnected and isomorphic graphs.

If all edges e_{ij} correspond to coordinates (x_i, y_j) on a Euclidean plane by $x_i = i, y_j = j$ for $1 < i \leq n$, $1 \leq j < n$, then there is a bijection between e_{ij} and (x_i, y_j). Its diagram is a lower triangle with $y = x - a$ for $a = 1, 2, \cdots, n-1$, and the graceful label $\theta(e)$ of an edge e is on the oblique line $y = x - a$.

For example, let $G = K_6$. Its diagram can be found in Fig.1.1.

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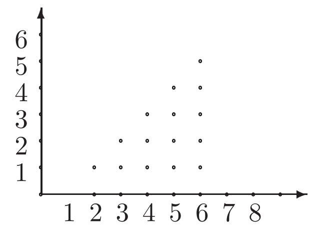

Fig.1.1

In this diagram, if $\theta(e) = 1$, then $\theta(e) \in \{|x-y| : 2-1, 3-2, 4-3, 5-4, 6-5\}$. If $\theta(e) = 2$, then $\theta(e) \in \{|x-y| : 3-1, 4-2, 5-3, 6-4\}$. $\cdots$, If $\theta(e) = 6-1$, then $\theta(e) \in \{|x-y| : 6-1\}$. In other words, there are 5 oblique lines on Fig.1 when $n = 6$. Suppose these lines are L_1, L_2, L_3, L_4, L_5. Let (x_{li}, y_{lj}) be a point on the plane with the coordinate (x_i, y_j) and l denotes $l-th$ oblique line. Then $\{(x_{16}, y_{11}) = (6,1)\} \in L_1$, $\{(x_{25}, y_{21}) = (5,1), (x_{26}, y_{22}) = (6,2)\} \in L_2$, $\{(x_{34}, y_{31}) = (4,1), (x_{35}, y_{32}) = (5,2), (x_{36}, y_{33}) = (6,3)\} \in L_3$, $\cdots$,$\{(x_{52}, y_{51}) = (2,1), (x_{53}, y_{52}) = (3,2), (x_{54}, y_{53}) = (4,3), (x_{55}, y_{54}) = (5,4), (x_{56}, y_{55}) = (6,5)\} \in L_5$. Moreover, we define

$$y_{11}(y_{21}+y_{22})\cdots(y_{n-1,1}+y_{n-1,2}+\cdots+y_{n-1,n-1}) = \sum y_{1j_1}y_{2j_2}\cdots y_{n-1,j_{n-1}}, \qquad (1)$$

$$x_{1,n}(x_{2,n-1}+x_{2,n})\cdots(x_{n-1,2}+x_{n-1,3}+\cdots+x_{n-1,n}) = \sum x_{1j_1}x_{2j_2}\cdots x_{n-1,j_{n-1}}. \qquad (2)$$

The expansion of these polynomials (1) and (2) both have $(n-1)!$ terms. Terms $\prod_{r=1}^{n-1} x_{s_r,i_r}$ and $\prod_{r=1}^{n-1} y_{s_r,j_r})$ in their expansion are called the *correspondent term pair*, denoted by $(x,y) = (\prod_{r=1}^{n-1} x_{s_r,i_r}, \prod_{r=1}^{n-1} y_{s_r,j_r})$. Then each pair (x,y) corresponds to a graceful graph as just explained.

In a labeling graph G, if a vertex labeling $v_i = n-i+1$ is replaced by $v_i = i$, then all edge labels are invariant. This kind of labeling are called *equivalent*, seeing in Fig 1.2 for details, in where, ($a \to a'$ and $b \to b'$).

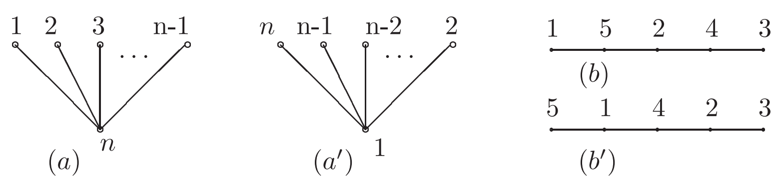

Fig.1.2

For instance, choose $n = 4$ in (1) and (2), i.e.,

$$\begin{aligned} &y_{11}(y_{21}+y_{22})(y_{31}+y_{32}+y_{33}) \\ &\quad = y_{11}y_{21}y_{31} + y_{11}y_{21}y_{32} + y_{11}y_{21}y_{33} + y_{11}y_{22}y_{31} + y_{11}y_{22}y_{32} + y_{11}y_{22}y_{33} \end{aligned}$$

$$x_{14}(x_{23}+x_{24})(x_{32}+x_{33}+x_{34})$$
$$= x_{14}x_{23}x_{32}+x_{14}x_{23}x_{33}+x_{14}x_{23}x_{34}+x_{14}x_{24}x_{32}+x_{14}x_{24}x_{33}+x_{14}x_{24}x_{34}$$

