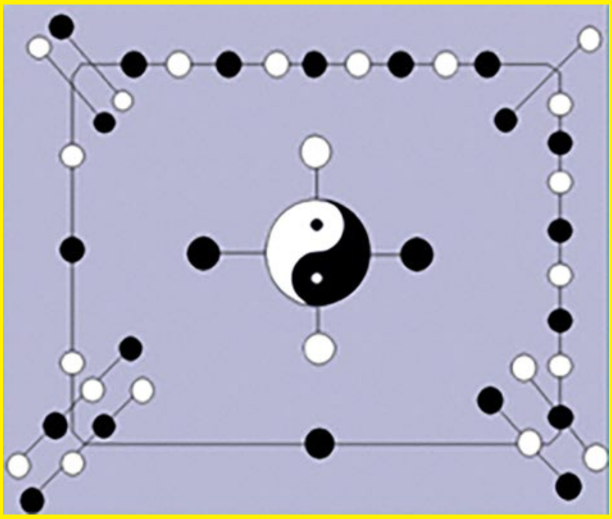
ISSN 1937 - 1055

VOLUME 1, 2023

INTERNATIONAL JOURNAL OF MATHEMATICAL COMBINATORICS



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THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

ACADEMY OF MATHEMATICAL COMBINATORICS & APPLICATIONS, USA

March, 2023

Vol.1, 2023 ISSN 1937-1055

International Journal of

Mathematical Combinatorics

(www.mathcombin.com)

Edited By

The Madis of Chinese Academy of Sciences and

Academy of Mathematical Combinatorics & Applications, USA

March, 2023

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Famous Words:

The man of science does not discover in order to know; He wants to know in order to discover.

By A.N. Whitehead, a British philosopher.

International J.Math. Combin. Vol.1(2023), 1-20

Grassmannians in the Lattice Points of Dilations of the Standard Simplex

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Abstract: A remarkable connection between the cohomology ring $\mathrm{H}^*(\mathrm{Gr}(d, d+r), \mathbb{Z})$ of the Grasssmannian $\mathrm{Gr}(d, d+r)$ and the lattice points of the dilation $r\Delta_d$ of the standard d-simplex is investigated. The natural grading on the cohomology induces different gradings of the lattice points of $r\Delta_d$. This leads to different refinements of the Ehrhart polynomial $L_{\Delta_d}(r)$ of the standard d-simplex. We study two of these refinements which are defined by the weights $(1, 1, \cdots, 1)$ and $(1, 2, \cdots, d)$. One of the refinements interprets the Poincaré polynomial $\mathrm{P}(\mathrm{Gr}(d, d+r), z)$ as the counting of the lattice points which lie on the slicing hyperplanes of the dilation $r\Delta_d$. Therefore, on the combinatorial level the Poincaré polynomial of the Grassmannian $\mathrm{Gr}(d, d+r)$ is a refinement of the Ehrhart polynomial $L_{\Delta_d}(r)$ of the standard d-simplex Δ_d.

Key Words: Cohomology ring, Grassmannian, partition, lattice polytope, simplex.

AMS(2010): 14M15, 14N15, 05E05.

§1. Introduction

Consider the diagonal sequence D_d of natural numbers realized from Pascal triangle below:

$$
\begin{array}{lccccccccccccc}
D_1 : & & & & & & & 1 & & & & & & \\
D_2 : & & & & & & 1 & & 1 & & & & & \\
D_3 : & & & & & 1 & & 2 & & 1 & & & & \\
D_4 : & & & & 1 & & 3 & & 3 & & 1 & & & \\
D_5 : & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
D_6 : & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
\iddots & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
& \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots
\end{array}
$$

Pascal Triangle

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[1]Supported by EPSRC GCRF grant EP/T001968/1, part of the Abram Gannibal Project.

[2]Received February 15, 2023, Accepted March 15, 2023.

One of the combinatorial interpretations of the terms of the sequence $D_d := \binom{r+d}{d}_{r=0}$, $d \in \mathbb{N}$, has to do with the counting of the lattice points associated with the dilations $r\Delta_d$ of the standard d-simplex Δ_d. By the standard d-simplex Δ_d we mean the convex hull of the set $\{\underline{0}, e_1, \cdots, e_d\}$ where $e_i' s$, $1 \leq i \leq d$ are the standard vectors in $\mathbb{R}^d$ and $\underline{0}$ is the origin. That is,

$$\Delta_d := \operatorname{conv}(\underline{0}, e_1, \ldots, e_d) = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot e_i \geq 0, \ \sum_{i=1}^{d} \mathbf{x} \cdot e_i \leq 1\} \tag{1.1}$$

and the dilation $r\Delta_d$, is given by

$$r\Delta_d = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot e_i \geq 0, \ \sum_{i=1}^{d} \mathbf{x} \cdot e_i \leq r, \quad r \in \mathbb{N}\}. \tag{1.2}$$

Lattice points are the points whose coordinates are integers. Asking for the lattice points on $r\Delta_d$ is tantamount to counting the integer solutions for the inequality

$$\sum_{i=1}^{d} \mathbf{x} \cdot e_i \leq r. \tag{1.3}$$

The number of lattice points on any given lattice polytope is well known. This is central theme of Ehrhart polynomials, [3], [6], [10], [11] and [16]. In fact the number of the lattice points on $r\Delta_d$ is given by

$$|r\Delta_d \cap \mathbb{Z}^d_{\geq 0}| = \binom{r+d}{d} \tag{1.4}$$

and its generating function by

$$\mathrm{P}(r\Delta_d, z) = \sum_{r=0}^{\infty} A_r z^r = \frac{1}{(1-z)^{d+1}}, \text{ where } \ A_r = \binom{r+d}{d}. \tag{1.5}$$

On the other hand, Grassmannians are ubiquitous in nature and they constitute one of the best understood algebraic varieties. They admit algebraic, combinatorial and geometric structures which are very elegant. Their classical cohomology theory has taken the center stage in algebraic combinatorics in recent years, see [4], [5], [7] , [8], [9], [10] and [12]. It turns out that the lattice points on $r\Delta_d$ encode some vital information about the indexing partitions of the Schubert varieties contained in the Grassmannian $Gr(d, d+r)$. This sheds more light on the cohomology ring of the Grassmannian. It is well known that the multiplicative generators of the cohomology of the Grassmannian $\mathrm{Gr}(d, d+r)$ are given by the special Schubert cycles σ_λ, see [3]. These cycles are indexed by one-row partitions $\lambda = (k), 1 \leq k \leq r$ and they constitute the total Chern class of the quotient bundle $\mathcal{Q}$, that is,

$$c(\mathcal{Q}) = 1 + \sigma_{\square} + \sigma_{\square\square} + \cdots + \sigma_{\square\square\cdots\square_{1\times r}}.$$

We study the monomials identified with the semi standard tableaux of these one-row Young diagrams and realize a natural graded polynomial $T_r(t)$ called dilation polynomial. This is our first refinement of the Ehrhart polynomial $L_\Delta(r)$ of the standard d-simplex Δ_d. It comes with the natural weight $(1, 1, \cdots, 1)$. The second refinement is the the Poincaré polynomial $\mathbf{P}(Gr(d, d+r), z)$ of the Grassmannian $\mathrm{Gr}(d, d+r)$ interpreted as the slicing of $r\Delta_d$ with hyperplanes with respect to the weight $(1, 2, \ldots, d)$. It is interesting to note that the natural grading on the cohomology of the Grassmannian $\mathrm{Gr}(d, d+r)$ induces different gradings of the lattice points of the dilation $r\Delta_d$ which give various refinements of the Ehrhart polynomial $L_\Delta(r)$. The paper is a generalisation of the previous studies in [1] and [2]. In section 2, we introduce a technique of counting lattice points by grading with respect to the weight $\mathbf{a} = (1, 1, \ldots, 1)$. This is just the slicing of the dilation $r\Delta_d$ into parallel regular $(d-1)$-simplices. The The polynomial

$$T_r^{(1,\ldots,1)}(t) = \sum_{k=0}^{r} \binom{k+d-1}{d-1} t^k \tag{1.6}$$

refines the Ehrhart polynomial $L_\Delta(r)$. We give a generating function for such polynomials as r grows. This grading allows us to establish in Section 3, a bijection between the lattice points of the dilation $r\Delta_d$ and the semi standard tableaux of row Young diagrams indexing the special Schubert cycles of the Grassmanninan $\mathrm{Gr}(d, d+r)$. By using another weight $\mathbf{h} = (1, 2, \ldots, d)$ which gives a different slicing of the simplex, we construct a polynomial

$$P_{r\Delta_d}^{(1,2,\cdots,d)}(z) = \left[\binom{k+d}{d}\right]_z \quad \text{for } 0 \le k \le r \tag{1.7}$$

which is a z-binomial coefficient. This gives a bijection between the lattice points in $r\Delta_d$ and partitions fitting into an $r \times d$ rectangle, and establishes that the grading given here to a lattice point eventually identifies this polynomial with the Poincaré polynomial of the Grassmannian $\mathrm{Gr}(d, d+r)$.

§2. The Dilation Polynomial $T_{r\Delta_d}, r \ge 1$

We define the lexicographical order $<_{\text{lex}}$ on the set $r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$ of lattice points on $r\Delta_d$ as follows: Let $\mathbf{a} = (a_1, \cdots, a_d)$ and $\mathbf{b} = (b_1, \cdots, b_d)$ be any two lattice points in $r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$. We say $\mathbf{a} <_{\text{lex}} \mathbf{b}$ if, in the integer coordinate difference $\mathbf{a} - \mathbf{b} \in \mathbb{Z}^d$, the leftmost nonzero entry is negative. As noted earlier, the set $r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$ of lattice points on $r\Delta_d$ is the integer solution set of the inequality (1.3). It turns out that the upper bound r in (1.3) defines a relation on the lattice points of the solution set which brings about the disjoint subdivisions of the integer solution set.

Proposition 2.1 *Let $\mathbf{a}$ and $\mathbf{b}$ be two lattice points in $r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$ such that $\mathbf{a} <_{\text{lex}} \mathbf{b}$. The relation $\mathbf{a} \sim \mathbf{b}$ defined by $\sum_{i=1}^d (a_i - b_i) = 0$ is an equivalence relation.*

The relation partitions the set $r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$ into disjoint equivalence classes. Notice that the integer solution set is complete with respect to the bound r in the sense that the sum of integer coordinates of the lattice points in $r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$ takes all the values of the integers in the closed interval $[0, r]$. Completeness is one of the beautiful properties of the standard d-simplex not all the lattice polytopes enjoy this feature.

Corollary 2.2 *Any two lattice points in $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ belong to the same class if and only if they share the same sum of their respective integer coordinates.*

Corollary 2.3 $|r\Delta_d \cap \mathbb{Z}^d_{\geq 0}/\sim| = r + 1$.

Proof This follows corollary 2 and the fact that r$\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ is complete

$$r\Delta_d \cap \mathbb{Z}^d_{\geq 0}/\sim := \{X_k : \sum_{i=1}^{d} x_i = k,\ 0 \leq k \leq r, \forall x = (x_1, \cdots, x_d) \in X_k\} \tag{2.1}$$

and hence, $|r\Delta_d \cap \mathbb{Z}^d_{\geq 0}/\sim| = r + 1$. □

Corollary 2.4 *The class of the origin $\underline{0} \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ is a singleton set.*

Proof The class of the origin denoted by X_0 is given by

$$X_0 = \{\underline{x} = (x_1, \cdots, x_d) \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0} : \sum_{i=1}^{d} x_i = 0\}. \tag{2.2}$$

Suppose that there is a lattice point $\mathbf{a}$ which belongs to X_0 such that $\mathbf{a}$ is not the origin. Since the origin $\underline{0}$ is $<_{\text{lex}}$ than every lattice point $\mathbf{a} \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$, so, $\underline{0} \sim \mathbf{a}$ implies that $\sum_{i=1}^{d}(0 - a_i) < 0$, This integer value is not in $[0, r]$, therefore, there is no lattice point $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ which is equivalent to the origin apart from itself hence $|X_0| = 1$. □

We now compute the size of each of the equivalence classes X_k such that $0 \leq k \leq r$.

Theorem 2.5 *Let $\mathcal{A} = r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ denote the set of lattice points on $r\Delta_d$ and let $X_k \subset \mathcal{A}$ be the collection of lattice points whose sum of their integer coordinates is k such that $0 \leq k \leq r$. Then $|X_k| = \binom{k+d-1}{d-1}$.*

Proof Notice that the chain of the following inclusions

$$\{(0, \cdots, 0)\} \subset \Delta_d \cap \mathbb{Z}^d_{\geq 0} \subset 2\Delta_d \cap \mathbb{Z}^d_{\geq 0} \cdots \subset r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$$

implies the following chain

$$\Delta_d \cap \mathbb{Z}^d_{>0} \subset 2\Delta_d \cap \mathbb{Z}^d_{>0} \subset \cdots \subset r\Delta_d \cap \mathbb{Z}^d_{>0}.$$

The subcollection X_k is given by

$$X_k = \{\underline{x} = (x_1, \cdots, x_d) \in \mathcal{A} : \quad \sum_{i=1}^{d} x_i = k, \quad 0 \leq k \leq r\},$$

$X_0 = \{(0, \cdots, 0)\}$ and so $|X_k| = 1$. Observe that

$$X_k = k\Delta_d \cap \mathbb{Z}^d_{\geq 0}/(k-1)\Delta_d \cap \mathbb{Z}^d_{\geq 0}, \quad 2 \leq k \leq r$$

In fact, $X_k's$ define a partition of the set $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ of the lattice points on $r\Delta_d$, that is,

$$\bigcap_{k=0}^{r} X_k = \emptyset, \qquad \bigcup_{k=0}^{r} X_k = \mathcal{A}$$

From Ehrhart theory, using (1.4),

$$|\Delta_d \cap \mathbb{Z}^d_{\geq 0}| = \binom{1+d}{d} = |X_0 \cup X_1|.$$

This implies that $|X_1| = d$. Similarly,

$$|2\Delta_d \cap \mathbb{Z}^d_{\geq 0}| = \binom{2+d}{d} = |X_0 \cup X_1 \cup X_2|,$$

which gives

$$|X_2| = \binom{2+d}{d} - d - 1 = \binom{1+d}{d-1}.$$

Continuing this way,

$$|X_k| = \binom{k+d}{d} - \sum_{j=1}^{k} \binom{k+d-j}{d} = \binom{k+d-1}{d-1}. \qquad \square$$

The disjoint union $\cup X_k$ of subcollections $X_k, \quad 0 \leq k \leq r$ of the set $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ of lattice points on $r\Delta_d$ defines a polynomial $T_r(t)$ of degree r in variable t given by

$$T_r(t) = \sum_{k=0}^{r} \binom{k+d-1}{d-1} t^k. \tag{2.3}$$

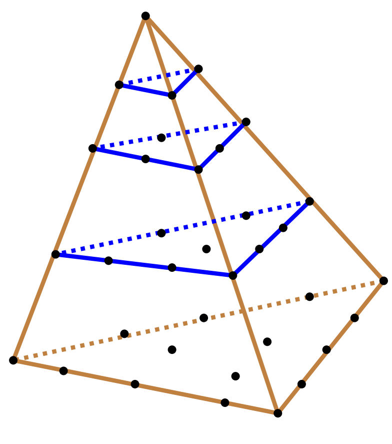

Figure 1 $T_4(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4$

We call $T_r(t)$ the dilation polynomial of degree r identified with the dilation $r\Delta_d$. This is precisely the slicing of $r\Delta_d$ with hyperplanes perpendicular to the direction $\mathbf{a} := (1, \cdots, 1)$ and enumerate all the lattice points in the different layers. That is,

$$\binom{k+d-1}{d-1} = \#\{v \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0} : v \cdot \mathbf{a} = k, 0 \leq k \leq r\}. \tag{2.4}$$

The dilation polynomial $T_4(t)$ for the 4th dilation of the standard 3-simplex is illustrated in Figure 1.

Remark 2.6 Dilation polynomials identified with $r\Delta_2$ and $r\Delta_3$ are called triangular and tetrahedral polynomials respectively.

Theorem 2.7 *Let $\mathcal{M} = \{T_r(t)\}_{r=0}$ be the sequence of dilation polynomials of lattice points counting on $r\Delta_d$ for $r \geq 0$. Then its generating series $G(t, z) = \sum_{r=0} T_r(t)z^r$ is given by*

$$G(t, z) = \frac{z}{(1-z)(1-tz)^d}.$$

Proof Notice from the equation (2.3) that

$$T_r(t) = T_{r-1}(t) + \frac{(r+1)\cdots(r+d-1)}{(d-1)!}t^r \quad \text{and} \quad \sum_{r\geq 0} \frac{(r+1)\cdots(r+d-1)}{(d-1)!}z^r = \frac{1}{(1-z)^d}.$$

$$G(t, z) = \sum_{r\geq 0} T_r(t)z^r = \sum_{r\geq 0} \left[T_{r-1}(t) + \tfrac{(r+1)\cdots(r+d-2)}{(d-1)!}t^{r-1}\right] z^r.$$

$$G(t, z) = zG(t, z) + \sum_{r\geq 1} \left[\tfrac{(r+1)\cdots(r+d-1)}{(d-1)!}t^{r-1}\right] z^r,$$

and so

$$G(t, z) = \frac{z}{(1-z)(1-tz)^d}. \qquad \square$$

§3. The Cohomology ring of Grassmannian $Gr(d, d + r)$

Let V be an n-dimensional complex vector space. The set of all maximal chains of subspaces in V is called the flag variety $\mathcal{F}\ell_n(\mathbb{C})$ of dimension $\frac{n(n-1)}{2}$. That is,

$$\mathcal{F}\ell_n(\mathbb{C}) := \{V_\bullet := \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = V \ \text{ such that } \ \dim V_i = i\}.$$

The Grassmannian $Gr(d, n)$ is the spacial case of the flag variety being the set of all d-dimensional subspaces in V. Its dimension is $d(n - d)$. There is a projection

$$\pi : \mathcal{F}\ell_n(\mathbb{C}) \longrightarrow Gr(d, n)$$

from the full flag variety $\mathcal{F}\ell_n(\mathbb{C})$ to the Grassmannian $Gr(d, n)$ with $\pi^{-1}(X_\lambda(F_\bullet)) = X_{w(\lambda)}(F_\bullet)$, where $X_\lambda(F_\bullet)$ is a Schubert variety in the Grassmannian $Gr(d,n)$ defined as the closure of the Schubert cell $C_\lambda(F_\bullet)$ given by

$$C_\lambda(F_\bullet) = \{V_d \in Gr(d,n) : \dim V_d \cap F_{n+i-\lambda_i} = i,\ 1 \le i \le d\},$$

with respect to the fixed flag $F_\bullet$:

$$F_\bullet := \{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_n = V \ \text{ such that } \ \dim F_i = i$$

The partition λ is called fitted in the sense that it has at most length d and each part cannot exceed $n-d$. The permutation $w(\lambda)$ identified with the partition $\lambda = (\lambda_1, \cdots, \lambda_d)$ is given by

$$w_i = i + \lambda_{d+1-i},\ 1 \le i \le d \text{ and } w_j < w_{j+1},\ d+1 \le j \le n. \tag{3.1}$$

This permutation is called Grassmannian in that it has a unique descent by definition. Every such permutation has the code $c(w(\lambda))$ of the form $(w_1 - 1, w_2 - 2, \cdots, w_d - d, 0, \ldots, 0)$ which can be represented by $(m_1, m_2, \cdots, m_d)$ by disregarding the string of zeros at the right hand. It turns out that the partition λ indexing the Schubert variety X_λ can be recovered from this code as $\lambda = (m_{i_1}, m_{i_2}, \cdots, m_{i_d})$ where $m_{i_1} \ge m_{i_2} \ge \cdots \ge m_{i_d}$ and $m_{i_p} \ne 0,\ 1 \le i_p \le d$. Recall that for any permutation w in the symmetric group S_n, the code $c(w)$ of w is the sequence $(c_1(w), \cdots, c_n(w))$ where $c_i(w) =\mid \{j : 1 \le i < j \le n \text{ and } w(i) > w(j)\} \mid$. For instance the code $c(w)$ of the permutation $w = 315426 \in S_6$ is $(2,0,2,1,0,0)$. The string of zeros at the right hand may be discarded. Notice that $c_i(w) \le n - i$. The length $\ell(w)$ of w is $\#\{(i,j) : w(i) > w(j), 1 \le i < j \le n\}$, the number of inversions in w, that is, the sum of integer coordinates of the code of w. It is well known that the cohomology ring of the Grassmannian $Gr(d,n)$ is generated by the Schubert cycles σ_λ. These are Poincaré dual of the fundamental classes in the homology of Schubert varieties. The Grassmannian $\mathrm{Gr}(d,n)$ admits many important vector bundles, most importantly there is a universal short exact sequence: $0 \longrightarrow \mathcal{S} \longrightarrow \mathbb{C}^n \times \mathrm{Gr}(d,n) \longrightarrow \mathcal{Q} \longrightarrow 0$ of bundles on $Gr(d,n)$ which makes it easy to compute the Chern class $c(\mathcal{Q})$ of the quotient bundle $\mathcal{Q}$ on the Grassmannian $Gr(d,n)$. Recall that $\mathcal{Q}$ is a globally generated vector bundle of rank $r := n-d$ and all its global sections are from the trivial bundle $\mathbb{C}^{d+r} \times \mathrm{Gr}(d, d+r)$. The total Chern class is the sum over all the one-row partitions inside the rectangle $\square_{r\times d}$. That is,

$$c(\mathcal{Q}) = 1 + \sigma_{\square} + \sigma_{\square\square} + \cdots + \sigma_{\square\square\cdots\square_{1\times r}}. \tag{3.2}$$

It turns out that the set of all one-row Young diagrams indexing the multiplicative generators of the cohomology of the Grassmannian $\mathrm{Gr}(d, d+r)$ is deeply connected with the lattice points of $r\Delta_d$. Let $\mathcal{C}_{d,r}$ be the set of row Young diagrams with at most r boxes and adjoin the empty set ϕ. That is,

$$\mathcal{C}_{d,r} = \{\square_{1\times k} :\ 1 \le k \le r\} \cup \emptyset.$$

The filling of the boxes of the row Young diagrams in $\mathcal{C}_{d,r}$ using the numbers in $[d] := \{1, \cdots, d\}$ is semi standard, that is, the numbers weakly increase from the left to the right. We denote the collection of all such fillings by $\mathcal{C}^d_{d,r}$ and call it the d-filling set of the dilation $r\Delta_d$. For instance, the 3-filling set $\mathcal{C}^3_{3,3}$ associated the second dilation $3\Delta_3$ of the standard 3-simplex is the following collection

$$\boxed{\,.\,},\ \boxed{1},\ \boxed{2},\ \boxed{3},\ \boxed{1}\boxed{1},\ \boxed{1}\boxed{2},\ \boxed{2}\boxed{2},\ \boxed{1}\boxed{3},\ \boxed{2}\boxed{3},$$
$$\boxed{3}\boxed{3},\ \boxed{1}\boxed{1}\boxed{1},\ \boxed{1}\boxed{1}\boxed{2},\ \boxed{1}\boxed{2}\boxed{2},\ \boxed{2}\boxed{2}\boxed{2},\ \boxed{1}\boxed{1}\boxed{3},$$
$$\boxed{1}\boxed{3}\boxed{3},\ \boxed{3}\boxed{3}\boxed{3},\ \boxed{2}\boxed{2}\boxed{3},\ \boxed{2}\boxed{3}\boxed{3},\ \boxed{1}\boxed{2}\boxed{3}.$$

These 20 semi standard Young tableaux can be organized in terms of their defining Young diagrams. It turns out that this arrangement can be expressed as a polynomial, given by $T_3(t) = 1 + 3t + 6t^2 + 10t^3$. This is the graded semi-standard polynomial of degree 3 illustrated in Figure 2.

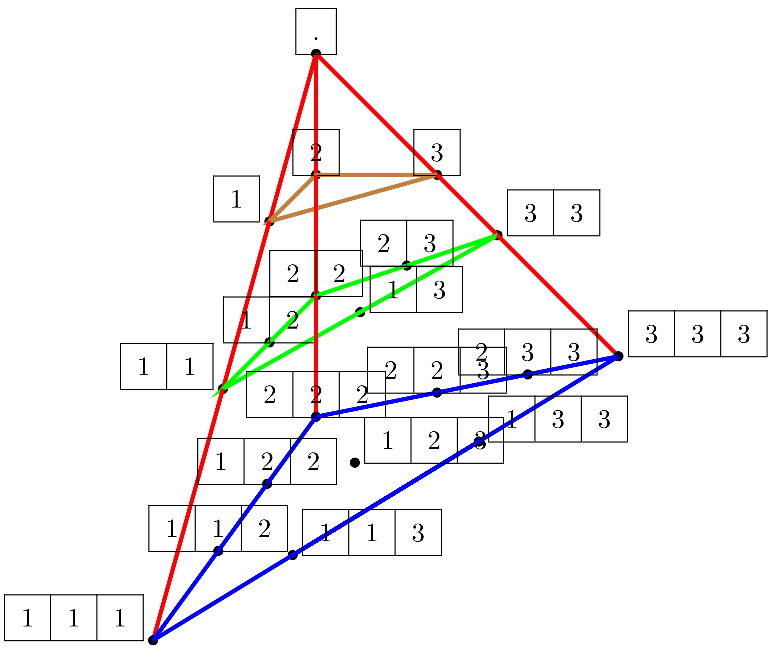

Figure 2. $T_3(t) = P_3(t) = 1 + 3t + 6t^2 + 10t^3$

Theorem 3.1 *(i) The size $L^d(r)$ of the d-filling set $\mathcal{C}^d_{d,r}$ is $\binom{r+d}{d}$ and the sequence $(L^d(r))_{r=0}^{\infty}$ of cardinalities as r grows is recorded by the generating function*

$$P(\mathcal{C}^d_{(d,r)}, z) = \frac{1}{(1-z)^{d+1}}.$$

(ii) More is true, there is a graded counting polynomial of the semi standard tableaux in $\mathcal{C}^d_{d,r}$ given by

$$P_r(t) = \sum_{k=0}^{r} \binom{k+d-1}{d-1} t^k$$

that is, a k-box row diagram gives $\binom{k+d-1}{d-1}$ semi standard Young tableaux. This has a generating function

$$G(t,z) = \frac{z}{(1-z)(1-tz)^d}.$$

Theorem 3.2 *There is a bijection $T \mapsto v(T)$ between the set $\mathcal{C}^d_{d,r}$ and the set $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ of the lattice points of the dilation $r\Delta_d$. Furthermore, the semi-standard polynomial $P_r(t)$ is precisely the dilation polynomial $T_r(t)$ identified with $r\Delta_d$.*

Proof To each semi standard tableau $T \in \mathcal{C}^d_{(d,r)}$ there exists a unique exponent vector $v(T) := (v(T)_1, \cdots, v(T)_d)$ in which the coordinate $v(T)_j$ is the number of appearances of j in T, $1 \leq j \leq d$. This is a bijection.

The number of semi standard fillings of each of the row diagram with shape $\lambda = (k), 0 \leq k \leq r$ using the elements of the set $\{1, \ldots, d\}$ has a well known closed formula. Notice that for a fixed point $\mathbf{a} = (1, \cdots, 1)$ the following identity holds

$$\prod_{1 \leq i < j \leq d} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \binom{k+d-1}{d-1} = \#\{v \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0} : v \cdot \mathbf{a} = k, 0 \leq k \leq r\}$$

Therefore, the semi-standard polynomial $P_r(t)$ can be viewed as the dilation polynomial $T_r(t)$. The bijection is a polynomial preserving map, see Figure 2. □

§4. Grassmannian Monomials

It is clear from the Theorem 3.1 that every standard tableau $T \in \mathcal{C}^d_{(d,r)}$ defines a monomial $\mathbf{t}^{v(T)}$ where $v(T) := (v(T)_1, \cdots, v(T)_d)$, that is,

$$\mathbf{t}^{v(T)} := \prod_{j=1}^{d} t_j^{\#\text{ times j appears in T}}, \quad \text{where } \ v(T) \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0}. \tag{4.1}$$

For instance, the monomial defined by $T = \boxed{1 \,|\, 1 \,|\, 2 \,|\, 3 \,|\, 3} \in \mathcal{C}^4_{(4,5)}$ is given by $\mathbf{t}^{\mathbf{a}} = t_1^2 t_2 t_3^2$ where $\mathbf{a} = (2,1,2,0)$. We call such monomials in $\mathcal{C}^d_{(d,r)}$ Grassmannian because they encode the data of indexing partitions of Schubert varieties in the Grassmannian $\mathrm{Gr}(d, d+r)$. We denote these monomials by W^r_d, that is,

$$W^r_d := \{t_1^{a_1} \cdots t_d^{a_d} : \sum_{i=1}^{d} a_i \leq r, \quad 0 \leq a_i \leq r\}$$

Proposition 4.1 *Let W^r_d and $W^{r'}_d$ be two Grassmannian monomial sets such that $r \leq r'$. Then $W^r_d \subseteq W^{r'}_d$.*

Proposition 4.2 *Every monomial $\mathbf{t}^{\mathbf{a}} \in \mathbb{Z}[t_1, \cdots, t_d]$ is Grassmannian.*

Proof It suffices to produce a Grassmannian set W^r_d containing $\mathbf{t}^{\mathbf{a}}$. By (4.1) there is a semi-standard tableau T which encodes the exponent vector $\mathbf{a}$ and this implies that there exists $r \in \mathbb{N}$ such that T is an element of the d- filling set $\mathcal{C}^d_{(d,r)}$, so $\mathbf{t}^{\mathbf{a}}$ belongs to the Grassmannian monomial set W^r_d. $\square$

Corollary 4.3 *If $r = \sum_{i=1}^d a_i$, where a_i is an integer coordinate of $\mathbf{a}$ then the Grassmannian set W^r_d is the smallest set containing the monomial $\mathbf{t}^{\mathbf{a}}$.*

It is important to quickly point out that the sum $P_r(t_1, \ldots, t_d)$ of all the monomials in W^r_d, that is,

$$P_r(t_1, \cdots, t_d) = \sum_{T \in \mathcal{C}^d_{(d,r)}} \prod_{j=1}^{d} t_j^{\#\text{ times j appears in T}} \tag{4.2}$$

is deeply connected with a polynomial representation (V, ρ) of the general linear group $GL_d(\mathbb{C})$ where $V := \bigoplus_{k=0}^r \operatorname{Sym}^k(\mathbb{C}^d)$ is the space of the direct sum of homogeneous symmetric polynomials of degree k in d variables. Let $\mathbb{C}[X] := \mathbb{C}[x_{11}, x_{12} \cdots, x_{dd}]$ be the ring of polynomial functions on $d \times d$ matrices. There is an action of $G = GL_d(\mathbb{C})$ on $\mathbb{C}[X]$ by conjugation. The character of the polynomial representation (V, ρ) is the polynomial $\chi_\rho \in \mathbb{C}[X]$ given by the trace of the matrix $\rho(X)$. Recall that the character χ_ρ of every polynomial representation (V, ρ) lies in the invariant ring $\mathbb{C}[X]^G$. Interested reader can consult [15] and [17].

Theorem 4.4 *The character χ_V of $V := \bigoplus_{k=0}^r \operatorname{Sym}^k(\mathbb{C}^d)$ as a polynomial representation ρ of the general linear group $GL_d(\mathbb{C})$ is $P_r(t_1, \cdots, t_d)$, that is,*

$$\chi_V = \sum_{T \in \mathcal{C}^d_{(d,r)}} \prod_{j=1}^{d} t_j^{\#\text{ times j appears in T}}$$

The sum ranging over all the semi standard fillings of the row diagrams with at most r boxes.

Proof Let $t_1, \cdots, t_d$ be eigenvalues of a generic $d \times d$ matrix X. The map $\mathbb{C}[X]^G \longrightarrow \mathbb{C}[t_1, t_2, \cdots, t_d]^{S_n}$ defined by$f \mapsto f(\operatorname{diag}(t_1, \ldots, t_d))$ is an isomorphism. Set $\lambda = (k)$ since $k's$ define the rows diagrams with at most r boxes, so the image of the character $f_\rho(X)$ is

$$\sum_{k=0}^{r} \frac{\det(t_i^{\lambda_i + d - j})_{1 \le i,j \le d}}{\det(t_i^{d-j})_{1 \le i,j \le d}}. \qquad \square$$

Corollary 4.5 *The dimension of the vector space $V := \bigoplus_{k=0}^r \operatorname{Sym}^k(\mathbb{C}^d)$ is $\chi_V(1, 1, \cdots, 1) := |r\Delta_d \cap \mathbb{Z}_{\ge 0}|$, the number of lattice points of the dilation $r\Delta_d$.*

Proof The Grassmannian set W^r_d spans the vector space $V := \bigoplus_{k=0}^r \operatorname{Sym}^k(\mathbb{C}^d)$. $\square$

Now to every monomial $\mathbf{t}^{\mathbf{a}} \in \mathbb{Z}[t_1, \cdots, t_d]$ we associate a weight $w_{\mathbf{a}}$ defined by

$$w_{\mathbf{a}} = \sum_{k=1}^{d} k a_k. \tag{4.3}$$

It turns out that $w_{\mathbf{a}}$ admits two important partitions $\lambda, \lambda^* \vdash w_{\mathbf{a}}$ which can be identified with the monomial $\mathbf{t}^{\mathbf{a}}$. These partitions, λ and λ^* are called α-partition and β-partition respectively. A partition $\lambda \vdash w_{\mathbf{a}}$ is said to be the α-partition of the monomial $t_1^{a_1} \cdots t_d^{a_d}$ if the number of parts of size i in λ is a_i, $1 \leq i \leq d$. The length $\ell(\lambda)$ of α-partition is $a_1 + \cdots + a_d$. The β partition $\lambda^* = (\lambda_1^*, \cdots, \lambda_d^*)$ of $w_{\mathbf{a}}$ is such that $\lambda_k^* = \sum_{i \geq k}^{d} a_i$, $1 \leq k \leq d$ and its length is d. For instance, the α-partition associated with the monomial $t_1^3 t_2^2 t_3^3 t_4^2 \in \mathbb{Z}[t_1, t_2, t_3, t_4]$ is (4,4,3,3,3,2,2,1,1,1) while its β partition i λ^* is $(10, 7, 5, 2)$. In fact, α and β partitions identified with the monomial $\mathbf{t}^{\mathbf{a}}$ can be realized in terms of the sum of the entries of the $d \times d$ upper triangular matrix $\mathrm{M}_{\mathbf{a}}$ associated with the exponent vector $\mathrm{a} = (a_1, \cdots, a_d)$ of the monomial, that is,

$$\mathrm{M}_{\mathbf{a}} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_d \\ & a_2 & a_3 & \cdots & a_d \\ & & a_3 & \cdots & a_d \\ & & & & \vdots \\ & & & & a_d \end{bmatrix} \tag{4.4}$$

The sum of the entries in the column k divided by k is the number of parts of size k in the α- partition λ of $w_{\mathbf{a}}$. The β partition $\lambda^* = (\lambda_1^* \cdots \lambda_d^*)$ of $w_{\mathbf{a}}$ is such that λ_k^* is the sum of the entries in the row k where $1 \leq k \leq d$. For instance, the 4×4 matrix $\mathrm{M}_{\mathbf{a}}$ corresponding to the monomial $t_1^3 t_2^2 t_3^3 t_4^2 \in \mathbb{Z}[t_1, t_2, t_3, t_4]$ is

$$\mathrm{M}_{\mathbf{a}} = \begin{bmatrix} 3 & 2 & 3 & 2 \\ & 2 & 3 & 2 \\ & & 3 & 2 \\ & & & 2 \end{bmatrix}$$

so the α-partition λ and the β-partition λ^* identified with the matrix $\mathrm{M}_{\mathbf{a}}$ are $1^3 2^2 3^3 4^2$ and $(10, 7, 5, 2)$ respectively.

Proposition 4.6 *Let λ be the α-partition of $w_{\mathbf{a}}$ associated with the monomial $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_d^{a_d} \in \mathbb{Z}[t_1, \cdots, t_d]$. Then its corresponding β-partition λ^* is the transpose of λ and vice versa.*

Proof Let $\lambda = (\lambda_1, \cdots, \lambda_{a_1 + \cdots + a_d})$ and $\lambda^* = (\lambda_1^*, \cdots, \lambda_d^*)$. It is obvious that these partitions satisfy the following identity

$$\sum_{k=1}^{a_1+\cdots+a_d} (2k-1)\lambda_k = \sum_{k=1}^{d} \lambda_k^{*2}. \qquad \square$$

It would be interesting to characterize and study all the monomials for which α-partition and β-partition coincide. This amounts to the characterization of all self conjugate partitions. Recall that for all $n \in \mathbb{N}$ such that $n > 2$ there is a bijection between the set of self conjugate partitions of n and the set of all distinct odd parts partitions of n. For instance, a square free monomial of the form $t_1 t_2 \cdots t_d$ admits the stair case partition $(d, d-1, \cdots, 1)$, this is deeply connected with the distribution of triangular numbers in the set $\mathbb{N}$ of natural numbers. We give a few other examples of such monomials.

Example 4.7 Some monomials following for which α and β-partitions coincide:

(*i*) All monomials of the form $t_{\frac{d}{2}}^{\frac{d}{2}} t_d^{\frac{d}{2}} \in \mathbb{Z}[t_1, t_2, \cdots, t_d]$ for even d;

(*ii*) All monomials of the form $t_1 t^{d-2} t_d \in \mathbb{Z}[t_1, t_2, \cdots, t_d]$;

(*iii*) All monomials of the form $t_1^{d-1} t_d \in \mathbb{Z}[t_1, t_2, \cdots, t_d]$;

(*iv*) All monomials of the form $t_{d-2} t_{d-1} t_d^{d-2} \in \mathbb{Z}[t_1, t_2, \cdots, t_d]$.

Lemma 4.8 *Let $\lambda^* = (\lambda_1, \cdots, \lambda_d)$ be the β-partition identified with the monomial $t_1^{a_1} \cdots t_d^{a_d} \in \mathbb{Z}[t_1, t_2, \cdots, t_d]$. Then the exponent vector $(a_1, \cdots, a_d)$ is equivalent to $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{d-1} - \lambda_d, \lambda_d)$.*

Proof It follows from the construction of the β partition λ^* from the exponent vector $(a_1, \cdots, a_d)$. □

Theorem 4.9 *Let $\mathbf{t}^{\mathbf{a}} \in W_d^r$ be a Grassmannian monomial associated with exponent vector $\mathbf{a} \in r\Delta_d \cap \mathbb{Z}_{\geq 0}^d$. If a partition λ^* is the β-partition identified with $\mathbf{t}^{\mathbf{a}}$ then the length $\ell(w(\lambda^*))$ of the Grassmannian permutation $w(\lambda^*)$ is the weight $w_{\mathbf{a}}$.*

Proof The code $c(w(\lambda^*))$ of Grassmannian permutation $w(\lambda^*)$ is of the form $(m_1, m_2, \cdots, m_d, 0, 0, \cdots, 0)$. The rearrangement of $m_1, m_2, \cdots, m_d$ in weakly decreasing order yields the fitted partition $\lambda^* = (\lambda_1^*, \cdots, \lambda_d^*)$. The sum of entries of the code $c(w) = (c_1(w), c_2(w), \cdots, c_n(w))$ of any permutation w is the length $\ell(w)$ of the partition, since each entry $c_i(w)$ is the number of inversions associated to the value w_i in the position i. Hence the length $\ell(w(\lambda^*))$ of $w(\lambda^*)$ is the size $|\lambda^*|$ of λ^*. Next we show that the weight $w_{\mathbf{a}}$ of the exponent vector $\mathbf{a} = (a_1, \cdots, a_d)$ of the Grassmannian monomial $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_d^{a_d}$ is $|\lambda^*|$. From Lemma 3.12 $a_i = \lambda_i^* - \lambda_{i+1}^*,\ \ 1 \leq i \leq d-1,\ \ a_d = \lambda_d^*$. Therefore, the weight $w_{\mathbf{a}} = \sum_{i=1}^{d-1} i(\lambda_i^* - \lambda_{i+1}^*) + d\lambda_d^* = |\lambda^*|$. □

Corollary 4.10 *Every β-partition λ^* identified with each of the monomials $\mathbf{t}^{\mathbf{a}} \in W_d^r$ fits into the $r \times d$ rectangle $\square_{r \times d}$.*

Proof It is sufficient to establish that the parts of λ^* cannot exceed r and the length $\ell(\lambda^*)$ of λ^* is d. Notice that the exponent vector $\mathbf{a}$ is a lattice point of $r\Delta_d$ and by definition $a_1 + \cdots + a_d \leq r$. Therefore each part λ_k^* of λ^* is at most r and length $\ell(\lambda^*)$ is d by the definition of λ^*. □

Corollary 4.11 *The set of β-partitions λ^* identified with monomials in W_d^r index the Schubert varieties in the Grassmannian $Gr(d, d+r)$, giving a bijection between lattice points in $r\Delta_d$ and partitions fitting into an $r \times d$ rectangle.*

The weight $w_{\mathbf{a}}$ defined in the equation (3.2) gives another refinement $P_{r\Delta_d}^h(z)$ of the Ehrhart polynomial of $r\Delta_d$ with respect to a fixed point $h = (1, 2, \cdots d)$.

$$P^h_{r\Delta_d}(z) = \sum_{m=0}^{dr} A_m z^m. \tag{4.5}$$

where $A_m = \#\{\mathbf{a} \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0} : \mathbf{a} \cdot h = m,\ 0 \leq m \leq dr\}$, that is, the number of exponent vectors $\mathbf{a}$ which share the weight m. We call $P^h_{r\Delta_d}(z)$ the weighted polynomial associated with the dilation $r\Delta_d$.