If $(x,y)=(x_{14}x_{23}x_{32}, y_{11}y_{21}y_{31})$, we get $x_{14}-y_{11}=3, x_{23}-y_{21}=2, x_{32}-y_{31}=1$. Hence (x,y) is correspondent to a graceful star graph.

If $(x,y)=(x_{14}x_{23}x_{33}, y_{11}y_{21}y_{32})$, we find $x_{14}-y_{11}=3, x_{23}-y_{21}=2, x_{33}-y_{32}=1$, which is correspondent to a graceful path graph.

If $(x,y)=(x_{14}x_{23}x_{34}, y_{11}y_{21}y_{33})$, we have $x_{14}-y_{11}=3, x_{23}-y_{21}=2, x_{34}-y_{33}=1$. It is correspondent to a graceful triangular graph.

Notice that by definition, these two labeling in pairs $(x,y)=(x_{14}x_{24}x_{32}, y_{11}y_{22}y_{31})$ and $(x,y)=(x_{14}x_{23}x_{34}, y_{11}y_{21}y_{33})$, $(x,y)=(x_{14}x_{24}x_{33}, y_{11}y_{22}y_{32})$ and $(x,y)=(x_{14}x_{23}x_{33}, y_{11}y_{21}y_{32})$, $(x,y)=(x_{14}x_{24}x_{34}, y_{11}y_{22}y_{33})$ and $(x,y)=(x_{14}x_{23}x_{32}, y_{11}y_{21}y_{31})$ are equivalent.

§2. The Enumeration of Graceful Trees

For enumerating graceful trees, a well-known result is useful.

Lemma 2.1([3]) *Let $T=\{t_1,t_2,\cdots,t_{n-1}\}$ be a set of $n-1$ involutions on $N=\{1,2,\cdots,n\}$. Then the product $t_1t_2\cdots t_{n-1}$ is an n-cyclic permutation if and only if (N,T) is a tree.*

From Lemma 2.1 we obtain a result in the following.

Theorem 2.1 *Let (x,y) be a correspondent term pair. If it is an n-cyclic permutation, then (x,y) corresponds to a graceful tree.*

Proof From the formulae (1) and (2), we have y_{11} and $x_{1n}\to(x_{1n},y_{11})$, y_{21} and $x_{2,n-1}\to(x_{2,n-1},y_{21})$, y_{22} and $x_{2,n}\to(x_{2,n},y_{22})$, $\cdots$,etc.. They satisfy $y=x-a, a=1,2,\cdots,n-1$. So $(x,y)=(\prod_{r=1}^{n-1}x_{s_r,i_r}, \prod_{r=1}^{n-1}y_{s_r,j_r})$, namely $\{\theta(x,y)\}=\{1,2,\cdots,n-1\}\}$. Now if it is n- cyclic permutation (not exist less than n), then it is correspondent to a connected graph of n vertices with $n-1$ edges by the Lemma 2.1. Therefore it is a graceful tree. □

Corollary 2.1 *A correspondent term pair (x,y) is a graceful tree only if*

$$\bigcup_{i=1}^{n-1}x_i\bigcup\bigcup_{j=1}^{n-1}y_j=\{1,2,\cdots,n\}.$$

Define a matrix A by

$$A=[a_{xy}],$$

where $a_{xy}=(x,y)$. This matrix shows that there are $(n-1)!/2$ labeling ways on graceful graphs, but in which $(n-2)!/2$ labeling ways are equivalent. We need to delete the pair $(2,n)$ in the matrix A. This is tantamount to cancel equivalent labeling. In addition, the three pairs $(1,n),(1,n-1)$ and $(n-1,n)$ consist of a 3-cyclic with an edge set $\{e_{1n},e_{1,n-1},e_{n-1,n}\}$. In other words, there are $(n-2)!/2$ graceful graphs contain 3-cyclic with edge $e_{n-1,n}$, correspondent to the pair $(n-1,n)$. Hence cancel the pair $(n-1,n)$ in the matrix A. So we get a new matrix A' from A.

According to the previous discussions, define a permutation

$$T(n)=\begin{pmatrix} y_1 & y_2 & \cdots & y_{n-1} \\ x_1 & x_2 & \cdots & x_{n-1}\end{pmatrix},$$

where $y_1=y_2=1, x_1=n, x_2=n-1$. Then we have the next result.

Theorem 2.2 *For an integer $n\geq 3$,*

(i) if $y_{i+1}=y_i$ or $y_{i+1}=y_i+1$ for all indexes i, then $T(n)$ corresponds to a graceful tree;

(ii) if there is an integer k such that $y_i=y_{i+1}$ and $y_{i+2}=y_i+1, y_{i+3}=y_i+2,\cdots,y_{i+k}=y_i+k-1$, rearrange y_j such that the j-th entry is $y'_j\leq j$ for $i+2\leq j\leq i+k$ and define $x'_j=y'_j+n-j$. Then the new pair (x',y') , namely

$$T'(n)=\begin{pmatrix} 1 & 1 & y'_3 & y'_4 & \cdots & y'_{n-1} \\ n & n-1 & x'_3 & x'_4 & \cdots & x'_{n-1}\end{pmatrix}$$

.

is still correspondent to a graceful tree.

Proof The case of $y_3=y_4=\cdots=y_{n-1}=1$ and $x_i=n-i+1, i=3,4,\cdots,n-1$ is trivial,which corresponds to a star tree.

We verify Theorem 2.2(i) in the first. When $y_1=y_2=1, x_1=n, x_2=n-1$, so v_1, v_{n-1} and v_n three vertices consist of a path. When $y_i=y_{i+1}, x_i=y_i+n-i$, then $x_{i+1}=y_i+n-i-1=x_i-1$. When $y_{i+1}=y_i+1, x_i=y_i+n-i$, then $x_{i+1}=x_i$. So for any integer $i, 1\leq i\leq n$, we know that $y_{i+1}=y_i+1\rightarrow x_{i+1}=x_i$; $y_{i+1}=y_i\rightarrow x_{i+1}=x_i-1$, i.e., $0\leq |y_{i+1}-y_i|\leq 1, 0\leq |x_{i+1}-x_i|\leq 1$ and $x_{n-1}-y_{n-1}=1$. Thereafter,

$$\bigcup_{i=1}^{n-1} y_i \bigcup \bigcup_{j=1}^{n-1} x_j=\{1,2,\cdots,n\}.$$

Because three vertices v_1, v_{n-1} and v_n consist of a path. When $y_3=y_2=y_1=1$, we obtain $x_3=n-2$. So v_{n-2} and v_1 are connected. Similarly, if $y_3=2, x_3=n-1$, v_2 and v_{n-1} are connected. In fact, for any integer $i, 1\leq i\leq n$, we have $y_{i+1}=y_i\rightarrow x_{i+1}=x_i+1$ or $y_{i+1}=y_i+1\rightarrow x_{i+1}=x_i$. If $y_{i+1}=y_i$, then y_{i+1} and y_i corresponds to same vertex v_s, x_{i+1} corresponds to vertex v_t, v_s and v_t are connected, by $x_i=y_i+n-i$. Similarly, if $x_{i+1}=x_i$, then x_{i+1} and x_i corresponds to same vertex v_t,y_{i+1} corresponds to vertex v_s, v_s and v_t are connected. we know that $T(n)$ corresponds to a graceful tree by Lemma 2.1.

For Theorem 2.2(ii), let $N=\{1,2,\cdots,n\}$. If $y_i=y_{i+1}, x_{i+1}=x_i-1$ and $y_{i+2},y_{i+3},\cdots,$ y_{i+k} are consecutive plus 1 of y_i, then $x_{i+2}=x_{i+3}=\cdots=x_{i+k}=x_{i+1}$. Since y_{i+1} does not participate in the rearrangement, we know that $x_{i+1}=y_{i+1}+n-i+1$. Notice that $y_{i+2},y_{i+3},\cdots,y_{i+k}$ participating in the rearrangement do not change these labels of n vertices. Namely, the labeling set $\{1,2,\cdots,n\}$ is not dependent on $x_{i+2},x_{i+3},\cdots,x_{i+k}$ by $x_{i+2}=x_{i+3}=\cdots=x_{i+k}=x_{i+1}$. In fact, $y_{i+2},y_{i+3},\cdots,y_{i+k}$ correspond to $k-1$ leaves of a tree, and