Lemma 4.12 *The polynomial $P^h_{r\Delta_d}(z) = \sum_{m=0}^{dr} A_m z^m$ specializes at $z = 1$ to the Ehrhart polynomial $L_{\Delta_d}(r)$.*

Remark 4.13 Notice that A_m is precisely the number of lattice points in the intersection of the dilation $r\Delta_d$ with the hyperplane H_m perpendicular to the direction $\mathbf{h} := (1, 2, \cdots, d)$. It is also interesting to note that the grading given here to a lattice point eventually identifies the weighted polynomial $P^h_{r\Delta_d}(z)$ with the Poincaré polynomial of the Grassmannian $\mathrm{Gr}(d, d+r)$.

Theorem 4.14 *Let $P^h_{r\Delta_2}(z)$ be the weighted polynomial of the lattice points of the dilation $r\Delta_d$. Then the Poincaré polynomial $P(Gr(d, d+r), t)$ of the Grassmannian $Gr(d, d+r)$ coincides with the weighted polynomial $P^h_{r\Delta_d}(z)$.*

Proof It is well known from the Borel presentation of the cohomology ring $H^*(Gr(d, d+r), \mathbb{Z})$ of the Grassmannian $\mathrm{Gr}(d, d+r)$ that the Poincaré polynomial $\mathrm{P}(\mathrm{Gr}(d, d+r), t)$ is given by the following Gaussian polynomial

$$\frac{(1-t)(1-t^2)\cdots(1-t^{d+r})}{(1-t)\cdots(1-t^d)(1-t)\cdots(1-t^r)}.$$

This is combinatorially simplified as

$$\sum_{\lambda \subseteq \square_{d\times r}} t^{|\lambda|}$$

where $|\lambda|$ is the number of boxes in the Young diagram of shape λ. The size $|\lambda|$ coincides with the length $\ell(w(\lambda))$ (the number of inversions) of the Grassmannian permutation $w(\lambda)$ identified with λ in the equation (3.1). Notice that $|\lambda| \leq dr$, therefore, It follows from the Theorem 4.9 that $|\lambda|$ is the weight $w_{\mathbf{a}}$ of the monomial $t^{\mathbf{a}} \in W^r_d$, $a \in r\Delta \cap \mathbb{Z}^d_{\geq 0}$, therefore, $\sum_{\lambda \subseteq \square_{2\times r}} t^{|\lambda|}$ is precisely the polynomial $\sum_{m=0}^{dr} A_m z^m$. □

Question 4.15 *Does the set $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ encode some data about the degree and the Hilbert polynomial of $Gr(d, d+r)$?*

The goal of this paper is the general study of some combinatorial geometry of the lattice points $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ associated with $r\Delta_d$. That is, we evoke some geometric information about these lattice points. In particular, we answer the following questions:

(*i*) *What kind of geometric information can be extracted from these integral solutions to (1.4)?*

(ii) What kind of combinatorial object parameterizes this solution set?

(iii) Is there an interesting polynomial P_r which keeps track of the integral points? In other words, Is there a generic polynomial of degree r whose exponents of its monomials with nonzero coefficients satisfy (1.3)*?*

Theorem 4.16 *Every linear polynomial function of the form $y = ax + 1$ such that $a \in \mathbb{N}$ is the fundamental polynomial of a certain standard d-simplex whose dimension is the slope of the polynomial function.*

Proof Consider the family $\mathcal{G}$ of'Cartesian graphs of all linear functions of the form $y = ax+1$ such that $a \in \mathbb{N}$. It is obvious that these graphs are parametrized by the x-intercepts since they all share the same y-intercept $(0, 1)$. Consider the sequence $\mathcal{E} = (-\frac{1}{a})_{a=1}$ of x-intercepts. $\mathcal{E}$ is strictly monotone decreasing and lies in the interval $[-1, 0)$. There is a bijection $a \mapsto -\frac{1}{a}$, between the sequence $\mathcal{K} = (\Delta_a)_{a=1}$ of standard a-simplices and the sequence $\mathcal{E}$ of x-intercepts of $\mathcal{G}$. As $\mathcal{K}$ diverges, $\mathcal{E}$ converges. $\square$

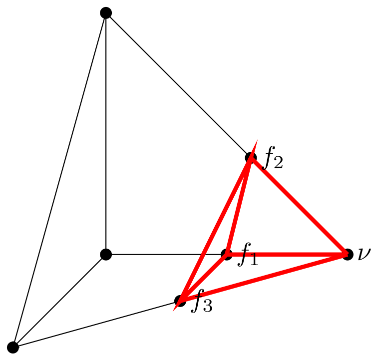

Figure 3. 3-Simplex

The sum of all the monomials in W_d^r is called the symbolic polynomial corresponding to the d-filling set $\mathcal{C}^d_{(d,r)}$. That is,

$$P_r(t_1, \cdots, t_d) = \sum_{T \in \mathcal{C}^d_{(d,r)}} \mathbf{t}^{wt(T)}. \tag{4.6}$$

For every lattice point $\mathbf{a} \in r\Delta_2 \cap \mathbb{Z}^2_{\geq 0}$, there is a corresponding monomial $\mathbf{t}^{\mathbf{a}}$ in the polynomial ring $\mathbb{Z}[t_1, t_2]$ given by $\mathbf{t}^{\mathbf{a}} := t_1^{a_1} t_2^{a_2}$. We call these monomials Grassmannian and denote their collection by W_2^r,that is,

$$W_2^r = \{t_1^{a_1} t_2^{a_2} \in \mathbb{Z}[t_1, t_2] : (a_1, a_2) \in r\Delta_2 \cap \mathbb{Z}^2_{\geq 0}\}. \tag{4.7}$$

To every monomial $\mathbf{t}^{\mathbf{a}} \in W_2^r$ we associate a weight $w_{\mathbf{a}}$ defined by

$$w_{\mathbf{a}} = \sum_{k=1}^{d} k a_k. \tag{4.8}$$

It turns out that $w_{\mathbf{a}}$ admits two important partitions $\lambda, \lambda^* \vdash w_{\mathbf{a}}$ which can be identified with the monomial $\mathbf{t}^{\mathbf{a}}$. These partitions, λ and λ^* are called α-partition and β-partition respectively. A partition $\lambda \vdash w_{\mathbf{a}}$ is said to be the α-partition of the monomial $t_1^{a_1} t_2^{a_2} \in W_2^r$ if the number of parts of size i in λ is a_i, $1 \leq i \leq 2$, while the β partition $\lambda^* = (\lambda_1^*, \lambda_2^*)$ of $w_{\mathbf{a}}$ is such that $\lambda_k^* = \sum_{i \geq k}^2 a_i$, $1 \leq k \leq 2$. This is not exclusively for only Grassmannian monomial, it is true for all monomials. For instance, given a monomial $t_1^2 t_2^3$. The corresponding *alpha*-partition λ and β-partition λ^* are (2,2,2,1,1) and (5,3) respectively.

Corollary 4.17 *The triangular polynomial* $T_r(t) = \sum_{c=0}^{r} (c+1) t^c$ *specialises at* $t = 1$ *to the Ehrhart polynomial* $\binom{r+2}{2}$.

We now give a combinatorial construction of a certain discrete object $\mathcal{C}_{d,r}^d$ identified with the lattice points of $r\Delta_d$ which we call the 3-filling set of the dilation. It describes certain fillings of a row Young diagram with the numbers from the set $[d] := \{1, \cdots, d\}$.

Theorem 4.18 The size $L^d(r)$ of the d-filling set $\mathcal{C}_{d,r}^d$ associated with the lattice points of the r^{th} dilation $r\Delta_d$ of the standard d-simplex is $\binom{r+d}{d}$. Moreover, the sequence $(L^d(r))_{r=0}^{\infty}$ as r grows is recorded by the generating function

$$P(\mathcal{C}_{(d,r)}^d, z) = \frac{1}{(1-z)^{d+1}}$$

Proof The size is given by $\sum_{k=0}^{r} \prod_{1 \leq i < j \leq d} \frac{\lambda_i - \lambda_j + j - i}{j - i}$ since these are the semi standard fillings of the Young diagrams of shapes $\lambda = (k)$ $0 \leq k \leq r$ using the numbers from the set $\{1, \cdots, d\}$ and hence $\binom{r+d}{d}$. The sequence $(L^d(r))_{r=0}^{\infty}$ is given by triangular numbers which is well known. It is obvious that the generating series is in the coefficient of the polynomial $\binom{r+d}{d}$, that is, the general term of the sequence. Therefore, it is given by

$$\sum_{r \geq 0} \binom{r+d}{d} z^r = \frac{1}{(1-z)^{d+1}}. \qquad \square$$

Corollary 4.19 *There is a bijection between the set* $\mathcal{C}_{d,r}^d$ *of semi standard fillings of the row Young diagrams with at most r boxes using the numbers from* $[d]$ *and the set* $r\Delta_d \cap \mathbb{Z}_{\geq 0}^d$ *of the lattice points in the* r^{th} *dilation of the standard d-simplex.*

This bijection can be clearly understood in the language of monomials. This is the subject of discussion in what follows.

The symbolic polynomial $P_r(t_1, \cdots, t_d)$ is deeply connected with a polynomial representation (V, ρ) of the general linear group $GL_d(\mathbb{C})$ where $V := \bigoplus_{k=0}^{r} \mathrm{Sym}^k(\mathbb{C}^d)$. The space of homogeneous symmetric polynomials of degree k in d variables is denoted by $\mathrm{Sym}^k(\mathbb{C}^d)$. Let $\mathbb{C}[X] := \mathbb{C}[x_{11}, x_{12} \cdots, x_{dd}]$ be the ring of polynomial functions on $d \times d$ matrices. There is an action of $G = GL_d(\mathbb{C})$ on $\mathbb{C}[X]$ by conjugation. The character of a polynomial representation (V, ρ) is the polynomial $\chi_\rho \in \mathbb{C}[X]$ given by the trace of the matrix $\rho(X)$. Recall that the character χ_ρ of every polynomial representation (V, ρ) lies in the invariant ring $\mathbb{C}[X]^G$.

Theorem 4.20 *The character χ_V of $V := \bigoplus_{k=0}^{r} \operatorname{Sym}^k(\mathbb{C}^d)$ as a polynomial representation ρ of the general linear group $GL_d(\mathbb{C})$ is the symbolic polynomial*

$$\chi_V = \sum_{T \in \mathcal{C}^d_{(d,r)}} \prod_{j=1}^{d} t_j^{\#\text{ times j appears in T}}$$

The sum ranging over all the semi standard fillings of the row diagrams with at most r boxes.

Proof Let $t_1, \cdots, t_d$ be eigenvalues of a generic $d \times d$ matrix X. The map $\mathbb{C}[X]^G \longrightarrow \mathbb{C}[t_1, t_2, \cdots, t_d]^{\mathcal{S}_n}$ defined by $f \mapsto f(\operatorname{diag}(t_1, \cdots, t_d))$ is an isomorphism. Set $\lambda = (k)$ since $k's$ define the rows diagrams with at most r boxes, so the image of the character $f_\rho(X)$ is

$$\sum_{k=0}^{r} \frac{\det(t_i^{\lambda_i + d - j})_{1 \le i,j \le d}}{\det(t_i^{d-j})_{1 \le i,j \le d}}.$$

□

Corollary 4.21 *The dimension of the vector space $V := \bigoplus_{k=0}^{r} \operatorname{Sym}^k(\mathbb{C}^d)$ given by the value of $\chi_V(1, 1, \cdots, 1)$ is the number of lattice points of the dilation $r\Delta_d$.*

Proof The number of semi standard tableaux $T \in \mathcal{C}^d_{(d,r)}$ defined by the set of k-box row diagrams with at most r boxes. That is,

$$\sum_{k=0}^{r} \prod_{1 \le i < j \le d} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

where $\lambda = (k)$ $0 \le k \le r$. This is precisely the number of monomials which constitute the character χ_V of each of these has coefficient 1. The value of $\chi_V(1, 1, \cdots, 1)$ is $\binom{r+d}{d}$. □

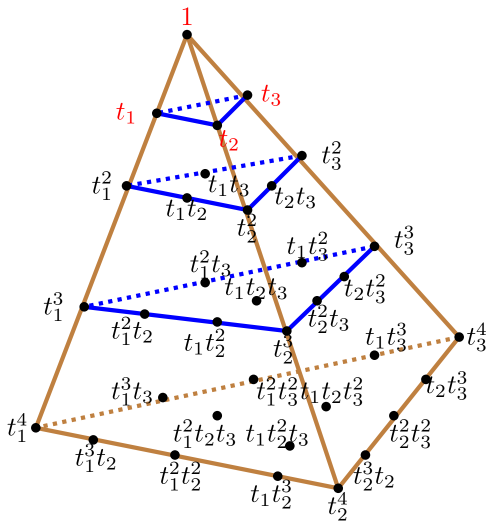

Figure 4. The monomial basis elements of W_3^4

The elements of W_d^r which span the vector space $V := \bigoplus_{k=0}^{r} \operatorname{Sym}^k(\mathbb{C}^d)$ encode the indexing partitions of the Schubert cycles of the cohomology ring of the Grassmanian $Gr(d, d+r)$ and therefore they are called Grassmannian monomials. This will dominate the discussion in what follows but we shall first describe in general how a monomial encodes information about partitions in the next section. Recall that partition λ of $n \in \mathbb{N}$ denoted $\lambda \vdash n$ is a list $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The length k of the partition λ is denoted by $\ell(\lambda)$ and each λ_i is called a part of the partition λ. Associated to every partition $\lambda \vdash n$ is its conjugate partition, $\lambda^t = (\lambda^t_1, \cdots, \lambda^t_m)$, which is also a partition of n where λ^t_i counts the parts of λ which are at least i. For example, the conjugate λ^t of the partition $\lambda = (4, 4, 3, 3, 3, 2, 2, 1, 1, 1)$ is given by $\lambda^t = (10, 7, 5, 2)$. A partition is said to be self conjugate if it coincides with its conjugate.

§5. The α and β Partitions of Monomial $t_1^{a_1} \cdots t_d^{a_d}$

Let $\mathbb{Z}[\mathbf{t}] := \mathbb{Z}[t_1, \cdots, t_d]$ be the polynomial ring over $\mathbb{Z}$ in the variables $t_1, \cdots, t_d$. We recall that by associating a monomial $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_d^{a_d}$ with its d-tuple exponent vector $\mathbf{a} = (a_1, \cdots, a_d) \in \mathbb{Z}^d_{\geq 0}$, a bijection between monomials in $\mathbb{Z}[t_1, \cdots, t_d]$ and exponent vectors in $\mathbb{Z}^d_{\geq 0}$ is realized.

We now construct the weighted polynomial Γ_{P_r} parameterized by the exponent vectors of the monomials the symbolic polynomial $P_r(t_1, \ldots, t_d)$ identified with the d-filling set $\mathcal{C}^d_{(d,r)}$. Recall that this is the character χ_V of the vector space $V := \bigoplus_{k=0}^r \mathrm{Sym}^k(\mathbb{C}^d)$ as a polynomial representation of the general linear group $GL_d(\mathbb{C})$ and notice that these exponent vectors are precisely $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$. Let the weight $w_{\mathbf{a}}$ defined in 5.1 be identified with each of the vectors. The identification realizes the weighted polynomial Ω_{P_r}

$$\Omega_{P_r}(z) = \sum_{m=0}^{dr} A_m z^m, \tag{5.1}$$

where A_m is number of exponent vectors of the monomials of $P_r(t_1, \cdots, t_d$ which share the same weight m. For instance, the weighted polynomial Ω_{P_3} parameterized by the exponent vectors of the symbolic polynomial $P_3(t_1, t_2, t_3)$ corresponding to $\mathcal{C}^3_{(3,3)}$ is given by

$$\Gamma_{P_3}(z) = 1 + z + 2z^2 + 3z^3 + 3z^4 + 3z^5 + 3z^6 + 2z^7 + z^8 + z^9.$$

This combinatorially defined polynomial from the lattice points of the dilation $r\Delta_d$ of the standard d-simplex Δ_d has an interesting interpretation in the cohomology of the Grassmannian $Gr(d, d+r)$.

The projection π induces a monomorphism π^* at the level of cohomology.

$$\pi^* : H^*(Gr(d, n), \mathbb{Z}) \longrightarrow H^*(\mathcal{F}\ell_n(\mathbb{C}), \mathbb{Z})$$

which takes cycle σ_λ to the cycle $\sigma_{w(\lambda)}$. The cohomology ring of the Grassmannian $Gr(d, n)$ is generated by the Schubert cycles σ_λ. These are Poincaré dual of the fundamental classes in the homology of Schubert varieties. Denote by Γ, the $\mathbb{Q}$-algebra of homogeneous symmetric functions in n variables $x_1, x_2, \cdots, x_n$. It well known that Γ is generated by Schur polynomials

s_λ among others, see [5], [8], [9] and [14]. By specializing $x_i = 0$ for $d+1 \le i \le n$, let Γ_d be the space of homogenous symmetric polynomials in variables $x_1, \cdots, x_d$, so Γ_d has the following presentation

$$\Gamma_d \cong \Gamma / \langle s_\lambda : \lambda \subsetneq \square_{d\times n-d}\rangle$$

The cohomology ring $H^*(Gr(d,n),\mathbb{Z})$ of the Grassmannian $Gr(d,n)$ by Borel presentation is given by

$$H^*(Gr(d,n),\mathbb{Z}) \cong \Gamma / \langle s_\lambda : \lambda \subsetneq \square_{d\times n-d}\rangle.$$

The interested readers may consult the following references [2], [4],[6],[7] and [12].

Recall that the Poincaré polynomial $P(X,t)$ associated with a given n-dimensional real manifold X is defined as

$$P(X,t) = \sum_{i=0}^{n} b_i(X)t^i$$

where $b_i(X) = \dim_{\mathbb{R}} H^i(X,\mathbb{R})$ is the i-th Betti number of X. This polynomial carries a lot of information about the topological and geometric invariants of X. It is well known that the cohomology ring $H^*(Gr(d,d+r),\mathbb{Z})$ has a polynomial description, that is,

$$H^*(Gr(d,d+r),\mathbb{Z}) \cong \mathbb{Z}[e'_1,\cdots,e'_d,e''_1,\ldots,e''_r]/\langle e_1,\cdots,e_{d+r}\rangle$$

where e'_i and e''_i are the i-th elementary symmetric functions in $x_1,\cdots,x_d$ and $x_{d+1},\cdots,x_{d+r}$ respectively and each x_i is the Chern class for the canonical bundle, so the Poncaré polynomial $P(Gr(d,d+r),t)$ is the following Gaussian polynomial

$$\frac{(1-t)(1-t^2)\cdots(1-t^{d+r})}{(1-t)\cdots(1-t^d)(1-t)\cdots(1-t^r)}$$

Theorem 5.1 *Let $P_w(z)$ be the weighted polynomial of the lattice points of the dilation $r\Delta_d$ of the standard d-simplex. Then the Poincaré polynomial $P(Gr(d,d+r),t)$ of the Grassmannian $Gr(d,d+r)$ coincides with the weighted polynomial $P_w(z)$.*

Example 5.2 The lattice points $(0,0,0),(1,0,0),(0,1,0),(2,0,0),(1,1,0),(0,0,1),(1,0,1)(0,1,1),$ $(0,0,2)(0,2,0),(3,0,0),(2,0,1),(1,2,0),(0,3,0),(0,2,1),(0,1,2),(0,0,3),(1,0,2),(2,0,1),(1,1,1)$ of $3\Delta_3$ graded by weights give the polynomial

$$1+t+3t^2+3t^3+3t^4+3t^5+3t^6+3t^7+t^8+t^9,$$

which is the Poincaré polynomial of the Grassmannian $Gr(3,6)$ so $3\Delta_3 \cap \mathbb{Z}^3_{\ge 0}$ encodes the Young poset of $Gr(3,6)$ shown in Figure 5.

Corollary 5.3 *Let λ^* be the β-partition identified with the monomial $\mathbf{t}^{\mathbf{a}} \in W^r_d$ then the length $\ell(w(\lambda^*))$ of the Grassmannian permutation $w(\lambda^*)$ is the weight $w_{\mathbf{a}}$ of the exponent vector $\mathbf{a} \in r\Delta_d \cap \mathbb{Z}^d_{\ge 0}$.*

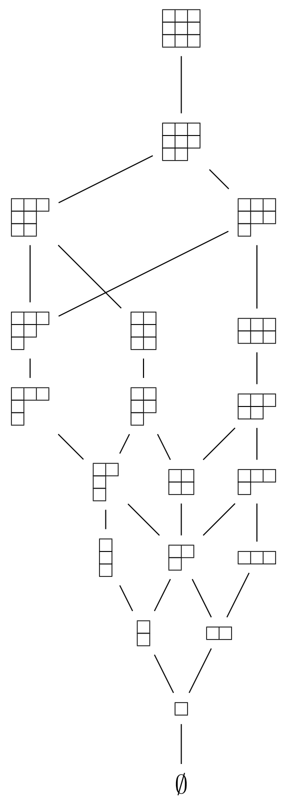

Figure 5

Acknowledgment

I would like to thank Dominic Bunett, Diane Maclagan and Mike Zabrocki for productive discussions during the preparation of the manuscript. I would also like to thank Balazs Szendröi for his hospitality and contributions during my visit to the University of Oxford where the work was carried out.

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International J.Math. Combin. Vol.1(2023), 21-38

Some New Ramanujan Type Series for $1/\pi$

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Abstract: In this paper, we derive some new Ramanujan-type series for $1/\pi$ as well as proved existing series, using Eisenstein series representations of the form $-P(q)+nP(q^n)$ and $P(q)+nP(q^n)$, along with Clausen's formulas.

Key Words: Eisenstein series, theta-functions, modular equations.

AMS(2010): 33C05, 33E05, 11F20, 11M36.

§1. Introduction

Ramanujan [43] recorded 17 hypergeometric series like representations for $1/\pi$ in which he gave the brief proof of first three series which are belong to the classical theory of elliptic functions. J. M. Borwein and P. B. Borwein were first proved all the 17 identities in 1987 [17]. Further they derived more series for $1/\pi$ [18], [19], [22]. Also many authors derived several new Ramanujan type series for $1/\pi$ as well as proved the existing identities in the subsequent years.

B. C. Berndt and H. H. Chan used Eisenstein series identities to prove Ramanujan type series for $1/\pi$ in their papers [12] and [13, where the latter one is coauthored with Wen-Chin Liaw. On the basis of the idea of above two papers and with the guidance of Chan, Baruah and Berndt used Eisenstein series identities of the form

$$-P(q^2)+nP(q^{2n}) \text{ and } P(q^2)+nP(q^{2n})$$

for $n=2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 9,\ 10,\ 13,\ 14,\ 15,\ 17,\ 18,\ 22$ and 25, to prove series of Ramanujan type series for $1/\pi$ in [3] and [4], by invoking the hints of Ramanujan. Further, Baruah and N. Nayak worked on Ramanujan type series for $1/\pi$ using Eisenstein series identities of the form $-P(-q)+nP(-q^n)$ and $P(-q)+nP(-q^n)$ for $n=3,\ 5,\ 7,\ 9,$ and 25. Motivated by this, using Clausen's formulas and Eisenstein series representations of the form $-P(q)+nP(q^n)$ and $P(q)+nP(q^n)$ for $n=2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9$ and 10, we proved 9 series out of 17 series that are recorded by Ramanujan in his famous paper [43] and some other existing series. Besides, we have recorded some new Ramanujan type series for $1/\pi$. A brief details of the existing identities which are

[1]Supported by Grant No. 191620127010/(CSIR-UGC NET DEC.2019) by the funding agency University Grants Commission, Government of India under Joint CSIR-UGC JRF scheme.

[2]Received February 8, 2022, Accepted March 16, 2023.

proved in the Sections 3-11 is given in the below table.

Sl. No.	Authors	Equations
1.	S. Ramanujan [43], [41]	(3.2), (4.3), (5.4), (6.1), (6.5), (8.4), (9.1), (9.6),(10.4)
2.	G. Bauer [7]	(3.2)
3.	J. Guillera [36]	(7.2)
4.	G. H. Hardy [39], [45]	(3.2)
5.	W. N. Bailey [2]	(3.2)
6.	J. M. Borwein and P. B. Borwein [17], [18]	(3.4), (7.5)
7.	B. C. Berndt, H. H. Chan and W. -C. Liaw [13]	(7.4), (9.5)
8.	N.D.Baruah and B.C.Berndt [3]	(3.1), (3.2), (3.3), (3.4), (4.1), (4.2), (4.4), (5.3), (5.4), (6.1), (6.2), (6.3), (6.4), (6.5), (7.1), (7.2), (7.4), (7.5), (8.3), (8.4), (9.1), (9.2), (9.3), (9.4), (9.5), (9.6), (10.2), (10.4)

The Section 2 contains preliminary definitions and results, in which (2.10) and (2.18) plays an important role in proving our results in the Sections 3-11, where (2.18) seems to be new.

§2. **Preliminaries**

Throughout the sequel, we use the following notation:

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$

where a and q are complex numbers with $|q| < 1$. For $|ab| < 1$, Ramanujan's general theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Further, Ramanujan [9, p36] considers following three special cases of $f(a,b)$:

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty},$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty},$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}.$$

After Ramanujan, we define

$$\chi(q) := (-q; q^2)_\infty.$$

The generalized hypergeometric functions ${}_pF_{p-1}$, $p \geq 1$, are defined by

$${}_pF_{p-1}[a_1,\ a_2, \cdots,\ a_p;\ b_1,\ b_2, \cdots,\ b_{p-1};\ x] := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_{p-1})_n} \frac{x^n}{n!},$$

where $|x| < 1$, $(a)_n := a(a+1)\cdots(a+n-1)$ and $(a)_0 := 1$. Ramanujan recorded the following identities in his Second Notebook [44] which give the relationship between hypergeometric series and theta functions. Moreover these identities are frequently used to derive our results. A proof of the below identities can be seen in [9, pp 120-124].

Lemma 2.1 *If*

$$q = e^{-y},\ y = -\pi \frac{{}_2F_1[\frac{1}{2},\ \frac{1}{2};\ 1\ ;\ 1-x\ ]}{{}_2F_1[\frac{1}{2},\ \frac{1}{2};\ 1\ ;\ x\ ]} \quad \textit{and} \quad z = {}_2F_1\left[\frac{1}{2},\ \frac{1}{2};\ 1\ ;\ x\ \right], \tag{2.1}$$

then

$$\varphi(q) = \sqrt{z}, \tag{2.2}$$

$$\varphi(-q) = \sqrt{z}(1-x)^{1/4}, \tag{2.3}$$

$$\psi(q) = \sqrt{\frac{z}{2}} \left(\frac{x}{q}\right)^{1/8}, \tag{2.4}$$

$$\psi(q^2) = \frac{\sqrt{z}}{2} \left(\frac{x}{q}\right)^{1/4}, \tag{2.5}$$

$$\psi(-q) = \sqrt{\frac{z}{2}} \left(\frac{x(1-x)}{q}\right)^{1/8}, \tag{2.6}$$

$$f(-q)=\frac{\sqrt{z}}{2^{1/6}}(1-x)^{1/6}\left(\frac{x}{q}\right)^{1/24}, \tag{2.7}$$

$$f(-q^2)=\frac{\sqrt{z}}{2^{1/3}}\left(\frac{x(1-x)}{q}\right)^{1/12}, \tag{2.8}$$

$$\chi(-q)=2^{1/6}(1-x)^{1/12}\left(\frac{q}{x}\right)^{1/24}, \tag{2.9}$$

and

$$\frac{dy}{dx}=-\frac{1}{x(1-x)z^2}. \tag{2.10}$$

Let $P(q)$ denote Ramanujan's Eisenstein series defined by

$$P(q):=1-24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, \quad |q|<1. \tag{2.11}$$

Further, Ramanujan [9, p.129] gave the representation for $P(q)$ in terms of $x,\ y$ and z:

$$P(q):=P(e^{-y})=(1-5x)z^2+12x(1-x)z\frac{dz}{dx}. \tag{2.12}$$

In the sequel, set

$$q:=e^{-\pi/\sqrt{n}}, \quad x_n:=x(e^{-\pi\sqrt{n}}) \ \text{ and } \ z_n:=z(e^{-\pi\sqrt{n}}). \tag{2.13}$$

From (2.2), (2.3), (2.5), (2.13) and [44, Entry 27, Chapter 16], we obtain that

$$x_{1/n}:=x(e^{-\pi/\sqrt{n}})=1-x_n \ \text{ and } \ z_{1/n}:=z(e^{-\pi/\sqrt{n}})=\sqrt{n}z_n. \tag{2.14}$$

The number x_n is called classical singular modulus. We often used the values of these numbers recorded by Ramanujan in [44]. For sometimes we borrow from [11] and [42]. Now employing (2.13) and (2.14) in (2.12) to obtain the following identities:

$$P(q):=P(e^{-\pi/\sqrt{n}})=(1-5x_{1/n})z_{1/n}^2+12x_{1/n}(1-x_{1/n})z_{1/n}\frac{dz_{1/n}}{dx_{1/n}}. \tag{2.15}$$

and

$$P(q^n):=P(e^{-\pi\sqrt{n}})=(1-5x_n)z_n^2+12x_n(1-x_n)z_n\frac{dz_n}{dx_n}. \tag{2.16}$$

The following theorem seems to be new and it produces the representations of the form $P(q)+nP(q^n)$, and with the help of Eisenstein series identities of the form $-P(q)+nP(q^n)$ [44, 47], we are able to derive some new Ramanujan-type series for $1/\pi$ as well as an alternate proof for the existing identities.

Theorem 2.2 *we have*

$$z_{1/n}\,\frac{dz_{1/n}}{dx_{1/n}} = -nz_n\,\frac{dz_n}{dx_n} + \frac{\sqrt{n}}{\pi x_n(1-x_n)} \tag{2.17}$$

and

$$P(e^{-\pi/\sqrt{n}}) + nP(e^{-\pi\sqrt{n}}) = \frac{12\sqrt{n}}{\pi} - 3nz_n^2. \tag{2.18}$$

Proof of (2.17) From (2.14), we have

$$z_{1/n}^2 = nz_n^2. \tag{2.19}$$

Differentiating (2.19) with respect to $x_{1/n}$ and using chain rule, we deduce that

$$2z_{1/n}\frac{dz_{1/n}}{dx_{1/n}} = 2nz_n\frac{dz_n}{dx_n}\frac{dx_n}{dx_{1/n}} + z_n^2\frac{dn}{dy}\frac{dy}{dx_{1/n}}. \tag{2.20}$$

From (2.14), we obtain that

$$\frac{dx_n}{dx_{1/n}} = -1. \tag{2.21}$$

From (2.1) and (2.13), we easily seen that

$$y = \frac{\pi}{\sqrt{n}}. \tag{2.22}$$

Differentiating (2.22) with respect to n, we find that

$$\frac{dn}{dy} = \frac{-2n\sqrt{n}}{\pi}. \tag{2.23}$$

Employing (2.14) in (2.10) to obtain

$$\frac{dy}{dx_{1/n}} = -\frac{1}{x_n(1-x_n)nz_n^2}. \tag{2.24}$$

Substituting (2.21), (2.23) and (2.24) into (2.20), we arrive at (2.17). □

Proof of (2.18) By employing (2.14) and (2.17) in (2.15), we find that

$$P(e^{-\pi/\sqrt{n}}) = n(-4+5x_n)z_n^2 - 12x_n(1-x_n)z_n\frac{dz_n}{dx_n} + \frac{12\sqrt{n}}{\pi}. \tag{2.25}$$

Then (2.18) follows from (2.16) and (2.25). □

Now our task is to obtain the relationship between Eisenstein series and ${}_3F_2$ hypergeometric series. To achieve this let us recall Clausen's formulas and Borwein's proofs [17, pp. 180-181]. Throughout the sequel, set

$$A_k := \frac{(\frac{1}{2})_k^3}{k!^3}, \quad B_k := \frac{(\frac{1}{4})_k(\frac{1}{2})_k(\frac{3}{4})_k}{k!^3} \text{ and } C_k := \frac{(\frac{1}{6})_k(\frac{1}{2})_k(\frac{5}{6})_k}{k!^3}. \tag{2.26}$$

If

$$X := 4x(1-x), \quad Y := \frac{4x}{(1-x)^2}, \quad U := \frac{x^2}{4(1-x)}, \quad V := \frac{4\sqrt{x}(1-x)}{(1+x)^2},$$
$$W := \frac{2\sqrt{X}}{1-X}, \quad L := \frac{27X^2}{(4-X)^3}, \quad \text{and} \quad M := \frac{27X}{(1-4X)^3},$$

then

$$z^2 = {}_3F_2\left[\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;X\right] = \sum_{k=0}^{\infty} A_k X^k, \quad 0 \le x \le \frac{1}{2}, \tag{2.27}$$

$$= \frac{1}{1-x}\,{}_3F_2\left[\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;-Y\right] = \frac{1}{1-x}\sum_{k=0}^{\infty}(-1)^k A_k Y^k, \quad 0 \le x \le 3-2\sqrt{2}, \tag{2.28}$$

$$= \frac{1}{\sqrt{1-x}}\,{}_3F_2\left[\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;-U\right] = \frac{1}{\sqrt{1-x}}\sum_{k=0}^{\infty}(-1)^k A_k U^k, \quad 0 \le x \le 2\sqrt{2}-2, \tag{2.29}$$

$$= \frac{1}{1+x}\,{}_3F_2\left[\frac{1}{4},\frac{1}{2},\frac{3}{4};1,1;V^2\right] = \frac{1}{1+x}\sum_{k=0}^{\infty} B_k V^{2k}, \quad 0 \le x \le 3-2\sqrt{2}, \tag{2.30}$$

$$= \frac{1}{1-2x}\,{}_3F_2\left[\frac{1}{4},\frac{1}{2},\frac{3}{4};1,1;-W^2\right] = \frac{1}{1-2x}\sum_{k=0}^{\infty}(-1)^k B_k W^{2k},$$
$$0 \le x \le \frac{1}{2}\left(1-2^{1/4}\sqrt{2-\sqrt{2}}\right), \tag{2.31}$$

$$= \frac{2}{\sqrt{4-X}}\,{}_3F_2\left[\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;L\right] = \frac{2}{\sqrt{4-X}}\sum_{k=0}^{\infty} C_k L^k, \quad 0 \le x \le \frac{1}{2}, \tag{2.32}$$

$$= \frac{1}{\sqrt{1-4X}}\,{}_3F_2\left[\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;-M\right] = \frac{1}{\sqrt{1-4X}}\sum_{k=0}^{\infty}(-1)^k C_k M^k, \quad 0 \le x \le \frac{1}{2}. \tag{2.33}$$

Differentiating (2.27) with respect to x, we find that

$$2z\frac{dz}{dx} = \sum_{k=0}^{\infty} A_k k X^{k-1} \cdot 4(1-2x). \tag{2.34}$$

Substituting (2.34) into (2.12) and using (2.27), we deduce that

$$P(q) = \sum_{k=0}^{\infty} \{6k(1-2x) + (1-5x)\} A_k X^k. \tag{2.35}$$

Setting $q = e^{-\pi\sqrt{n}}$ in (2.35), we obtain that

$$P(e^{-\pi\sqrt{n}}) = \sum_{k=0}^{\infty} \left\{ \frac{6k(1+x_n)+x_n}{1-x_n} + (1-5x_n) \right\} A_k X_n^k, \tag{2.36}$$

where $X_n = 4x_n(1-x_n)$. Similarly, differentiating each of (2.28)-(2.33) with respect to x, and proceeding as above, we deduce that

$$P(e^{-\pi\sqrt{n}}) = \frac{1+x_n}{1-x_n} \sum_{k=0}^{\infty} (6k+1)\,(-1)^k A_k Y_n^k, \tag{2.37}$$

$$= \frac{1}{\sqrt{1-x_n}} \sum_{k=0}^{\infty} \{6k(2-x_n) + 1 - 2x_n\}\,(-1)^k A_k U_n^k, \tag{2.38}$$

$$= \frac{1}{(1+x_n)^2} \sum_{k=0}^{\infty} \left\{6k(x_n^2 - 6x_n + 1) + x^2 - 10x_n + 1\right\} (-1)^k B_k V_n^{2k}, \tag{2.39}$$

$$= \frac{-1}{(1-2x_n)^2} \sum_{k=0}^{\infty} \left\{6k(4x_n^2 - 4x_n - 1) + 2x_n^2 - 5x_n - 1\right\} (-1)^k B_k W_n^{2k}, \tag{2.40}$$

$$= \sum_{k=0}^{\infty} \left\{ \frac{2(1-5x_n)}{\sqrt{4-X_n}} + \frac{3k(4x_n^3 - 6x_n^2 - 6x_n + 4) + 6x_n^3 - 9x_n^2 + 3x_n}{(1-x_n+x_n^2)^{\frac{3}{2}}} \right\} C_k L_n^k, \tag{2.41}$$

$$= \sum_{k=0}^{\infty} \left\{ \frac{1-5x_n}{\sqrt{1-4X_n}} + \frac{6k(64x_n^3 - 9x_n^2 + 30x_n + 1) + 96x_n^3 - 144x_n^2 + 48x_n}{(1-16x_n+16x_n^2)^{\frac{3}{2}}} \right\} \times (-1)^k C_k M_n^k, \tag{2.42}$$

where $X_n := 4x_n(1-x_n)$, $Y_n := \dfrac{4x_n}{(1-x_n)^2}$, $U_n := \dfrac{x_n^2}{4(1-x_n)}$, $V := \dfrac{4\sqrt{x_n}(1-x_n)}{(1+x_n)^2}$, $W_n :=$

$\dfrac{2\sqrt{X_n}}{1-X_n}$, $L_n := \dfrac{27X_n^2}{(4-X_n)^3}$ and $M_n := \dfrac{27X_n}{(1-4X_n)^3}$. Put $n=1$ in (2.18), we obtain that

$$P(e^{-\pi}) = \frac{6}{\pi} - \frac{3}{2}z_1^2,$$

Employing (2.27), we find that

$$\frac{6}{\pi} = P(e^{-\pi}) + \frac{3}{2}\sum_{k=0}^{\infty} A_k, \tag{2.43}$$

where $x_1 = \frac{1}{2}$ and $X_1 = 1$. The series (2.43) seems to be new and this is similar to the series recorded by Ramanujan in [8, p. 256].

§3. **Example:** $n = 2$

Theorem 3.1 *We have*

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \left\{(8-5\sqrt{2})k + 3 - 2\sqrt{2}\right\} A_k(2\sqrt{2}-2)^{3k}, \tag{3.1}$$

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}(-1)^k(4k+1)A_k, \tag{3.2}$$

$$\frac{2\sqrt{\sqrt{2}-1}}{\pi} = \sum_{k=0}^{\infty} \left\{(4\sqrt{2}-2)k + \sqrt{2}-1\right\}(-1)^k A_k \left(\frac{\sqrt{2}-1}{2}\right)^{3k}, \tag{3.3}$$

$$\frac{5\sqrt{5}}{\pi} = \sum_{k=0}^{\infty}(28k+3)C_k\left(\frac{3}{5}\right)^{3k}. \tag{3.4}$$

Proof From Entry 13(viii) in Chapter 17 of Ramanujan's second notebook [44] (Also [9, p.127]), we see that

$$-P(q) + 2P(q^2) = (1+x)z^2. \tag{3.5}$$

Setting $q = e^{-\pi/\sqrt{2}}$ in (3.5), then using (2.14) and the value of the singular modulus $x_2 = (\sqrt{2}-1)^2$ [11, p. 281], we find that

$$-P(e^{-\pi/\sqrt{2}}) + 2P(e^{-\pi\sqrt{2}}) = 2(-1+2\sqrt{2})z_2^2. \tag{3.6}$$

Setting $n = 2$ in (2.18), we obtain that

$$P(e^{-\pi/\sqrt{2}}) + 2P(e^{-\pi\sqrt{2}}) = \frac{12\sqrt{2}}{\pi} - 6z_2^2. \tag{3.7}$$

Adding (3.6) and (3.7), we immediately deduce that

$$P(e^{-\pi\sqrt{2}}) = \frac{3\sqrt{2}}{\pi} - (2 - \sqrt{2})z_2^2. \tag{3.8}$$

By employing (2.27) in (3.8), one can rewrite (3.8) as

$$P(e^{-\pi\sqrt{2}}) = \frac{3\sqrt{2}}{\pi} - (2 - \sqrt{2})\sum_{k=0}^{\infty} A_k X_2^k, \tag{3.9}$$

where $X_2 = (2\sqrt{2} - 2)$. Now, setting $n = 2$ in (2.36), then with the aid of (2.27) and the value of the singular modulus $x_2 = (\sqrt{2} - 1)^2$ [44, p. 214], [11, p. 281], we easily obtain that

$$P(e^{-\pi\sqrt{2}}) = \sum_{k=0}^{\infty}(-14 + 10\sqrt{(2)} + (-30 + 24\sqrt{(2)})k)A_k X_2^k, \tag{3.10}$$

From (3.9) and (3.10), we arrive at (3.1). Similarly the proofs of (3.2), (3.3) and (3.4) are follows, by employing (2.28), (2.29) and (2.32) in (3.8) and setting $n = 2$ in (2.37), (2.38) and (2.41), respectively. □

§4. **Example:** $n = 6$

Theorem 4.1 *We have*

$$\frac{\sqrt{6} + \sqrt{2} + 1}{\pi} = \sum_{k=0}^{\infty}\left\{\left(6\sqrt{3} + 3\sqrt{6} - 6\right)k + 2\sqrt{3} + \sqrt{6} - 3 - \sqrt{2}\right\}(-1)^k A_k \times \left(8(\sqrt{2}+1)^2(\sqrt{3} - \sqrt{2})^3(2 - \sqrt{3})^3\right)^k, \tag{4.1}$$

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}(-1)^k \left\{(12\sqrt{2} - 12)k + 4\sqrt{4} - 5\right\} A_k(\sqrt{2} - 1)^{4k}, \tag{4.2}$$

$$\frac{2\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}(8k + 1)B_k\frac{1}{9^k}. \tag{4.3}$$

Proof From [47], we have

$$-P(q)+6P(q^6)=f_1f_2f_3f_6\left(\frac{5\chi^6(-q^3)}{\chi^6(-q)}-\frac{q\chi^6(-q)}{\chi^6(-q^3)}\right). \tag{4.4}$$

This is

$$-P(q)+6P(q^6)=\frac{\chi^9(-q)\chi(-q^3)}{\chi^8(-q^2)}\frac{f^2(q^6)}{f^2(q^2)}\left(\frac{5\chi^6(-q^3)}{\chi^6(-q)}-\frac{q\chi^6(-q)}{\chi^6(-q^3)}\right)\varphi^4(q). \tag{4.5}$$

If $q=e^{-\pi/\sqrt{6}}$, then we obtain from [6, Theorem 4.1] that

$$\frac{f^2(e^{-\pi\sqrt{6}})}{f^2(e^{-2\pi/\sqrt{6}})}=\frac{(\sqrt{2}+1)^{1/3}}{\sqrt{3}}. \tag{4.6}$$

Setting $q=e^{-\pi/\sqrt{6}}$ in (4.5), using (2.9), (4.6), (2.14) and the values of the singular moduli $x_{3/2}=2\left(-3-2\sqrt{2}+2\sqrt{3}+\sqrt{6}\right)\left(\sqrt{2}-1\right)^2\left(\sqrt{3}+\sqrt{2}\right)^2$ [42] and $x_6=\left(2-\sqrt{3}\right)^2\left(\sqrt{3}-\sqrt{2}\right)^2$ [11, p. 282], we deduce that

$$-P(e^{-\pi/\sqrt{6}})+6P(e^{-\pi\sqrt{6}})=6(15+12\sqrt{2}-8\sqrt{3}-6\sqrt{6})z_6^2. \tag{4.7}$$

Setting $n=2$ in (2.18), we see that

$$P(e^{-\pi/\sqrt{6}})+6P(e^{-\pi\sqrt{6}})=\frac{12\sqrt{6}}{\pi}-18z_6^2. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$P(e^{-\pi\sqrt{6}})=(6+6\sqrt{2}-4\sqrt{3}-6\sqrt{6})z_6^2. \tag{4.9}$$

As in the previous Section, by employing (2.27), (2.28) and (2.30) in (4.9) and setting $n=6$ in (2.36), (2.37) and (2.39), we easily deduce the identities (4.1), (4.2) and (4.3), respectively.□

§5. **Example:** $n=9$

Theorem 5.1 *We have*

$$\frac{2}{(3\sqrt{3}-5)a\pi}=\sum_{k=0}^{\infty}\left(12k+3-\sqrt{3}\right)A_k\left(2-\sqrt{3}\right)^{4k}, \tag{5.1}$$

$$\frac{1+(6\sqrt{3}-10)a}{\pi}=\sum_{k=0}^{\infty}(-1)^k\left[\left\{9+(30-18\sqrt{3})a\right\}k+3+(6-4\sqrt{3})a\right]$$

$$\times A_k \frac{2^k(\sqrt{3}-1)^{4k}(a-1-\sqrt{3})^{2k}}{\{1+(6\sqrt{3}-10)a\}^k}, \tag{5.2}$$

$$\frac{2\sqrt{2+(12\sqrt{3}-20)a}}{\pi} = \sum_{k=0}^{\infty}(-1)^k \left[\left\{18+(36\sqrt{3}-60)a\right\}k+3+(10\sqrt{3}-18)a\right] \times A_k\left(\frac{(\sqrt{3}+1)(\sqrt{2}-3^{1/4})^3}{4}\right)^{2k}, \tag{5.3}$$

$$\frac{16}{\pi\sqrt{3}} = \sum_{k=0}^{\infty}(-1)^k(28k+3)B_k\frac{1}{48^k}, \tag{5.4}$$

where $a=\sqrt{3+2\sqrt{3}}$.

Proof On page 475 and page 345 of [9], we have

$$-P(q)+9P(q^9) = \frac{8f_3^6}{f_1^2f_9^2}\left\{f_1^6+9qf_1^3f_9^3+27q^2f_9^6\right\}^{1/3} \tag{5.5}$$

and

$$1+9q\frac{f_9^3}{f_1^3} = \left\{1+27q\frac{f_3^{12}}{f_1^{12}}\right\}^{1/3}. \tag{5.6}$$

The above identity can be written as

$$f_3^6 = \frac{f_1^6}{\sqrt{27q}}\left\{\left(1+9q\frac{f_9^3}{f_1^3}\right)^3-1\right\}^{1/2}. \tag{5.7}$$

Setting $q=e^{-\pi/3}$ in (5.5), then using (2.7), (5.7), (2.14) and the value of the singular modulus $x_9=\frac{1}{2}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4\left(\sqrt{4+2\sqrt{3}}-\sqrt{3+2\sqrt{3}}\right)^2$ [11, p. 290], we find that

$$-P(e^{-\pi/3})+9P(e^{-3\pi}) = 18\sqrt{3+2\sqrt{3}}(\sqrt{3}-1)Z_9^2. \tag{5.8}$$

Setting $n=9$ in (2.18), we see that

$$P(e^{-\pi/3})+9P(e^{-3\pi}) = \frac{36}{\pi}-27z_9^2. \tag{5.9}$$

From (5.8) and (5.9), we obtain that

$$P(e^{-3\pi}) = \frac{2}{\pi}\left\{\sqrt{3+2\sqrt{3}}(\sqrt{3}-1)-\frac{3}{2}\right\}z_9^2. \tag{5.10}$$

Now, employing (2.27), (2.28), (2.29) and (2.31) in (5.10) and setting $n=9$ in (2.36), (2.37), (2.38) and (2.40), we arrive at (5.1), (5.2), (5.3) and (5.4), respectively. □

The proofs of the Sections 6-11 follow along the similar lines as those in previous sections, so we do not record the proofs.

§6. **Example:** $n = 3$

Theorem 6.1 *We have*

$$\frac{4}{\pi} = \sum_{k=0}^{\infty} (6k+1)\, A_k \frac{1}{4^k}, \tag{6.1}$$

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} (-1)^k \left\{(15\sqrt{3}-24)k + 6\sqrt{3} - 10\right\} A_k 2^k (\sqrt{3}-1)^{6k}, \tag{6.2}$$

$$\frac{4\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} (-1)^k \left\{(30-6\sqrt{3})k + 7 - 3\sqrt{3}\right\} A_k \frac{(2-\sqrt{3})^{3k}}{2^{4k}}, \tag{6.3}$$

$$\frac{8\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} \left\{(85\sqrt{3}-135)k + 8\sqrt{3} - 12\right\} B_k \left(\frac{8\sqrt{2}}{51\sqrt{3}-75}\right)^{2k+1}, \tag{6.4}$$

$$\frac{5\sqrt{5}}{2\sqrt{3}\pi} = \sum_{k=0}^{\infty} (11k+1) C_k \left(\frac{4}{125}\right)^k. \tag{6.5}$$

We note that $-P(e^{-\pi/\sqrt{3}}) + 3P(e^{-\pi\sqrt{3}}) = \frac{9\sqrt{3}}{2} z_3^2$ and $x_3 = \frac{2-\sqrt{3}}{4}$.

§7. **Example:** $n = 4$

Theorem 7.1 *We have*

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \left\{(48\sqrt{2}-66)k + 20\sqrt{2} - 28\right\} A_k (1584\sqrt{2} - 2240)^k, \tag{7.1}$$

$$\frac{2\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} (-1)^k (6k+1) A_k \frac{1}{8^k}, \tag{7.2}$$

$$\frac{2\sqrt{3\sqrt{2}-4}}{\pi} = \sum_{k=0}^{\infty}(-1)^k\left\{(24\sqrt{2}-30)k+8\sqrt{2}-11\right\}A_k\left(\frac{(\sqrt{2}-1)^6}{16\sqrt{2}}\right)^k, \tag{7.3}$$

$$\frac{9}{2\pi} = \sum_{k=0}^{\infty}(7k+1)B_k\left(\frac{32}{81}\right)^k, \tag{7.4}$$

$$\frac{\sqrt{33}}{\pi} = \sum_{k=0}^{\infty}(126k+10)C_k\left(\frac{2}{11}\right)^{3k+1}. \tag{7.5}$$

We note that $-P(e^{-\pi/2})+3P(e^{-2\pi}) = 12z_4^2$ and $x_4 = (\sqrt{2}-1)^4$.

§8. **Example:** $n = 5$

Theorem 8.1 *We have*

$$\frac{\sqrt{2}}{b\pi} = \sum_{k=0}^{\infty}\left\{(5+\sqrt{5})k+1\right\}A_k(\sqrt{5}-2)^{2k}, \tag{8.1}$$

$$\begin{aligned}\frac{1}{\pi} = \sum_{k=0}^{\infty}&\left[\left\{80+35\sqrt{5}-(30\sqrt{2}+14\sqrt{10})b\right\}k+34+15\sqrt{5}-(13\sqrt{2}+6\sqrt{10})b\right]\\ &\times(-1)^kA_k8^k\left\{617+276\sqrt{5}-(485+217\sqrt{5})\frac{b}{\sqrt{2}}\right\}^k,\end{aligned} \tag{8.2}$$

$$\begin{aligned}\frac{8}{\pi} = \sum_{k=0}^{\infty}&\left[2\left\{(15+5\sqrt{5})b-7\sqrt{10}-5\sqrt{2}\right\}k+(9+3\sqrt{5})b-7\sqrt{2}-5\sqrt{10}\right](-1)^k\\ &\times A_k\left(\frac{\sqrt{5}-1}{4}\right)^{3k}\left(\frac{b^2}{2}-\frac{b}{\sqrt{2}}\right)^{6k},\end{aligned} \tag{8.3}$$

$$\frac{8}{\pi} = \sum_{k=0}^{\infty}(-1)^k(20k+3)B_k\frac{1}{4^k}, \tag{8.4}$$

$$\frac{2(-5+4\sqrt{5})^{3/2}}{b\sqrt{10}\pi} = \sum_{k=0}^{\infty}\left\{(142-58\sqrt{5})k+21-9\sqrt{5}\right\}C_k\left(\frac{27(-9875+4420\sqrt{5})}{55^3}\right)^k, \tag{8.5}$$

where $b = \sqrt{\sqrt{5}+1}$.

We note that $-P(e^{-\pi/\sqrt{5}})+5P(e^{-\pi\sqrt{5}}) = \dfrac{b(15-\sqrt{5})}{\sqrt{2}} z_5^2$ and the singular modulus for $n=5$ is $x_5 = \dfrac{1}{2}\left(\dfrac{\sqrt{5}-1}{2}\right)^3\left(\dfrac{b^2}{2}-\dfrac{b}{\sqrt{2}}\right)^2$.

§9. **Example:** $n=7$

Theorem 9.1 *We have*

$$\frac{16}{\pi} = \sum_{k=0}^{\infty}(42k+5)A_k\frac{1}{2^{6k}}, \tag{9.1}$$

$$\frac{1}{\pi} = \sum_{k=0}^{\infty}(-1)^k\left\{(255\sqrt{7}-672)k+112\sqrt{7}-296\right\}A_k(32-12\sqrt{7})^{3k}, \tag{9.2}$$

$$\frac{8\sqrt{2}}{\pi} = \sum_{k=0}^{\infty}(-1)^k\left\{(255\sqrt{7}-672)k+112\sqrt{7}-296\right\}A_k\left(\frac{8-3\sqrt{7}}{4}\right)^{3k}, \tag{9.3}$$

$$\frac{29241}{\pi} = \sum_{k=0}^{\infty}\left\{(76160-455\sqrt{7})k+6728-784\sqrt{7}\right\}B_k\left(\frac{8\sqrt{2}(325+119\sqrt{7})}{29241}\right)^{2k}, \tag{9.4}$$

$$\frac{9\sqrt{7}}{\pi} = \sum_{k=0}^{\infty}(65k+8)(-1)^kB_k\left(\frac{16}{63}\right)^{2k}, \tag{9.5}$$

$$\frac{85\sqrt{85}}{18\pi\sqrt{3}} = \sum_{k=0}^{\infty}(133k+8)C_k\left(\frac{4}{85}\right)^{3k}. \tag{9.6}$$

We note that $-P(e^{-\pi/\sqrt{7}})+7P(e^{-\pi\sqrt{7}}) = \dfrac{75\sqrt{7}}{8}z_7^2$ and $x_7 = \dfrac{8-3\sqrt{7}}{16}$.

§10. **Example:** $n=10$

Theorem 10.1 *We have*

$$\frac{310}{(680-480\sqrt{2}+304\sqrt{5}-215\sqrt{10})\pi} = \sum_{k=0}^{\infty}\left(930+220k-50\sqrt{2}+16\sqrt{5}-29\sqrt{10}\right)$$

$$\times A_k \left\{(3+\sqrt{5})(2+\sqrt{5})(3\sqrt{2}-\sqrt{5}-2)\right\}^{3k}, \qquad (10.1)$$

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}(-1)^k \left\{(60-24\sqrt{5})k + 23 - 10\sqrt{5}\right\} A_k(\sqrt{5}-2)^{4k}, \qquad (10.2)$$

$$\frac{\sqrt{10}\sqrt{102\sqrt{10}-144\sqrt{5}+228\sqrt{2}-322}}{\pi} = \sum_{k=0}^{\infty}\left\{\left(1020\sqrt{10}-1440\sqrt{5}+2280\sqrt{2}-3210\right)k + 407\sqrt{10}-576\sqrt{5}+910\sqrt{2}-1285\right\} \times (-1)^k A_k \left(\frac{207\sqrt{10}-288\sqrt{5}+450\sqrt{2}-647}{8}\right)^k, \qquad (10.3)$$

$$\frac{9}{2\sqrt{2}\pi} = \sum_{k=0}^{\infty}(10k+1)B_k \frac{1}{9^{2k}}. \qquad (10.4)$$

We note that $x_{10} = 323 - 228\sqrt{2} + 144\sqrt{5} - 102\sqrt{10}$ and $-P(e^{-\pi/\sqrt{10}}) + 10P(e^{-\pi\sqrt{10}}) = \left(2550 - 1800\sqrt{2} + 1152\sqrt{5} - 804\sqrt{10}\right) z_{10}^2$.

§11. **Example:** $n = 8$

Theorem 11.1 *We have*

$$\frac{7}{2\sqrt{2}\pi} = \sum_{k=0}^{\infty}\left[\left\{(560+392\sqrt{2})c - 1575 - 1120\sqrt{2}\right\}k + (248+174\sqrt{2})c - 700 - 497\sqrt{2}\right] \times A_k 16^k \left\{(4490+3175\sqrt{2})c - 12756 - 9020\sqrt{2}\right\}^k, \qquad (11.1)$$

$$\frac{14}{c\pi} = \sum_{k=0}^{\infty}(-1)^k \left(14k + 3 - \sqrt{2}\right) A_k \left(\frac{5\sqrt{2}-7}{8}\right)^k, \qquad (11.2)$$

$$\frac{7\sqrt{2}\sqrt{(10+7\sqrt{2})c - 28 - 20\sqrt{2}}}{\pi} = \sum_{k=0}^{\infty}\left[\left\{(560+392\sqrt{2})c - 1554 - 1120\sqrt{2}\right\}k + (216+152\sqrt{2})c - 609 - 434\sqrt{2}\right] \times (-1)^k A_k \left(\frac{(320+225\sqrt{2})c - 908 - 640\sqrt{2}}{32}\right)^k, \qquad (11.3)$$

$$\frac{343}{\pi(32-13\sqrt{2})} = \sum_{k=0}^{\infty}(70k+12-3\sqrt{2})B_k\frac{2^{5k}(325\sqrt{2}-457)^k}{7^{4k}}, \tag{11.4}$$

where $c = \sqrt{1+5\sqrt{2}}$.

We note that $x_8 = 113 + 80\sqrt{2} - 4c(7\sqrt{2}-10)$ and

$$-P(e^{-\pi/\sqrt{8}}) + 10P(e^{-\pi\sqrt{8}}) = \left\{600 - 416\sqrt{2} - \left(\frac{1408+1024\sqrt{2}}{7}\right)c\right\}z_8^2.$$

Acknowledgement

I would like to thank Vasuki K. R. for his guidance and support during the preparation of paper.

Data Availability

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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International J.Math. Combin. Vol.1(2023), 39-53

Some Generalized Inequalities for Functions of Bounded Variation Involving Weighted Area Balance Functions

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Abstract: In this study, we first prove an identity for the integrable functions involving weighted area balance function. Then, using this equality, some generalized inequalities for mappings of bounded variation are obtained. Moreover, some generalized inequalities for Lipschitzian functions are given. The results in this paper generalize the inequalities obtained in [10] and [19].

Key Words: Function of bounded variation, Riemann-Stieltjes integrals, area balance functions.

AMS(2010): 26D15, 26A45, 26D10.

§1. Introduction

Let $P : \kappa_1 = \varkappa_0 < \varkappa_1 < \cdots < \varkappa_n = \kappa_2$ be any partition of $[\kappa_1, \kappa_2]$ and let $\Delta\omega(\varkappa_i) = \omega(\varkappa_{i+1}) - \omega(\varkappa_i)$. Then ω is said to be of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta\omega(\varkappa_i)|$$

is bounded for all such partitions [4].

Let ω be of bounded variation on $[\kappa_1, \kappa_2]$, and $\sum \Delta\omega(P)$ denote the sum $\sum_{i=1}^{n} |\Delta\omega(\varkappa_i)|$ corresponding to the partition P of $[\kappa_1, \kappa_2]$. The number

$$\bigvee_{\kappa_1}^{\kappa_2}(\omega) := \sup\left\{\sum \Delta F(P) : P \in P([\kappa_1, \kappa_2])\right\},$$

is called the total variation of ω on $[\kappa_1, \kappa_2]$. Here $P([\kappa_1, \kappa_2])$ denotes the family of partitions of $[\kappa_1, \kappa_2]$ [4]. For a function of bounded variation $\omega : [\kappa_1, \kappa_2] \to \mathbb{C}$ we define the *cumulative variation function* $CVF : [\kappa_1, \kappa_2] \to [0, \infty)$ by

$$CVF(\xi) := \bigvee_{\kappa_1}^{\xi}(\omega)$$

[1]Received December 10, 2022, Accepted March 16, 2023.

the total variation ω on the interval $[\kappa_1,\xi]$. It is known that the CVF is monotonic nondecreasing on $[\kappa_1,\kappa_2]$ and is continuous in a point $c\in[\kappa_1,\kappa_2]$ if and only if the generating function ω is continuing in that point.

Integral inequalities for functions of bounded variation have potential applications in mathematical sciences. They have applications in numerical integration, probability and optimization theory, stochastic, statistics, information and integral operator theory. In the past, many authors have worked on some inequalities for functions of bounded variation. see for example ([1]-[12], [14]-[21]).

In [18], Dragomir give the following lemma which will be used frequently in our paper.

Lemma 1.1 *Let $\digamma,\omega:[\kappa_1,\kappa_2]\to\mathbb{C}$. If $\digamma$ is a continuous on $[\kappa_1,\kappa_2]$ and ω is of bounded variation on $[\kappa_1,\kappa_2]$, then*

$$\left|\int_{\kappa_1}^{\kappa_2}\digamma(\xi)d\omega(\xi)\right|\le\int_{\kappa_1}^{\kappa_2}|\digamma(\xi)|\,d\left(\bigvee_{\kappa_1}^{\xi}(\omega)\right)\le\max_{\xi\in[\kappa_1,\kappa_2]}|\digamma(\xi)|\bigvee_{\kappa_1}^{\kappa_2}(\omega).\tag{1.1}$$

In [19], Dragomir introduce the following area balance function:

Let $\digamma:[\kappa_1,\kappa_2]\to\mathbb{C}$ be Lebesgue integrable function. Then we define the function $AB_{\digamma}(\kappa_1,\kappa_2,.):[\kappa_1,\kappa_2]\to\mathbb{C}$ by

$$AB_{\digamma}(\kappa_1,\kappa_2,\varkappa)=\frac{1}{2}\left[\int_{\varkappa}^{\kappa_2}\digamma(\xi)d\xi-\int_{\kappa_1}^{\varkappa}\digamma(\xi)d\xi\right].\tag{1.2}$$

Moreover, Dragomir proved the following inequalities involving area balance function.

Theorem 1.1 *Let $\digamma:[\kappa_1,\kappa_2]\to\mathbb{C}$ be a function of bounded variation on $[\kappa_1,\kappa_2]$. Then*

$$\begin{aligned}
&\left|AB_{\digamma}(\kappa_1,\kappa_2,\varkappa)-\left(\frac{\kappa_1+\kappa_2}{2}-\varkappa\right)\digamma(\varkappa)\right|\\
&\le AB_{\bigvee_{\kappa_1}^{\cdot}(\digamma)}(\kappa_1,\kappa_2,\varkappa)-\left(\frac{\kappa_1+\kappa_2}{2}-\varkappa\right)\bigvee_{\kappa_1}^{\varkappa}(\digamma)\\
&=\frac{1}{2}\left[\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\xi}^{\varkappa}(\digamma)\right)d\xi+\int_{\varkappa}^{\kappa_2}\left(\bigvee_{\varkappa}^{\xi}(\digamma)\right)d\xi\right]\\
&\le\frac{1}{2}\left[(\varkappa-\kappa_1)\bigvee_{\kappa_1}^{\varkappa}(\digamma)+(\kappa_2-\varkappa)\bigvee_{\varkappa}^{\kappa_2}(\digamma)\right]\\
&\le\frac{1}{2}\times\begin{cases}\left[\frac{1}{2}(\kappa_2-\kappa_1)+\frac{1}{2}\left|\varkappa-\frac{\kappa_1+\kappa_2}{2}\right|\right]\bigvee_{\kappa_1}^{\kappa_2}(\digamma)\\ \left[\frac{1}{2}\bigvee_{\kappa_1}^{\kappa_2}(\digamma)\xi+\frac{1}{2}\left|\bigvee_{\kappa_1}^{\varkappa}(\digamma)-\bigvee_{\varkappa}^{\kappa_2}(\digamma)\right|\right](\kappa_2-\kappa_1)\end{cases}
\end{aligned}\tag{1.3}$$

for any $\varkappa\in[\kappa_1,\kappa_2]$.

Moreover, Dragomir prove some inequalities for the area balance of absolutely continuous functions in [20]. Delevar and Dragomir give some weighted trapezoidal inequalities related to the area balance of a function in [13]. On the other hand, in [10] Budak and Pehlivan establish some generalizations of the results in [19]. In this paper we obtain some new generalized weighted inequalities by the weighted area balance functions.

§2. Main Results

First we recall the following weighted area functions given in [10]:

Let $\varpi : [\kappa_1, \kappa_2] \to [0, \infty)$ be Lebesgue integrable and let $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a function of bounded variation on $[\kappa_1, \kappa_2]$. Then we define the weighted area balance function $WAB_F(\kappa_1, \kappa_2, .; \varpi) : [\kappa_1, \kappa_2] \to \mathbb{R}$ by

$$WAB_F(\kappa_1, \kappa_2, \varkappa; \varpi) := \frac{1}{2}\left[\int_{\varkappa}^{\kappa_2} F(\xi)\varpi(\xi)d\xi - \int_{\kappa_1}^{\varkappa} F(\xi)\varpi(\xi)d\xi\right].$$

Throughout the paper, we denote the weighted area balance function $AB_F(\kappa_1, \kappa_2, \varkappa; \varpi)$ by $WAB_F(\kappa_1, \kappa_2, \varkappa)$. First we prove the following generalized identity involving the weighted area balance function.

Lemma 2.1 *Let $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a function of bounded variation on $[\kappa_1, \kappa_2]$. Then we have the following identity*

$$\begin{aligned} & WAB_F(\kappa_1, \kappa_2, \varkappa) - \lambda AB_{\varpi}(\kappa_1, \kappa_2, \varkappa)F(\varkappa) \\ & -\frac{(1-\lambda)}{2}\left[\left(\int_{\varkappa}^{\kappa_2} \varpi(\eta)d\eta\right)F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa} \varpi(\eta)d\eta\right)F(\kappa_1)\right] \\ = & \frac{1}{2}\int_{\kappa_1}^{\kappa_2} K_{\varpi}^{\lambda}(\xi, \varkappa)dF(\xi) \end{aligned} \tag{2.1}$$

for all $\varkappa \in [\kappa_1, \kappa_2]$ and $\lambda \in [0,1]$ where $K_{\varpi}^{\lambda}(\xi, \varkappa) : [\kappa_1, \kappa_2] \times [\kappa_1, \kappa_2] \to \mathbb{R}$ is defined by

$$K_{\varpi}^{\lambda}(\xi, \varkappa) = \begin{cases} \lambda \displaystyle\int_{\kappa_1}^{\xi} \varpi(\eta)d\eta + (1-\lambda)\int_{\xi}^{\varkappa} \varpi(\eta)d\eta, & \kappa_1 \le \xi < \varkappa \\ \lambda \displaystyle\int_{\xi}^{\kappa_2} \varpi(\eta)d\eta + (1-\lambda)\int_{\varkappa}^{\xi} \varpi(\eta)d\eta, & \varkappa \le \xi \le \kappa_2 \end{cases}$$

and the integrals in the right hand side are taken in the Riemann-Stieltjes sense.

Proof From the definition of $K_{\varpi}^{\lambda}(\xi,\varkappa)$, we have

$$\begin{aligned}
&\int_{\kappa_1}^{\kappa_2} K_{\varpi}^{\lambda}(\xi,\varkappa)dF(\xi)\\
&=\int_{\kappa_1}^{\varkappa}\left[\lambda\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta+(1-\lambda)\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right]dF(\xi)\\
&\quad+\int_{\varkappa}^{\kappa_2}\left[\lambda\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta+(1-\lambda)\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right]dF(\xi)\\
&=\lambda\int_{\kappa_1}^{\varkappa}\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)dF(\xi)+(1-\lambda)\int_{\kappa_1}^{\varkappa}\left(\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right)dF(\xi)\\
&\quad+\lambda\int_{\varkappa}^{\kappa_2}\left(\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta\right)dF(\xi)+(1-\lambda)\int_{\varkappa}^{\kappa_2}\left(\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right)dF(\xi). \qquad (2.2)
\end{aligned}$$

Using the integration by parts for Riemann-Stieltjes integrals, we get

$$\begin{aligned}
&\int_{\kappa_1}^{\varkappa}\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)dF(\xi)\\
&=\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)F(\xi)\Bigg|_{\kappa_1}^{\varkappa}-\int_{\kappa_1}^{\varkappa}F(\xi)d\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)\\
&=\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\varkappa)-\int_{\kappa_1}^{\varkappa}F(\xi)\varpi(\xi)d\xi. \qquad (2.3)
\end{aligned}$$

Similarly, we have

$$\int_{\kappa_1}^{\varkappa}\left(\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right)dF(\xi)=\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)-\int_{\kappa_1}^{\varkappa}F(\xi)\varpi(\xi)d\xi, \qquad (2.4)$$

$$\int_{\varkappa}^{\kappa_2}\left(\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta\right)dF(\xi)=\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\varkappa)+\int_{\varkappa}^{\kappa_2}F(\xi)\varpi(\xi)d\xi \qquad (2.5)$$

and

$$\int_{\varkappa}^{\kappa_2}\left(\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right)dF(\xi)=\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2)+\int_{\varkappa}^{\kappa_2}F(\xi)\varpi(\xi)d\xi. \qquad (2.6)$$

If we substitute the equalities (2.3) – (2.6) in (2.2), then we obtain

$$
\begin{aligned}
&\int_{\kappa_1}^{\kappa_2} K_{\varpi}^{\lambda}(\xi,\varkappa)dF(\xi) \\
&= \lambda\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\varkappa) - \lambda\int_{\kappa_1}^{\varkappa}F(\xi)\varpi(\xi)d\xi \\
&\quad +(1-\lambda)\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1) - (1-\lambda)\int_{\kappa_1}^{\varkappa}F(\xi)\varpi(\xi)d\xi \\
&\quad -\lambda\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\varkappa) + \lambda\int_{\varkappa}^{\kappa_2}F(\xi)\varpi(\xi)d\xi \\
&\quad -(1-\lambda)\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2) + (1-\lambda)\int_{\varkappa}^{\kappa_2}F(\xi)\varpi(\xi)d\xi \\
&= -\lambda f(\varkappa)\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta - \int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right) + \lambda\left(\int_{\varkappa}^{\kappa_2}F(\xi)\varpi(\xi)d\xi - \int_{\kappa_1}^{\varkappa}F(\xi)\varpi(\xi)d\xi\right) \\
&\quad +(1-\lambda)\left(\int_{\varkappa}^{\kappa_2}F(\xi)\varpi(\xi)d\xi - \int_{\kappa_1}^{\varkappa}F(\xi)\varpi(\xi)d\xi\right) \\
&\quad -(1-\lambda)\left[\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right] \\
&= 2WAB_F(\kappa_1,\kappa_2,\varkappa) - 2\lambda AB_{\varpi}(\kappa_1,\kappa_2,\varkappa)F(\varkappa) \\
&\quad -(1-\lambda)\left[\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right]
\end{aligned}
$$

which completes the proof. □

Remark 2.1 If we take $\lambda = 1$ and $\lambda = 0$ in Lemma 2.1, then Lemma 2.1 reduces to Lemma 2 and Lemma 3 in [10], respectively.

Corollary 2.1 *If we choose $\varpi(\varkappa) = 1$ for all $\varkappa \in [\kappa_1, \kappa_2]$ in Lemma 2.1, then we have the following identity*

$$
\begin{aligned}
&WAB_F(\kappa_1,\kappa_2,\varkappa) - \lambda\left(\frac{\kappa_1+\kappa_2}{2} - \varkappa\right)F(\varkappa) \\
&\quad -\frac{(1-\lambda)}{2}\left[\frac{\kappa_2 f(\kappa_2)+\kappa_1 f(\kappa_1)}{2} - \frac{F(\kappa_1)+F(\kappa_2)}{2}\varkappa\right] \\
&= \frac{1}{2}\int_{\kappa_1}^{\kappa_2}\left[\lambda p(\varkappa,\xi) + (1-\lambda)\left|\varkappa-\xi\right|\right]dF(\xi) \qquad (2.7)
\end{aligned}
$$

for all $\varkappa \in [\kappa_1, \kappa_2]$ *and* $\lambda \in [0,1]$ *where* $p(\xi, \varkappa) : [\kappa_1, \kappa_2] \times [\kappa_1, \kappa_2] \to \mathbb{R}$ *is defined by*

$$p(\xi, \varkappa) = \begin{cases} \xi - \kappa_1 & \kappa_1 \le \xi < \varkappa \\ \\ \kappa_2 - \xi & \varkappa \le \xi \le \kappa_2. \end{cases}$$

Remark 2.2 If we take $\lambda = 1$ and $\lambda = 0$ in Corollary 2.1, then the equality (2.7) reduces to the equalities (2.1) and (2.2) of Theorem 1 in [19], respectively.

Theorem 2.2 *If* $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ *is a function of bounded variation on* $[\kappa_1, \kappa_2]$, *then we have*

$$\begin{aligned} &\Bigg| WAB_F(\kappa_1, \kappa_2, \varkappa) - \lambda AB_\varpi(\kappa_1, \kappa_2, \varkappa) F(\varkappa) \\ &\quad - \frac{(1-\lambda)}{2} \left[\left(\int_{\varkappa}^{\kappa_2} \varpi(\eta) d\eta \right) F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa} \varpi(\eta) d\eta \right) F(\kappa_1) \right] \Bigg| \\ &\le \frac{1}{2} \lambda \int_{\kappa_1}^{\varkappa} \left(\bigvee_{\xi}^{\varkappa} (F) \right) \varpi(\xi) d\xi + \frac{1}{2} \lambda \int_{\varkappa}^{\kappa_2} \left(\bigvee_{\varkappa}^{\xi} (F) \right) \varpi(\xi) d\xi \\ &\quad + \frac{1}{2} (1-\lambda) \int_{\kappa_1}^{\varkappa} \varpi(\xi) \left(\bigvee_{\kappa_1}^{\xi} (F) \right) d\xi + \frac{1}{2} (1-\lambda) \int_{\varkappa}^{\kappa_2} \varpi(\xi) \left(\bigvee_{\xi}^{\kappa_2} (F) \right) d\xi \\ &\le \frac{1}{2} \left[\bigvee_{\kappa_1}^{\varkappa} (F) \int_{\kappa_1}^{\varkappa} \varpi(\xi) d\xi + \bigvee_{\varkappa}^{\kappa_2} (F) \int_{\varkappa}^{\kappa_2} \varpi(\xi) d\xi \right] \\ &\le \frac{1}{2} \begin{cases} \left[\frac{1}{2} \int_{\kappa_1}^{\kappa_2} \varpi(\xi) d\xi + \frac{1}{2} \left| \int_{\kappa_1}^{\varkappa} \varpi(\xi) d\xi - \int_{\varkappa}^{\kappa_2} \varpi(\xi) d\xi \right| \right] \bigvee_{\kappa_1}^{\kappa_2} (F), \\ \left[\frac{1}{2} \bigvee_{\kappa_1}^{\kappa_2} (F) \xi + \left| \bigvee_{\kappa_1}^{\varkappa} (F) - \bigvee_{\varkappa}^{\kappa_2} (F) \right| \right] \int_{\kappa_1}^{\kappa_2} \varpi(\xi) d\xi \end{cases} \end{aligned} \tag{2.8}$$

for all $\varkappa \in [\kappa_1, \kappa_2]$ *and* $\lambda \in [0,1]$.