$min\{x_i\} = x_{n-1} > max\{y_i\} = y_{n-1}, i = 1, 2, \cdots, n$. If y_{i+r} is replaced by $y_{i+r-j}(1 \leq j \leq r-2)$ for $2 \leq i \leq k$, then $y_{i+r} > y_{i+r-j}$. We obtain $x'_{i+r-j} > x_{i+r-j} = x_{i+1}$, since there exists an $x_s = x'_{i+r-j}(x_1 \geq x_s \geq x_{i+1})$ correspondent to a vertex of a tree, which does not change these y_{i+r} correspondent to leaves. If y_{i+r} is replaced by $y_{i+r+j}(1 \leq j \leq k-r)$, we obtain $x'_{i+r+j} = y_{i+r} + n - i - r - j < x_{i+r+j} = x_{i+1}$. If $x'_{i+r+j} \geq x_{n-1}$, there exists an $x_s = x'_{i+r+j}(x_{i+1} \geq x_s \geq x_{n-1})$. Now if $x'_{i+r+j} \leq y_{n-1}$, then there still exists a $y_t = x'_{i+r+j}$. Both of them do not change these y_{i+r} correspondent to leaves. Therefore,

$$T'(n) = \begin{pmatrix} 1 & 1 & y'_3 & y'_4 & \cdots & y'_{n-1} \\ n & n-1 & x'_3 & x'_4 & \cdots & x'_{n-1} \end{pmatrix}.$$

still corresponds to a graceful tree. □

According to Theorem 2.2, the rearrangement on y_i enable us to get new graceful tree, is not equivalent to the original tree. We enumerate all rearrangement labeling on graceful trees in the following.

Let $T(1^2, 2^2, 3^2, \cdots, k^{r_0})$ denote a permutation

$$\begin{pmatrix} 1 & 1 & y_3 & y_4 & \cdots & y_{n-1} \\ n & n-1 & x_3 & x_4 & \cdots & x_{n-1} \end{pmatrix},$$

in which, $y_1 = y_2, y_3 = y_4, \cdots, y_{2i-1} = y_{2i} = i$ for $i \leq k$. Let $E(T_n)$ denote the number of all non-equivalent graceful trees of n vertices, and $E(T_n, k^{r_0})$ denote the number of permutations on $k+1, k+2, \cdots, n-k-r_0+1$ satisfying $y_i \leq i$ and $x_i = y_i + n - i$ for $k+1 \leq i \leq n-k-r_0+1$. Applying Theorem 2.2 we find the following result.

Theorem 2.3 *For any integer $n > 2$, let $E(T_n, K) = \sum_{1 \leq k \leq \frac{n}{2}} E(T_n, k^{r_0})$. If $n \equiv 0(mod 2)$, then*

$$\begin{aligned} E(T_n, K) &= \sum_{i=2}^{\alpha} i^{n-3i+2}(i^{i-1} - 1) \cdot (i-2)! \\ &+ \sum_{i=1}^{\beta-1} (\alpha+i)((\alpha+i)^{\alpha-2i+1} - 1) \cdot (\alpha+i-2)! \\ &+ \sum_{i=1}^{\gamma} (2i-1) \cdot (\frac{n}{2} - i)! + (\alpha-1)\sum_{i=0}^{\lambda} i! \\ &+ \sum_{i=1}^{\beta} i(\alpha - 2i + \rho + 2) \cdot (\alpha - 2i + \rho)!, \end{aligned} \tag{3}$$

where,

$$\begin{cases} \alpha = \frac{n}{3}, \beta = \frac{n}{6}, \gamma = \frac{n}{6}, \lambda = \frac{n}{3} - 1, \rho = 0, & if\ n \equiv 0(mod 6); \\ \alpha = \frac{n-1}{3}, \beta = \frac{n+2}{6}, \gamma = \frac{n-4}{6}, \lambda = \frac{n-1}{3}, \rho = 1, & if\ n \equiv -2(mod 6); \\ \alpha = \frac{n+1}{3}, \beta = \frac{n-2}{6}, \gamma = \frac{n-2}{6}, \lambda = \frac{n-2}{3}, \rho = -1, & if\ n \equiv 2(mod 6). \end{cases}$$