Proof Taking the modulus identity (2.1), we have

$$\begin{aligned} &\Bigg| WAB_F(\kappa_1, \kappa_2, \varkappa) - \lambda AB_\varpi(\kappa_1, \kappa_2, \varkappa) F(\varkappa) \\ &\quad - \frac{(1-\lambda)}{2} \left[\left(\int_{\varkappa}^{\kappa_2} \varpi(\eta) d\eta \right) F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa} \varpi(\eta) d\eta \right) F(\kappa_1) \right] \Bigg| \\ &= \frac{1}{2} \left| \int_{\kappa_1}^{\kappa_2} K_\varpi^\lambda(\xi, \varkappa) dF(\xi) \right| \\ &= \frac{1}{2} \left| \int_{\kappa_1}^{\varkappa} \left[\lambda \int_{\kappa_1}^{\xi} \varpi(\eta) d\eta + (1-\lambda) \int_{\xi}^{\varkappa} \varpi(\eta) d\eta \right] dF(\xi) \right. \end{aligned}$$

$$
\begin{aligned}
&+\int_{\varkappa}^{\kappa_2}\left[\lambda\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta+(1-\lambda)\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right]dF(\xi)\Bigg| \\
&\le\frac{1}{2}\left|\int_{\kappa_1}^{\varkappa}\left[\lambda\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta+(1-\lambda)\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right]dF(\xi)\right| \\
&+\frac{1}{2}\left|\int_{\varkappa}^{\kappa_2}\left[\lambda\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta+(1-\lambda)\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right]dF(\xi)\right|.
\end{aligned}
$$

Using Lemma 1.1, we get

$$
\begin{aligned}
&\Bigg|WAB_F(\kappa_1,\kappa_2,\varkappa)-\lambda AB_{\varpi}(\kappa_1,\kappa_2,\varkappa)F(\varkappa) \\
&\quad-\frac{(1-\lambda)}{2}\left[\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2)-\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right]\Bigg| \\
&\le\frac{1}{2}\left[\int_{\kappa_1}^{\varkappa}\left|\lambda\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta+(1-\lambda)\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right|d\left(\bigvee_{\kappa_1}^{\xi}(F)\right)\right] \\
&\quad+\frac{1}{2}\left[\int_{\varkappa}^{\kappa_2}\left|\lambda\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta+(1-\lambda)\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right|d\left(\bigvee_{\xi}^{\kappa_2}(F)\right)\right] \\
&=\frac{1}{2}\left[\int_{\kappa_1}^{\varkappa}\left(\lambda\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta+(1-\lambda)\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right)d\left(\bigvee_{\kappa_1}^{\xi}(F)\right)\right] \\
&\quad+\frac{1}{2}\left[\int_{\varkappa}^{\kappa_2}\left(\lambda\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta+(1-\lambda)\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right)d\left(\bigvee_{\xi}^{\kappa_2}(F)\right)\right] \\
&=\frac{1}{2}\left[\lambda\int_{\kappa_1}^{\varkappa}\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)d\left(\bigvee_{\kappa_1}^{\xi}(F)\right)+(1-\lambda)\int_{\kappa_1}^{\varkappa}\left(\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right)d\left(\bigvee_{\kappa_1}^{\xi}(F)\right)\right] \\
&\quad+\frac{1}{2}\left[\lambda\int_{\varkappa}^{\kappa_2}\left(\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta\right)d\left(\bigvee_{\varkappa}^{\kappa_2}(F)\right)+(1-\lambda)\int_{\varkappa}^{\kappa_2}\left(\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right)d\left(\bigvee_{\varkappa}^{\kappa_2}(F)\right)\right].
\end{aligned}
\tag{2.9}
$$

By utilizing the integration by parts for Riemann-Stieltjes integrals, we obtain

$$
\int_{\kappa_1}^{\varkappa}\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)d\left(\bigvee_{\kappa_1}^{\xi}(F)\right)=\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)\bigvee_{\kappa_1}^{\xi}(F)\Bigg|_{\kappa_1}^{\varkappa}-\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\kappa_1}^{\xi}(F)\right)d\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)
$$

$$
\begin{aligned}
&= \left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)\bigvee_{\kappa_1}^{\varkappa}(F) - \int_{\kappa_1}^{\varkappa}\left(\bigvee_{\kappa_1}^{\xi}(F)\right)\varpi(\xi)d\xi \\
&= \int_{\kappa_1}^{\varkappa}\left[\bigvee_{\kappa_1}^{\varkappa}(F) - \bigvee_{\kappa_1}^{\xi}(F)\right]\varpi(\xi)d\xi = \int_{\kappa_1}^{\varkappa}\left(\bigvee_{\xi}^{\varkappa}(F)\right)\varpi(\xi)d\xi, \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
\int_{\kappa_1}^{\varkappa}\left(\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right)d\left(\bigvee_{\kappa_1}^{\xi}(F)\right) &= \left(\int_{\xi}^{\varkappa}(\varpi(\eta)d\eta\right)\bigvee_{\kappa_1}^{\xi}(F)\Bigg|_{\kappa_1}^{\varkappa} - \int_{\kappa_1}^{\varkappa}\left(\bigvee_{\kappa_1}^{\xi}(F)\right)d\left(\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right) \\
&= \left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)\bigvee_{\kappa_1}^{\varkappa}(F) + \int_{\kappa_1}^{\varkappa}\left(\bigvee_{\kappa_1}^{\xi}(F)\right)d\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right) \\
&= \left(\int_{\kappa_1}^{\varpi}\varpi(\eta)d\eta\right)\bigvee_{\kappa_1}^{\varkappa}(F) + \int_{\kappa_1}^{\varkappa}\left(\bigvee_{\kappa_1}^{\xi}(F)\right)\varpi(\xi)d\xi \\
&= \int_{\kappa_1}^{\varkappa}\varpi(\xi)\bigvee_{\kappa_1}^{\xi}(F)d\xi, \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
\int_{\varkappa}^{\kappa_2}\left(\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta\right)d\left(\bigvee_{\xi}^{\kappa_2}(F)\right) &= \left(\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta\right)\bigvee_{\xi}^{\kappa_2}(F)\Bigg|_{\varkappa}^{\kappa_2} - \int_{\varkappa}^{\kappa_2}\left(\bigvee_{\xi}^{\kappa_2}(F)\right)d\left(\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta\right) \\
&= -\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)\bigvee_{\varkappa}^{\kappa_2}(F) + \int_{\varkappa}^{\kappa_2}\left(\bigvee_{\xi}^{\kappa_2}(F)\right)\varpi(\xi)d\xi \\
&= \int_{\varkappa}^{\kappa_2}\left(\bigvee_{\xi}^{\kappa_2}(F) - \bigvee_{\varkappa}^{\kappa_2}(F)\right)\varpi(\xi)d\xi = \int_{\varkappa}^{\kappa_2}\left(\bigvee_{\varkappa}^{\xi}(F)\right)\varpi(\xi)d\xi \qquad (2.12)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\varkappa}^{\kappa_2}\left(\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right)d\left(\bigvee_{\xi}^{\kappa_2}(F)\right) &= \left(\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right)\bigvee_{\xi}^{\kappa_2}(F)\Bigg|_{\varkappa}^{\kappa_2} - \int_{\varkappa}^{\kappa_2}\left(\bigvee_{\xi}^{\kappa_2}(F)\right)d\left(\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right) \\
&= \left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)\bigvee_{\varkappa}^{\kappa_2}(F) - \int_{\varkappa}^{\kappa_2}\varpi(\xi)\left(\bigvee_{\xi}^{\kappa_2}(F)\right)d\xi \\
&= \int_{\varkappa}^{\kappa_2}\varpi(\xi)\left(\bigvee_{\varkappa}^{\kappa_2}(F) - \bigvee_{\xi}^{\kappa_2}(F)\right)d\xi = \int_{\varkappa}^{\kappa_2}\varpi(\xi)\bigvee_{\xi}^{\kappa_2}(F)d\xi. \qquad (2.13)
\end{aligned}
$$

By substituting the equalities (2.10)-(2.13) in (2.9), we establish

$$\left|WAB_F(\kappa_1,\kappa_2,\varkappa)-\lambda AB_F(\kappa_1,\kappa_2,\varkappa)-\frac{(1-\lambda)}{2}\left[\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2)-\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right]\right|$$
$$\leq\frac{1}{2}\left[\lambda\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\xi}^{\varkappa}(F)\right)\varpi(\xi)d\xi+(1-\lambda)\int_{\kappa_1}^{\varkappa}\varpi(\xi)\left(\bigvee_{\kappa_1}^{\xi}(F)\right)d\xi\right]$$
$$+\frac{1}{2}\left[\lambda\int_{\varkappa}^{\kappa_2}\left(\bigvee_{\varkappa}^{\xi}(F)\right)\varpi(\xi)d\xi+(1-\lambda)\int_{\varkappa}^{\kappa_2}\varpi(\xi)\left(\bigvee_{\xi}^{\kappa_2}(F)\right)d\xi\right]$$

which gives first inequality in (2.8). By using the facts that

$$\bigvee_{\xi}^{\varkappa}(F)\leq\bigvee_{\kappa_1}^{\varkappa}(F),\quad\bigvee_{\kappa_1}^{\xi}(F)\leq\bigvee_{\kappa_1}^{\varkappa}(F),\quad\xi\in[\kappa_1,\varkappa],$$
$$\bigvee_{\varkappa}^{\xi}(F)\leq\bigvee_{\varkappa}^{\kappa_2}(F),\quad\bigvee_{\xi}^{\kappa_2}(F)\leq\bigvee_{\varkappa}^{\kappa_2}(F),\quad\xi\in[\varkappa,\kappa_2]$$

we obtain

$$\frac{1}{2}\left[\lambda\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\xi}^{\varkappa}(F)\right)\varpi(\xi)d\xi+(1-\lambda)\int_{\kappa_1}^{\varkappa}\varpi(\xi)\left(\bigvee_{\kappa_1}^{\xi}(F)\right)d\xi\right]$$
$$+\frac{1}{2}\left[\lambda\int_{\varkappa}^{\kappa_2}\left(\bigvee_{\varkappa}^{\xi}(F)\right)\varpi(\xi)d\xi+(1-\lambda)\int_{\varkappa}^{\kappa_2}\varpi(\xi)\left(\bigvee_{\varkappa}^{\kappa_2}(F)\right)d\xi\right]$$
$$\leq\frac{1}{2}\left[\lambda\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\kappa_1}^{\varkappa}(F)\right)\varpi(\xi)d\xi+(1-\lambda)\int_{\kappa_1}^{\varkappa}\varpi(\xi)\bigvee_{\kappa_1}^{\varkappa}(F)d\xi\right]$$
$$+\frac{1}{2}\left[\lambda\int_{\varkappa}^{\kappa_2}\left(\bigvee_{\varkappa}^{\kappa_2}(F)\right)\varpi(\xi)d\xi+(1-\lambda)\int_{\varkappa}^{\kappa_2}\varpi(\xi)\bigvee_{\varkappa}^{\kappa_2}(F)d\xi\right]$$
$$=\frac{1}{2}\left[\lambda\bigvee_{\kappa_1}^{\varkappa}(F)\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi+(1-\lambda)\bigvee_{\kappa_1}^{\varkappa}(F)\int_{\varkappa}^{\varkappa}\varpi(\xi)d\xi\right]$$
$$+\frac{1}{2}\left[\lambda\bigvee_{\varkappa}^{\kappa_2}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi+(1-\lambda)\bigvee_{\varkappa}^{\kappa_2}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right]$$
$$=\bigvee_{\kappa_1}^{\varkappa}(F)\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi+\bigvee_{\varkappa}^{\kappa_2}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi.$$

This proves the second inequality in (2.8).

Notice that the last inequality in (2.8) is obvious from the fact that

$$\max\{p,q\}=\frac{p+q}{2}+\frac{1}{2}|p-q|$$

for $p,q\in\mathbb{R}$. □

Remark 2.3 If we take $\lambda=1$ and $\lambda=0$ in Theorem 2.2, then we have the following inequalities, respectively

$$\begin{aligned}
&|AB_F(\kappa_1,\kappa_2,\varkappa)-\lambda AB_\varpi(\kappa_1,\kappa_2,\varkappa)F(\varkappa)|\\
&\le \frac{1}{2}\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\xi}^{\varkappa}(F)\right)\varpi(\xi)d\xi+\frac{1}{2}\int_{\varkappa}^{\kappa_2}\left(\bigvee_{\varkappa}^{\xi}(F)\right)\varpi(\xi)d\xi\\
&\le \frac{1}{2}\left[\bigvee_{\kappa_1}^{\varkappa}(F)\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi+\bigvee_{\varkappa}^{\kappa_2}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right]\\
&\le \frac{1}{2}\begin{cases}\left[\frac{1}{2}\int_{\kappa_1}^{\kappa_2}\varpi(\xi)d\xi+\frac{1}{2}\left|\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi-\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right|\right]\bigvee_{\kappa_1}^{\kappa_2}(F),\\[2ex]
\left[\frac{1}{2}\bigvee_{\kappa_1}^{\kappa_2}(F)\xi+\left|\bigvee_{\kappa_1}^{\varkappa}(F)-\bigvee_{\varkappa}^{\kappa_2}(F)\right|\right]\int_{\kappa_1}^{\kappa_2}\varpi(\xi)d\xi\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\left|AB_F(\kappa_1,\kappa_2,\varkappa)-\frac{1}{2}\left[\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2)-\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right]\right|\\
&\le \frac{1}{2}\int_{\kappa_1}^{\varkappa}\varpi(\xi)\left(\bigvee_{\kappa_1}^{\xi}(F)\right)d\xi+\frac{1}{2}\int_{\varkappa}^{\kappa_2}\varpi(\xi)\left(\bigvee_{\xi}^{\kappa_2}(F)\right)d\xi\\
&\le \frac{1}{2}\left[\bigvee_{\kappa_1}^{\varkappa}(F)\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi+\bigvee_{\varkappa}^{\kappa_2}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right]\\
&\le \frac{1}{2}\begin{cases}\left[\frac{1}{2}\int_{\kappa_1}^{\kappa_2}\varpi(\xi)d\xi+\frac{1}{2}\left|\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi-\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right|\right]\bigvee_{\kappa_1}^{\kappa_2}(F),\\[2ex]
\left[\frac{1}{2}\bigvee_{\kappa_1}^{\kappa_2}(F)\xi+\left|\bigvee_{\kappa_1}^{\varkappa}(F)-\bigvee_{\varkappa}^{\kappa_2}(F)\right|\right]\int_{\kappa_1}^{\kappa_2}\varpi(\xi)d\xi\end{cases}
\end{aligned}$$

which are proved by Budak and Pehlivan in [10].

Corollary 2.2 *If we choose $\varpi(\varkappa)=1$ for all $\varkappa\in[\kappa_1,\kappa_2]$ in Theorem 2.2, then we have the* *following inequality*

$$
\begin{aligned}
&\Bigg| WAB_F(\kappa_1,\kappa_2,\varkappa) - \lambda\left(\frac{\kappa_1+\kappa_2}{2}-\varkappa\right)F(\varkappa) \\
&\quad - \frac{(1-\lambda)}{2}\left[\frac{\kappa_2 f(\kappa_2)+\kappa_1 f(\kappa_1)}{2} - \frac{F(\kappa_1)+F(\kappa_2)}{2}\varkappa\right]\Bigg| \\
&\le \frac{\lambda}{2}\left[\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\xi}^{\varkappa}(F)\right)d\xi + \int_{\varkappa}^{\kappa_2}\left(\bigvee_{\varkappa}^{\xi}(F)\right)d\xi\right] \\
&\quad + \frac{(1-\lambda)}{2}\left[\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\kappa_1}^{\xi}(F)\right)d\xi + \int_{\varkappa}^{\kappa_2}\left(\bigvee_{\xi}^{\kappa_2}(F)\right)d\xi\right] \\
&\le \frac{1}{2}\left[\bigvee_{\kappa_1}^{\varkappa}(F)\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi + \bigvee_{\varkappa}^{\kappa_2}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right] \\
&\le \frac{1}{2}\begin{cases} \left[\frac{1}{2}(\kappa_2-\kappa_1) + \left|\varkappa - \frac{\kappa_1+\kappa_2}{2}\right|\right]\bigvee_{\kappa_1}^{\kappa_2}(F), \\[2ex] \left[\frac{1}{2}\bigvee_{\kappa_1}^{\kappa_2}(F)\xi + \left|\bigvee_{\kappa_1}^{\varkappa}(F) - \bigvee_{\varkappa}^{\kappa_2}(F)\right|\right](\kappa_2-\kappa_1) \end{cases}
\end{aligned}
$$

for all $\varkappa \in [\kappa_1,\kappa_2]$ *and* $\lambda \in [0,1]$.

Remark 2.4 If $\lambda = 1$ in Corollary 2.2, the inequalities (2.7) reduce to the inequalities (1.3).

Remark 2.5 If we take $\lambda = 0$ in Corollary 2.2, then the inequality (2.7) reduces to the equalities (3.9) of Theorem 3.4 in [19].

§3. Inequalities for Lipschitzian Functions

In this section, we obtain some inequalities for Lipschitzian functions. First we give the following important fact:

If ω is Lipschitzian with the constant $L > 0$; i.e.

$$|\omega(\xi) - \omega(\eta)| \le L\,|\xi - \eta| \text{ for any } \xi,\eta \in (\kappa_1,\kappa_2)$$

then, it is well known that for any Riemann integrable function $g : [\kappa_1,\kappa_2] \to \mathbb{R}$ the Riemann-Stieltjes integral $\int_{\kappa_1}^{\kappa_2} g(\xi)d\omega(\xi)$ exist and

$$\left|\int_{\kappa_1}^{\kappa_2} g(\xi)d\omega(\xi)\right| \le L\int_{\kappa_1}^{\kappa_2}|g(\xi)|\,d\xi. \tag{3.1}$$

Theorem 3.1 *If ϖ is bounded on $[\kappa_1, \kappa_2]$, i.e.*

$$\|\varpi\|_\infty = \sup_{\xi \in [\kappa_1, \kappa_2]} |\varpi(\xi)| < \infty,$$

and if : $[\kappa_1, \kappa_2] \to \mathbb{R}$ is a Lipschitzian with the constant $L > 0$ on $[\kappa_1, \kappa_2]$, then we have

$$\begin{aligned} &\Bigg| WAB_F(\kappa_1, \kappa_2, \varkappa) - \lambda AB_\varpi(\kappa_1, \kappa_2, \varkappa) F(\varkappa) \\ &\quad - \frac{(1-\lambda)}{2} \left[\left(\int_\varkappa^{\kappa_2} \varpi(\eta) d\eta \right) F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa} \varpi(\eta) d\eta \right) F(\kappa_1) \right] \Bigg| \\ &\leq \frac{L \|\varpi\|_\infty}{2} \left[\frac{(\kappa_2 - \kappa_1)^2}{4} + \left(\varkappa - \frac{\kappa_1 + \kappa_2}{2} \right)^2 \right] \end{aligned} \tag{3.2}$$

for all $\varkappa \in [\kappa_1, \kappa_2]$.

Proof By taking modulus in the inequality (2.1) and by using the inequality (3.1), we obtain

$$\begin{aligned} &\Bigg| WAB_F(\kappa_1, \kappa_2, \varkappa) - \lambda AB_\varpi(\kappa_1, \kappa_2, \varkappa) F(\varkappa) \\ &\quad - \frac{(1-\lambda)}{2} \left[\left(\int_\varkappa^{\kappa_2} \varpi(\eta) d\eta \right) F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa} \varpi(\eta) d\eta \right) F(\kappa_1) \right] \Bigg| \\ &\leq \frac{1}{2} \left[\lambda \left| \int_{\kappa_1}^{\varkappa} \left(\int_{\kappa_1}^{\xi} \varpi(\eta) d\eta \right) dF(\xi) \right| + (1-\lambda) \left| \int_{\kappa_1}^{\varkappa} \left(\int_{\xi}^{\varkappa} \varpi(\eta) d\eta \right) dF(\xi) \right| \right] \\ &\quad + \frac{1}{2} \left[\lambda \left| \int_{\varkappa}^{\kappa_2} \left(\int_{\xi}^{\kappa_2} \varpi(\eta) d\eta \right) dF(\xi) \right| + (1-\lambda) \left| \int_{\varkappa}^{\kappa_2} \left(\int_{\varkappa}^{\xi} \varpi(\eta) d\eta \right) dF(\xi) \right| \right] \\ &\leq \frac{L}{2} \left[\lambda \int_{\kappa_1}^{\varkappa} \left| \int_{\kappa_1}^{\xi} \varpi(\eta) d\eta \right| d\xi + (1-\lambda) \int_{\kappa_1}^{\varkappa} \left| \int_{\xi}^{\varkappa} \varpi(\eta) d\eta \right| d\xi \right] \\ &\quad + \frac{L}{2} \left[\lambda \int_{\varkappa}^{\kappa_2} \left| \int_{\xi}^{\kappa_2} \varpi(\eta) d\eta \right| d\xi + (1-\lambda) \int_{\varkappa}^{\kappa_2} \left| \int_{\varkappa}^{\xi} \varpi(\eta) d\eta \right| d\xi \right]. \end{aligned}$$

Since ϖ is bounded on $[\kappa_1, \kappa_2]$, we have

$$\begin{aligned} &\Bigg| WAB_F(\kappa_1, \kappa_2, \varkappa) - \lambda AB_\varpi(\kappa_1, \kappa_2, \varkappa) F(\varkappa) \\ &\quad - \frac{(1-\lambda)}{2} \left[\left(\int_\varkappa^{\kappa_2} \varpi(\eta) d\eta \right) F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa} \varpi(\eta) d\eta \right) F(\kappa_1) \right] \Bigg| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{L}{2}\left[\lambda \|\varpi\|_{\infty,[\kappa_1,\varkappa]} \int_{\kappa_1}^{\varkappa} (\xi-\kappa_1)\,d\xi + (1-\lambda)\|\varpi\|_{\infty,[\kappa_1,\varkappa]} \int_{\kappa_1}^{\varkappa} (\varkappa-\xi)\,d\xi\right] \\
&\quad + \frac{L}{2}\left[\lambda \|\varpi\|_{\infty,[\varkappa,\kappa_2]} \int_{\varkappa}^{\kappa_2} (\kappa_2-\xi)\,d\xi + (1-\lambda)\|\varpi\|_{\infty,[\varkappa,\kappa_2]} \int_{\varkappa}^{\kappa_2} (\xi-\varkappa)\,d\xi\right] \\
&= \frac{L}{4}\left[\|\varpi\|_{\infty,[\kappa_1,\varkappa]} (\varkappa-\kappa_1)^2 + \|\varpi\|_{\infty,[\varkappa,\kappa_2]} (\kappa_2-\varkappa)^2\right].
\end{aligned}
$$

Using the facts that

$$
\|\varpi\|_{\infty,[\kappa_1,\varkappa]} \leq \|\varpi\|_{\infty} \quad \text{and} \quad \|\varpi\|_{\infty,[\varkappa,\kappa_2]} \leq \|\varpi\|_{\infty}
$$

for all $\varkappa \in [\kappa_1, \kappa_2]$, we obtain

$$
\begin{aligned}
&\Bigg| WAB_F(\kappa_1,\kappa_2,\varkappa) - \lambda AB_\varpi(\kappa_1,\kappa_2,\varkappa)F(\varkappa) \\
&\quad - \frac{(1-\lambda)}{2}\left[\left(\int_{\varkappa}^{\kappa_2} \varpi(\eta)d\eta\right)F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa} \varpi(\eta)d\eta\right)F(\kappa_1)\right]\Bigg| \\
&= \frac{L\|\varpi\|_\infty}{4}\left[(\varkappa-\kappa_1)^2 + (\kappa_2-\varkappa)^2\right] \\
&= \frac{L\|\varpi\|_\infty}{2}\left[\frac{(\kappa_2-\kappa_1)^2}{4} + \left(\varkappa - \frac{\kappa_1+\kappa_2}{2}\right)^2\right]
\end{aligned}
$$

which completes the proof. □

Remark 3.1 If $\lambda = 1$ or $\lambda = 0$ in Theorem 3.1, we have respectively the following inequalities,

$$
\begin{aligned}
&|AB_F(\kappa_1,\kappa_2,\varkappa) - \lambda AB_\varpi(\kappa_1,\kappa_2,\varkappa)F(\varkappa)| \\
&\leq \frac{L\|\varpi\|_\infty}{2}\left[\frac{(\kappa_2-\kappa_1)^2}{4} + \left(\varkappa - \frac{\kappa_1+\kappa_2}{2}\right)^2\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| AB_F(\kappa_1,\kappa_2,\varkappa) - \frac{1}{2}\left[\left(\int_{\varkappa}^{\kappa_2} \varpi(\eta)d\eta\right)F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa} \varpi(\eta)d\eta\right)F(\kappa_1)\right]\right| \\
&\leq \frac{L\|\varpi\|_\infty}{2}\left[\frac{(\kappa_2-\kappa_1)^2}{4} + \left(\varkappa - \frac{\kappa_1+\kappa_2}{2}\right)^2\right]
\end{aligned}
$$

which are proved by Budak and Pehlivan in [10].

Corollary 3.1 *Under assumptions of Theorem* 3.1 *with* $\varpi(\varkappa) = 1$ *for all* $\varkappa \in [\kappa_1, \kappa_2]$, *we have* *the following inequality*

$$\begin{aligned}
&\left| WAB_F(\kappa_1,\kappa_2,\varkappa) - \lambda\left(\frac{\kappa_1+\kappa_2}{2}-\varkappa\right)F(\varkappa) \right. \\
&\left. -\frac{(1-\lambda)}{2}\left[\frac{\kappa_2 F(\kappa_2)+\kappa_1 F(\kappa_1)}{2}-\frac{F(\kappa_1)+F(\kappa_2)}{2}\varkappa\right]\right| \\
&\leq \frac{L}{2}\left[\frac{(\kappa_2-\kappa_1)^2}{4}+\left(\varkappa-\frac{\kappa_1+\kappa_2}{2}\right)^2\right]. \qquad (3.3)
\end{aligned}$$

Remark 3.2 Ifc$\lambda = 1$ or $\lambda = 0$ in Corollary 3.3, then the inequality (3.3) reduces to the equalities (4.2) of Theorem 4.1 and 4.5 of Theorem 4.2 in [19], respectively.

Corollary 3.2 *Let* $\varkappa = \frac{\kappa_1+\kappa_2}{2}$ *and let* $\varpi : [\kappa_1,\kappa_2] \to [0,\infty)$ *be symmetric about* $\varkappa = \frac{\kappa_1+\kappa_2}{2}$ *(i.e.* $\varpi(\varkappa) = \varpi(\kappa_1+\kappa_2-\varkappa)$*) in Theorem* 3.1*. Then we have*

$$\left| WAB_F\left(\kappa_1,\kappa_2,\frac{\kappa_1+\kappa_2}{2}\right) - (1-\lambda)\frac{F(\kappa_2)-F(\kappa_1)}{4}\int_{\kappa_1}^{\kappa_2}\varpi(\eta)d\eta \right| \leq \frac{L}{8}(\kappa_2-\kappa_1)^2\|\varpi\|_\infty .$$

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

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International J.Math. Combin. Vol.1(2023), 54-61

Monophonic Graphoidal Covering Number of Corona Product Graph of Some Standard Graphs with the Wheel

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Abstract: A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called a monophonic path if it is a chordless path. A monophonic graphoidal cover of a graph G is a collection ψ_m of monophonic paths in G such that every vertex of G is an internal vertex of at most one monophonic path in ψ_m and every edge of G is in exactly one monophonic path in ψ_m. The minimum cardinality of a monophonic graphoidal cover of G is called the monophonic graphoidal covering number of G and is denoted by $\eta_m(G)$. In this paper, we find the monophonic graphoidal covering number of corona product of wheel with some standard graphs.

Key Words: Graphoidal cover, Smarandachely graphoidal cover, monophonic path, monophonic graphoidal cover, monophonic graphoidal covering number.

AMS(2010): 05C70.

§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [6]. The concept of graphoidal cover was introduced by Acharya and Sampathkumar [2] and further studied in [1, 3, 7, 8].

A *graphoidal cover* of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:

(i) Every path in ψ has at least two vertices;

(ii) Every vertex of G is an internal vertex of at most one path in ψ;

(iii) Every edge of G is in exactly one path in ψ.

The minimum cardinality of a graphoidal cover of G is called the *graphoidal covering number* of G and is denoted by $\eta(G)$.

The collection ψ is called an *acyclic graphoidal cover* of G if no member of ψ is cycle; it

[1]Received February 16, 2022, Accepted March 17, 2023.

is called a *geodesic graphoidal cover* if every member of ψ is a shortest path in G. The minimum cardinality of an acyclic (geodesic) graphoidal cover of G is called the *acyclic (geodesic) graphoidal covering number* of G and is denoted by $\eta_a(\eta_g)$. The acyclic graphoidal covering number and geodesic graphoidal covering number are studied in [4,5]. Generally, a *Smarandachely graphoidal cover* $\mathscr{C}(G, k, \mathscr{P})$ of graph G is the union of subgraphs with property $\mathscr{P}$, hold with every vertex $v \in V(G)$ is in at most k subgraphs and every edge is in exactly one subgraph with property $\mathscr{P}$. Certainly, let $\mathscr{P} = \{\text{path}, \text{cycle}\}$ or $\mathscr{P} = \{\text{path}\}$ and $k = 1$. Then, a Smarandachely graphoidal cover $\mathscr{C}(G, 1, \mathscr{P})$ is respectively the graphoidal cover of G or acyclic graphoidal cover of G.

A *chord* of a path P is an edge joining any two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ *of a vertex* v *in* G is $e_m(v) = max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius* is $rad_m(G) = min\{e_m(v) : v \in V(G)\}$ and the *monophonic diameter* is $diam_m(G) = max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [10, 11].

A *monophonic graphoidal cover* of a graph G is a collection ψ_m of monophonic paths in G such that every vertex of G is an internal vertex of at most one monophonic path in ψ_m and every edge of G is in exactly one monophonic path in ψ_m. The minimum cardinality of a monophonic graphoidal cover of G is called the *monophonic graphoidal covering number* of G and is denoted by $\eta_m(G)$. The monophonic graphoidal covering number was introduced [12] and studied in [13,14].

Product graphs have been used to generate mathematical models of complex networks which inherits properties of real networks. By using basic graphs, corona graphs are defined by taking corona product of the basic graphs.

Definition 1.1 *The corona of two graphs G and H is the graph $G \circ H$ formed from one copy of G and $|V(G)|$ copies of H, where the i^{th} vertex of G is adjacent to every vertex in the i^{th} copy of H.*

§2. Monophonic Graphoidal Covering Number on Corona Product of Wheel with Some Standard Graphs

Theorem 2.1 *For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 5)$, $\eta_m(W_n) = n$.*

Proof Let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v\}$ and $V(C_{n-1}) = \{u_1, u_2, \cdots, u_{n-1}\}$ and let $P_1 : u_1, u_2, \cdots, u_{n-2}$, $P_2 : u_1, u_{n-1}, u_{n-2}$, $P_3 : u_1, v, u_{n-2}$ and $P_{i+2} : v, u_i$ $(2 \leq i \leq n-3$ and $i = n-1)$. It is clear that $\psi_m = \{P_1, P_2, \cdots, P_{n-1}, P_{n+1}\}$ is a minimum monophonic graphoidal cover of W_n. Hence $\eta_m(W_n) = n$. □

Theorem 2.2 *(i) If $G = P_r \circ W_n$, then $\eta_m(G) = 2nr - 1$;*

(ii) If $G = W_n \circ P_r$, then $\eta_m(G) = n(r + 2) - 2$.

Proof Let $P : u_1, u_2, \cdots, u_r$ be a path of order r and let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \cdots, v_n\}$.

(i) Let G be the corona product of P_r and W_n. The graph G is shown in Figure 1. Let $M_1 : v_{1,1}, u_1, u_2, \cdots, u_r, v_{r,1}$; $M_{i+1} : v_{i,2}, v_{i,3}, \cdots, v_{i,n-1}$ $(1 \leq i \leq r)$; $M'_i : v_{i,2}, v_{i,n}, v_{i,n-1}$ $(1 \leq i \leq r)$; $M''_i : v_{i,2}, v_{i,1}, v_{i,n-1}$ $(1 \leq i \leq r)$ and $S_1 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{n} \{(u_i, v_{i,j})\} - \{(u_1, v_{1,1}), (u_r, v_{r,1})\}$, $S_2 = \bigcup_{i=1}^{r} (\bigcup_{j=3}^{n} \{(v_{i,1}, v_{i,j})\} - \{(v_{i,1}, v_{i,n-1})\})$.

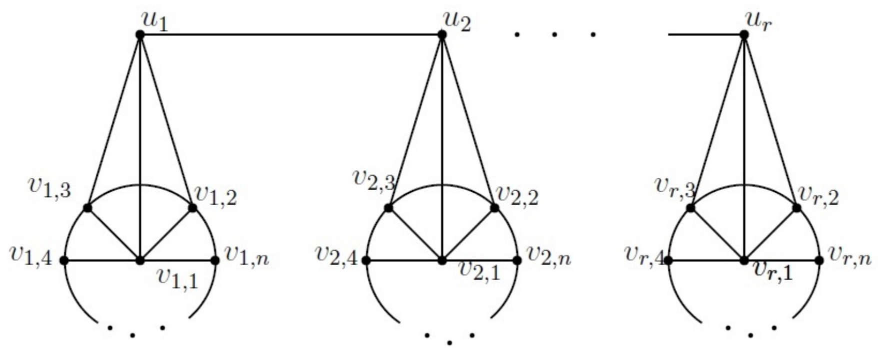

Figure 1

It is clear that $\psi_m = S_1 \cup S_2 \cup \{M_1, M_2, \ldots, M_{r+1}, M'_1, M'_2, \ldots, M'_r, M''_1, M''_2, \cdots, M''_r\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = (nr - 2) + r(n - 3) + (3r + 1) = 2nr - 1$.

(ii) Let G be the corona product of W_n and P_r. The graph G is shown in Figure 2.

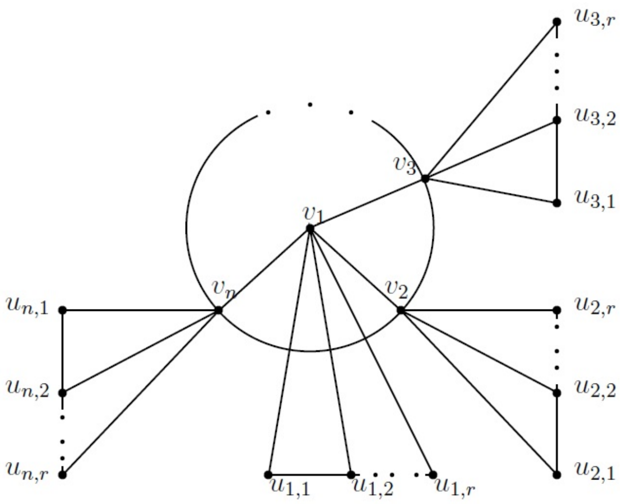

Figure 2

Let $M_1 : u_{2,1}, v_2, v_3, \cdots, v_{n-1}, u_{n-1,1}$; $M_2 : v_2, v_n, v_{n-1}$; $M_3 : v_2, v_1, v_{n-1}$; $M_{i+1} : v_1, v_i$ $(3 \leq i \leq n-2)$; $M_n : v_1, v_n$; $M'_i : u_{i,1}, u_{i,2}, \cdots, u_{i,r}$ $(1 \leq i \leq n)$ and $S = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_{n-1}, u_{n-1,1})\}$.

It is clear that $\psi_m = S \cup \{M_1, M_2, \ldots, M_n, M'_1, M'_2, \ldots, M'_n\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = (nr-2) + 2n = n(r+2) - 2$. $\square$

Theorem 2.3 *(i) If $G = C_r \circ W_n$, then $\eta_m(G) = 2rn$;*

(ii) If $G = W_n \circ C_r$, then $\eta_m(G) = n(r+3) - 2$.

Proof Let $C_r : u_1, u_2, \cdots, u_r, u_1$ be a cycle of order r and let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \cdots, v_n\}$.

(i) Let G be the corona product of C_r and P_n.

Case 1. $r = 3$.

The graph G in this case is shown in Figure 3.

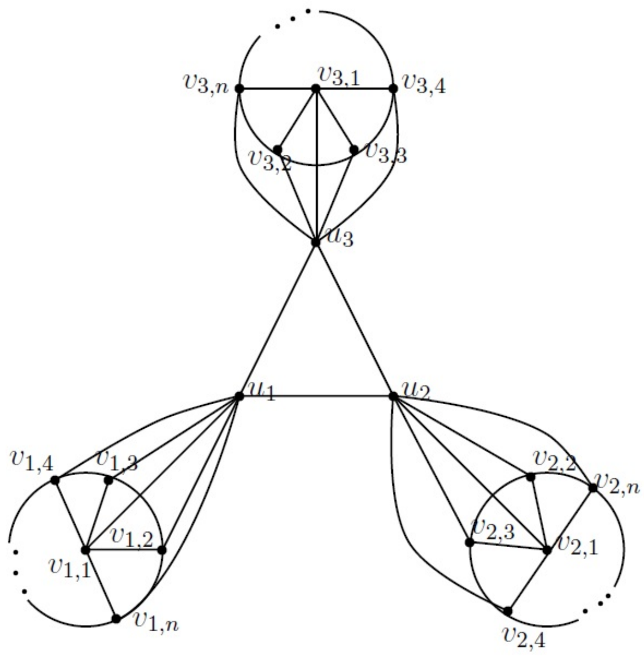

Figure 3

Let $M_1 : v_{1,1}, u_1, u_2$; $M_2 : v_{2,1}, u_2, u_3$; $M_3 : v_{3,1}, u_3, u_1$; $M_{i+3} : v_{i,2}, v_{i,3}, \cdots, v_{i,n-1}$ $(1 \leq i \leq 3)$; $M'_i : v_{i,2}, v_{i,n}, v_{i,n-1}$ $(1 \leq i \leq 3)$; $M'_{i+3} : v_{i,2}, v_{i,1}, v_{i,n-1}$ $(1 \leq i \leq 3)$ and $S_1 = \bigcup_{i=1}^{3}(\bigcup_{j=1}^{n}\{(u_i, v_{i,j})\} - \{(u_i, v_{i,1})\})$, $S_2 = \bigcup_{i=1}^{3}(\bigcup_{j=3}^{n}\{(v_{i,1}, v_{i,j})\} - \{(v_{i,1}, v_{i,n-1})\})$.

It is clear that every M_i $(1 \leq i \leq 6)$ and M'_i $(1 \leq i \leq 6)$ are monophonic paths and every element in $S_1 \cup S_2$ is a monophonic path. Hence $\psi_m = S_1 \cup S_2 \cup \{M_1, M_2, \cdots, M_6, M'_1, M'_2, \ldots, M'_6\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = 3n - 3 + 3(n-3) + 12 = 6n$.

Case 2. $r > 3$.

Let $M_1 : v_{i,1}, u_1, u_2, \ldots, u_{r-1}, v_{r-1,1}$; $M_2 : u_1, u_r, u_{r-1}$; $M_{i+2} : v_{i,2}, v_{i,3}, \cdots, v_{i,n-1}$ $(1 \leq i \leq r)$; $M'_i : v_{i,2}, v_{i,n}, v_{i,n-1}$ $(1 \leq i \leq r)$; $M''_i : v_{i,2}, v_{i,1}, v_{i,n-1}$ $(1 \leq i \leq r)$ and $S_1 = (\bigcup_{i=1}^{r}\bigcup_{j=1}^{n}(u_i, v_{i,j})) - \{(u_1, v_{1,1}), (u_{r-1}, v_{r-1,1})\}$, $S_2 = \bigcup_{i=1}^{r}(\bigcup_{j=3}^{n}(v_{i,1}, v_{i,j}) - \{(v_{i,1}, v_{i,n-1})\})$.

It is clear that every M_i $(1 \le i \le r+2)$, M'_i $(1 \le i \le r)$ and M''_i $(1 \le i \le r)$ is a monophonic path and every element in $S_1 \cup S_2$ is a monophonic path. Hence $\psi_m = S_1 \cup S_2 \cup \{M_1, M_2, \cdots, M_{r+2}, M'_1, M'_2, \ldots, M'_r, M''_1, M''_2, \cdots, M''_r\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = (rn-2) + r(n-3) + (3r+2) = 2rn$.

(ii) Let G be the corona product of W_n and C_r. The graph G in this case is shown in Figure 4. Let $M_1 : u_{2,1}, v_2, v_3, \cdots, v_{n-1}, u_{n-1,1}$; $M_2 : v_2, v_n, v_{n-1}$; $M_3 : v_2, v_1, v_{n-1}$; $M'_i : u_{i,1}, u_{i,2}, \cdots, u_{i,r-1}$ $(1 \le i \le n)$; $M''_i : u_{i,1}, u_{i,r}, u_{i,r-1}$ $(1 \le i \le n)$ and $S_1 = \bigcup_{i=3}^{n}(v_1, v_i) - \{(v_1, v_{n-1})\}$, $S_2 = \bigcup_{i=1}^{n}\bigcup_{j=1}^{r}(v_i, u_{i,j}) - \{(v_2, u_{2,1})(v_{n-1}, u_{n-1,1})\}$.

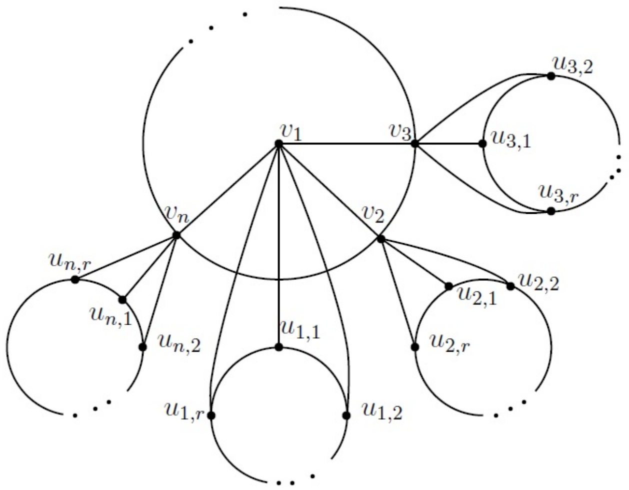

Figure 4

It is clear that every M_i $(1 \le i \le 3)$, M'_i $(1 \le i \le n)$ and M''_i $(1 \le i \le n)$ are monophonic paths and every element in $S_1 \cup S_2$ is a monophonic path in G. Hence $\psi_m = S_1 \cup S_2 \cup \{M_1, M_2, M_3, M'_1, M'_2, M'_3, \cdots, M'_n, M''_1, M''_2, \cdots, M''_n\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = (n-3) + (nr-2) + (2n+3) = n(r+3) - 2$. □

Theorem 2.4 (i) *If* $G = K_r \circ W_n$, *then* $\eta_m(G) = \frac{r}{2}(r + 4n - 11)$;

(ii) *If* $G = W_n \circ K_r$, *then* $\eta_m(G) = n(r^2 + r + 2) - 10$.

Proof Let K_r be the complete graph of order r with the vertex set $\{u_1, u_2, \cdots, u_r\}$ and let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \ldots, v_n\}$.

(i) Let G be the corona product of K_r and W_n. The graph G is shown in Figure 5. Let $M_i : v_{i,1}, u_i, u_{i+1}$ $(1 \le i \le r-1)$; $M_r : v_{r,1}, u_r, u_1$; $N_i : v_{i,2}, v_{i,3}, \cdots, v_{i,n-1}$ $(1 \le i \le r)$; $N'_i : v_{i,2}, v_{i,n}, v_{i,n-1}$ $(1 \le i \le r)$; $N''_i : v_{i,2}, v_{i,1}, u_{i,n-1}$ $(1 \le i \le r)$ and

$$S_1 = \bigcup_{i=1}^{r}\bigcup_{j=2}^{n}(u_i, v_{i,j}),$$
$$S_2 = \bigcup_{i=1}^{r}(\bigcup_{j=3}^{n}(v_{i,1}, v_{i,j}) - \{(v_{i,1}, v_{i,n-1})\},$$

$$S_3 \quad = \quad E(K_r) - \{(u_1, u_2), (u_2, u_3), \cdots, (u_{r-1}, u_r), (u_r, u_1)\}.$$

It is clear that every M_i, N_i, N_i', N_i'', for $1 \le i \le r$, are monophonic paths and every element in $S_1 \cup S_2 \cup S_3$ is a monophonic path. Hence,

$$\psi_m = S_1 \bigcup S_2 \bigcup S_3 \bigcup \{M_1, M_2, \cdots, M_r, N_1, N_2, \cdots, N_r, N_1', N_2', \cdots, N_r', N_1'', N_2'', \cdots, N_r''\}$$

is a minimum monophonic graphoidal cover of G and hence

$$\eta_m(G) = r(n-1) + r(n-3) + \frac{r(r-1)}{2} - r + 4r = \frac{r}{2}(r + 4n - 11).$$

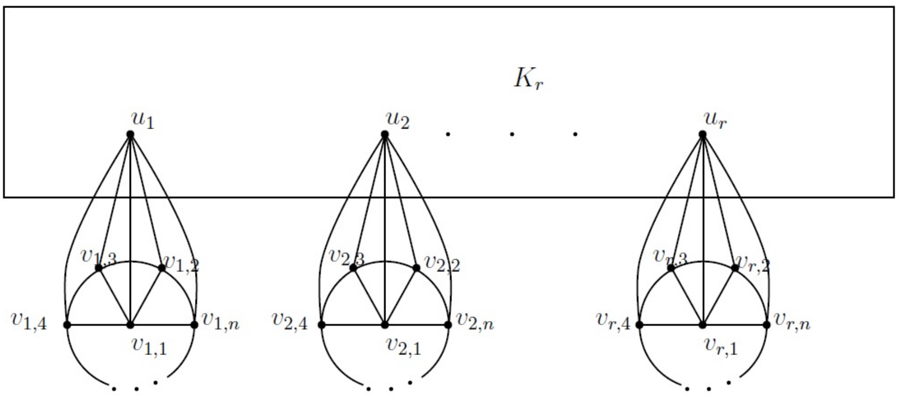

Figure 5

(ii) Let G be the corona product of W_n and K_r, which is shown in Figure 6.