If $n \equiv 1(mod2)$*, then*

$$
\begin{aligned}
E(T_n, K) &= \sum_{i=2}^{\alpha'} i^{n-3i+2}(i^{i-1}-1)\cdot(i-2)! \\
&+ \sum_{i=1}^{\beta'}((\alpha'+i)^{\alpha'-2i}-1)\cdot(\alpha'+i-2)! \\
&+ \sum_{i=1}^{\gamma'}(2i-1)\cdot(\frac{n-1}{2}-i)! + (\alpha'-1)\sum_{i=0}^{\lambda'} i! \\
&+ \sum_{i=1}^{\beta'} i(\alpha'-2i+\rho'+2)\cdot(\alpha'-2i+\rho')! + \beta' + 1, \qquad (4)
\end{aligned}
$$

where,

$$
\begin{cases}
\alpha' = \frac{n+1}{3}, \beta' = \frac{n-5}{6}, \gamma' = \frac{n+1}{6}, \lambda' = \frac{n+1}{3}-2, \rho' = -1, & if\ n \equiv -1(mod6); \\
\alpha' = \frac{n}{3}, \beta' = \frac{n-3}{6}, \gamma' = \frac{n-3}{6}, \lambda' = \frac{n}{3}-1, \rho' = 0, & if\ n \equiv 3(mod6); \\
\alpha' = \frac{n+2}{3}, \beta' = \frac{n-7}{6}, \gamma' = \frac{n-1}{6}, \lambda' = \frac{n-4}{3}, \rho' = -2, & if\ n \equiv 2(mod6).
\end{cases}
$$

Proof Let $k = 1, r_0 = 2$. Then

$$
T(1^2) = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & \cdots & n-2 \\ n & n-1 & n-1 & n-1 & n-1 & \cdots & n-1 \end{pmatrix}.
$$

In fact, it is correspondent to a graceful tree(see Fig.2.1 below).

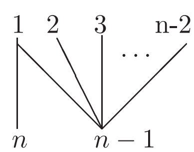

Fig.2.1

If $y_3 \neq 2$, then $y_3 = 3$ because $x_i = y_i + n - i$ and $max\{x_i\} = n$. Similarly, if $y_4 \neq 2$ too, then $y_4 = 4$. If there is an integer r,$3 \leq r \leq n-1$ such that $y_r = 2$, then $y_i = i, x_i = n$ for $3 \leq i < r$. In other word, only $y_3 = 2$ or $y_3 = 3$, and y_4 is one element of the set $\{2,3,4\} - \{y_3\}$, y_5 is one element of the set $\{2,3,4,5\} - \{y_3, y_4\}$, $\cdots$. Continuing this process, y_{n-1} is uniquely determined at the final. Hence the number of permutations is $2 \times 2 \times 2 \times \cdots \times 2 \times 1 = 2^{n-4}$. When

$$
T(1^3) = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & \cdots & n-3 \\ n & n-1 & n-2 & n-2 & n-2 & \cdots & n-2 \end{pmatrix},
$$

then choose an element y_4 in the set $\{2,3,4\}$, an element y_5 in the set $\{2,3,4,5\} - \{y_4\}$, $\cdots$. Continuing in this manner, y_{n-2} and y_{n-1} are 2 selectable. So the number of such permutations is $3 \times 3 \times 3 \times \cdots \times 3 \times 2! = 3^{n-6} \cdot 2!$.

Similarly, When

$$T(1^4) = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 3 & \cdots & n-4 \\ n & n-1 & n-2 & n-3 & n-3 & n-3 & \cdots & n-3 \end{pmatrix},$$

we have $E(T_n, 1^4) = 4 \times 4 \times 4 \times \cdots \times 4 \times 3! = 4^{n-8} \cdot 3!$ and generally,

$$E(T_n, 1^r) = \begin{cases} r^{n-2r}(r-1)!, & \begin{cases} 2 \le r \le \frac{n}{2} - 1, & \text{n is even;} \\ 2 \le r \le \frac{n-1}{2}, & \text{n is odd.} \end{cases} \\ (n-r-1)!, & \begin{cases} \frac{n}{2} - 1 < r \le n-1, & \text{n is even;} \\ \frac{n-1}{2} < r \le n-1, & \text{n is odd.} \end{cases} \end{cases} \tag{5}$$

In general, if $k + \lceil \frac{k}{2} \rceil \le \frac{n}{2} - 1$

$$E(T_n, k^r) = \sum_{r=k+1}^{\frac{n}{2} - \lceil \frac{k}{2} \rceil} r^{n-2r-k+1} \cdot (r-1)! + \sum_{r=\frac{n}{2} - \lceil \frac{k}{2} \rceil + 1}^{n-k} (n-k-r)!, \text{n is even;}$$

$$E(T_n, k^r) = \sum_{r=k+1}^{\lceil \frac{n-k}{2} \rceil} r^{n-2r-k+1} \cdot (r-1)! + \sum_{r=\lceil \frac{n-k}{2} \rceil + 1}^{n-k} (n-k-r)!, \text{n is odd.} \tag{6}$$

If $k + \lceil \frac{k}{2} \rceil > \frac{n}{2} - 1$

$$E(T_n, k^r) = \begin{cases} \sum_{r=1}^{n-2k} (n-2k-r)!, & \text{n is even;} \\ \sum_{r=1}^{n-2k-1} (n-2k-r)!, & \text{n is odd.} \end{cases} \tag{7}$$

By (6) and (7), when n is even, define

$$f(k) = \sum_{r=k+1}^{\frac{n}{2} - \lceil \frac{k}{2} \rceil} r^{n-2r-k+1} \cdot (r-1)!$$

with $k \in \{\frac{n}{3} - 1, \frac{n-1}{3} - 1, \frac{n+1}{3} - 1\}$. Then we know that

(a) if $n \equiv 0(mod 6)$, $k = \frac{n}{3} - 1$, then

$$f(\frac{n}{3} - 1) = (\frac{n}{3})^2 (\frac{n}{3} - 1)!;$$

(b) if $n \equiv -2(mod 6)$, $k = \frac{n-1}{3} - 1$, then

$$f(\frac{n-1}{3} - 1) = (\frac{n-1}{3})^3 (\frac{n-1}{3} - 1)! + (\frac{n-1}{3} + 1)(\frac{n-1}{3})!;$$

(c) if $n \equiv 2(mod 6)$, $k = \frac{n}{3} - 1$, then

$$f(\frac{n+1}{3} - 1) = (\frac{n+1}{3})(\frac{n+1}{3} - 1)!.$$

Whence we obtain that

$$\sum_{i=2}^{r<\frac{n+1}{3}} (i^{n-2i}+i^{n-2i-1}+i^{n-2i-2}+\cdots+i^{n-3i+2})(i-1)! = \sum_{i=2}^{r<\frac{n+1}{3}} i^{n-3i+2}(i^{i-1}-1)(i-2)!. \quad (8)$$

When $n \equiv 0(mod6)$,

$$\sum_{i=1}^{\frac{n}{6}-1}((\frac{n}{3}+i)^1+(\frac{n}{3}+i)^2+(\frac{n}{3}+i)^3+\cdots+(\frac{n}{3}+i)^{\frac{n}{3}-2i+1})(\frac{n}{3}+i-1)!$$
$$=\sum_{i=1}^{\frac{n}{6}-1}(\frac{n}{3}+i)((\frac{n}{3}+i)^{(\frac{n}{3}-2i+1)}-1)(\frac{n}{3}+i-2)!.$$

We obtain that

$$\sum_{k+\lceil\frac{k}{2}\rceil\leq\frac{n}{2}-1}\sum_{r=k+1}^{\frac{n}{2}-\lceil\frac{k}{2}\rceil} r^{n-2r-k+1}(r-1)!$$
$$=\sum_{i=2}^{\frac{n}{3}} i^{n-3i+2}(i^{i-1}-1)(i-2)!+\sum_{i=1}^{\frac{n}{6}-1}(\frac{n}{3}+i)((\frac{n}{3}+i)^{(\frac{n}{3}-2i+1)}-1)(\frac{n}{3}+i-2)!. \quad (9)$$

Similarly, when $n \equiv -2(mod6)$,

$$\sum_{k+\lceil\frac{k}{2}\rceil\leq\frac{n}{2}-1}\sum_{r=k+1}^{\frac{n}{2}-\lceil\frac{k}{2}\rceil} r^{n-2r-k+1}(r-1)!$$
$$=\sum_{i=2}^{\frac{n-1}{3}} i^{n-3i+2}(i^{i-1}-1)(i-2)!$$
$$+\sum_{i=1}^{\frac{n-4}{6}}(\frac{n-1}{3}+i)((\frac{n-1}{3}+i)^{(\frac{n-1}{3}-2i+1)}-1)(\frac{n-1}{3}+i-2)!, \quad (10)$$