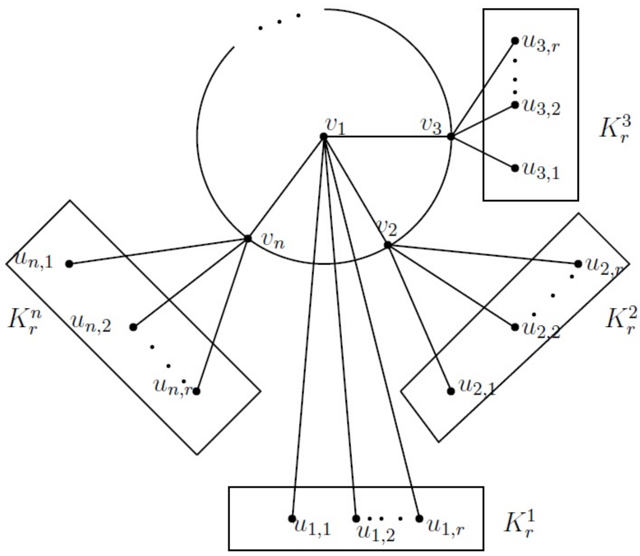

Figure 6

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{n-1}, u_{n-1,1}$; $M_2 : v_2, v_n, v_{n-1}$; $M_3 : v_2, v_1, v_{n-1}$ and

$$S_1 = \bigcup_{i=3}^{n}(v_1, v_i) - \{(v_1, v_{n-1})\},$$
$$S_2 = \bigcup_{i=1}^{n}\bigcup_{j=1}^{r}(v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_{n-1}, u_{n-1,1})\},$$
$$S_3 = \bigcup_{i=1}^{n} E(K_r^i).$$

It is clear that every M_1, M_2 and M_3 are monophonic paths and every element in $S_1 \cup S_2 \cup S_3$ is a monophonic path. Hence $\psi_m = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3\}$ is a minimum monophonic graphoidal cover of G and hence

$$\eta_m(G) = (n-3) + (nr-2) + n(\frac{r(r-1)}{2}) = n(r^2 + r + 2) - 10. \qquad \square$$

Theorem 2.5 *If $G = W_r \circ W_s$, then $\eta_m(G) = r(2s+4) - 2$.*

Proof Let $W_r = K_1 + C_{r-1}$ be a wheel with $V(K_1) = \{u_1\}$ and $V(C_{r-1}) = \{u_1, u_2, \cdots, u_r\}$ and let $W_s = K_1 + C_{s-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{s-1}) = \{v_1, v_2, \cdots, v_s\}$. The graph G is shown in Figure 7. Let $M_1 : v_{2,1}, u_2, u_3, \cdots, u_{r-1}, v_{r-1,1}$; $M_2 : u_2, u_r, u_{r-1}$; $M_3 : u_2, u_1, u_{r-1}$; $M_i' : v_{i,2}, v_{i,3}, \cdots, v_{i,s-1}$ $(1 \le i \le r)$; $M_i'' : v_{i,2}, v_{i,s}, v_{i,s-1}$ $(1 \le i \le r)$; $M_i''' : v_{i,2}, v_{i,1}, v_{i,s-1}$ $(1 \le i \le r)$ and $S_1 = \bigcup_{i=3}^{r}(u_1, u_i) - \{(u_1, u_{r-1})\}$, $S_2 = \bigcup_{i=1}^{r}(\bigcup_{j=3}^{s}(v_{i,1}, v_{i,j}) - \{(v_{i,1}, v_{i,s-1})\})$, $S_3 = \bigcup_{i=1}^{r}\bigcup_{j=1}^{s}(u_i, v_{i,j}) - \{(u_2, v_{2,1}), (u_{r-1}, v_{r-1,1})\}$.

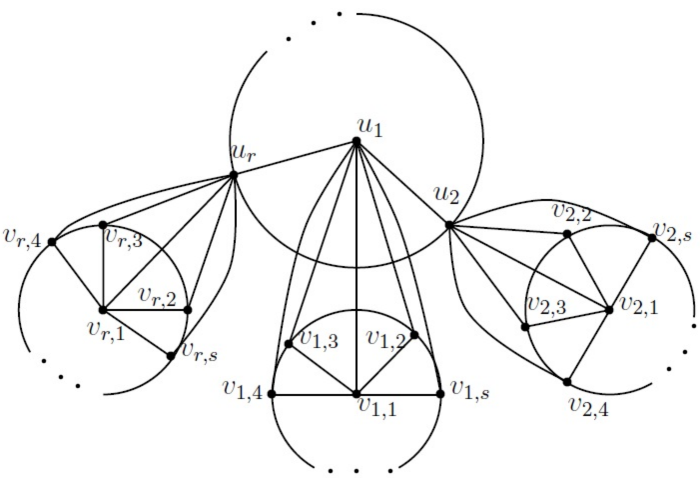

Figure 7

It is clear that $\psi_m = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M_1', M_2', \cdots, M_r', M_1'', M_2'', \cdots, M_r'', M_1''', M_2''', \cdots, M_r'''\}$ is a minimum monophonic graphoidal cover of G and so

$$\eta_m(G) = (3r+3) + (r-3) + r(s-3) + (rs-2) = r(2s+4) - 2. \qquad \square$$

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International J.Math. Combin. Vol.1(2023), 62-73

On Skew Randić Sum Eccentricity Energy of Digraphs

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Abstract: In this paper we introduce the concept of skew Randić sum eccentricity energy of digraphs. We then obtain upper and lower bounds for skew Randić sum eccentricity energy of digraphs. Then we compute the skew Randić sum eccentricity of some digraphs such as star digraph, complete bipartite digraph, $(S_m \wedge P_2)$ digraph and crown digraph.

Key Words: Digraph, skew-adjacency matrix of graph, skew Randić sum eccentricity energy, Smarandachely sum eccentricity energy.

AMS(2010): 05C50.

§1. Introduction

In [1], we have introduced the Randić sum eccentricity energy of a simple graph G as follows. The Randić sum eccentricity energy adjacency matrix of G is a $n \times n$ matrix $A_{rse} = (a_{ij})$, where

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{e(v_i)+e(v_j)}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent,} \end{cases}$$

where $e(v_i)$ is the eccentricity of the vertex v_i. The Randić sum eccentricity energy of G is defined as the sum of absolute values of the eigenvalues of the Randić sum eccentricity energy adjacency matrix of G. Generally, a Smarandachely sum eccentricity energy adjacency matrix of G is a $n \times n$ matrix $A^s_{rse} = (a^s_{ij})$ with

$$a^s_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt[d+1]{e^d(v_i)+e^d(v_j)}}, & \text{if the distance of vertices } v_i \text{ and } v_j \text{ is } d, \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not connected} \end{cases}$$

which characterizes the non-homogeneity of vertices on a graph by eccentricity. Certainly, the matrix A_{rse} characterizes vertices of G in case of homogeneity which is a submatrix of A^s_{rse}.

In 2010, Adiga, Balakrishnan and Wasin So [5] introduced the skew energy of a digraph

[1]Received December 02, 2022, Accepted March 18, 2023.

as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. The skew-adjacency matrix of D is the $n \times n$ matrix $S(D) = (s_{ij})$ where $s_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(D)$, $s_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(D)$ and $s_{ij} = 0$ otherwise. Hence $S(D)$ is a skew symmetric matrix of order n and all its eigenvalues are of the form $i\lambda$ where $i = \sqrt{-1}$ and λ is a real number. The skew energy of G is the sum of the absolute values of eigenvalues of $S(D)$.

Motivated by these works, we introduce the concept of skew Randić sum eccentricity energy of a digraph as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. Then the skew Randić sum eccentricity adjacency matrix of D is the $n \times n$ matrix $A_{srse} = (a_{ij})$ where

$$a_{ij} = \begin{cases} \frac{1}{\sqrt{e(v_i)+e(v_j)}}, & \text{if } (v_i, v_j) \in \Gamma(D), \\ -\frac{1}{\sqrt{e(v_i)+e(v_j)}}, & \text{if } (v_j, v_i) \in \Gamma(D), \\ 0, & \text{otherwise.} \end{cases}$$

Then, the skew Randić sum eccentricity energy $E_{srse}(D)$ of D is defined as the sum of the absolute values of eigenvalues of A_{srse}.

For example Let D be the directed circle on 4 vertices with the arc set $\{(1,2), (2,3), (3,4), (4,1)\}$. Then

$$A_{srse} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

Then, the characteristic equation is given by $\lambda^4 + \lambda^2$. The eigenvalues are i, 0, 0, $-i$ and skew Randić sum eccentricity energy of D is 2.

In Section 2 of this paper we obtain the upper and lower bounds for skew Randić sum eccentricity energy of digraphs. In Section 3 we compute the skew Randić sum eccentricity energy of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ digraph and a crown digraph.

§2. Upper and Lower Bounds for Skew Randić Sum Eccentricity Energy

Theorem 2.1 *Let D be a simple digraph of order n. Then*

$$E_{srse}(D) \leq \sqrt{2n \sum_{j \sim k} \left(\frac{1}{e(v_i) + e(v_j)} \right)}.$$

Proof Let $i\lambda_1, i\lambda_2, i\lambda_3, \cdots, i\lambda_n$, be the eigenvalues of A_{srse}, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq$

$\cdots \geq \lambda_n$. Since

$$\sum_{j=1}^{n}(i\lambda_j)^2 = tr(A_{srse}^2) = -\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}^2 = -2\sum_{j\sim k}\left(\frac{1}{e(v_i)+e(v_j)}\right),$$

we have

$$\sum_{j=1}^{n}|\lambda_j|^2 = 2\sum_{j\sim k}\left(\frac{1}{e(v_i)+e(v_j)}\right). \tag{1}$$

Applying the Cauchy-Schwartz inequality

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} a_j^2\right)\cdot\left(\sum_{j=1}^{n} b_j^2\right)$$

with $a_j = 1$, $b_j =\mid \lambda_j \mid$, we obtain

$$E_{srse}(D) = \sum_{j=1}^{n}|\lambda_j| = \sqrt{(\sum_{j=1}^{n}|\lambda_j|)^2} \leq \sqrt{n\sum_{j=1}^{n}|\lambda_j|^2} = \sqrt{2n\sum_{j\sim k}\left(\frac{1}{e(v_i)+e(v_j)}\right)}.$$

This completes the proof. $\square$

Theorem 2.2 *Let D be a simple digraph of order n. Then*

$$E_{srse}(D) \geq \sqrt{2\sum_{j\sim k}\left(\frac{1}{e(v_i)+e(v_j)}\right) + n(n-1)p^{\frac{2}{n}}},\ where\ \ p = |det A_{srse}| = \prod_{j=1}^{n}|\lambda_j|. \tag{2}$$

Proof Notice that

$$(E_{srse}(D))^2 = \left(\sum_{j=1}^{n}|\lambda_j|\right)^2 = \sum_{j=1}^{n}|\lambda_j|^2 + \sum_{1\leq j\neq k\leq n}|\lambda_j||\lambda_k|.$$

By arithmetic-geometric mean inequality, we get

$$\begin{aligned}
\sum_{1\leq j\neq k\leq n}|\lambda_j||\lambda_k| &= |\lambda_1|(|\lambda_2| + |\lambda_3| + \cdots + |\lambda_n|) \\
&\quad +|\lambda_2|(|\lambda_1| + |\lambda_3| + \cdots + |\lambda_n|) + \cdots \\
&\quad +|\lambda_n|(|\lambda_1| + |\lambda_2| + \cdots + |\lambda_{n-1}|) \\
&\geq n(n-1)(|\lambda_1||\lambda_2|...|\lambda_n|)^{\frac{1}{n}}(|\lambda_1|^{n-1}|\lambda_2|^{n-1}...|\lambda_n|^{n-1})^{\frac{1}{n(n-1)}} \\
&= n(n-1)(\prod_{j=1}^{n}|\lambda_j|)^{\frac{1}{n}}(\prod_{j=1}^{n}|\lambda_j|)^{\frac{1}{n}} = n(n-1)\left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{2}{n}}.
\end{aligned}$$

Thus,

$$(E_{srse}(D))^2 \geq \sum_{j=1}^{n} |\lambda_j|^2 + n(n-1)\left(\prod_{j=1}^{n} |\lambda_j|\right)^{\frac{2}{n}}.$$

From the equation (1), we get

$$(E_{srse}(D))^2 \geq 2\sum_{j\sim k}\left(\frac{1}{e(v_i)+e(v_j)}\right) + n(n-1)p^{\frac{2}{n}},$$

which gives (2). □

§3. Skew Randić Sum Eccentricity Energies of Some Families of Graphs

We begin with some basic definitions and notations.

Definition 3.1([3]) *A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.*

Definition 3.2([3]) *A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.*

Definition 3.3([2]) *The crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and edge set $\{u_iv_j; 1 \leq i, j \leq n, i \neq j\}$. S_n^0 is therefore S_n^0 coincides with complete bipartite graph $K_{n,n}$ with the horizontal edges removed.*

Definition 3.4([4]) *The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) | v_iv_k \in E(S_m)$ and $v_jv_l \in E(P_2)$ and $1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.*

Now we compute skew Randić sum eccentricity energies of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ and a crown digraph.

Theorem 3.5 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $K_{m,n}$ complete bipartite digraph be respectively given by*

$$\begin{aligned} V(D) &= \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}, \\ \Gamma(D) &= \{(u_i, v_j) \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}. \end{aligned}$$

Then, the skew Randić sum eccentricity energy of the complete bipartite digraph is $\sqrt{mn}$.

Proof The skew Randić sum eccentricity matrix of complete bipartite digraph is given by

$$A_{srse} = \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ -\frac{1}{2} & \cdots & -\frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2} & \cdots & -\frac{1}{2} & 0 & \cdots & 0 \end{pmatrix}$$

with a characteristic polynomial

$$|\lambda I - A_{srse}| = \begin{vmatrix} \lambda I_m & -\frac{1}{2}J^T \\ \frac{1}{2}J & \lambda I_n \end{vmatrix},$$

where J is an $n \times m$ matrix with all the entries are equal to 1. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_m & -\frac{1}{2}J^T \\ \frac{1}{2}J & \lambda I_n \end{vmatrix} = 0,$$

which can be written as

$$|\lambda I_m| \left| \lambda I_n - \left(\frac{1}{2}J\right) \frac{I_m}{\lambda} \left(-\frac{1}{2}J^T\right) \right| = 0.$$

On simplification, we obtain

$$\frac{\lambda^{m-n}}{(4)^n} \left|(4)\lambda^2 I_n + JJ^T\right| = 0,$$

which can be written as

$$\frac{\lambda^{m-n}}{(4)^n} P_{JJ^T}(4\lambda^2) = 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix JJ^T. Thus, we have

$$\frac{\lambda^{m-n}}{(4)^n}(4\lambda^2 + mn)(4\lambda^2)^{n-1} = 0,$$

which is same as

$$\lambda^{m+n-2}(\lambda^2 + \frac{mn}{4}) = 0.$$

Therefore, the spectrum of $K_{m,n}$ is given by

$$Spec\,(K_{m,n}) = \begin{pmatrix} 0 & i\sqrt{\frac{mn}{4}} & -i\sqrt{\frac{mn}{4}} \\ m+n-2 & 1 & 1 \end{pmatrix}.$$

Hence, the skew Randić sum eccentricity energy of complete bipartite digraph is

$$E_{srse}(K_{m,n}) = \sqrt{mn},$$

as desired. □

Theorem 3.6 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of S_n star digraph be respectively given by*

$$V(D) \quad = \quad \{v_1, v_2, \cdots, v_n\}, \quad \Gamma(D) = \{(v_1, v_j) \,|\, 2 \le j \le n\}$$

Then, the skew Randić sum eccentricity energy of D is

$$E_{srse}(S_n) = 2\sqrt{\frac{n-1}{3}}.$$

Proof The skew Randić sum eccentricity matrix of the star digraph D is given by

$$A_{srse} = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \cdots & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with a characteristic polynomial

$$\begin{aligned} |\lambda I - A_{srse}| &= \begin{vmatrix} \lambda & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \cdots & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \lambda & 0 & \cdots & 0 \\ \frac{1}{\sqrt{3}} & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & \lambda \end{vmatrix} \\ &= \left(\frac{1}{\sqrt{3}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}, \end{aligned}$$

where $\mu = \lambda\sqrt{3}$. Then

$$|\lambda I - A_{srse}| = \phi_n(\mu)\left(\frac{1}{\sqrt{3}}\right)^n,$$

where

$$\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}.$$

Using the properties of the determinants, we obtain

$$\phi_n(\mu) = (\mu\phi_{n-1}(\mu) + \mu^{n-2})$$

after some simplifications. Iterating this, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2 + (n-1)).$$

Therefore

$$|\lambda I - A_{srse}| = \left(\frac{1}{\sqrt{3}}\right)^n \left[\left((3)\lambda^2 + (n-1)\right)(\lambda\sqrt{3})^{n-2}\right].$$

Thus, the characteristic equation is given by

$$\lambda^{n-2}\left(\lambda^2 + \frac{n-1}{3}\right) = 0.$$

Hence,

$$Spec\,(S_n) = \begin{pmatrix} 0 & i\sqrt{\frac{n-1}{3}} & -i\sqrt{\frac{n-1}{3}} \\ n-2 & 1 & 1 \end{pmatrix}$$

and the skew Randić sum eccentricity energy of S_n is

$$E_{srse}(S_n) = 2\sqrt{\frac{n-1}{3}}.$$

□

Theorem 3.7 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $(S_m \wedge P_2)(m > 1)$ digraph be respectively given by*

$$\begin{aligned} V(D) &= \{v_1, v_2, \cdots, v_{2m+2}\}, \\ \Gamma(D) &= \{(v_1, v_j), (v_{m+2}, v_k) \,|\, 2 \le k \le m+1, m+3 \le j \le 2m+2\}. \end{aligned}$$

Then, the skew Randić sum eccentricity energy of D is

$$E_{srse}(D) = 4\sqrt{\frac{n-1}{3}}.$$

Proof The skew Randić sum eccentricity matrix of $(S_m \wedge P_2)$ digraph is given by

$$A_{srse} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \gamma & \cdots & \gamma \\ 0 & 0 & \cdots & 0 & -\gamma & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -\gamma & 0 & \cdots & 0 \\ 0 & \gamma & \cdots & \gamma & 0 & 0 & \cdots & 0 \\ -\gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n},$$

where $m+1=n$ and $\gamma = \frac{1}{\sqrt{3}}$. Then, its characteristic polynomial is given by

$$|\lambda I - A_{srse}| = \begin{vmatrix} \lambda & 0 & \cdots & 0 & 0 & -\gamma & \cdots & -\gamma \\ 0 & \lambda & \cdots & 0 & \gamma & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & \gamma & 0 & \cdots & 0 \\ 0 & -\gamma & \cdots & -\gamma & \lambda & 0 & \cdots & 0 \\ \gamma & 0 & \cdots & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}.$$

Hence, the characteristic equation is given by

$$\left(\frac{1}{\sqrt{3}}\right)^{2n} \begin{vmatrix} \Lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} = 0,$$

where $\Lambda = \sqrt{3}\lambda$.

Let

$$\phi_{2n}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n}$$

$$= (-1)^{2n+2n}\Lambda \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{(2n-1)\times(2n-1)}$$

$$+(-1)^{2n+1} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+1} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

Using the properties of the determinants, we obtain

$$\Psi_{2n-1}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda)$$

after some simplifications, where

$$\Theta_n(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 1 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{n\times n}.$$

Then,

$$\phi_{2n}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as above, we obtain

$$\begin{aligned} \phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+1}\Psi_{2n-2}(\Lambda) + (-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda) \\ &= \Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda). \end{aligned}$$

Proceeding like this, we obtain at the $(n-1)^{th}$ step

$$\phi_{2n}(\Lambda) = (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda),$$

where,

$$\xi_{n+1}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)}.$$

$$\begin{aligned} \phi_{2n}(\Lambda) &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\Lambda\Theta_n(\Lambda) \\ &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n}\Theta_n(\Lambda) \\ &= ((n-1)\Lambda^{n-2} + \Lambda^n)\Theta_n(\Lambda). \end{aligned}$$

Using the properties of the determinants, we obtain

$$\Theta_n(\Lambda) = (n-1)\Lambda^{n-2} + \Lambda^n.$$

Therefore

$$\phi_{2n}(\Lambda) = ((n-1)\Lambda^{n-2} + \Lambda^n)^2.$$

Hence characteristic equation becomes

$$\left(\frac{1}{\sqrt{3}}\right)^{2n}\phi_{2n}(\Lambda) = 0,$$

which is same as

$$\left(\frac{1}{\sqrt{3}}\right)^{2n}((n-1)\Lambda^{n-2}+\Lambda^{n})^2=0$$

and can be reduced to

$$\lambda^{2n-4}((n-1)+(3)\lambda^2)^2=0.$$

Therefore

$$Spec\,(D)=\begin{pmatrix} 0 & i\sqrt{\frac{n-1}{3}} & -i\sqrt{\frac{n-1}{3}} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence, the skew Randić sum eccentricity energy of $(S_m \wedge P_2)$ digraph is

$$E_{srse}(D)=4\sqrt{\frac{n-1}{3}}.$$

□

Theorem 3.8 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $S_n^0(n>2)$ crown digraph be respectively given by*

$$V(D)=\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n\},\quad \Gamma(D)=\{(u_i,v_j)\,|\,1\le i\le n, 1\le j\le n, i\ne j\}.$$

Then, the skew Randić sum eccentricity energy of the crown digraph is $2(n-1)$.

Proof The skew Randić sum eccentricity matrix of crown digraph is given by

$$A_{srse}=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & X & \cdots & X \\ 0 & 0 & \cdots & 0 & X & 0 & \cdots & X \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & X & X & \cdots & 0 \\ 0 & -X & \cdots & -X & 0 & 0 & \cdots & 0 \\ -X & 0 & \cdots & -X & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -X & -X & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $X=\frac{1}{\sqrt{4}}$. Its characteristic polynomial is

$$|\lambda I-A_{srse}|=\begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{4}}K^T \\ \frac{1}{\sqrt{4}}K & \lambda I_n \end{vmatrix},$$

where K is an $n\times n$ matrix. Hence, the characteristic equation is given by

$$\begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{4}}K^T \\ \frac{1}{\sqrt{4}}K & \lambda I_n. \end{vmatrix}=0,$$

which is the same as
$$|\lambda I_n| \left| \lambda I_n - \left(\frac{K}{\sqrt{4}}\right) \frac{I_n}{\lambda} \left(-\frac{K^T}{\sqrt{4}}\right) \right| = 0$$
and can be written as
$$\frac{1}{(4)^n} P_{KK^T}((4)\lambda^2) = 0,$$
where $P_{KK^T}(\lambda)$ is the characteristic polynomial of the matrix KK^T. Thus, we have
$$\frac{1}{(4)^n}[4\lambda^2 + (n-1)^2][4\lambda^2 + 1]^{n-1} = 0,$$
which is same as
$$\left(\lambda^2 + \frac{(n-1)^2}{4}\right) \left(\lambda^2 + \frac{1}{4}\right)^{n-1} = 0.$$
Therefore
$$Spec\,(S_n^0) = \begin{pmatrix} i\sqrt{\frac{(n-1)^2}{4}} & -i\sqrt{\frac{(n-1)^2}{4}} & i\frac{1}{\sqrt{4}} & -i\frac{1}{\sqrt{4}} \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$
Hence, the skew Randić sum-eccentricity energy of crown digraph is
$$E_{srse}(S_n^0) = 2(n-1)$$
as desired. □

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International J.Math. Combin. Vol.1(2023), 74-80

Note on Full Signed Graphs and Full Line Signed Graphs

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Abstract: In this paper, we introduced the new notions full signed graph and full line signed graph of a signed graph and its properties are obtained. Also, we obtained the structural characterizations of these notions. Further, we presented some interesting switching equivalent characterizations.

Key Words: Signed graphs, neutrosophic signed graph, balance, switching, full signed graph, full line signed graph.

AMS(2010): 05C22.

§1. Introduction

For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [1]. The non-standard will be given in this paper as and when required.

To model individuals' preferences towards each other in a group, Harary [2] introduced the concept of signed graphs in 1953. A signed graph $S = (G, \sigma)$ is a graph $G = (V, E)$ whose edges are labeled with positive and negative signs (i.e., $\sigma : E(G) \to \{+, -\}$). The vertices of a graph represent people and an edge connecting two nodes signifies a relationship between individuals. The signed graph captures the attitudes between people, where a positive (negative edge) represents liking (disliking). A neutrosophic signed graph $S^N = (G, \sigma, H)$ for a subgraph $H \subset G$ with property $\mathscr{P}$ is such a graph that $G \backslash H$ is a signed graph but H is indefinite for those of uncertainties in reality. Certainly, if there are no indefinite subgraph in G, it must be a signed graph. An unsigned graph is a signed graph with the signs removed. Similar to an unsigned graph, there are many active areas of research for signed graphs.

The sign of a cycle (this is the edge set of a simple cycle) is defined to be the product of the signs of its edges; in other words, a cycle is positive if it contains an even number of negative edges and negative if it contains an odd number of negative edges. A signed graph S is said to be balanced if every cycle in it is positive. A signed graph S is called totally unbalanced if every cycle in S is negative. A chord is an edge joining two non adjacent vertices in a cycle.

A *marking* of S is a function $\zeta : V(G) \to \{+, -\}$. Given a signed graph S one can easily

[1]Received November 11, 2022, Accepted March 18, 2023.

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define a marking ζ of S as follows: For any vertex $v \in V(S)$,

$$\zeta(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking ζ of S is called *canonical marking* of S. For more new notions on signed graphs refer the papers (see [6, 8, 9, 13-17, 17-26]).

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1 *A signed graph S is balanced if and only if either of the following equivalent conditions is satisfied:*

(i) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary [2]).*

(ii) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$ (Sampathkumar [6]).*

Switching S with respect to a marking ζ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs.

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be *weakly isomorphic* (see [28]) or *cycle isomorphic* (see [29]) if there exists an isomorphism $\phi : G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known.

Theorem 1.2(T. Zaslavsky [29]) *Given a graph G, any two signed graphs in $\psi(G)$, where $\psi(G)$ denotes the set of all the signed graphs possible for a graph G, are switching equivalent if and only if they are cycle isomorphic.*

§2. Full Signed Graph of a Signed Graph

Let $G = (V, E)$ be a graph and the full graph $\mathcal{FG}(G)$ of G is a graph whose vertex is the union of vertices, edges and blocks of G in which two vertices are adjacent if the corresponding members of G are adjacent or incident (see [4]). Let $G = (V, E)$ be a graph. Then G is a connected graph if and only if $\mathcal{FG}(G)$ is connected.

Motivated by the existing definition of complement of a signed graph, we now extend the notion of full graphs to signed graphs as follows: The *full signed graph* $\mathcal{FS}(S) = (\mathcal{FG}(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{FG}(G)$ and sign of any edge uv is $\mathcal{FS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called a full signed graph, if $S \cong \mathcal{FS}(S')$ for some signed graph S'. The following result restricts the class of full signed graphs.

Theorem 2.1 *For any signed graph $S = (G, \sigma)$, its full signed graph $\mathcal{FS}(S)$ is balanced.*

Proof Since sign of any edge $e = uv$ in $\mathcal{FS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S, by Theorem 1.1, $\mathcal{FS}(S)$ is balanced. □

For any positive integer k, the k^{th} iterated full signed graph, $\mathcal{FS}^k(S)$ of S is defined as follows:

$$\mathcal{FS}^0(S) = S,\ \mathcal{FS}^k(S) = \mathcal{FS}(\mathcal{FS}^{k-1}(S)).$$

Corollary 2.2 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $\mathcal{FS}^k(S)$ is balanced.*

Corollary 2.3 *For any two signed graphs S_1 and S_2 with the same underlying graph, $\mathcal{FS}(S_1) \sim \mathcal{FS}(S_2)$.*

The following result characterize signed graphs which are full signed graphs.

Theorem 2.4 *A signed graph $S = (G, \sigma)$ is a full signed graph if, and only if, S is balanced signed graph and its underlying graph G is a full graph.*

Proof Suppose that S is balanced and G is a full graph. Then there exists a graph G' such that $\mathcal{FG}(G') \cong G$. Since S is balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge uv in S satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $\mathcal{FS}(S') \cong S$. Hence S is a full signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a full signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that

$$\mathcal{FS}(S') \cong S.$$

Hence, G is the full graph of G' and by Theorem 2.1, S is balanced. □

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [3] as follows:

$\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the $\mathcal{FS}(S)$ is balanced (Theorem 2.1). We now examine, the conditions under which negation $\eta(S)$ of $\mathcal{FS}(S)$ is balanced.

Proposition 2.5 *Let $S = (G, \sigma)$ be a signed graph. If $\mathcal{FG}(G)$ is bipartite then $\eta(\mathcal{FS}(S))$ is balanced.*

Proof Since, by Theorem 2.1, $\mathcal{FS}(S)$ is balanced, it follows that each cycle C in $\mathcal{FS}(S)$ contains even number of negative edges. Also, since $\mathcal{FG}(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{FS}(S)$ is also even. Hence $\eta(\mathcal{FS}(S))$ is balanced. □

§3. Full Line Signed Graph of a Signed Graph

Let $G = (V, E)$ be a graph and the full line graph $\mathcal{FLG}(G)$ of a graph G is a graph and $V(\mathcal{FLG}(G))$ is the union of the set of vertices, edges and blocks of G in which two vertices are joined by an edge in $\mathcal{SFL}(G)$ if the corresponding vertices and edges of G are adjacent or the corresponding members of G are incident (See [5]).

Motivated by the existing definition of complement of a signed graph, we now extend the notion of full line graphs to signed graphs as follows: The *full line signed graph* $\mathcal{FLS}(S) = (\mathcal{FLG}(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{FLG}(G)$ and sign of any edge uv is $\mathcal{FLS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called a full line signed graph, if $S \cong \mathcal{FLS}(S')$ for some signed graph S'. The following result restricts the class of full line signed graphs.

Theorem 3.1 *For any signed graph $S = (G, \sigma)$, its full line signed graph $\mathcal{FLS}(S)$ is balanced.*

Proof Since sign of any edge $e = uv$ in $\mathcal{FLS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S, by Theorem 1.1, $\mathcal{FLS}(S)$ is balanced. □

For any positive integer k, the k^{th} iterated full line signed graph, $\mathcal{FLS}^k(S)$ of S is defined as follows:

$$\mathcal{FLS}^0(S) = S,\ \mathcal{FLS}^k(S) = \mathcal{FLS}(\mathcal{FLS}^{k-1}(S)).$$

Corollary 3.2 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $\mathcal{FLS}^k(S)$ is balanced.*

Corollary 3.3 *For any two signed graphs S_1 and S_2 with the same underlying graph, $\mathcal{FLS}(S_1) \sim \mathcal{FLS}(S_2)$.*

The following result characterize signed graphs which are full line signed graphs.

Theorem 3.4 *A signed graph $S = (G, \sigma)$ is a full line signed graph if, and only if, S is balanced signed graph and its underlying graph G is a full line graph.*

Proof Suppose that S is balanced and G is a full line graph. Then there exists a graph G' such that $\mathcal{FLG}(G') \cong G$. Since S is balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge uv in S satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $\mathcal{FLS}(S') \cong S$. Hence S is a full line signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a full line signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $\mathcal{FLS}(S') \cong S$. Hence, G is the full line graph of G' and by Theorem 2.1, S is balanced. □

For a signed graph $S = (G, \sigma)$, the $\mathcal{FLS}(S)$ is balanced (Theorem 3.1). We now examine, the conditions under which negation $\eta(S)$ of $\mathcal{FLS}(S)$ is balanced.

Proposition 3.5 *Let $S = (G, \sigma)$ be a signed graph. If $\mathcal{FLG}(G)$ is bipartite then $\eta(\mathcal{FLS}(S))$ is balanced.*

Proof Since, by Theorem 3.1, $\mathcal{FLS}(S)$ is balanced, it follows that each cycle C in $\mathcal{FLS}(S)$ contains even number of negative edges. Also, since $\mathcal{FLG}(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{FLS}(S)$ is also even. Hence $\eta(\mathcal{FLS}(S))$ is balanced. $\Box$

§4. Switching Equivalence of Full Signed Graphs and Full Line Signed Graphs

In [5], the authors remarked that $\mathcal{FLG}(G)$ and $\mathcal{FG}(G)$ are isomorphic if and only if G is a block. We now give a characterization of signed graphs whose full signed graphs are switching equivalent to their full line signed graphs.

Theorem 4.1 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FS}(S) \sim \mathcal{FLS}(S)$ if and only if G is a block.*

Proof Suppose $\mathcal{FS}(S) \sim \mathcal{FLS}(S)$. This implies that $\mathcal{FG}(G) \cong \mathcal{FLG}(G)$ and hence, G is a block. Conversely, suppose that G is a block. Then

$$\mathcal{FG}(G) \cong \mathcal{FLG}(G).$$

Now, if S any signed graph with G is a block, by Theorems 2.1 and 3.1, $\mathcal{FS}(S)$ and $\mathcal{FLS}(S)$ are balanced and hence, the result follows from Theorem 1.2. This completes the proof. $\Box$

In view of the negation operator introduced by Harary [3], we have the following cycle isomorphic characterizations.

Corollary 4.2 *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $\eta(\mathcal{FS}(S_1)) \sim \eta(\mathcal{FS}(S_2))$ if G_1 and G_2 are isomorphic.*

Corollary 4.3 *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $\eta(\mathcal{FLS}(S_1)) \sim \eta(\mathcal{FLS}(S_2))$ if G_1 and G_2 are isomorphic.*

Corollary 4.4 *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $\mathcal{FS}(\eta(S_1))$ and $\mathcal{FS}(\eta(S_2))$ are cycle isomorphic if G_1 and G_2 are isomorphic.*

Corollary 4.5 *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $\mathcal{FLS}(\eta(S_1))$ and $\mathcal{FLS}(\eta(S_2))$ are cycle isomorphic if G_1 and G_2 are isomorphic.*

Corollary 4.6 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FS}(\eta(S)) \sim \mathcal{FLS}(S)$ if and only if G is a block.*

Corollary 4.7 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FS}(S) \sim \mathcal{FLS}(\eta(S))$ if and only if G is a block.*

Corollary 4.8 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FS}(\eta(S)) \sim \mathcal{FLS}(\eta(S))$ if and only if G is a block.*

Acknowledgements

The authors would like to thank the referees for their invaluable comments and suggestions which led to the improvement of the manuscript.

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International J.Math. Combin. Vol.1(2023), 81-86

On Nano $\lambda\psi g$-Irresolute Functions in Nano Topological Spaces

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Abstract: In this paper we introduce Nano $\lambda\psi g$-irresolute functions and discussed some of their properties. Also we investigate the relationships between the other existing Nano irresolute functions.

Key Words: $N\lambda\psi g$-closed sets, $N\lambda\psi g$-irresolute function.

AMS(2010): 54A05, 54C10.

§1. Introduction

In 1991, Balachandran [1] et.al, introduced and studied the notations of generalized continuous functions. Different types of generalizations of continuous functions were studied by various authors in the recent development of topology. Continous function is one of the main functions in topology. Lellis Thivagar [4] introduced Nano topological space with respect to a subset X of a universe which is defined in terms of lower and upper approximations of X. The elements of Nano topological space are called Nano open sets. He has also defined Nano closed sets, Nano-interior and Nano closure of a set. He also introduced the weak forms of Nano open sets, namely Nano-open sets, Nano semi open sets and Nano preopen sets. He also defined Nano continuous functions, Nano open mapping, Nano closed mapping and Nano Homeomorphism. M.K.R.S.Veerakumar [10] was introduced the notion of ψ closed sets in topological spaces. Maki [6] introduced the notion of Λ-sets in topological spaces in 1986. Λ-set is a set A which is equal to its kernel, i.e., to the intersection of all open supersets of A. N.R.Santhi Maheswari and P.Subbulakshmi [7], [8], [10] introduced Nano $\Lambda_\psi(A)$ sets, nano Λ^*_ψ(A) sets, nano Λ_ψ-set and nano Λ^*_ψ-set in nano topological spaces and we also introduce Nano (Λ,ψ)-closed sets , Nano (Λ,ψ)-Open sets and Nano $\lambda\psi$ generalized Closed sets and Nano $\lambda\psi g$-continuous functions in nano topological spaces. In this paper we introduce Nano $\lambda\psi g$-irresolute functions and discussed some of their properties. Also we investigate the relationships between the other existing Nano irresolute functions.

[1]Received December 11, 2022, Accepted March 19, 2023.

§2. Preliminaries

Definition 2.1([7]) *Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Then U is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be in discernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$*

(1) *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is $L_R(X) = U_{x\in U}\{R(X) : R(X) \subseteq X\}$, where $R(X)$ denotes the equivalence class determined by $X \in U$;*

(2) *The upper approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $U_R(X)$.That is $U_R(X) = U_{x\in U}\{R(X) : R(X) \cap X = \phi\}$;*

(3) *The boundary of the region of X with respect to R is the set of all objects, which can be classified neither as X nor as not X with respect to R and it is denoted by $B_R(X)$. That is $B_R(X) = U_R(X) - L_R(X)$.*

Lemma 2.2([4]) *If (U, R) is an approximation space and $X, Y \subseteq U$, then*

(1) $L_R(X) \subseteq X \subseteq U_R(X)$;
(2) $L_R(\phi) = U_R(\phi) = \phi$;
(3) $L_R(U) = U_R(U) = U$;
(4) $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$;
(5) $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$;
(6) $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$;
(7) $L_R(X \cap Y) = L_R(X) \cap L_R(Y)$;
(8) $L_R(X) \subseteq L_R(Y)$*and* $U_R(X) \subseteq U_R(Y)$ *whenever* $X \subseteq Y$;
(9) $U_R(X^c) = [L_R(X)]^c$ *and* $L_R(X^c) = [U_R(X)]^c$;
(10) $U_R(U_R(X)) = L_R(U_R(X)) = U_R(X)$;
(11) $L_R(L_R(X)) = U_R(L_R(X)) = L_R(X)$

Definition 2.3([4]) *Let U be the Universe and R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. $\tau_R(X)$ satisfies the following axioms:*

(1) *U and $\phi \in \tau_R(X)$;*
(2) *The union of elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$;*
(3) *The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.*

We call $(U, \tau_R(X))$ is a Nano topological space. The elements of $\tau_R(X)$ are called a open sets and the complement of a Nano open set is called Nano closed sets.

Throughout this paper $(U, \tau_R(X))$ is a nano topological space with respect to X where $X \subseteq U$, R is an equivalence relation on U, U/R denotes the family of equivalence classes of U by R.

Definition 2.4([4]) *If $(U, \tau_R(X))$ is a nano topological space with respect to X. Where $X \subseteq G$ and if $A \subseteq G$, then*

(1) *The Nano interior of the set A is defined as the union of all Nano open subsets contained in A and is denoted by $Nint(A)$, $Nint(A)$ is the largest Nano open subset of A;*

(2) *The Nano closure of the set A is defined as the intersection of all Nano closed sets containing A and is denoted by $Ncl(A)$. $Ncl(A)$ is the smallest Nano closed set containing A.*

Definition 2.5([4]) *Let $(U, \tau_R(X))$ be a Nano topological space and $A \subseteq G$. Then, A is said to be*

(i) *Nano semi-open (briefly Ns-open) if $A \subseteq Ncl(Nint(A)$;*

(ii) *Nano α-open (briefly $N\alpha$-open) if $A \subseteq Nint(Ncl(Nint(A))$;*

(iii) *Nano regular-open (briefly Nr-open) if $A = Nint(Ncl(A)$.*

The complements of the above mentioned open sets are called their respective closed sets.