and when $n \equiv 2(mod6)$,

$$+\sum_{k+\lceil\frac{k}{2}\rceil\leq\frac{n}{2}-1}\sum_{r=k+1}^{\frac{n}{2}-\lceil\frac{k}{2}\rceil} r^{n-2r-k+1}(r-1)!$$
$$=\sum_{i=2}^{\frac{n+1}{3}} i^{n-3i+2}(i^{i-1}-1)(i-2)!$$
$$+\sum_{i=1}^{\frac{n-8}{6}}(\frac{n+1}{3}+i)((\frac{n+1}{3}+i)^{(\frac{n+1}{3}-2i+1)}-1)(\frac{n+1}{3}+i-2)!. \quad (11)$$

Now let

$$f_1(k) = \sum_{r=\frac{n}{2}-\lceil\frac{k}{2}\rceil+1}^{n-k} (n-k-r)!.$$

Similarly, we get that

(a) When $n \equiv 0(mod6)$, $k = \frac{n}{3} - 1$,

$$\sum_{r=\frac{n}{2}-\lceil\frac{k}{2}\rceil+1}^{n-k} (n-k-r)! = \sum_{i=1}^{\frac{n}{3}-1} f_1(i)$$
$$= \sum_{i=1}^{\frac{n}{6}} (2i-1)(\frac{n}{2}-i)! + (\frac{n}{3}-1) \sum_{i=0}^{\frac{n}{3}-1} i!. \tag{12}$$

(b) When $n \equiv -2(mod6)$, $k = \frac{n-1}{3} - 1$,

$$\sum_{r=\frac{n}{2}-\lceil\frac{k}{2}\rceil+1}^{n-k} (n-k-r)! = \sum_{i=1}^{\frac{n-1}{3}-1} f_1(i)$$
$$= \sum_{i=1}^{\frac{n-4}{6}} (2i-1)(\frac{n}{2}-i)! + (\frac{n-1}{3}-1) \sum_{i=0}^{\frac{n-1}{3}-1} i!. \tag{13}$$

(c) When $n \equiv 2(mod6)$, $k = \frac{n+1}{3} - 1$,

$$\sum_{r=\frac{n}{2}-\lceil\frac{k}{2}\rceil+1}^{n-k} (n-k-r)! = \sum_{i=1}^{\frac{n+1}{3}-1} f_1(i)$$
$$= \sum_{i=1}^{\frac{n-2}{6}} (2i-1)(\frac{n}{2}-i)! + (\frac{n+1}{3}-1) \sum_{i=0}^{\frac{n+1}{3}-1} i!. \tag{14}$$

When $k + \lceil\frac{k}{2}\rceil > \frac{n}{2} - 1$. Let

$$f_2(k) = \sum_{r=1}^{n-2k} (n-2k-r)!.$$

We know that

(a)When $n \equiv 0(mod6)$, $k > \frac{n}{3} - 1$,

$$\sum f_2(k \geq \frac{n}{3}) = \sum_{i=1}^{\frac{n}{6}} i(\frac{n}{3}-2i+2)(\frac{n}{3}-2i)!. \tag{15}$$

(b) When $n \equiv -2(mod6)$, $k > \frac{n-1}{3} - 1$,

$$\sum f_2(k \geq \frac{n-1}{3}) = \sum_{i=1}^{\frac{n+2}{6}} i(\frac{n-1}{3} - 2i + 3)(\frac{n-1}{3} - 2i + 1)!. \quad (16)$$

(c) When $n \equiv 2(mod6)$, $k > \frac{n+1}{3} - 1$,

$$\sum f_2(k \geq \frac{n+1}{3}) = \sum_{i=1}^{\frac{n-2}{6}} i(\frac{n+1}{3} - 2i + 1)(\frac{n+1}{3} - 2i - 1)!. \quad (17)$$

To sum up, we obtain (3) by formulae $(9), (10), (11), (12), (13), (14), (15), (16)$ and (17).

Similarly, the discussion for the case $n \equiv 1(mod2)$ can be divided into three subcases, i.e., $n \equiv -1(mod6), k = \frac{n-2}{3}$, $n \equiv 3(mod6), k = \frac{n}{3} - 1$ and $n \equiv 1(mod6), k = \frac{n-4}{3}$, and the formula (4) can be found as the formula (3). □

For example, $E(T_6, K) = 10$ when $n = 6$. We obtain 10 non-equivalent graceful trees by permutations following.