Definition 2.6([8]) *Let A be a subset of a Nano topological space $(U, \tau_R(X))$. A subset $N\Lambda_\psi(A)$ is defined as $N\Lambda_\psi(A) = \cap\{H/A \subseteq H$ and $H \in N\psi O(U, \tau_R(X)\}$.*

Definition 2.7([8]) *A subset A of a Nano topological space $(U, \tau_R(X))$ is called a $N\Lambda_\psi$-set if $A = N\Lambda_\psi(A)$. The set of all $N\Lambda_\psi$ -sets is denoted by $N\Lambda_\psi(U, \tau_R(X))$.*

Definition 2.8([9]) *Let A be a subset of a Nano topological space $(U, \tau_R(X))$. A subset $N(\Lambda, \psi)$ closed if $A = B \cap C$, where B is $N\Lambda_\psi$ set and C is a $N\psi$ closed set.*

Definition 2.9 *Let $(U, \tau_R(X))$ be a Nano topological space and $A \subseteq G$. Then A is said to be*

(1) *Nano sg-closed (briefly Nsg-closed) [2] if $Nscl(A) \subseteq G$ whenever $A \subseteq G$ and G is Nano-semi open in U;*

(2) *Nano ψ-closed (briefly $N\psi$-closed) [10] if $Nscl(A) \subseteq G$ whenever $A \subseteq G$ and G is Nano-sg open in U;*

(3) *Nano $\lambda\psi$ generalized closed (briefly $N\lambda\psi g$-closed) [8] if $N\psi cl(A) \subseteq G$, whenever $A \subseteq G$ and G is $N(\Lambda, \psi)$- open in U.*

Remark 2.10([9]) We have known the conclusions following:

(1) Every Nano closed set is $N\lambda\psi g$-closed;

(2) Every Ns-closed set is $N\lambda\psi g$-closed;

(3) Every Nr-closed set is $N\lambda\psi g$-closed;

(4) Every $N\alpha$-closed set is $N\lambda\psi g$-closed;

(5) Every $N\psi$-closed set is $N\lambda\psi g$-closed.

Definition 2.11 *The function $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ is said to be*

(1) *Nr-continuous [3] if the inverse image of every Nano closed set in $(V, \tau'_R(Y))$ is Nr-closed in $(U, \tau_R(X))$;*

(2) *Nano-continuous [5] if the inverse image of every Nano closed set in $(V, \tau'_R(Y))$ is Nano closed in $(U, \tau_R(X))$;*

(3) *Ns-continuous [2] if the inverse image of every nano closed set in $(V, \tau'_R(Y))$ is Ns-closed in $(U, \tau_R(X))$;*

(4) *$N\alpha$-continuous [3] if the inverse image of every Nano closed set in $(V, \tau'_R(Y))$ is $N\alpha$-closed in $(U, \tau_R(X))$;*

(5) *$N\psi$-continuous [9] if the inverse image of every Nano closed set in $(V, \tau'_R(Y))$ is $N\psi$-closed in $(U, \tau_R(X))$;*

(6) *$N\lambda\psi g$-continuous [8] if the inverse image of every Nano closed set in $(V, \tau'_R(Y))$ is $N\lambda\psi g$-closed in $(U, \tau_R(X))$.*

Definition 2.12 *The function $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ is said to be*

(1) *Nr-irresolute [3] if the inverse image of every Nr-closed set in $(V, \tau'_R(Y))$ is Nr-closed in $(U, \tau_R(X))$;*

(2) *Ns-irresolute [2] if the inverse image of every Ns-closed set in $(V, \tau'_R(Y))$ is Ns-closed in $(U, \tau_R(X))$;*

(3) *$N\alpha$-irresolute [3] if the inverse image of every $N\alpha$-closed set in $(V, \tau'_R(Y))$ is $N\alpha$-closed in $(U, \tau_R(X))$;*

(4) *$N\psi$-irresolute [2] if the inverse image of every $N\psi$-closed set in $(V, \tau'_R(Y))$ is $N\psi$-closed in $(U, \tau_R(X))$.*

§3. $N\lambda\psi g$ -Irresolute Functions

In this section, we introduce and study a new concept of $N\lambda\psi g$-irresolute functions in Nano topological spaces.

Definition 3.1 *A function $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ is said to be $N\lambda\psi g$-irresolute if $f^{-1}(G)$ is a $N\lambda\psi g$-open set in $(U, \tau_R(X))$ for every $N\lambda\psi g$-open set G in $(V, \tau'_R(Y))$.*

Example 3.2 Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b, c, d\}\}$ and $X = \{b, c\}$.Then $\tau_R(X) = \{U, \phi, \{b, c, d\}\}$. Let $V = \{a, b, c, d\}$ with $V/R' = \{\{b\}, \{d\}, \{a, c\}\}$ and $Y = \{a, b\}$. Then $\tau'_R(Y) = \{V, \phi, \{b\}, \{a, c\}, \{a, b, c\}\}$. Define a mapping $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ as, $f(a) = a$, $f(b) = b$, $f(c) = d$, $f(d) = c$. Here the inverse image of every $N\lambda\psi g$- closed set in $(V, \tau'_R(Y))$ is $N\lambda\psi g$-closed set in $(U, \tau_R(X))$. Hence $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ is $N\lambda\psi g$-irresolute.

Theorem 3.3 *Let $(U, \tau_R(X))$, $(V, \tau'_R(Y))$ and $(W, \tau'_R(Z))$ be Nano topological spaces. If $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ and $g : (V, \tau'_R(Y)) \to (W, \tau'_R(Z))$ are two functions. If f is $N\lambda\psi g$-irresolute and g is $N\lambda\psi g$-continuous then $g \circ f$ is $N\lambda\psi g$-continuous.*

Proof Let G be a Nano closed set in $(W, \tau'_R(Z))$. Since g is $N\lambda\psi g$-continuous, $g^{-1}(G)$ is a $N\lambda\psi g$-closed set in $(V, \tau'_R(Y))$. Since f is $N\lambda\psi g$-irresolute, $f^{-1}(g^{-1}(G))$ is a $N\lambda\psi g$-closed set in $(U, \tau_R(X))$. Thus $(g \circ f)^{-1}(G)$ is $N\lambda\psi g$-closed in U, for every $N\lambda\psi g$ closed set G in $(W, \tau'_R(Z))$. Hence the composition $g \circ f : (U, \tau_R(X)) \to (W, \tau'_R(Z))$ is $N\lambda\psi g$-continuous. □

Theorem 3.4 *Let $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ and $g : (V, \tau'_R(Y)) \to (W, \tau''_R(Z))$ be two maps. If f and g are both $N\lambda\psi g$-irresolute then $g \circ f$ is $N\lambda\psi g$-continuous.*

Proof Let G be Nano closed in $(W, \tau''_R(Z))$. Since every Nano closed sets is $N\lambda\psi g$-closed. Since g is $N\lambda\psi g$-irresolute, $g^{-1}(G)$ is $N\lambda\psi g$-open in $N\lambda\psi g$-closed $(W, \tau''_R(Z))$. Since f is

$N\lambda\psi g$-irresolute, $f^{-1}(g^{-1}(G))$ is $N\lambda\psi g$-closed in U. Thus $(g\circ f)^{-1}(G) = f^{-1}(g^{-1}(G))$ is a $N\lambda\psi g$-closed set in $(U, \tau_R(X))$, for every Nano closed set G in $(W, \tau''_R Z))$. Hence $g\circ f$ is $N\lambda\psi g$ - continuous. □

Theorem 3.5 *Let $(U, \tau_R(X))$, $(V, \tau'_R(Y))$ and $(W, \tau"_R(Z))$ be Nano topological spaces. If $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ and $g : (V, \tau'_R(Y)) \to (W, \tau''_R(Z))$ be two functions. If f and g are both $N\lambda\psi g$-irresolute then $g\circ f$ is $N\lambda\psi g$ - irresolute.*

Proof Let G be $N\lambda\psi g$-closed in $(W, \tau''_R Z))$. Since g is $N\lambda\psi g$-irresolute, $g^{-1}(G)$ is $N\lambda\psi g$-closed in $(W, \tau''_R(Z))$. Since f is $N\lambda\psi g$-irresolute, $f^{-1}(g^{-1}(G))$ is $N\lambda\psi g$-closed in $(U, \tau_R(X))$. Thus $(g\circ f)^{-1}(G) = f^{-1}(g^{-1}(G))$ is $N\lambda\psi g$-closed set in $(U, \tau_R(X))$, for every $N\lambda\psi g$-closed set in $(W, \tau''_R(Z))$. Hence $g\circ f$ is $N\lambda\psi g$-irresolute. □

Theorem 3.6 *A function $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ is $N\lambda\psi g$-irresolute if and only if the inverse image $f^{-1}(G)$ is $N\lambda\psi g$-closed set in $(U, \tau_R(X))$, for every $N\lambda\psi g$-closed set in $(V, \tau'_R(Y))$.*

Proof Let G be $N\lambda\psi g$-closed in $(V, \tau'_R(Y))$. Then $V - G$ is $N\lambda\psi g$-open in $(V, \tau'_R(Y))$. Since f is $N\lambda\psi g$-irresolute, $f^{-1}(V - G)$ is $N\lambda\psi g$-open in $(U, \tau_R(X))$. But $f^{-1}(V - G) = U - f^{-1}(G)$. Hence $f^{-1}(G)$ is $N\lambda\psi g$-closed in $(U, \tau_R(X))$.

Conversely, assume that inverse image $f^{-1}(G)$ is $N\lambda\psi g$-closed in $(U, \tau_R(X))$, for every $N\lambda\psi g$-closed set G in $(V, \tau'_R(Y))$. Let F be $N\lambda\psi g$-open in $(V, \tau'_R(Y))$. Then $V - F$ is $N\lambda\psi g$-closed in $(V, \tau'_R(Y))$. By assumption, $f^{-1}(V - F)$ is $N\lambda\psi g$-closed in $(U, \tau_R(X))$. But $f^{-1}(V - F) = U - f^{-1}(F)$. Then $f^{-1}(F)$ is $N\lambda\psi g$-open in $(U, \tau_R(X))$. Hence f is $N\lambda\psi g$ - irresolute. □

Theorem 3.7 *A function $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ is $N\lambda\psi g$-irresolute if and only if $f(N\lambda\psi gcl(F)) \subseteq N\lambda\psi gcl(f(F))$ for every subset F of $(U, \tau_R(X))$.*

Proof Suppose f is $N\lambda\psi g$-irresolute. Let $F \subseteq U$. Then $f(F) \subseteq V$. Hence $N\lambda\psi gcl(f(F))$ is $N\lambda\psi g$-closed in V. Since f is $N\lambda\psi g$-irresolute, $f^{-1}(N\lambda\psi gcl(f(F)))$ is $N\lambda\psi g$-closed in $(U, \tau_R(X))$. Since $f(F) \subseteq N\lambda\psi gcl(f(F))$, which implies $F \subseteq f^{-1}(N\lambda\psi gcl(f(F)))$. Since $N\lambda\psi gcl(F)$ is the smallest $N\lambda\psi g$-closed set containing F, $N\lambda\psi gcl(F) \subseteq f^{-1}(N\lambda\psi gcl(f(F)))$. Hence $f(N\lambda\psi gcl(F)) \subseteq N\lambda\psi gcl(f(F))$.

Conversely, assume that $f(N\lambda\psi gcl(F)) \subseteq N\lambda\psi gcl(f(F))$, for every subset F of U. Let G be $N\lambda\psi g$-closed in V. Now, $f^{-1}(G) \subseteq U$. Hence $f(N\lambda\psi gcl(f^{-1}(G))) \subseteq N\lambda\psi gcl(f(f^{-1}(G))) = N\lambda\psi gcl(G)$, which implies $N\lambda\psi gcl(f^{-1}(G)) \subseteq f^{-1}(N\lambda\psi gcl(G)) = f^{-1}(G)$ that implies $f^{-1}(G)$ is $N\lambda\psi g$-closed in U, for every $N\lambda\psi g$-closed set G in V. Hence f is $N\lambda\psi g$-irresolute. □

Theorem 3.8 *A function $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ is $N\lambda\psi g$-irresolute if and only if $f^{-1}(N\lambda\psi gint(G)) \subseteq N\lambda\psi gint(f^{-1}(G))$, for every subset G of $(V, \tau'_R(Y))$.*

Proof Let f be $N\lambda\psi g$-irresolute. Let $G \subseteq V$. Then $N\lambda\psi gint(G)$ is $N\lambda\psi g$-open in V. Since f is $N\lambda\psi g$-irresolute, $f^{-1}(N\lambda\psi gint(G))$ is $N\lambda\psi g$-open in $(U, \tau_R(X))$. Hence $N\lambda\psi gint(f^{-1}(N\lambda\psi gint(G))) = f^{-1}(N\lambda\psi gint(G))$. Since $G \subseteq V$, $N\lambda\psi gint(G) \subseteq G$ always. Hence $f^{-1}(N\lambda\psi gint(G)) = N\lambda\psi gint(f^{-1}(N\lambda\psi gint(G))) \subseteq N\lambda\psi gint(f^{-1}(G))$. Thus $f^{-1}(N\lambda\psi gint(G)) \subseteq N\lambda\psi gint(f^{-1}(G))$.

Conversely, let $f^{-1}(N\lambda\psi gint(G)) \subseteq N\lambda\psi gint(f^{-1}(G))$, for every subset G of V. Let F be $N\lambda\psi g$-open in V and hence $N\lambda\psi gint(F) = F$. By our assumption, $f^{-1}(F) \subseteq N\lambda\psi gint(f^{-1}(F))$. But $N\lambda\psi gint(f^{-1}(F)) \subseteq f^{-1}(F)$. Hence $f^{-1}(F) = N\lambda\psi gint(f^{-1}(F))$. Then $f^{-1}(F)$ is $N\lambda\psi g$-open in U, for every subset F of V. Hence f is $N\lambda\psi g$-irresolute. □

Theorem 3.9 *A function $f : (U, \tau_R(X)) \to (V, \tau'_R(Y))$ is $N\lambda\psi g$-irresolute if and only if $N\lambda\psi gcl(f^{-1}(G)) \subseteq f^{-1}(N\lambda\psi gcl(G))$, for every subset G of V.*

Proof Suppose f is $N\lambda\psi g$-irresolute. Let $G \subseteq V$, then $N\lambda\psi gcl(G)$ is $N\lambda\psi g$-closed in V. Since f is irresolute, $f^{-1}(N\lambda\psi gcl(G))$ is $N\lambda\psi g$-closed in U. Thus $N\lambda\psi gcl(f^{-1}(N\lambda\psi gcl(G))) = f^{-1}(N\lambda\psi gcl(G))$. Since $G \subseteq N\lambda\psi gcl(G)$, then $f^{-1}(G) \subseteq f^{-1}(N\lambda\psi gcl(G))$. Now, $N\lambda\psi gcl(f^{-1}(G)) \subseteq N\lambda\psi gcl(f^{-1}(N\lambda\psi gcl(G))) = f^{-1}(N\lambda\psi gcl(G))$ which implies $N\lambda\psi gcl(f^{-1}(G)) \subseteq f^{-1}(N\lambda\psi gcl(G))$, for every subset G of V.

Conversely, let $N\lambda\psi gcl(f^{-1}(G)) \subseteq f^{-1}(N\lambda\psi gcl(G))$, for every subset G of V. Let F be $N\lambda\psi g$-closed in V and hence $N\lambda\psi gcl(F) = F$. By our assumption, $N\lambda\psi gcl(f^{-1}(F)) \subseteq f^{-1}(F)$. But $f^{-1}(F) \subseteq N\lambda\psi gcl(f^{-1}(F))$. Hence $f^{-1}(F) = N\lambda\psi gcl(f^{-1}(F))$. Then $f^{-1}(F)$ is $N\lambda\psi g$-closed in U, for every subset F of V. Hence f is $N\lambda\psi g$-irresolute. □

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International J.Math. Combin. Vol.1(2023), 87-97

Bounds of the Radio Number of Stacked-Book Graph with Odd Paths

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Abstract: A stacked-book graph $G_{m,n}$ is obtained from the Cartesian product of a star graph S_m and a path P_n, where m and s are the orders of the star graph and the path respectively. Obtaining the radio number of a graph is a rigorous process, which is dependent on diameter of G and positive difference of non-negative integer labels $f(u)$ and $f(v)$ assigned to any two u, v in the vertex set $V(G)$ of G. This paper obtains tight upper and lower bounds of the radio number of $G_{m,n}$ where the path P_n has an odd order. The case where P_n has an even order has been investigated.

Key Words: Radio labeling, Smarandachely radio labeling, direct product of graphs, cross product of graphs, star, path.

AMS(2010): 05C35.

§1. Introduction

All graphs mentioned in this work are simple and undirected. The vertex and edge sets of a graph G are designated as $V(G)$ and $E(G)$ respectively and $e = uv \in E(G)$ connects $u, v \in V(G)$ while $d(u, v)$ denotes the shortest distance between $u, v \in V(G)$. We represent the diameter of G as diam(G).

The radio labeling, which often aims to solve signal interference problems in a wireless network, was first suggested in 1980 by Hale [7] and it is described as follows: Suppose that f is a non negative integer function on $V(G)$ and that f satisfies the radio labeling condition, $|f(u) - f(v)| \geq \mathrm{diam(G)} + 1 - \mathrm{d(u,v)}$ for every pair $u, v \in V(G)$. The span$f$ of f is $f_{max}(G) - f_{min}(G)$, where f_{max} and f_{min} are largest and lowest labels, respectively, assigned on $V(G)$ and the lowest value of span$f$ is the radio number, $rn(G)$, of G. Generally, let $V_1 \subset V(G)$ be a subset of vertices in G with property $\mathscr{P}$. If a labeling f satisfying the radio labeling condition for vertices in $V(G) \backslash V_1$ but $|f(u) - f(v)| < \mathrm{diam(G)} + 1 - \mathrm{d(u,v)}$ for every pair $u, v \in V_1$, then f is called a Smarandachely radio labeling of G and span$f$ of f is denoted by span$^S f$. Clearly,

[1]Received December 13, 2022, Accepted March 20, 2023.

$\text{span}^S f$=span$f$ if $V_1 = \emptyset$, i.e., the case of radio labeling on G. It is established that to obtain the radio numbers of graphs is hard. However, for certain graphs, the radio numbers have been obtained. Recent results on radio number include those on middle graph of path [3], trees, [4] and edge-joint graphs [12]. Liu and Zhu [11] showed that for path, P_n, $n \geq 3$,

$$rn(P_n) = \begin{cases} 2k(k-1)+1 & \text{if } n = 2k; \\ 2k^2+2 & \text{if } n = 2k+1. \end{cases}$$

Liu and Zhu's results compliment those obtained by Chatrand et. al. in [5] and [6] about the same graph. Liu and Xie worked on square graphs. In [9], they obtained $rn(P_n^2)$ of square of path as follows:

$$rn(P_n^2) = \begin{cases} k^2+2 & \text{if } n \equiv 1(\text{mod } 4), \text{n} \geq 9; \\ k^2+1 & \text{if otherwise.} \end{cases}$$

Other results on squares of graphs include those obtained for C_n^2 in [10], where C_n is a cycle of order n. On Cartesian products graphs, Jiang [8] solved the radio number problem for $(P_m \Box P_n)$, where for $m, n > 2$, and obtains $rn(P_m \Box P_n) = \frac{mn^2+nm^2-n}{2} - mn - m + 2$, for m odd and n even. Saha and Panigrahi [13] worked on Cartesian products of Cycles while Ajayi and Adefokun in [1] and [2] probe on the radio number of the Cartesian product of path and star graph called the stacked-book graph $G = S_m \Box P_n$. They observed in [1] that $rn(S_m \Box P_n) \leq n^2 m + 1$, a result the authors noted, citing a existing result in [8], is not a tight bound.) In [1], they obtained improve the results for path P_n, where n is even.

In this paper, we investigate further on the radio number of stacked-book graphs, $S_m \Box P_n$, in the case where n is odd and combined with [2], we improve the weak bounds obtained in [1].

§2. Preliminaries

Let S_m be a star of order $m \geq 3$ and let v_1 be the center vertex of S_m and $v_2, v_3, \cdots, v_m$ are adjacent to v_1 and let P_n be a path containing n vertices starting from u_1 to u_n. Furthermore, $P = u \xrightarrow{a} v \xrightarrow{b} w$ represents a path of length $a+b$, for which $d(u,v) = a$ and $d(u,w) = b$, where a and b are positive integers. If a stacked-book graph is obtained from the Cartesian product $G_{m,n} = S_m \Box P_n$ of S_m and P_n, then $V(G_{m,n})$ is the Cartesian product of $V(S_m)$ and $V(P_n)$, for which if $u_i v_j \in V(G_{m,n})$, then $u_i \in V(S_m)$, $v_j \in E(P_n)$, while, if $u_i v_j u_k v_i$ forms an edge in $E(G_{m,n})$, then $u_i = u_k$ and $v_j v_l \in E(P_n)$ or $v_j = v_l$ and $u_i u_k \in E(S_m)$.

Some of the following are adopted from [2].

Definition 2.1 *Where it is convenient, we denote $u_i v_j$ as u_{ij} and edge $u_i v_j u_k v_l$ as $u_{ij} u_{kl}$.*

Remark 2.1 Stacked-book graph $G_{m,n}$ contains n number of S_m stars, which can be expressed as the set $\{S_{m(i)} : 0 \leq i \leq n\}$.

Definition 2.2 *For $G_{m,n} = S_m \Box P_n$, $V_{(i)} \subset V(G_{m,n})$ is the set of vertices on $S_{m(i)}$ stated as $V_{(i)} = u_1 v_i, u_2 v_i, \cdots, u_m v_i$.*

Remark 2.2 We must mention that u_1v_i in the set in the last definition is the center vertex of $S_m(i)$.

Definition 2.3 *Let $G_{m,n} = S_m \square P_n$, n odd, the pair $S_{m(i)}, S_{m(i+\frac{n-1}{2})}$ is a subgraph $G''(i) \subseteq G_{m,n}$, which is induced by V_i and $V_{i+\frac{n-1}{2}}$, where $i \notin \{1, \frac{n+1}{2}, n\}$.*

Remark 2.3 It can be seen that with n odd, every $G_{m,n}$ contains $\frac{n-2}{3}$ number of $G''(i)$ subgraphs and the diameter $diam(G''(i))$ of $G''(i)$ is $\frac{n+3}{3}$.

Next, we introduce the following definitions:

Definition 2.4 *Let $G_{m,n} = S_m \square P_n$. Then, $\bar{G}_{m,n} \subseteq G_{m,n}$ is a subgraph of $G_{m,n}$ induced by the stars $S_{m(1)}, S_{m(\frac{n+1}{2})}, S_n$.*

We now define a class of paths $P'(i)$.

Definition 2.5 *Let $\{P'(t)\}_{t=1}^m$ be a class of paths in $G_{m,n}$, where $P'(t) := v_{j(1)} \xrightarrow{\alpha} v_{k(\frac{n+1}{2})} \xrightarrow{\beta} v_{l(n)}$, such that $j \neq k \neq l$, $v_{j(1)} \in V_{(i)}, v_{k(\frac{n+1}{2})} \in V_{(\frac{n+1}{2})}$ and $v_{l(n)} \in V_{(n)}$ and $1 \leq j, k, l \leq m$.*

It can be verified that $\{P'(t)\}_{t=1}^m$ contains two other sub-classes defined without loss of generality as follows:

$P'_1(t) = \{v_{1(1)} \xrightarrow{\frac{n+1}{2}} v_{3(\frac{n+1}{2})} \xrightarrow{\frac{n+3}{2}} v_{2(n)}, v_{2(1)} \xrightarrow{\frac{n+1}{2}} v_{1(\frac{n+1}{2})} \xrightarrow{\frac{n+1}{2}} v_{3(n)}, v_{3(1)} \xrightarrow{\frac{n+3}{2}} v_{2(\frac{n+1}{2})} \xrightarrow{\frac{n+1}{2}} v_{1(n)}\}$

$P'_2(t) = v_{a(1)} \xrightarrow{\frac{n+3}{2}} v_{b(\frac{n+1}{2})} \xrightarrow{\frac{n+3}{2}} v_{c(n)}$, $a \neq b \neq c$, $4 \leq a, b, c \leq m$. Clearly, $|P'_1(t)| = 3$ and $|P'_2(t)| = m - 2$.

§3. Results

In the next results, we establish a lower bound of the radio number $rn(G_{m,n})$ of a stacked-book graph $G_{m,n}$.

Lemma 3.1 *Let f be the radio labeling function on $G_{m,n}$, n odd, and let*

$$V_{(\frac{n+1}{2})} = \left\{v_{1(\frac{n+1}{2})}, v_{2(\frac{n+1}{2})}, v_{3(\frac{n+1}{2})}, \left\{v_{d(\frac{n+1}{2}):4 \leq d \leq m}\right\}\right\}$$

be the vertex set of the mid vertices in $P(t) \subseteq \{P'(t)\}_{t=1}^m$. Now, let $v \in V_{\frac{n+1}{2}}$ be some vertex in $V_{\frac{n+1}{2}}$. If $f(v)$ is f_{max} on $V(P(t))$, then

$$f(v) = \begin{cases} \frac{n+5}{2} & \text{if } v \in \left\{v_{1(\frac{n+1}{2})}, v_{2(\frac{n+1}{2})}, v_{3(\frac{n+1}{2})}\right\}; \\ \frac{n+3}{2} & \text{otherwise.} \end{cases}$$

Proof Since $P(t) \subset G_{m,n}$, then, radio labeling of any vertex on $V(P(t))$ is based on $diam(G_{m,n})$ and for $u, v \in V(P(t))$, $d(u, v) = k$, where k is the distance between u and v in

$G_{m,n}$. We consider the three paths in $P_1'(t)$ next.

Case 1(a) For $P_1'(1) := v_{1(1)} \xrightarrow{\frac{n+1}{2}} v_{3(\frac{n+1}{2})} \xrightarrow{\frac{n+3}{2}} v_{2(n)}$, let $f(v_{1(1)}) = 0$. Now $d(v_{1(1)}, v_{2(n)}) = n$. Therefore $f(v_{2(n)}) \geq f(v_{1(1)}) + diam(G_{m,n}) + 1 - n$=2. Also, $d(v_{2(n)}, v_{3(\frac{n+1}{2})}) = \frac{n+3}{2}$ and thus, $f(v_{3(\frac{n+1}{2})}) \geq f(v_{2(n)}) + dim(G_{m,n}) + 1 - \frac{n+3}{2} \geq \frac{n+5}{2}$. (It should be note that if we set $f(v_{2(n)}) = 0$, then, $f(v_{3(\frac{n+1}{2})}) \geq \frac{n+7}{2}$.)

Case 1(b) For $P_1'(2) := v_{2(1)} \xrightarrow{\frac{n+1}{2}} v_{1(\frac{n+1}{2})} \xrightarrow{\frac{n+1}{2}} v_{3(n)}$, let $f(v_{2(1)}) = 0$, then $d(v_{2(1)}, v_{3(n)}) = n+1$ and thus, $f(v_3(n)) \geq n + 2 - (n+1) = 1$. Likewise, $d(v_{3(n)}, v_{1(\frac{n+1}{2})}) = \frac{n+2}{2}$ and therefore, $f(v_{3(n)}) \geq n + 3 - (\frac{n+1}{2}) = \frac{n+5}{2}$.

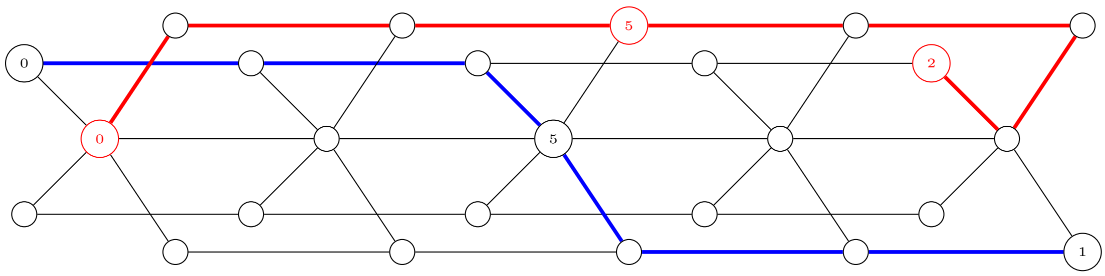

Figure 1 Illustration of Case 1(a) and (b) in a $G_{5,5}$ stacked-book graph

Case 1(c) Now for $P_1'(3) := v_{3(1)} \xrightarrow{\frac{n+3}{2}} v_{2(\frac{n+1}{2})} \xrightarrow{\frac{n+1}{2}} v_{1(n)}$, we assume $f(v_{1(n)}) = 0$. Also, $d(v_{3(1)}, v_1(n)) = n + 1$ in $G_{m,n}$. Thus, $f(v_{3(1)}) \geq 2$ and since $d(v_{3(1)}, v_{2\frac{n+1}{2}}) \geq \frac{n+3}{2}$, then, $f(v_{2(\frac{n+1}{2})}) \geq 2 + n + 2 - (\frac{n+3}{2}) \geq \frac{n+5}{2}$.

Next we consider the paths in $P_2'(t)$.

Case 2. Every path in $P_2'(t)$ are geometrically similar and are of the form $P_2'(4) = v_{a(1)} \xrightarrow{\frac{n+3}{2}} v_{b(\frac{n+1}{2})} \xrightarrow{\frac{n+3}{2}} v_{n(n)}$, such that $d(v_{a(1)}, v_{c(n)}) = n + 1$ and $d(v_{c(n)}, v_b(\frac{n+1}{2})) = \frac{n+3}{2}$, in $G_{m,n}$ and for all $a \neq b \neq c \neq m$, without loss of generality. Thus, suppose that $f(v_{a(1)}) = 0$, then $f(v_c(n)) \geq 1$ and $f(v_{b(\frac{n+1}{2})}) \geq \frac{n+3}{2}$. $\square$

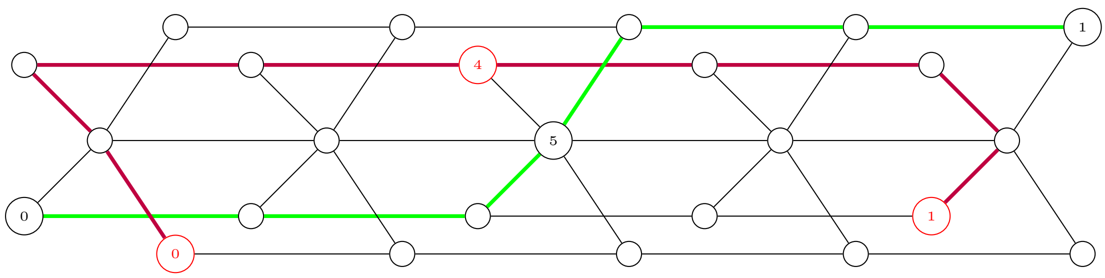

Figure 2 Illustration of Case 1(c) and Case 2 in a $G_{5,5}$ stacked-book graph

Remark 3.1 In (a) and (c) of Case 1, if the respective center vertices $v_{1(1)}$ and $v_{1(n)}$ of stars $S_{(1)}$ and $S_{(n)}$ are labeled $f(v_{1(1)}) = f(v_{1(n)}) = 0$, the radio labels on the mid vertices of their paths would be at least $\frac{n+7}{2}$.

Remark 3.2 For the m paths in $\{P't\}_{t=1}^{m}$, the sum of all the radio labels on the center vertices ($span(f)$) of f on $P'(t)$ is $3(\frac{n+5}{2}) + (m-3)(\frac{n+3}{2}) = \frac{1}{2}(mn + 3m + 6)$.

Next, we obtained a lower bound for $\{P(t)\}_{t=1}^{m}$.

Remark 3.3 From Remark 3.2, we notice that for optimum labeling of the three vertices on each of the paths in $\{P(t)\}_{t=1}^{m}$, the end vertex, which closest to the mid vertex is most suitable to be labeled first. These are $v_{1(1)} \in P_1'(1), v_1(n) \in P_1'(3)$ and any end vertex in the remaining paths. We refer to each of these ends vertices as initial label vertex.

Lemma 3.2 *Let $G(*)$ be a subgraph of $G_{m,n}$, induced by all the end point vertices and the midpoint vertices of $\{P_1'(t)\}_{t=1}^{m}$ i.e $S_{m(1)}, S_{m(\frac{n+1}{2})}, S_{m(n)}$. Then $rn(G(*)) \geq \frac{1}{2}(2mn+4m-n+5)$ in $G_{m,n}$.*

Proof Let v_1 and v_2 be center vertices on $S_{m(1)}$ and $S_{m(n)}$ respectively. There exist vertices $u_\alpha, u_\beta \in S_{m(\frac{n+1}{2})}$, $\alpha \neq \beta$, u_α, u_β not center vertices of $S_{m(\frac{n+1}{2})}$ such that $d(v_1, u_\alpha) = d(v_2, u_\beta) = \frac{n+1}{2}$. Also, there exists a subset $A = \{\omega_r\}$ in $S_{m(1)}$, (or $S_{m(n)}$) such that $|A| = m-3$, and $B = \{x_s\}$ in $S_{m(\frac{n+1}{2})}$, $|B| = m-1$, such that for $r \neq s$, $d(\omega_r, x_s) = \frac{n+3}{2}$. Now, the sum of $span(f)$ of f for all the pair (ω_r, x_s) will be $(m-1)(\frac{n+1}{2}) = \frac{1}{2}(mn + m - n - 1)$ and thus,

$$\begin{aligned} rn(G(*)) &\geq \frac{1}{2}(mn + m - n - 1) + \frac{1}{2}mn + 3m + 6] \\ &\geq \frac{1}{2}(2mn + 4m - n + 5). \end{aligned}$$

This completes the proof. □

We extend the result in Lemma 3.2 in other to obtain a lower bound for the radio number of stacked book graph $G_{m,n}$, with off $n \geq 5$.

Definition 3.1 *Let $G_{m,n}$ be a stacked-book graph with odd n, $n \geq 5$, and $m \geq 4$. Also, let $G(*)$ as defined earlier. A subgraph $G(**)$ of $G_{m,n}$ as $G(**) = G_{m,n} \backslash G(*)$.*

Remark 3.4 We can see that $G(**)$ is a subgraph of $G_{m,n}$, induced by $\{S_{m(i)}\}_{i=2}^{n-1}$, $i \neq \frac{n+1}{2}$.

Definition 3.2 *The subgraph of $G(**)$, induced by $S_{m(t)}$ and $S_{m(t+\frac{n-1}{2})}$ is denoted by $G''(t)$.*

Remark 3.5 It should be noted that $G(**) \subset G_{m,n}$ contains exactly $\frac{n-3}{2}$ $G''(t)$ subgraphs.

Remark 3.6 Let $G''(t)$ be induced by $S_{m(t)}$ and $S_{m(t+\frac{n-1}{2})}$ and let $V(S_{m(t)}) = \{u_1, u_2, \cdots, u_m\}$ and $V(S_{m(t+\frac{n-1}{2})}) = \{v_1, v_2, \cdots, v_m\}$ be the vertex sets of $S_{m(t)}$ and $S_{m(t+\frac{n-1}{2})}$ where u_1 and v_1 are the respective center vertices. It can be seen that, $d(u_i, v_j) \in \{\frac{n+1}{2}, \frac{n+3}{2}\}$, where $i \neq j$.

Remark 3.7 For $i \neq j$, $d(u_1, v_j) = d(u_j, v_1) = \frac{n+1}{2}$ and for $i \neq j$, $i, j \neq 1$, $d(u_i, v_j) = \frac{n+3}{2}$.

Now we obtain a lower bound value for the radio number labeling of $G''(t)$ in $G_{m,n}$.

Lemma 3.3 *Let $G''(t) \subset G_{m,n}$, with $m \geq 4$ and $n \geq 5$, n odd, be a subgraph of $G_{m,n}$. Then*

$$rn(G''(t)) \geq mn + m - \frac{1}{2}(n-3).$$

Proof Let u_1 and v_1 be center vertices of $S_{m(t)}$ and $S_{m(t+\frac{n-1}{2})}$. By Remark 3.7 above, $d(u_1, v_i) = d(u_i, v_1) = \frac{n+1}{2}$, is the shortest distance between the center vertex of a star in $G''(t)$ and a non-center vertex in the other star in $G''(t)$. It is optimal, therefore to label the center vertices as f_{min} and f_{max}. Now, without loss of generality, set $f_{min} = f(v_1) = 0$. Since $d(v_1, u_i) = \frac{n+1}{2}$, $i \in \{2, 3, \cdots, m\}$. Therefore

$$f(u_i) \geq f(v_1) + diam(G_{m,n}) + 1 - d(u_i, v_1).$$

Let $i = 2$. Thus,

$$\begin{aligned} f(u_2) &\geq 0 + n + 2 - \frac{n+1}{2} \\ &\geq \frac{n+3}{2}. \end{aligned}$$

Now $d(u_2, v_3) = \frac{n+3}{2}$ and therefore,

$$\begin{aligned} f(v_3) &\geq \frac{n+3}{2} + n + 2 - \frac{n+3}{2} \\ &\geq \frac{n+3}{2} + \frac{n+1}{2}. \end{aligned}$$

Also, for $d(v_3, u_4) = \frac{n+3}{2}$, $f(v_4) \geq \frac{n+3}{2} + 2(\frac{n+1}{2})$. By continuing the iteration, we have

$$f(v_m) \geq \frac{n+3}{2} + 2m - 3(\frac{n+1}{2}).$$

Lastly,

$$\begin{aligned} f_{max} = f(u_1) &\geq 2(\frac{n+3}{2}) + (2m-3)(\frac{n+1}{2}) \\ &= mn + m - \frac{1}{2}(n-3). \end{aligned}$$

□

Next we extend the last result to obtain a lower bound for $G(**)$.

Lemma 3.4 *For $G(**) \subset G_{m,n}$, $rn(G(**)) \geq \frac{1}{2}(mn^2 - 2mn - 3m + 2n - 12)$.*

Proof From Lemma 3.3, the *span*(f) of f on $G''(t) = mn + m - \frac{1}{2}(n-3)$. For $G''(t)$, $f_{max} = mn + m - \frac{1}{2}(n-3)$. Let $t = 2$ and let $u_1 = v_{m(2)} \in S_{m(2)}$ and $v_1' = v_{m(2+\frac{n+1}{2})} \in S_{m(2+\frac{n+1}{2})}$, be center vertices of $S_{m(2)}$ and $S_{m(2+\frac{n+1}{2})}$. Now, $d(u_i, v_1') = \frac{n+1}{2}$. Thus,

$$f(v_1') \geq f(u_1) + n + 2 - \frac{n+1}{2} = f(u_1) + \frac{n+3}{2}.$$

This implies that for $G''(3)$, induced by $S_{m(3)}$ and $S_{m(2+\frac{n+1}{2})}$, $f_{min} = f(u_1) + \frac{n+3}{2}$, and $f_{max} = f(v_1'')$, where v'' is the center vertex of $S_{m(3)}$. From the precedure in Lemma 3.3, there are $\frac{n-3}{2}$ $G''(t)$ subgraphs in $G_{m,n}$. Therefore, f_{max} of $G(**)$ is $f(v_1^{(k)}) \in S_{(m(\frac{n-1}{2}))}$, where $f(v_1^{(k)})$ is the center vertex of $S_{m(\frac{n-1}{2})}$. Following the iteration,

$$\begin{aligned} f(v_1^{(k)}) &\geq \frac{n-3}{2}[mn + m - \frac{1}{2}(n-3)] + \frac{n-5}{2}\left(\frac{n+3}{2}\right) \\ &\geq \frac{1}{2}(mn^2 - 2mn - 3m + 2n - 12). \end{aligned}$$

This completes the proof. □

Now we establish a lower bound for the radio number of $G_{m,n}$.

Theorem 3.1 *Let $G_{m,n}$ be a stacked-book graph with $m \geq 4$ and $n \geq 5$. Then,*

$$rn(G_{m,n}) \geq \frac{mn^2 + m + 2n - 4}{2}.$$

Proof From Lemmas 3.3 and 3.4,

$$rn(G(*)) \geq m(n+2) - \frac{1}{2}(n-5) \text{ and } rn(G(**)) \geq \frac{1}{2}(mn^2 - 2mn - 3n + 2n - 12).$$

Now, since $G_{m,n} = G(*) \cup G(**)$, suppose that u_1 is the center vertex of $S_{m(\frac{n-1}{2})}$ and $v_1 \in S_{m(n)}$ is the center vertex of $S_{m(n)}$. Clearly $d(u_1, v_i) = \frac{n+1}{2}$. Now, $f(u_1) = f_{max}$ of $G(**)$ and

$$f(u_1) \geq \frac{1}{2}(mn^2 - 2mn - 3m + 2n - 12).$$

Therefore,

$$\begin{aligned} f(v_1) &\geq f(u_i) + n + 2 - \frac{(n+1)}{2} \\ &\geq \frac{1}{2}(mn^2 - 2mn - 3m + 2n - 12) + n + 2 - \frac{(n+1)}{2} \\ &\geq \frac{1}{2}(mn^2 - 2mn - 3m + n - 13) + n + 2. \end{aligned}$$

For $G(*)$, set $f(v_1) = f_{min}$. Thus, $rn(G_{m,n}) \geq f(v_1) + rn(G(*))$ and hence,

$$\begin{aligned} rn(G_{m,n}) &\geq \frac{1}{2}(mn^2 - 2mn - 3m + n - 13) + n + 2 + m(n+2) - \frac{1}{2}(n-5) \\ &\geq \frac{mn^2 + m + 2n - 4}{2}. \end{aligned}$$

This completes the proof. □

Next, we investigate the upper bound of a stacked-book graph. The technique involves manual radio labeling of subgraphs $G(*)$ and $G(**)$ and merging the results.