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 4 \\ 6 & 5 & 5 & 5 & 5 \end{pmatrix} \to \{e_{16}, e_{15}, e_{25}, e_{35}, e_{45}\}; \begin{pmatrix} 1 & 1 & 2 & 4 & 3 \\ 6 & 5 & 5 & 6 & 4 \end{pmatrix} \to \{e_{16}, e_{15}, e_{25}, e_{46}, e_{34}\};$$
$$\begin{pmatrix} 1 & 1 & 3 & 2 & 4 \\ 6 & 5 & 6 & 4 & 5 \end{pmatrix} \to \{e_{16}, e_{15}, e_{36}, e_{24}, e_{45}\}; \begin{pmatrix} 1 & 1 & 3 & 4 & 2 \\ 6 & 5 & 6 & 6 & 3 \end{pmatrix} \to \{e_{16}, e_{15}, e_{36}, e_{46}, e_{23}\};$$
$$\begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 6 & 5 & 4 & 4 & 4 \end{pmatrix} \to \{e_{16}, e_{15}, e_{14}, e_{24}, e_{34}\}; \begin{pmatrix} 1 & 1 & 1 & 3 & 2 \\ 6 & 5 & 4 & 5 & 3 \end{pmatrix} \to \{e_{16}, e_{15}, e_{14}, e_{35}, e_{23}\};$$
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 6 & 5 & 4 & 3 & 3 \end{pmatrix} \to \{e_{16}, e_{15}, e_{14}, e_{13}, e_{23}\}; \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 6 & 5 & 4 & 3 & 3 \end{pmatrix} \to \{e_{16}, e_{15}, e_{14}, e_{13}, e_{12}\};$$
$$\begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 6 & 5 & 5 & 4 & 4 \end{pmatrix} \to \{e_{16}, e_{15}, e_{25}, e_{24}, e_{34}\}; \begin{pmatrix} 1 & 1 & 2 & 2 & 2 \\ 6 & 5 & 5 & 4 & 3 \end{pmatrix} \to \{e_{16}, e_{15}, e_{25}, e_{24}, e_{23}\}.$$

When n is a large number, $E(T_n) >> E(T_n, K)$. Of course, there exist a lot of isomorphic trees in the previous enumeration. We have verified the number of non-isomorphic graceful paths P_n for $n \leq 13$ vertices in the following table.

n	2	3	4	5	6	7	8	9	10	11	12	13
E(P_n)	1	1	1	2	6	8	10	30	74	162	330	760

References

[1] Dénes, J., The representation of a permutation as the product of a minimal number of transpositions, and its connection with the theory of graph, *Magyar. Tud. Akad, Mat, Kutató Int. Közl.*, 4, 1959, 63-71.

[2] Joseph A.Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, Jan 3,2007.

[3] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs, Proc.* (1966), 349-355.

[4] Y.C.Yang and G.X.Wang, On the gracefulness of two graph families, *J.Beijing Polytechnic University*, Vol.11, 4(1985), 59-66.

[5] Y.C.Yang and G.X.Wang, On the gracefulness of $C_n \times P_2$, *J. Math. Res. & Exp.*, Vol.12, 1(1992), 143-148.

[6] Y.C.Yang and G.X.Wang, An initial research on the gracefulness of $P_n \cup P_m$, *J. Beijing Polytechnic University*, Vol.19, 4(1993), 15-26.

[7] Y.C.Yang and G.X.Wang, On the gracefulness of $P_n \cup P_m$, *J. Guangxi University*, Vol.18, 2(1993), 87-94.

[8] Y.C.Yang and G.X.Wang, On the gracefulness of the union of two stars and three stars, *Combinatorics, Graph Theory, Algorithms and Applications (Beijing, 1993)*, 417-424, World Sci. Publishing, River Edge, NJ, 1994.

[9] Y.C.Yang and G.X.Wang, On the gracefulness of product graph $C_{4n+2} \times P_{4m+3}$, *Combinatorics, Graph Theory, Algorithms and Applications (Beijing, 1993)*, 425-431, World Sci. Publishing, River Edge, NJ, 1994.

[10] Y.C.Yang and G.X.Wang, Generalized graceful graphs with applications, *J. Taiyuan Institute of Machinery*, Vol.15 (Supp. 1994), 6-15.

We think in generalities, but we live in detail.

By A.N. Whitehead, a British mathematician.

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[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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