Lemma 3.5 *Let $G(**) \subset G_{m,n}$, with n-odd. Then, $rn(G(**)) \leq \frac{1}{2}(mn^2 - 2mn + 2n - 3m - 12)$.*

Proof From earlier definition, if n is odd, then, $G_{m,n} = G(*) \cup G(**)$, where $G(**)$ contains $\frac{n-3}{2}$ $G''(t)$ graphs. Let $G''(\frac{n-1}{2})$ be induced by $S_{m(\frac{n-1}{2})}$ and $S_{m(n-1)}$, n-odd. Let $V(S_{m(\frac{n+1}{2})}) = \left\{v_{\frac{n-1}{1}(i)}\right\}_{i=1}^{m}$, $V(S_{m(n-1)}) = \{v_{n-1}\}_{i=1}^{m}$, where $v_{\frac{n-1}{2}(1)}$, $v_{n-1(1)}$ are center vertices and $d(v_{\frac{n-1}{2}(j)}, v_{n-1(j)}) = \frac{n-1}{2}$, for all $1 \leq j \leq m$, $d(v_{\frac{n-1}{2}(1)}), v_{n-1(j)} = d(v_{n-1(1)}, v_{\frac{n-1}{2}(j)}) = \frac{n+1}{2}$ and $d(v_{\frac{n-1}{2}(j)}, v_{n-1(k)}) = \frac{n+3}{2}$. Now, let $f(v_{\frac{n-1}{2}(1)}) = 0$. Since $d(v_{\frac{n-1}{2}(1)}, v_{n-1(2)}) = \frac{n+1}{2}$, then set $f(v_{n-1(k)}) = \frac{n+3}{2}$. Let $f(v_{\frac{n-1}{2}(1)}) = 0$. Since $d(v_{\frac{n-1}{2}(1), v_{n-1(2)}}) = \frac{n+1}{2}$, then set $f(v_{n-1(2)}) = \frac{n+3}{2}$, $d(v_{n-1(2)}, v_{\frac{n-1}{2}(3)}) = \frac{n+3}{2}$ and thus,

$$f(v_{n-1}(4)) = \frac{n+3}{2} + 2\frac{n+1}{2}.$$

Thus, by continuously alternating the process, it gets to the case where $d(v_{\frac{n-1}{2}(m)}, d(v_{n-1(m-1)})) = \frac{n+3}{2}$. Thus,

$$f(\frac{n-1}{2}(m)) = \frac{n+3}{2} + m - 2(\frac{n+1}{2})$$

and since $d(v_{\frac{n-1}{2}(m)}, v_{n-1(3)}) = \frac{n+3}{2}$,

$$f(v_{n-1(3)}) = \frac{n+3}{2} + (m-1)(\frac{n+1}{2}), \quad f(v_{\frac{n-1}{2}(2)}) = \frac{n+3}{2} + m(\frac{n+1}{2}).$$

Depending on the size of m, the labeling continues until

$$f(v_{\frac{n-1}{2}}) = \frac{n+3}{2} + 2m - 3\frac{(n+3)}{2} + 2(2m-3)(\frac{n+1}{2})$$

is attained and finally, $d(v_{\frac{n-1}{2}(m-1)}, v_{n-1(1)}) = \frac{n+1}{2}$ and thus, $f(v_{n-1(1)}) = \frac{2(n+3)}{2} + 2m - 3 + \frac{n+1}{2}$. (By following the same argument, it is easy to obtain similar result for m-even.) Now, $d(v_{n-1(1)}, v_{\frac{n-3}{2}(1)}) = \frac{n+1}{2}$, where $v_{\frac{n-3}{2}(1)}$ is the center vertex of $G''_{m(\frac{n-3}{2})}$. Therefore,

$$\begin{aligned} f_{min}(G''(\frac{n-3}{2})) &= f(v_{\frac{n-3}{2}(1)}) = f(v_{n-1(1)}) + n + 2 - \frac{n+2}{2} \\ &= f(v_{n-1(1)}) + \frac{n+3}{2} = \frac{3(n+3)}{2} + 2m - 3(\frac{n+1}{2}) \end{aligned}$$

and

$$\begin{aligned} f_{max}(G''(\frac{n-3}{2}) &= f(v_{n-2(1)}) = f(v_{\frac{n-3}{2}(1)}) + \frac{2(n+3)}{2} + \frac{(2m-3)(n+1)}{2} \\ &= \frac{5(n+3)}{2} + 2(2m-3)\frac{(n+1)}{2}, \end{aligned}$$

which is $f_{max}(G''(\frac{n-3}{2}))$. Now, the process is extended to $G''(2)$, for which

$$\begin{aligned} f(\frac{n+3}{2}) &= \frac{(n-5)(n+3)}{4} + \frac{(n-3)(n+3)}{2} + \frac{(n-2)(2m-3)(n+1)}{4} \\ &= \frac{1}{2}(mn^2 - 2mn + 2n - 3m - 12). \end{aligned}$$

□

Remark 3.8 It can be observed that for the optimal radio labeling of $G(*)$, $f_{max}(G(*))$ is $f(v_{\frac{n+1}{2}(1)})$, the label on the center vertex of $S_{m(\frac{n+1}{2})}$. Since for v_α, v_β in $S_{m(1)}$ and $S_{m(n)}$ respectively, $\alpha, \beta \neq 1$, $d(v_{\frac{n+1}{2}(1)}, v_\alpha) = d(v_{\frac{n+1}{2}(1)}, v_\beta) = \frac{n+1}{2}$, which is less than $\frac{n+3}{2}$, the value of $d(v_{\frac{n+1}{2}(k)}, v_\alpha)$, where $k \neq \alpha$, $k, \alpha \neq 1$, and v_α either belongs to $S_{m(1)}$ or $S_{m(n)}$. Thus, we manually label $G(*)$, such that $v_{\frac{n+1}{2}(1)}$ gets the last label and thus, $f(v_{\frac{n+1}{2}(1)}) = f_{max}(G(*))$.

Next, we consider some necessary conditions for establishing the upper bound of $G(*)$.

Lemma 3.6 *Let $G(*) \subset G_{m,n}$ be a subset of $G_{m,n}$, induced by $S_{m(1)}, S_{m(\frac{n+1}{2})}$ and $S_{m(n)}$. If $v_{1(1)}$ (or $v_{n(1)}$) and $v_{\frac{n+2}{2}(1)}$ are the center vertices of $S_{m(1)}$ (or $S_{m(n)}$) and $S_{m(\frac{n+1}{2})}$ respectively, and $f_{min}(G(*)) \neq f(v_{1(1)})$ (or $f(v_{n(1)})$), and $f_{max}(G(*)) \neq f(v_{\frac{n+1}{2}})$ (or vice versa), then, $|f_{min}(G(*)) - f_{max}(G(*))| \neq rn(G(*))$.*

Proof Without loss of generality, select $v_{1(1)}$ over $v_{n(1)}$. Suppose that $f(v_{1,1})$ and $f(v_{\frac{n+1}{2}(1)})$ are not $f_{min}(G(*))$ and $f_{max}(G(*))$ respectively. Let $v_\alpha \in V(S_{m(1)})$, $v_\beta \in V(S_{m(\frac{n+1}{2})})$, and $v_\gamma \in V(S_{m(n)})$ be non-center vertices, and let the set of the following vertices, $\{v_\alpha, v_\beta, v_\gamma, v_{1(1)}, v_{\frac{n+1}{2}(1)}\}$ be X, and let $H = V(G(*))\backslash X$ be the subgraph of $G(*)$ induced by $V(G(*)) - X$, and such that the radio number of H is positive integer p. Without loss of generality, let there be some $v_k \in V(H)$, where $v_k = v_{n(i)} \in S_{m(n)}, \gamma \neq i$ and $d(v_k, v_\beta) = \frac{n+3}{2}$, there exist a radio numbering sequence $v_k \to v_\beta, \to v_{1(1)} \to v_\gamma \to v_{\frac{n+1}{2}(1)} \to v_\alpha$. Suppose that $f(v_k)$ is the $f_max(H)$, that is, $f(v_k) = p$. Since $d(v_k, v_\beta) = \frac{n+3}{2}$, then $f(v_\beta) = p + \frac{n+1}{2}$ and likewise, it is observed that the radio labeling sequence yields $f_{max}(G(*)) = p+2n+7$. Now, suppose on the contrary, that $f(v_{1(1)})$ and $f(v_{\frac{n+1}{2}(1)})$ are $f_{min}(G(*))$ and $f_{max}(G(*))$ respectively. Let $v_{k(0)}$ be the vertex in H, which holds the least radio label. Obviously $v_{k(0)} \neq v_k$ and since $|V(G(*))| - |V(H)| \equiv 3 \text{mod } 1$, then $v_{k(0)}$ is a is also a vertex on the same star as v_k, this time, $S_{m(n)}$. Thus,if $v_{k(0)}$ is also not a center vertex, then, $d(v_{1(1)}, v_{k(0)}) = n$. Let $f(v_{1(1)}) = 0$. Now, we have the radio labeling sequence: $v_{1(1)} \to (v_{k(0)} \to \cdots \to v_k) \to v_\beta \to v_\alpha \to v_\gamma \to v_{\frac{n+1}{2}(1)}$. Since $d(v_{k(0)}, v_{1(1)}) = n$, then, $f(v_{k(0)}) = 2$ and since $|f_{min}(H) - f_{max}(H)| = p$, then $f(v_k) = 2 + p$. Labeling the sequence, afterwards, we have

$$f_{max}(G(*)) = f(v_{\frac{n+1}{2}(1)}) = p + \frac{3n+11}{2},$$

which is less than $p + 2n + 7$. □

Remark 3.9 It is noted that $v_{1(1)}$ (or $v_{n(1)(1)}$) and $v_{\frac{n+1}{2}}$ can be $f_{min}(G(*))$ and $f_{max}G(*)$ interchangeably. However, they both will have to be used for these roles. It is trivial to show that optimal radio labeling will not be attained if just one of them is used.

Next we obtain an upper bound for $G(*)$, based on Lemma 3.6.

Theorem 3.2 *For $G(*) \subset G_{m,n}$, $m \geq 5$, $rn(G(*)) \leq \frac{1}{2}(2mn + 4m - n + 7)$.*

Proof From Lemma 3.6, for $v_{1(1)} \in S_{m(1)}$, let $f(v_{1(1)}) = 0$. There exist $m - 1$ vertices of $S_{m(n)}$, such that for each $v_{n(i)} \in V(S_{m(n)})$, $i \neq 1$, $d(v_{1(1)}, v_{n(i)}) = n$. Thus, without loss of generality, let the first vertex be $v_{n(2)}$. Then, $f(v_{n(2)}) = 2$. Likewise, there exists $m - 1$, non-center vertex on $S_{m(\frac{n+1}{2})}$, and for each $v_{\frac{n+1}{2}(j)}$, $j \neq 1$, $d(v_{n(2)}, v_{\frac{n+1}{2}(j)}) = \frac{n+3}{2}$, where $j \neq 2$. So, now, let $j = 3$, then,

$$f(v_{\frac{n+1}{2}(3)}) = 2 + n + 2 - \frac{n+3}{2} = 2 + \frac{n+1}{2}.$$

In similar way,

$$f(v_{1(4)}) = 2 + \frac{n+1}{2} + \frac{n+1}{2}.$$

Now, we label $v_{n(1)}$, which is at distance n from $v_{1(4)}$ as $f(v_{n(1)}) = 4 + \frac{n+1}{2} + \frac{n+1}{2}$. Now, two of the center vertices are labeled. For, say, $v_{\frac{n+1}{2}(5)}$,

$$f(v_{\frac{n+1}{2}(5)}) = 4 + \frac{n+1}{2} + \frac{n+1}{2} + \frac{n+3}{2}.$$

It can be seen that for each of $S_{m(1)}$, $S_{m(\frac{n+1}{2})}$ and $S_{m(n)}$, there are $m-2$ vertices left to be labeled. This is now done by adding $\frac{(n+1)}{2}$ and 1 in alternating manner to the cumulative label values, such that we have

$$f(v_{1(6)}) = 4 + 3(\frac{n+1}{2}) + \frac{n+3}{2} \text{ and } f(v_{n(7)}) = 5 + 3(\frac{n+1}{2}) + \frac{n+3}{2}.$$

Thus by continuing the iteration until it gets to

$$f(v_{\frac{n+1}{2}(1)}) = (m+2) + (2m-3)(\frac{n+1}{2}) + 2(\frac{n+3}{2}) = \frac{1}{2}(2mn + 4m - n + 7). \qquad \square$$

Next, we merged the last results to obtain an upper bound for the radio number of a stacked-book graph $G_{m,n}$, where $m \geq 5$.

Theorem 3.3 *Let $m \geq 5$. Then, $rn(G_{m,n}) \leq \frac{1}{2}(mn^2 + 2n + m - 2)$.*

Proof Recall that $G = G(*) \cup G(**)$. From Lemma 3.5, where $G(**)$ is labeled, we see that for $G(**)$, $f_{max}(G(**)) = f(v_{\frac{n+3}{2}(1)})$. For $G(*) \in G_{m,n}$, we see in Lemma 3.2 that $f(v_{1(1)}) = f_{min}$. Clearly, $d(v_{\frac{n+3}{2}(1)}, v_{1(1)}) = \frac{n+1}{2}$. Thus, for $v_{1(1)} \in G_{m,n}$,

$$f(v_{1(1)}) = f(v_{\frac{n+3}{2}(1)}) + \frac{n+3}{2} = \frac{1}{2}(mn^2 - 2mn + 2n - 3m - 12) + \frac{n+3}{2} = \frac{1}{2}(mn^2 - 2mn + 3n - 3m - 9).$$

Thus by Lemma 3.2,

$$f_{max}(G_{m,n}) = f(v_{1(1)}) + f_{max}(G(**)) = \frac{1}{2}(mn^2 + 2n + m - 2). \qquad \square$$

Remark 3.10 We observe that the result in Theorem 3.3 that the there is just a difference of of 1 between this upper bound and the lower bound established earlier in the work. It is believed that the lower bound can be improved to coincide with the upper bound.

A radio labeling of $G_{5,5}$ is shown in Figure 3, where it is demonstrated that $rn(G_{5,5}) \leq 69$.

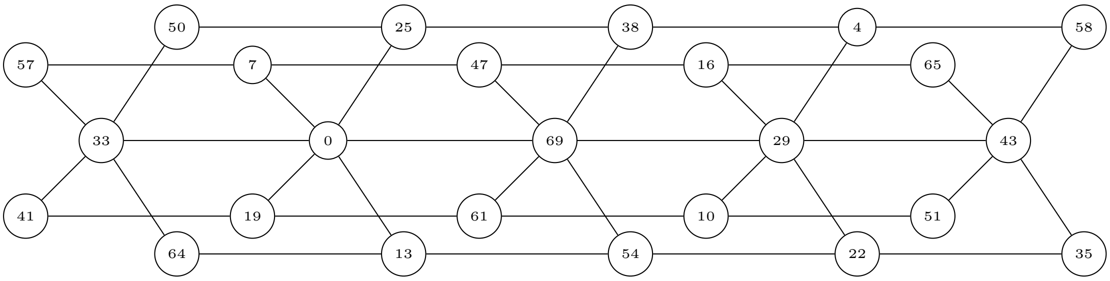

Figure 3 A $G_{5,5}$ stacked-book graph

§4. Conclusion

This work has greatly improved results obtained in [1] and extended the outcomes of [1] to the odd-path factor of the stacked-book graph class. It is safe now to say that this work and [2] have provided a tight bounds for the radio number of the general stacked-book graph. Further work to obtain the exact value of the radio number for stacked-book graph should be considered.

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International J.Math. Combin. Vol.1(2023), 98-110

On Laplacian of Skew-Quotient of Randić and Sum-Connectivity Energy of Digraphs

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Abstract: In this paper we introduce the concept of Laplacian of skew-quotient of Randić and sum-connectivity energy of directed graphs. Then we compute the Laplacian of skew-quotient of Randić and sum-connectivity energy of some graphs such as star digraph, complete bipartite digraph, the $(S_m \wedge P_2)$ digraph and a crown digraph.

Key Words: Laplacian, Skew-quotient of Randić sum-connectivity energy, Smarandachely sum-connectivity adjacency matrix.

AMS(2010): 05C50.

§1. Introduction

In [3], we introduce the concept of skew-quotient of Randić and sum-connectivity energy of a digraph as follows. Let a and b be two nonnegative real numbers with $a \neq 0$ and D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. Then, the skew-quotient of Randić and sum-connectivity adjacency matrix of D is the $n \times n$ matrix $A_{sqrs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} \dfrac{1}{\sqrt{\frac{a(d_i+d_j)}{b(d_i d_j)}}}, & \text{if } (v_i, v_j) \in \Gamma(D), \\ -\dfrac{1}{\sqrt{\frac{a(d_i+d_j)}{b(d_i d_j)}}}, & \text{if } (v_j, v_i) \in \Gamma(D), \\ 0, & \text{otherwise.} \end{cases}$$

Then, the skew-quotient of Randić and sum-connectivity energy $E_{sqrs}(D)$ of D is defined as the sum of the absolute values of eigenvalues of A_{sqrs}. Generally, a skew-quotient of Smarandachely sum-connectivity adjacency matrix of D is a $n \times n$ matrix $A^S_{sqrs} = (a^S_{ij})$ with entries

[1]Received December 11, 2022, Accepted March 21, 2023.

$$a_{ij}^S = \begin{cases} \dfrac{1}{\sqrt[\rho+1]{\frac{a(d_i^\rho+d_j^\rho)}{b(d_i^\rho d_j^\rho)}}}, & \text{if the directed distacne from } v_i \text{ to } v_j \text{ is } \rho, \\ -\dfrac{1}{\sqrt[\rho+1]{\frac{a(d_i^\rho+d_j^\rho)}{b(d_i^\rho d_j^\rho)}}}, & \text{if the directed distacne from } v_j \text{ to } v_i \text{ is } \rho, \\ 0, & \text{otherwise} \end{cases}$$

which characterizes the non-homogeneity of vertices on a digraph by directed distance. Certainly, the matrix A_{sqrs} characterizes vertices of G in case of homogeneity which is a submatrix of A_{sqrs}^S.

In 2004, D. Vukičević and Gutman [6] have defined the Laplacian matrix of the graph G, denoted by $L = (L_{ij})$, as a square matrix of order n whose elements are defined by

$$L_{ij} = \begin{cases} \delta_i, & \text{if } i = j, \\ -1, & \text{if } i \neq j \text{ and the vertices } v_i, v_j \text{ are adjacent}, \\ 0, & \text{if } i \neq j \text{ and the vertices } v_i, v_j \text{ are not adjacent}, \end{cases}$$

where δ_i is the degree of vertex v_i. The eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$ of L, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are called the Laplacian eigenvalues of G. In 2006, Gutman and B. Zhou [2] have defined the Laplacian energy of $LE(G)$ of G as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|,$$

where m is number of edges and n is number of vertices of G.

Motivated by these works, we introduce the Laplacian of skew Quotient of Randić and sum-connectivity energy of a digraph G as follows. Let a and b be two nonnegative real number with $a \neq 0$. The Laplacian of skew quotient of Randić and sum-connectivity adjacency matrix of G is the $n \times n$ matrix $A_{lsqrs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} \delta_i, & \text{if } i = j, \\ \dfrac{1}{\sqrt{\frac{a(d_i+d_j)}{b(d_i d_j)}}}, & \text{if} (v_i, v_j) \in \Gamma(D), \\ -\dfrac{1}{\sqrt{\frac{a(d_i+d_j)}{b(d_i d_j)}}}, & \text{if} (v_j, v_i) \in \Gamma(D), \\ 0, & \text{if the vertices} \, v_i \text{ and} \, v_j \text{ are not adjacent.} \end{cases}$$

where δ_i is the degree of vertex v_i. Where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are called the eigenvalues of A_{lsqrs}. Then, the Laplacian of quotient of Randić and sum-connectivity energy of G is

$$E_{lsqrs}(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|,$$

where m is number of edges and n is number of vertices of G.

In Section 2 we compute the laplacian of skew-quotient of Randić and sum-connectivity energy of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ digraph and a crown digraph.

§2. Laplacian of Skew-Quotient of Randić and Sum-Connectivity Energies of Some Families of Graphs

We begin with some basic definitions and notations.

Definition 2.1([4]) *A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.*

Definition 2.2([4]) *A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.*

Definition 2.3([1]) *The Crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and edge set $\{u_i v_j; 1 \leq i, j \leq n, i \neq j\}$. S_n^0 is therefore S_n^0 coincides with complete bipartite graph $K_{n,n}$ with the horizontal edges removed.*

Definition 2.4([5]) *The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) | v_i v_k \in E(S_m)$ and $v_j v_l \in E(P_2)$ and $1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.*

Now we compute Laplacian of skew- quotient of Randić and sum-connectivity energies of some directed graphs such as complete bipartite digraph, star digraph, the $(S_m \wedge P_2)$ digraph and a crown digraph.

Theorem 2.5 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $K_{n,n}$ complete bipartite digraph be respectively given by*

$$V(D) = \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\} \text{ and } \Gamma(D) = \{(u_i, v_j) \,|\, 1 \leq i \leq n, 1 \leq j \leq n\}.$$

Then, the Laplacian of skew-quotient of Randić and sum-connectivity energy of the complete bipartite digraph is

$$2\sqrt{\frac{(n^2)^2}{a(n+n)}}.$$

Proof The Laplacian of skew-quotient of Randić and sum-connectivity matrix of complete bipartite digraph is given by

$$A_{lsqrs} = \begin{pmatrix} n & 0 & \cdots & 0 & \gamma & \gamma & \cdots & \gamma \\ 0 & n & \cdots & 0 & \gamma & \gamma & \cdots & \gamma \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n & \gamma & \gamma & \cdots & \gamma \\ -\gamma & -\gamma & \cdots & -\gamma & n & 0 & \cdots & 0 \\ -\gamma & -\gamma & \cdots & -\gamma & 0 & n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & -\gamma & \cdots & -\gamma & 0 & 0 & \cdots & n \end{pmatrix},$$

where $\gamma = \dfrac{1}{\sqrt{\frac{a(n+n)}{b(n^2)}}}$. Then its characteristic polynomial is

$$\begin{aligned} |\lambda I - A_{lsqrs}| &= \begin{vmatrix} \lambda-n & 0 & \cdots & 0 & -\gamma & -\gamma & \cdots & -\gamma \\ 0 & \lambda-n & \cdots & 0 & -\gamma & -\gamma & \cdots & -\gamma \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda-n & -\gamma & -\gamma & \cdots & -\gamma \\ \gamma & \gamma & \cdots & \gamma & \lambda-n & 0 & \cdots & 0 \\ \gamma & \gamma & \cdots & \gamma & 0 & \lambda-n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & \gamma & \cdots & \gamma & 0 & 0 & \cdots & \lambda-n \end{vmatrix} \\ &= \begin{vmatrix} \Lambda I_n & -\frac{1}{\sqrt{\frac{a(n+n)}{b(n^2)}}} J^T \\ \frac{1}{\sqrt{\frac{a(n+n)}{b(n^2)}}} J & \Lambda I_n \end{vmatrix}, \end{aligned}$$

where $\Lambda = \lambda - n$, J is an $n \times n$ matrix with all the entries are equal to 1. Hence the characteristic equation is given by

$$\begin{vmatrix} \Lambda I_m & -\frac{1}{\sqrt{\frac{a(n+n)}{b(n^2)}}} J^T \\ \frac{1}{\sqrt{\frac{a(n+n)}{b(n^2)}}} J & \Lambda I_n \end{vmatrix} = 0.$$

This can be written as

$$|\Lambda I_n| \left| \Lambda I_n - \left(\frac{1}{\sqrt{\frac{a(n+n)}{b(n^2)}}} J \right) \frac{I_n}{\Lambda} \left(-\frac{1}{\sqrt{\frac{a(n+n)}{b(n^2)}}} J^T \right) \right| \quad = \quad 0.$$

On simplification, we obtain

$$\frac{\Lambda^{n-n}}{\frac{a(n+n)}{b(n^2)}}\left|\left(\frac{a(n+n)}{b(n^2)}\right)\Lambda^2 I_n + JJ^T\right| = 0,$$

which can be written as

$$\frac{\Lambda^{n-n}}{\frac{a(n+n)}{b(n^2)}}P_{JJ^T}\left(-\left(\frac{a(n+n)}{b(n^2)}\right)\Lambda^2\right) = 0,$$

where $P_{JJ^T}(\Lambda)$ is the characteristic polynomial of the matrix ${}_nJ_n$. Thus, we have

$$\frac{\Lambda^{n-n}}{\frac{a(n+n)}{b(n^2)}}\left(\frac{a(n+n)}{b(n^2)}\Lambda^2 + n^2\right)\left(\frac{a(n+n)}{b(n^2)}\Lambda^2\right)^{n-1} = 0,$$

which is same as

$$\Lambda^{n+n-2}\left(\Lambda^2 + \frac{n^2}{\frac{a(n+n)}{b(n^2)}}\right) = 0.$$

Hence,

$$Spec\,(D) = \begin{pmatrix} n & n + i\sqrt{\frac{n^2}{\frac{a(n+n)}{b(n^2)}}} & n - i\sqrt{\frac{n^2}{\frac{a(n+n)}{b(n^2)}}} \\ n+n-2 & 1 & 1 \end{pmatrix}.$$

Hence, the Laplacian of skew-quotient of Randić and sum-connectivity energy of the complete bipartite digraph is

$$E_{lsqrs}(D) = 2\sqrt{\frac{b(n^2)^2}{a(n+n)}},$$

as desired. □

Theorem 2.6 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of S_n star digraph be respectively given by*

$$\begin{aligned} V(D) &= \{v_1, v_2, \cdots, v_n\}, \\ \Gamma(D) &= \{(v_1, v_j)\,|\,2 \le j \le n\}. \end{aligned}$$

Then, the laplacian of skew-quotient of Randić and sum-connectivity energy of D is

$$\begin{aligned} E_{lsqrs}(S_n) = \; & \frac{(n-2)^2}{n} + \left|\frac{n^2 - 2(n-1)}{n} + i\sqrt{\frac{n^3a - 4(n-1)(an - b(n-1))}{2na}}\right| \\ & + \left|\frac{n^2 - 2(n-1)}{n} - i\sqrt{\frac{n^3a - 4(n-1)(an - b(n-1))}{2na}}\right|. \end{aligned}$$

Proof The Laplacian of skew-quotient of Randić and sum-connectivity matrix of the star digraph D is given by

$$A_{lsqrs}=\begin{pmatrix} n-1 & \frac{1}{\sqrt{\frac{an}{b(n-1)}}} & \frac{1}{\sqrt{\frac{an}{b(n-1)}}} & \cdots & \frac{1}{\sqrt{\frac{an}{b(n-1)}}} & \frac{1}{\sqrt{\frac{an}{b(n-1)}}} \\ -\frac{1}{\sqrt{\frac{an}{b(n-1)}}} & 1 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{\frac{an}{b(n-1)}}} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{\frac{an}{b(n-1)}} & 0 & 0 & \cdots & 1 & 0 \\ -\frac{1}{\frac{an}{b(n-1)}} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Hence the characteristic polynomial is given by

$$\begin{aligned} |\lambda I - A_{lsqrs}| &= \begin{vmatrix} \lambda-(n-1) & -\frac{1}{\sqrt{\frac{an}{b(n-1)}}} & -\frac{1}{\sqrt{\frac{an}{b(n-1)}}} & \cdots & -\frac{1}{\sqrt{\frac{an}{b(n-1)}}} \\ \frac{1}{\sqrt{\frac{an}{b(n-1)}}} & \lambda-1 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{\frac{an}{b(n-1)}}} & 0 & \lambda-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{\frac{an}{b(n-1)}}} & 0 & 0 & \cdots & \lambda-1 \end{vmatrix} \\ &= \left(\frac{1}{\sqrt{\frac{an}{b(n-1)}}}\right)^n \begin{vmatrix} \gamma & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}, \end{aligned}$$

where $\mu=(\lambda-1)\sqrt{\frac{an}{b(n-1)}}$, $\gamma=(\lambda-(n-1))\sqrt{\frac{an}{b(n-1)}}$. Then,

$$|\lambda I - A_{lsqrs}| = \phi_n(\mu)\left(\sqrt{\frac{b(n-1)}{a(n)}}\right)^n$$

where

$$\phi_n(\mu)=\begin{vmatrix} \gamma & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}.$$

Using the properties of the determinants, we obtain after some simplifications

$$\phi_n(\mu) = (\mu\phi_{n-1}(\mu) + \mu^{n-2}).$$

Iterating this, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu\gamma + (n-1)).$$

Therefore

$$|\lambda I - A_{lsqrs}| = \left(\sqrt{\frac{b(n-1)}{an}}\right)^n \left[\left(\left(\frac{an}{b(n-1)}\right)(\lambda-1)(\lambda-(n-1)) + (n-1)\right) \times \left((\lambda-1)\sqrt{\frac{b(n-1)}{an}}\right)^{n-2}\right].$$

Thus the characteristic equation is given by

$$(\lambda-1)^{n-2}\left((\lambda-1)(\lambda-(n-1)) + \frac{b(n-1)^2}{an}\right) = 0.$$

Hence

$$Spec(D) = \begin{pmatrix} 1 & n + i\sqrt{\frac{n^3a-4(n-1)(an-b(n-1))}{2na}} & n - i\sqrt{\frac{n^3a-4(n-1)(an-b(n-1))}{2na}} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Hence the Laplacian of the quotient of Randić and sum-connectivity energy of S_n is

$$E_{lsqrs}(S_n) = \frac{(n-2)^2}{n} + \left|\frac{n^2-2(n-1)}{n} + i\sqrt{\frac{n^3a-4(n-1)(an-b(n-1)}{2na}}\right| + \left|\frac{n^2-2(n-1)}{n} - i\sqrt{\frac{n^3a-4(n-1)(an-b(n-1)}{2na}}\right|.$$

This completes the proof. □

Theorem 2.7 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of S_n^0 crown digraph be respectively given by $V(D) = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$, $\Gamma(D) = \{(u_i, v_j) \mid 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$. Then the Laplacian of skew-quotient of Randić and sum-connectivity energy of the crown digraph is*

$$4(n-1)\sqrt{\frac{(n-1)b}{a2}}.$$

Proof The Laplacian of skew-quotient of Randić and sum-connectivity matrix of crown digraph is given by

$$A_{lsqrs}=\begin{pmatrix} n-1 & 0 & \cdots & 0 & 0 & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & \cdots & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} \\ 0 & n-1 & \cdots & 0 & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & 0 & \cdots & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n-1 & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & \cdots & 0 \\ 0 & -\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & \cdots & -\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & n-1 & 0 & \cdots & 0 \\ -\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & 0 & \cdots & -\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & 0 & n-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & -\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}} & \cdots & 0 & 0 & 0 & \cdots & n-1 \end{pmatrix}.$$

Its characteristic polynomial is

$$|\lambda I - A_{lsqrs}| = \begin{vmatrix} (\lambda-(n-1))I_n & -\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}}K^T \\ \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}}K & (\lambda-(n-1)I_n \end{vmatrix},$$

where K is an $n\times n$ matrix. Hence the characteristic equation is given by

$$\begin{vmatrix} \Lambda I_n & -\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}}K^T \\ \sqrt{\frac{b((n-1)^2)}{a(2(n-1))}}K & \Lambda I_n. \end{vmatrix} = 0,$$

where $\Lambda = (\lambda-(n-1)$, this is same as

$$|\Lambda I_n|\left|\Lambda I_n - \left(\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}}K\right)\frac{I_n}{\Lambda}\left(-\sqrt{\frac{b((n-1)^2)}{a(2(n-1))}}K^T\right)\right| = 0,$$

which can be written as

$$\left(\frac{b((n-1)^2)}{a(2(n-1))}\right)^n P_{KK^T}\left(\left(\frac{a(2(n-1))}{b((n-1)^2)}\right)\Lambda^2\right) = 0,$$

where $P_{KK^T(\Lambda)}$ is the characteristic polynomial of the matrix KK^T. Thus we have

$$\left(\frac{b((n-1)^2)}{a(2(n-1))}\right)^n [\frac{a(2(n-1))}{b((n-1)^2)})\Lambda^2 + (n-1)^2][\frac{a(2(n-1))}{b((n-1)^2)})\Lambda^2 + 1]^{n-1} = 0,$$

which is same as

$$\left(\Lambda^2 + \frac{b(n-1)^3}{2a}\right)\left(\Lambda^2 + \frac{b(n-1)}{a2}\right)^{n-1} = 0.$$

Therefore

$$Spec\,(S_n^0) = \begin{pmatrix} i\sqrt{\frac{b(n-1)^3}{2a}} + (n-1) & -i\sqrt{\frac{b(n-1)^3}{2a}} + (n-1) & i\sqrt{\frac{b(n-1)}{2a}} + (n-1) & -i\sqrt{\frac{b(n-1)}{2a}} + (n-1) \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

Hence the Laplacian of the quotient of Randić and sum-connectivity energy of crown graph is

$$E_{lqrs}(S_n^0) = 4(n-1)\sqrt{\frac{(n-1)b}{a2}},$$

as desired. □

Theorem 2.8 *Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $(S_m \wedge P_2)$ digraph be respectively given by*

$$\begin{aligned} V(D) &= \{v_1, v_2, \cdots, v_{2m+2}\}, \\ \Gamma(D) &= \{(v_1, v_j), (v_{m+2}, v_k) \,|\, 2 \le k \le m+1, m+3 \le j \le 2m+2\}. \end{aligned}$$

Then the laplacian of skew-quotient of Randić and sum-connectivity energy of D is

$$\begin{aligned} \frac{(2n-4)(2-n)}{n} &+ 2\left|\frac{n^2-2(n-1)}{n} + i\sqrt{\frac{n^3a-4(n-1)(an-b(n-1))}{2na}}\right| \\ &+ 2\left|\frac{n^2-2(n-1)}{n} - i\sqrt{\frac{n^3a-4(n-1)(an-b(n-1))}{2na}}\right|. \end{aligned}$$

Proof The Laplacian of skew-quotient of Randić and sum-connectivity matrix of $(S_m \wedge P_2)$ digraph is given by

$$A_{lsqrs} = \begin{pmatrix} n-1 & 0 & \cdots & 0 & 0 & \sqrt{\frac{b(n-1)}{an}} & \cdots & \sqrt{\frac{b(n-1)}{an}} \\ 0 & 1 & \cdots & 0 & -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ 0 & \sqrt{\frac{b(n-1)}{an}} & \cdots & \sqrt{\frac{b(n-1)}{an}} & n-1 & 0 & \cdots & 0 \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \end{pmatrix}_{2n\times 2n},$$

where $m+1 = n$. Its characteristic polynomial is given by

$$|\lambda I - A_{lsqrs}| = \begin{vmatrix} \lambda-(n-1) & 0 & \cdots & 0 & 0 & -\sqrt{\frac{b(n-1)}{an}} & \cdots & -\sqrt{\frac{b(n-1)}{an}} \\ 0 & \lambda-1 & \cdots & 0 & \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda-1 & \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ 0 & -\sqrt{\frac{b(n-1)}{an}} & \cdots & -\sqrt{\frac{b(n-1)}{an}} & \lambda-(n-1) & 0 & \cdots & 0 \\ \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & \lambda-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda-1 \end{vmatrix}_{2n\times 2n}.$$

Hence the characteristic equation is given by

$$\left(\sqrt{\frac{b(n-1)}{an}}\right)^{2n}\begin{vmatrix} \gamma & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \gamma & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} = 0,$$

where

$$\Lambda = \sqrt{\frac{na}{(n-1)b}}(\lambda-1) \quad \text{and} \quad \gamma = \sqrt{\frac{na}{(n-1)b}}(\lambda-(n-1)).$$

Let

$$\begin{aligned} \phi_{2n}(\Lambda) &= \begin{vmatrix} \gamma & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \gamma & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n}. \\ &= (-1)^{2n+2n}\Lambda \begin{vmatrix} \gamma & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \gamma & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{(2n-1)\times(2n-1)} \end{aligned}$$

$$+(-1)^{2n+2}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \gamma & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+2}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \gamma & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

Using the properties of the determinants, we obtain

$$\Psi_{2n-1}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda)$$

after some simplifications, where

$$\Theta_n(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 1 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \gamma \end{vmatrix}_{n\times n}.$$

Then

$$\phi_{2n}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as above, we obtain

$$\begin{aligned}\phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+2}\Psi_{2n-2}(\Lambda) + (-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda)\\ &= \Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda).\end{aligned}$$

Proceeding like this, we obtain at the $(n-1)^{th}$ step

$$\phi_{2n}(\Lambda) = (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda),$$

where

$$\xi_{n+1}(\Lambda) = \begin{vmatrix} \gamma & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)}.$$

$$\begin{aligned}\phi_{2n}(\Lambda) &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\gamma\Theta_n(\Lambda)\\ &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\gamma\Theta_n(\Lambda)\\ &= (\Lambda^{n-1}\gamma + (n-1)\Lambda^{n-2})\Theta_n(\Lambda).\end{aligned}$$

Using the properties of the determinants, we obtain

$$\Theta_n(\Lambda) = \Lambda^{n-1}\gamma + (n-1)\Lambda^{n-2}.$$

Therefore

$$\phi_{2n}(\Lambda) = (\Lambda^{n-1}\gamma + (n-1)\Lambda^{n-2})^2.$$

Hence, the characteristic equation becomes

$$\left(\sqrt{\frac{b(n-1)}{an}}\right)^{2n}\phi_{2n}(\Lambda) = 0,$$

which is same as

$$\left(\sqrt{\frac{b(n-1)}{an}}\right)^{2n}(\Lambda^{n-1}\gamma + (n-1)\Lambda^{n-2})^2 = 0$$

and can be reduced to

$$\lambda^{2n-4}((\frac{na}{b(n-1)}(\lambda-1)(\lambda-(n-1)) + (n-1))^2 = 0.$$

Therefore

$$Spec\left((S_m \wedge P_2)\right) = \begin{pmatrix} 1 & n + i\sqrt{\frac{n^3a - i4(n-1)(an-b(n-1))}{2na}} & n - i\sqrt{\frac{n^3a - i4(n-1)(an-b(n-1))}{2na}} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence the Laplacian of skew Quotient of Randić and sum-connectivity energy of $(S_m \wedge P_2)$ graph is

$$\begin{aligned} E_{lsqrs}((S_m \wedge P_2)) &= \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| \\ &= \frac{(2n-4)(2-n)}{n} \\ &\quad +2\left| \frac{n^2 - 2(n-1)}{n} + i\sqrt{\frac{n^3a - 4(n-1)(an-b(n-1))}{2na}} \right| \\ &\quad +2\left| \frac{n^2 - 2(n-1)}{n} - i\sqrt{\frac{n^3a - 4(n-1)(an-b(n-1)))}{2na}} \right|. \end{aligned}$$

This completes the proof. □

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On Line-Block Signed Graphs

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Abstract: In this paper we introduced the new notion called line-block signed graph of a signed graph and its properties are studied. Also, we obtained the structural characterization of this new notion and presented some switching equivalent characterizations.

Key Words: Signed graphs, neutrosophic signed graph, balance, switching, line-block signed graph.

AMS(2010): 05C22.

§1. Introduction

For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [1]. The non-standard will be given in this paper as and when required.

Given a graph $G = (V, E)$, the line-block graph of $G = (V, E)$, denoted $\mathcal{LBG}(G)$, is defined to be that graph with $V(\mathcal{LBG}(G)) = E(G) \cup B$, where B is set of blocks of G and any two vertices in $V(\mathcal{LBG}(G))$ are joined by an edge if, and only if, the corresponding blocks are adjacent or one corresponds to a block of G and other to a line incident with it (see [4]).

To model individuals' preferences towards each other in a group, Harary [2] introduced the concept of signed graphs in 1953. A signed graph $S = (G, \sigma)$ is a graph $G = (V, E)$ whose edges are labeled with positive and negative signs (i.e., $\sigma : E(G) \to \{+, -\}$). The vertices of a graph represent people and an edge connecting two nodes signifies a relationship between individuals. The signed graph captures the attitudes between people, where a positive (negative edge) represents liking (disliking). A neutrosophic signed graph $S^N = (G, \sigma, H)$ for a subgraph $H \subset G$ with property $\mathscr{P}$ is such a graph that $G \setminus H$ is a signed graph but H is indefinite for those of uncertainties in reality. Certainly, if there are no indefinite subgraph in G, it must be a signed graph. An unsigned graph is a signed graph with the signs removed. Similar to an unsigned graph, there are many active areas of research for signed graphs.

The sign of a cycle (this is the edge set of a simple cycle) is defined to be the product of the signs of its edges; in other words, a cycle is positive if it contains an even number of negative edges and negative if it contains an odd number of negative edges. A signed graph S is said to be balanced if every cycle in it is positive. A signed graph S is called totally unbalanced if

[1]Received December 10, 2022, Accepted March 22, 2023.

every cycle in S is negative. A chord is an edge joining two non adjacent vertices in a cycle.

A *marking* of S is a function $\zeta : V(G) \to \{+,-\}$. Given a signed graph S one can easily define a marking ζ of S as follows: For any vertex $v \in V(S)$,

$$\zeta(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking ζ of S is called *canonical marking* of S. For more new notions on signed graphs refer the papers (see [5, 9-13, 13-22]).

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1 *A signed graph S is balanced if and only if either of the following equivalent conditions is satisfied:*

(i) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary [2]).*

(ii) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$ (Sampathkumar [6]).*

Switching S with respect to a marking ζ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs.

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be *weakly isomorphic* (see []23) or *cycle isomorphic* (see [24]) if there exists an isomorphism $\phi : G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (see [24]).

Theorem 1.2(T. Zaslavsky [24]) *Given a graph G, any two signed graphs in $\psi(G)$, where $\psi(G)$ denotes the set of all the signed graphs possible for a graph G, are switching equivalent if and only if they are cycle isomorphic.*

§2. Line-Block Signed Graph of a Signed Graph

Motivated by the existing definition of complement of a signed graph, we now extend the notion of line-block graphs to signed graphs as follows: The *line-block signed graph* $\mathcal{LBS}(S) = (\mathcal{LBG}(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{LBG}(G)$ and sign of any edge uv is $\mathcal{LBS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called a line-block signed graph, if $S \cong \mathcal{LBS}(S')$ for some signed graph S'. The following result restricts the class of line-block signed graphs.

Theorem 2.1 *For any signed graph $S = (G, \sigma)$, its line-block signed graph $\mathcal{LBS}(S)$ is balanced.*

Proof Since sign of any edge $e = uv$ in $\mathcal{LBS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S, by Theorem 1.1, $\mathcal{LBS}(S)$ is balanced. $\square$

For any positive integer k, the k^{th} iterated line-block signed graph, $\mathcal{LBS}^k(S)$ of S is defined as follows:

$$\mathcal{LBS}^0(S) = S,\ \mathcal{LBS}^k(S) = \mathcal{LBS}(\mathcal{LBS}^{k-1}(S)).$$

Corollary 2.2 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $\mathcal{LBS}^k(S)$ is balanced.*

In [4], the authors remarked that $\mathcal{LBG}(G)$ and G are isomorphic if and only if G is K_2. We now characterize the signed graphs and its line block signed graphs are cycle isomorphic.

Theorem 2.3 *For any signed graph $S = (G, \sigma)$, the line-block signed graph $\mathcal{LBS}(S)$ and S are cycle isomorphic if and only if the underlying of S is is isomorphic to K_2 and S is balanced.*

Proof Suppose $\mathcal{LBS}(S) \sim S$. This implies, $\mathcal{LBG}(G) \cong G$ and hence G is isomorphic to K_2. Then $\mathcal{LBS}(S)$ is balanced and hence if S is unbalanced and its line-block signed graph $\mathcal{LBS}(S)$ being balanced can not be switching equivalent to S in accordance with Theorem 1.2. Therefore, S must be balanced.

Conversely, suppose that S balanced signed graph with the underlying graph G is isomorphic to K_2. Then, since $\mathcal{LBS}(S)$ is balanced as per Theorem 2.1 and since $\mathcal{LBG}(G) \cong G$, the result follows from Theorem 1.2 again. $\square$

Corollary 2.4 *Let $S = (G, \sigma)$ be a connected signed graph. Then the n^{th}-iterated line-block signed graph $\mathcal{LBS}^n(S)$, $n \geq 1$ and S are cycle isomorphic if and only if the underlying of S is isomorphic to K_2 and S is balanced.*

Corollary 2.5 *Let $S = (G, \sigma)$ be any signed graph with no isolated vertices, the n^{th}-iterated line-block signed graph $\mathcal{LBS}^n(S)$, $n \geq 1$ and S are cycle isomorphic if and only if the underlying of S is isomorphic to mK_2, $m \geq 1$ and S is balanced.*

The following result characterize signed graphs which are line-block signed graphs.

Theorem 2.6 *A signed graph $S = (G, \sigma)$ is a line-block signed graph if, and only if, S is balanced signed graph and its underlying graph G is a line-block graph.*

Proof Suppose that S is balanced and G is a line-block graph. Then there exists a graph G' such that $\mathcal{LBG}(G') \cong G$. Since S is balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge uv in S satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $\mathcal{LBS}(S') \cong S$. Hence S is a line-block signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a line-block signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $\mathcal{LBS}(S') \cong S$. Hence, G is the line-block graph of G' and by Theorem 2.1, S is balanced. $\square$

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [3] as follows: $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the $\mathcal{LBS}(S)$ is balanced (Theorem 2.1). We now examine, the conditions under which negation $\eta(S)$ of $\mathcal{LBS}(S)$ is balanced.

Proposition 2.7 *Let $S = (G, \sigma)$ be a signed graph. If $\mathcal{LBG}(G)$ is bipartite then $\eta(\mathcal{LBS}(S))$ is balanced.*

Proof Since, by Theorem 2.1, $\mathcal{LBS}(S)$ is balanced, it follows that each cycle C in $\mathcal{LBS}(S)$ contains even number of negative edges. Also, since $\mathcal{LBG}(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{LBS}(S)$ is also even. Hence $\eta(\mathcal{LBS}(S))$ is balanced. $\square$

Acknowledgements

The authors would like to thank the referees for their invaluable comments and suggestions which led to the improvement of the manuscript.

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Pair Difference Cordial Labeling of Subdivision of Wheel and Comb Graphs

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Abstract: In this paper, we discuss about the pair difference cordial labeling behavior of subdivision of wheel and comb graphs.

Key Words: Pair difference cordial labeling, pair difference cordial graph, Smarandachely pair difference cordial labeling, Smarandachely pair difference cordial graph, cycle, path, wheel, subdivision.

AMS(2010): 05C78.

§1. Introduction

We consider only finite, undirected and simple graphs. The notion of pair difference cordial labeling of graphs was introduced in [4]. Pair difference cordial labeling behaviour of several graphs like path, cycle, star, Mirror graph, Shadow graph,double fan, mangolian tent, grid etc have been investigated in [4-10]. In this we investigate the pair difference cordial labeling behaviour of subdivision of wheel and comb graphs. Terms not defined here follow from Harary [2,3].

§2. Preliminaries

Definition 2.1([7]) *A subdivision graph $S(G)$ of a graph G is obtained by replacing each edge uv by a path uvw.*

Definition 2.2([4]) *Let $G = (V, E)$ be a (p, q) graph. Define*

$$\rho = \begin{cases} \frac{p}{2}, & \textit{if } p \textit{ is even} \\ \frac{p-1}{2}, & \textit{if } p \textit{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm\rho\}$ called the set of labels. Consider a mapping $f : V \longrightarrow L$

[1]Received December 30, 2022, Accepted March 22, 2023.

by assigning different labels in L to the different elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd.The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ *such that* $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$. *Otherwise, it is called a Smarandachely pair difference cordial labeling if* $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \geq 2$, *where* Δ_{f_1} *and* $\Delta_{f_1^c}$ *respectively denote the number of edges labeled with* 1 *and number of edges not labeled with* 1.

A graph G for which there exists a pair difference cordial labeling or Smarandachely pair difference cordial labeling is called a pair difference cordial graph or Smarandachely pair difference cordial graph.

Theorem 2.3([7]) *A wheel* W_n *is pair difference cordial if and only if* n *is even.*

§3. Main Results

Theorem 3.1 *A subdivision of the wheel* $W_n, S(W_n)$ *is pair difference cordial for all values of* $n \geq 3$.

Proof Let us take the vertex set and edge set of $S(W_n)$ as follows: $V(S(W_n)) = \{a, a_i, b_i, u_i : 1 \leq i \leq n\}$ and $E(S(W_n)) = \{aa_i, a_ib_i, b_iu_i : 1 \leq i \leq n\} \cup \{u_ib_{i+1} : 1 \leq i \leq n-1\} \cup \{b_nu_1\}$. This graph has $3n + 1$ vertices and $4n$ edges.

Case 1. n is even.

Assign the labels $1, 4, 7, \cdots, \frac{3n-4}{2}$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$ and assign the labels $2, 5, 8, \cdots, \frac{3n-2}{2}$ to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n}{2}}$ respectively. Next assign the labels $-1, -4, -7, \cdots, -(\frac{3n-4}{2})$ to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_n$ respectively and assign the labels $-2, -5, -8, \cdots, -(\frac{3n-2}{2})$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, b_{\frac{n+6}{2}}, \cdots, b_n$. Now we assign the labels $3, 6, 9, \cdots, \frac{3n}{2}$ respectively to the vertices $u_1, u_2, u_3, \cdots, u_{\frac{n}{2}}$ and assign the labels $-3, -6, -9, \cdots, -(\frac{3n}{2})$ to the vertices $u_{\frac{n+2}{2}}, u_{\frac{n+4}{2}}, u_{\frac{n+6}{2}}, \cdots, u_n$ respectively. Finally assign the label 1 to the vertex a. Clearly in this case $\Delta_{f_1^c} = \Delta_{f_1} = 2n$.

Case 2. n is odd.

Assign the labels $1, 4, 7, \cdots, \frac{3n-7}{2}$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n-1}{2}}$ and assign the labels $2, 5, 8, \cdots, \frac{3n-5}{2}$ to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n-1}{2}}$ respectively. Next assign the labels $-1, -4, -7, \cdots, -(\frac{3n-7}{2})$ to the vertices $a_{\frac{n+1}{2}}, a_{\frac{n+3}{2}}, a_{\frac{n+5}{2}}, \cdots, a_{n-1}$ respectively and assign the labels $-2, -5, -8, \cdots, -(\frac{3n-5}{2})$ to the vertices $b_{\frac{n+1}{2}}, b_{\frac{n+3}{2}}, b_{\frac{n+5}{2}}, \cdots, b_{n-1}$. Now we assign the labels $3, 6, 9, \cdots, \frac{3n-3}{2}$ respectively to the vertices $u_1, u_2, u_3, \cdots, u_{\frac{n-1}{2}}$ and assign the labels $-3, -6, -9, \cdots, -(\frac{3n-3}{2})$ to the vertices $u_{\frac{n+1}{2}}, u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \cdots, u_{n-1}$ respectively. Finally assign the label $\frac{3n-1}{2}, -(\frac{3n+1}{2}), -(\frac{3n-1}{2}), \frac{3n+1}{2}$ to the vertices a, a_n, b_n, u_n . Clearly in this case $\Delta_{f_1^c} = \Delta_{f_1} = 2n$. □

Theorem 3.2 *A subdivision of the spokes of wheel* W_n *is pair difference cordial for all values of* $n \geq 3$.

Proof Let G_s be the subdivision of the spokes of the wheel W_n with the vertex set $V(G_s) =$

$\{a, a_i, b_i : 1 \leq i \leq n\}$ and edge set $E(G_s) = \{aa_i, a_ib_i : 1 \leq i \leq n\} \cup \{b_ib_{i+1} : 1 \leq i \leq n-1\} \cup \{b_na_1\}$. Here the graph G_s has $2n+1$ vertices and $3n$ edges.

Case 1. $n \equiv 0(mod4)$.

Assign the labels $2, 6, 10, \cdots, n-2$ respectively to the vertices $a_1, a_3, a_5, \cdots, a_{\frac{n-2}{2}}$ and assign the labels $5, 9, 13, \cdots, n-3$ to the vertices $a_2, a_4, a_6, \cdots, a_{\frac{n-4}{2}}$ respectively. Next assign the labels $-1, -5, -9, \cdots, -(n-3)$ to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+6}{2}}, a_{\frac{n+10}{2}}, \cdots, a_{n-1}$ respectively and assign the labels $-4, -8, -12, \cdots, -n$ to the vertices $a_{\frac{n+4}{2}}, a_{\frac{n+8}{2}}, a_{\frac{n+12}{2}}, \cdots, a_n$. Now we assign the labels $3, 7, 11, \cdots, n-1$ respectively to the vertices $b_1, b_3, b_5, \cdots, b_{\frac{n-2}{2}}$ and assign the labels $4, 8, 12, \cdots, n$ to the vertices $b_2, b_4, b_6, \cdots, b_{\frac{n-4}{2}}$ respectively. Next assign the labels $-2, -6, -10, \cdots, -(n-2)$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+6}{2}}, b_{\frac{n+10}{2}}, \cdots, b_{n-1}$ respectively and assign the labels $-3, -7, -11, \cdots, -(n-1)$ to the vertices $b_{\frac{n+4}{2}}, b_{\frac{n+8}{2}}, b_{\frac{n+12}{2}}, \cdots, b_n$. Finally assign the labels $1, n-1, n$ to the vertices $a, a_{\frac{n}{2}}, b_{\frac{n}{2}}$.

Case 2. $n \equiv 1(mod4)$.

Assign the labels $2, 6, 10, \cdots, n-3$ respectively to the vertices $a_1, a_3, a_5, \cdots, a_{\frac{n-3}{2}}$ and assign the labels $5, 9, 13, \cdots, n$ to the vertices $a_2, a_4, a_6, \cdots, a_{\frac{n-1}{2}}$ respectively. Next assign the labels $-1, -5, -9, \cdots, -(n-4)$ to the vertices $a_{\frac{n+1}{2}}, a_{\frac{n+5}{2}}, a_{\frac{n+9}{2}}, \cdots, a_{n-2}$ respectively and assign the labels $-4, -8, -12, \cdots, -(n-1)$ to the vertices $a_{\frac{n+3}{2}}, a_{\frac{n+7}{2}}, a_{\frac{n+11}{2}}, \cdots, a_{n-1}$. Now we assign the labels $3, 7, 11, \cdots, n-2$ respectively to the vertices $b_1, b_3, b_5, \cdots, b_{\frac{n-3}{2}}$ and assign the labels $4, 8, 12, \cdots, n-1$ to the vertices $b_2, b_4, b_6, \cdots, b_{\frac{n-1}{2}}$ respectively. Next assign the labels $-2, -6, -10, \cdots, -(n-3)$ to the vertices $b_{\frac{n+1}{2}}, b_{\frac{n+5}{2}}, b_{\frac{n+9}{2}}, \cdots, b_{n-2}$ respectively and assign the labels $-3, -7, -11, \cdots, -(n-2)$ to the vertices $b_{\frac{n+3}{2}}, b_{\frac{n+7}{2}}, b_{\frac{n+11}{2}}, \cdots, b_{n-1}$. Finally assign the labels $1, -(n-1), -n$ to the vertices a, a_n, b_n.

Case 3. $n \equiv 2(mod4)$.

Assign the labels $2, 6, 10, \cdots, n-4$ respectively to the vertices $a_1, a_3, a_5, \cdots, a_{\frac{n-4}{2}}$ and assign the labels $5, 9, 13, \cdots, n-1$ to the vertices $a_2, a_4, a_6, \cdots, a_{\frac{n-2}{2}}$ respectively. Next assign the labels $-1, -5, -9, \cdots, -(n-1)$ to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+6}{2}}, a_{\frac{n+10}{2}}, \cdots, a_n$ respectively and assign the labels $-4, -8, -12, \cdots, -(n-2)$ to the vertices $a_{\frac{n+4}{2}}, a_{\frac{n+8}{2}}, a_{\frac{n+12}{2}}, \cdots, a_{n-1}$. Now we assign the labels $3, 7, 11, \cdots, n-3$ respectively to the vertices $b_1, b_3, b_5, \cdots, b_{\frac{n-4}{2}}$ and assign the labels $4, 8, 12, \cdots, n-2$ to the vertices $b_2, b_4, b_6, \cdots, b_{\frac{n-2}{2}}$ respectively. Next assign the labels $-2, -6, -10, \cdots, -n$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+6}{2}}, b_{\frac{n+10}{2}}, \cdots, b_n$ respectively and assign the labels $-3, -7, -11, \cdots, -(n-3)$ to the vertices $b_{\frac{n+4}{2}}, b_{\frac{n+8}{2}}, b_{\frac{n+12}{2}}, \cdots, b_{n-1}$. Finally assign the labels $1, n-1, n$ to the vertices $a, a_{\frac{n}{2}}, b_{\frac{n}{2}}$.

Case 4. $n \equiv 3(mod4)$.

Assign the labels $2, 6, 10, \cdots, n-5$ respectively to the vertices $a_1, a_3, a_5, \cdots, a_{\frac{n-5}{2}}$ and assign the labels $5, 9, 13, \cdots, n-2$ to the vertices $a_2, a_4, a_6, \cdots, a_{\frac{n-3}{2}}$ respectively. Next assign the labels $-1, -5, -9, \cdots, -(n-2)$ to the vertices $a_{\frac{n+1}{2}}, a_{\frac{n+5}{2}}, a_{\frac{n+9}{2}}, \cdots, a_{n-1}$ respectively and assign the labels $-4, -8, -12, \cdots, -(n-3)$ to the vertices $a_{\frac{n+3}{2}}, a_{\frac{n+7}{2}}, a_{\frac{n+11}{2}}, \cdots, a_{n-2}$. Now we assign the labels $3, 7, 11, \cdots, n-4$ respectively to the vertices $b_1, b_3, b_5, \cdots, b_{\frac{n-5}{2}}$ and assign the labels $4, 8, 12, \cdots, n-3$ to the vertices $b_2, b_4, b_6, \cdots, b_{\frac{n-3}{2}}$ respectively. Next assign the labels

$-2, -6, -10, \cdots, -(n-1)$ to the vertices $b_{\frac{n+1}{2}}, b_{\frac{n+5}{2}}, b_{\frac{n+9}{2}}, \cdots, b_{n-1}$ respectively and assign the labels $-3, -7, -11, \cdots, -(n-4)$ to the vertices $b_{\frac{n+3}{2}}, b_{\frac{n+7}{2}}, b_{\frac{n+11}{2}}, \cdots, b_{n-2}$. Finally assign the labels $1, -n, , -(n-1)$ respectively to the vertices a, a_n, b_n and assign the labels $n, n-1$ to the vertices $a_{\frac{n-1}{2}}, b_{\frac{n-1}{2}}$ respectively.

Table 1 given below establishes that this vertex labeling gives subdivision of spoke of the wheel is pair difference cordial.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
$n \equiv 0 \pmod 4$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$
$n \equiv 2 \pmod 4$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$

Table 1

This completes the proof. □

Theorem 3.3 *A subdivision of the rim edges of the wheel W_n is pair difference cordial for all values of $n \geq 3$.*

Proof Let G_r be the subdivision of rim edges of the wheel graph with the vertex set $V(G_r) = \{a, a_i, b_i : 1 \leq i \leq n\}$ and edge set

$$E(G_r) = \{aa_i : 1 \leq i \leq n\} \cup \{a_ib_i, a_{i+1}b_i : 1 \leq i \leq n-1\} \cup \{b_na_1\}.$$

Certainly, the graph G_r has $2n+1$ vertices and $3n$ edges.

Case 1. $3 \leq n \leq 11$.

Tables 2 and 3 shows that subdivision of rim edges of the wheel is pair difference cordial for all values of $3 \leq n \leq 11$. Assign the label 1 to the vertex a.

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a
3	2	4	-2									1
4	2	4	-1	-3								1
5	2	4	-1	-3	-5							1
6	2	4	6	-2	-4	-5						1
7	2	4	6	-1	-3	-5	-6					1
8	2	4	6	8	-2	-4	-7	-8				1
9	2	4	6	8	-1	-3	-5	-8	-9			1
10	2	4	6	8	10	-2	-4	-6	-10	-7		1
11	2	4	6	8	10	-1	-3	-5	-7	-11	-8	1

Table 2

n	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
3	3	-1	-3								
4	3	4	-2	-4							
5	3	5	-2	-4	4						
6	3	5	-1	-3	-6						
7	3	5	7	-2	-4	-7	-8				
8	3	5	7	-1	-3	-5	-6	8			
9	3	5	7	9	-2	-4	-6	-7	9		
10	3	5	7	9	-1	-3	-5	-8	-9	10	
11	3	5	7	9	11	-2	-4	-6	-9	-10	11

Table 3

Case 2. $n \equiv 0(mod 4), n \geq 12$.

Assign the labels $2, 4, 6, \cdots, n$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$ and assign the labels $-2, -4, -6, \cdots, -(\frac{n+4}{2})$ to the vertices $a_{n+22}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_{\frac{3n+4}{4}}$ respectively and assign the labels $-(\frac{n+6}{2})$,$-(\frac{n+8}{2})$,$-(\frac{n+10}{2}), \cdots, -(\frac{3n+4}{4})$ respectively to the vertices $a_{\frac{3n+8}{4}}$, $a_{\frac{3n+12}{4}}$, $a_{\frac{3n+16}{4}}, \cdots, a_n$. Now we assign the labels $3, 5, 7, \cdots, n-1$ respectively to the vertices b_1, b_2, b_3, $\cdots, b_{\frac{n-2}{2}}$. Next assign the labels $-1, -3, -5, \cdots, -(\frac{n+2}{2})$ to the vertices $b_{n2}, b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, \cdots, b_{\frac{3n}{4}}$ respectively and assign the labels $-(\frac{3n+8}{4})$,$-(\frac{3n+12}{4})$,$-(\frac{3n+16}{4}), \cdots, -n$ respectively to the vertices $b_{\frac{3n+4}{4}}$, $b_{\frac{3n+8}{4}}$, $b_{\frac{3n+12}{4}}, \cdots, b_{n-1}$. Next assign the labels $1, n$ respectively to the vertices a, b_n.

Case 3. $n \equiv 1(mod 4), n \geq 13$.

Assign the labels $2, 4, 6, \cdots, n-1$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n-1}{2}}$ and assign the labels $-1, -3, -5, \cdots, -(\frac{n+5}{2})$ to the vertices $a_{n+12}, a_{\frac{n+3}{2}}, a_{\frac{n+5}{2}}, \cdots, a_{\frac{3n+5}{4}}$ respectively and assign the labels $-(\frac{n+7}{2})$,$-(\frac{n+9}{2})$,$-(\frac{n+11}{2}), \cdots, -(\frac{3n+5}{4})$ respectively to the vertices $a_{\frac{3n+9}{4}}$, $a_{\frac{3n+13}{4}}$, $a_{\frac{3n+17}{4}}, \cdots, a_n$. Now we assign the labels $3, 5, 7, \cdots, n$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n-1}{2}}$. Next assign the labels $-2, -4, -6, \cdots, -(\frac{n+3}{2})$ to the vertices $b_{n+12}, b_{\frac{n+3}{2}}, b_{\frac{n+5}{2}}, \cdots, b_{\frac{3n+1}{4}}$ respectively and assign the labels $-(\frac{3n+9}{4})$,$-(\frac{3n+13}{4})$,$-(\frac{3n+17}{4}), \cdots$, $-n$ respectively to the vertices $b_{\frac{3n+5}{4}}$, $b_{\frac{3n+9}{4}}$, $b_{\frac{3n+13}{4}}, \cdots, b_{n-1}$. Next assign the labels $1, n$ respectively to the vertices a, b_n.

Case 4. $n \equiv 2(mod 4), n \geq 14$.

Assign the labels $2, 4, 6, \cdots, n$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$ and assign the labels $-2, -4, -6, \cdots, -(\frac{n+2}{2})$ to the vertices $a_{n+22}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_{\frac{3n+2}{4}}$ respectively and assign the labels $-(\frac{n+4}{2})$,$-(\frac{n+6}{2})$,$-(\frac{n+8}{2}), \cdots, -(\frac{3n+2}{4})$ respectively to the vertices $a_{\frac{3n+6}{4}}$, $a_{\frac{3n+10}{4}}$, $a_{\frac{3n+14}{4}}, \cdots, a_n$. Now we assign the labels $3, 5, 7, \cdots, n-1$ respectively to the vertices b_1, b_2, b_3, $\cdots, b_{\frac{n-2}{2}}$. Next assign the labels $-1, -3, -5, \cdots, -(\frac{n+2}{2})$ to the vertices $b_{n2}, b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, \cdots, b_{\frac{3n}{4}}$ respectively and assign the labels $-(\frac{3n+2}{4})$,$-(\frac{3n+6}{4})$,$-(\frac{3n+10}{4}), \cdots, -n$ respectively to the vertices $b_{\frac{3n+2}{4}}$, $b_{\frac{3n+6}{4}}$, $b_{\frac{3n+10}{4}}, \cdots, b_{n-1}$. Next assign the labels $1, 1$ respectively to the vertices a, b_n.

Case 5. $n \equiv 3(mod4), n \geq 15$.

Assign the labels $2, 4, 6, \cdots, n-1$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n-1}{2}}$ and assign the labels $-1, -3, -5, \cdots, -(\frac{n+3}{2})$ to the vertices $a_{n+12}, a_{\frac{n+3}{2}}, a_{\frac{n+5}{2}}, \cdots, a_{\frac{3n+3}{4}}$ respectively and assign the labels $-(\frac{n+5}{2})$,$-(\frac{n+7}{2})$,$-(\frac{n+9}{2})$, $\cdots$, $-(\frac{3n+3}{4})$ respectively to the vertices $a_{\frac{3n+7}{4}}$, $a_{\frac{3n+11}{4}}$, $a_{\frac{3n+15}{4}}, \cdots, a_n$. Now we assign the labels $3, 5, 7, \cdots, n$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n-1}{2}}$. Next assign the labels $-2, -4, -6, \cdots, -(\frac{n+1}{2})$ to the vertices $b_{n+12}, b_{\frac{n+3}{2}}, b_{\frac{n+5}{2}}, \cdots, b_{\frac{3n-1}{4}}$ respectively and assign the labels $-(\frac{3n+7}{4})$,$-(\frac{3n+11}{4})$,$-(\frac{3n+15}{4}), \cdots,$ $-n$ respectively to the vertices $b_{\frac{3n+3}{4}}$, $b_{\frac{3n+7}{4}}$, $b_{\frac{3n+11}{4}}, \cdots, b_{n-1}$. Next assign the labels $1, n$ respectively to the vertices a, b_n.

Table 4 given below establishes that this vertex labeling gives subdivision of rim edges of the wheel is pair difference cordial.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
$n \equiv 0 \pmod 4$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$
$n \equiv 2 \pmod 4$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$

Table 4

A pair difference cordial labeling on subdivision of rim edges of the wheel W_5 is shown in Figure 1.

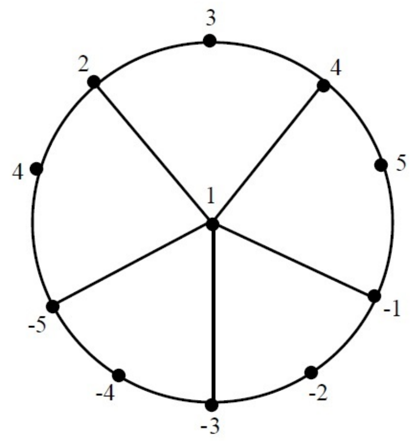

Figure 1

This completes the proof. □

Theorem 3.4 *A subdivision of comb $P_n \odot K_1$ is pair difference cordial for all values of $n \geq 2$.*

Proof Let the vertex set and edge set be $V(P_n \odot K_1) = \{a_i, b_i, c_i : 1 \leq i \leq n\} \cup \{d_i : 1 \leq i \leq n-1\}$ and $E(P_n \odot K_1) = \{a_ib_i, b_ic_i : 1 \leq i \leq n\} \cup \{a_id_i, d_ia_{i+1} : 1 \leq i \leq n-1\}$. There are $4n-1$ vertices and $4n-2$ edges.
There are four cases arises.

Case 1. n is even.

Assign the labels $3, 6, 9, \cdots, \frac{3n}{2}$ to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$ respectively and assign the labels $2, 5, 8, \cdots, \frac{3n-2}{2}$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n}{2}}$. Next assign the labels $1, 4, 7, \cdots, \frac{3n-4}{2}$ to the vertices $c_1, c_2, c_3, \cdots, c_{\frac{n}{2}}$ respectively and assign the labels $-3, -6, -9, \cdots, -(\frac{3n-6}{2})$ respectively to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_{n-1}$. Now we assign the labels $-2, -5, -8, \cdots, -(\frac{3n-8}{2})$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, b_{\frac{n+6}{2}}, \cdots, b_{n-1}$ and assign the labels $-1, -4, -7, \cdots, -(\frac{3n-4}{2})$ to the vertices $c_{\frac{n+2}{2}}, c_{\frac{n+4}{2}}, c_{\frac{n+6}{2}}, \cdots, c_n$. Next assign the labels $\frac{3n+2}{2}, \frac{3n+4}{2}, \frac{3n+6}{2}, \cdots, 2n-1$ respectively to the vertices $d_1, d_2, d_3, \cdots, d_{\frac{n-2}{2}}$ and assign the labels $-(\frac{3n+2}{2}), -(\frac{3n+4}{2}), -(\frac{3n+6}{2}), \cdots, -(2n-1)$ to the vertices $d_{\frac{n}{2}}, d_{\frac{n+2}{2}}, d_{\frac{n+4}{2}}, \cdots, d_{n-2}$ respectively. Finally assign the labels $-(\frac{3n}{2}), -(\frac{3n-2}{2}), 1$ respectively to the vertices b_n, a_n, u_{n-1}.

Case 2. $n = 3$.

Assign the labels $1, 2, 3, -1, -2, -3, 4, -4, -4$ respectively to the vertices $c_1, b_1, a_1, c_2, b_2, a_2, c_3, b_3, a_3$ and assign the labels $5, -5$ to the vertices d_1, d_2 respectively.

Case 3. $n = 5$.

Assign the labels $1, 2, 3, 4, 5, 6$ respectively to the vertices $c_1, b_1, a_1, c_2, b_2, a_2$ and assign the labels $-1, -2, -3, -4, -5, -6$ to the vertices $c_3, b_3, a_3, c_4, b_4, a_4$ respectively. Next assign the labels $7, -7, 8, -8$ respectively to the vertices d_1, d_2, d_3, d_4 and assign the labels $-9, -9, 9$ to the vertices a_5, b_5, c_5 respectively.

Case 4. n is odd $n \geq 7$.

Assign the labels $3, 6, 9, \cdots, \frac{3n-3}{2}$ to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n-1}{2}}$ respectively and assign the labels $2, 5, 8, \cdots, \frac{3n-9}{2}$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n-1}{2}}$. Next assign the labels $1, 4, 7, \cdots, \frac{3n-4}{2}$ to the vertices $c_1, c_2, c_3, \cdots, c_{\frac{n}{2}}$ respectively and assign the labels $-3, -6, -9, \cdots, -(\frac{3n-6}{2})$ respectively to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_{n-1}$. Now we assign the labels $-2, -5, -8, \cdots, -(\frac{3n-8}{2})$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, b_{\frac{n+6}{2}}, \cdots, b_{n-1}$ and assign the labels $-1, -4, -7, \cdots, -(\frac{3n-4}{2})$ to the vertices $c_{\frac{n+2}{2}}, c_{\frac{n+4}{2}}, c_{\frac{n+6}{2}}, \cdots, c_n$. Next assign the labels $\frac{3n+2}{2}, \frac{3n+4}{2}, \frac{3n+6}{2}, \cdots, 2n-1$ respectively to the vertices $d_1, d_2, d_3, \cdots, d_{\frac{n-2}{2}}$ and assign the labels $-(\frac{3n+2}{2}), -(\frac{3n+4}{2}), -(\frac{3n+6}{2}), \cdots, -(2n-1)$ to the vertices $d_{\frac{n}{2}}, d_{\frac{n+2}{2}}, d_{\frac{n+4}{2}}, \cdots, d_{n-2}$ respectively. Finally assign the labels $-(\frac{3n}{2}), -(\frac{3n-2}{2}), 1$ respectively to the vertices b_n, a_n, u_{n-1}.

In all the cases, we have $\Delta_{f_1} = \Delta_{f_1^c} = 2n - 1$. □

Theorem 3.5 *A subdivision of the edges of the path P_n in the comb $P_n \odot K_1$ is pair difference cordial for all values of $n \geq 2$.*

Proof Let G be subdivision of the edges of the path P_n in the comb graph $P_n \odot K_1$. Let the vertex set and edge set be $V(G) = \{a_i, b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq n-1\}$ and $E(G) = \{a_ib_i : 1 \leq i \leq n\} \cup \{a_ic_i, c_ia_{i+1} : 1 \leq i \leq n-1\}$, which has $3n - 1$ vertices and $3n - 2$ edges. There are two cases arises.

Case 1. n is even.

Subcase 1.1 $n = 2$.

Assign the labels $1, 2, 1, -2, -1$ respectively to the vertices b_1, a_1, c_1, a_2, b_2 respectively.

Subcase 1.2 $n \geq 4$.

Assign the labels $3, 6, 9, \cdots, \frac{3n-6}{2}$ to the vertices $b_2, b_3, b_4, \cdots, b_{\frac{n}{2}}$ respectively and assign the labels $1, 4, 7, \cdots, \frac{3n-4}{2}$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$. Next assign the labels $5, 8, 11, \cdots, \frac{3n-2}{2}$ to the vertices $c_1, c_2, c_3, \cdots, c_{\frac{n-2}{2}}$ respectively and assign the labels $-1, -3, -5, \cdots, -n+1$ respectively to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_n$. Now we assign the labels $-2, -4, -6, \cdots, -n$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, b_{\frac{n+6}{2}}, \cdots, b_n$ and assign the labels $-(n+1), -(n+2), -(n+3) \cdots, -(\frac{3n-2}{2})$ to the vertices $c_{\frac{n}{2}}, c_{\frac{n+2}{2}}, c_{\frac{n+4}{2}}, \cdots, c_{n-2}$. Finally assign the labels $2, -(\frac{3n-2}{2})$ respectively to the vertices b_1, c_{n-1}.

Case 2. n is odd.

Subcase 2.1 $n = 3$.

Assign the labels $1, 2, -1, -2, 3, -3, 4, -4$ respectively to the vertices $b_1, a_1, b_2, a_2, b_3, a_3, c_1, c_2$.

Subcase 2.2 $n \geq 5$.

Assign the labels $3, 6, 9, \cdots, \frac{3n-3}{2}$ to the vertices $c_1, c_2, c_3, \cdots, c_{\frac{n-3}{2}}$ respectively and assign the labels $1, 4, 7, \cdots, \frac{3n-7}{2}$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n-1}{2}}$. Next assign the labels $2, 5, 8, \cdots, \frac{3n-5}{2}$ to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n-1}{2}}$ respectively and assign the labels $-1, -3, -5, \cdots, -n$ respectively to the vertices $a_{\frac{n+1}{2}}, a_{\frac{n+3}{2}}, a_{\frac{n+5}{2}}, \cdots, a_n$. Now we assign the labels $-2, -4, -6, \cdots, -n-1$ to the vertices $b_{\frac{n+1}{2}}, b_{\frac{n+3}{2}}, b_{\frac{n+5}{2}}, \cdots, b_n$ and assign the labels $-(n+2), -(n+3), -(n+4), \cdots, -(\frac{3n-1}{2})$ to the vertices $c_{\frac{n+1}{2}}, c_{\frac{n+3}{2}}, c_{\frac{n+5}{2}}, \cdots, c_{n-1}$. Finally assign the labels $\frac{3n-11}{2}$ to the vertex $c_{\frac{n-1}{2}}$.

Table 5 given below establishes that this vertex labeling gives subdivision of the edges of the path P_n in the comb graph is pair difference cordial.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$\frac{3n-4}{2}$	$\frac{3n}{2}$
n is even	$\frac{3n-2}{2}$	$\frac{3n-2}{2}$

Table 5

This completes the proof. □

Theorem 3.6 *A subdivision of the pendant edges of the comb $P_n \odot K_1$ in the comb is pair difference cordial for all values of $n \geq 2$.*

Proof Let G be subdivision of the pendant edges of the path P_n in the comb $P_n \odot K_1$. Let the vertex set and edge set be $V(G) = \{a_i, b_i, c_i : 1 \leq i \leq n\}$ and $E(G) = \{a_ib_i, b_ic_i : 1 \leq i \leq n\}$, which has $3n-1$ vertices and $3n-2$ edges.

Case 1. n is even.

Subcase 1.1 $n = 2$.

Assign the labels $1, 2, 1, -2, -1$ respectively to the vertices b_1, a_1, c_1, a_2, b_2 respectively.

Subcase 1.2 $n \geq 4$.

Assign the labels $3, 6, 9, \cdots, \frac{3n-6}{2}$ to the vertices $b_2, b_3, b_4, \cdots, b_{\frac{n}{2}}$ respectively and assign the labels $1, 4, 7, \cdots, \frac{3n-4}{2}$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$. Next assign the labels $5, 8, 11, \cdots, \frac{3n-2}{2}$ to the vertices $c_1, c_2, c_3, \cdots, c_{\frac{n-2}{2}}$ respectively and assign the labels $-1, -3, -5, \cdots, -n+1$ respectively to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_n$. Now we assign the labels $-2, -4, -6, \cdots, -n$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, b_{\frac{n+6}{2}}, \cdots, b_n$ and assign the labels $-(n+1), -(n+2), -(n+3) \cdots, -(\frac{3n-2}{2})$ to the vertices $c_{\frac{n}{2}}, c_{\frac{n+2}{2}}, c_{\frac{n+4}{2}}, \cdots, c_{n-2}$. Finally assign the labels $2, -(\frac{3n-2}{2})$ respectively to the vertices b_1, c_{n-1}.

Case 2. n is odd.

Subcase 2.1 $n = 3$.

Assign the labels $1, 2, -1, -2, 3, -3, 4, -4$ respectively to the vertices $b_1, a_1, b_2, a_2, b_3, a_3, c_1, c_2$.

Subcase 2.2 $n \geq 5$.

Assign the labels $3, 6, 9, \cdots, \frac{3n-3}{2}$ to the vertices $c_1, c_2, c_3, \cdots, c_{\frac{n-3}{2}}$ respectively and assign the labels $1, 4, 7, \cdots, \frac{3n-7}{2}$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n-1}{2}}$. Next assign the labels $2, 5, 8, \cdots, \frac{3n-5}{2}$ to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n-1}{2}}$ respectively and assign the labels $-1, -3, -5, \cdots, -n$ respectively to the vertices $a_{\frac{n+1}{2}}, a_{\frac{n+3}{2}}, a_{\frac{n+5}{2}}, \cdots, a_n$. Now we assign the labels $-2, -4, -6, \cdots, -n-1$ to the vertices $b_{\frac{n+1}{2}}, b_{\frac{n+3}{2}}, b_{\frac{n+5}{2}}, \cdots, b_n$ and assign the labels $-(n+2), -(n+3), -(n+4), \cdots, -(\frac{3n-1}{2})$ to the vertices $c_{\frac{n+1}{2}}, c_{\frac{n+3}{2}}, c_{\frac{n+5}{2}}, \cdots, c_{n-1}$. Finally assign the labels $\frac{3n-11}{2}$ to the vertex $c_{\frac{n-1}{2}}$.

Table 6 given below establishes that this vertex labeling gives subdivision of the edges of the path P_n in the comb graph is pair difference cordial.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$\frac{3n-4}{2}$	$\frac{3n}{2}$
n is even	$\frac{3n-2}{2}$	$\frac{3n-2}{2}$

Table 6

A subdivision of the pendant edges of the comb $P_7 \odot K_1$ is pair difference cordial is shown in Figure 2.

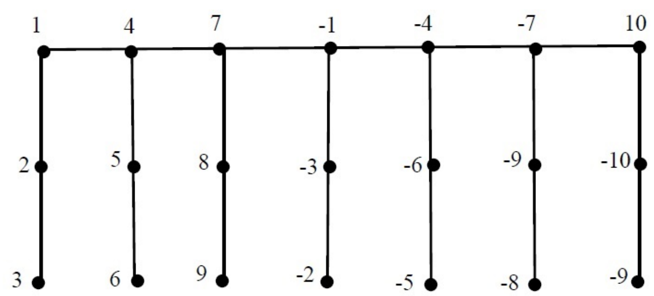

Figure 2

This completes the proof. $\square$

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Famous Words

The branch developing of science is easy to achieve local achievements on scientific subjects which created really the western industries but only now, it has left the nature with many worldwide problems in the passed more than 300 years, also affecting the human survival. Different from the developing of western science, Chinese science has its own distinctive characters. It emphasizes the unity of humans with the nature and holds the law of thing evolving with the whole life cycle of thing. Certainly, it is not easy to achieve local scientific achievements and can not bring the industrial revolution into being but it will not also bring the crisis to human existence. — Extracted from *Combinatorial Theory on the Universe*, a book of Dr.Linfan Mao on mathematics with philosophy of science, which systematically discusses the recognition of humans from the local to the whole, published by Global Knowledge-Publishing House in 2023.

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[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

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[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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March 2023

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An International Journal on Mathematical Combinatorics

ISSN 1937-1055