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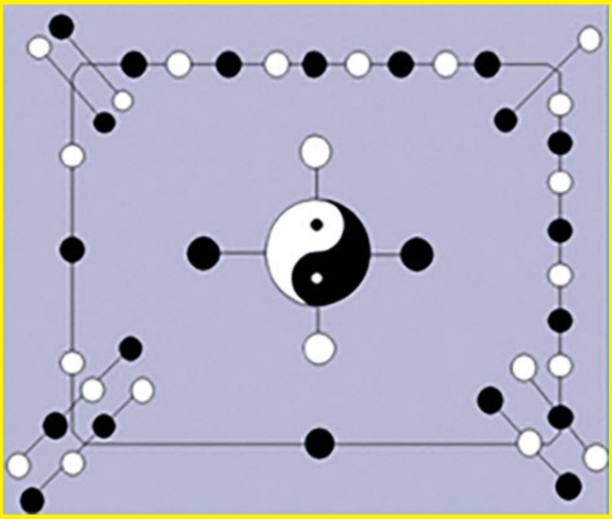

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Aims and Scope: The *mathematical combinatorics* is a subject that applying combinatorial notion to all mathematics and all sciences for understanding the reality of things in the universe, motivated by *CC Conjecture* of Dr.L.F. MAO on mathematical sciences. The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly, which publishes original research papers and survey articles in all aspects of mathematical combinatorics, Smarandache multi-spaces, Smarandache geometries, non-Euclidean geometry, topology and their applications to other sciences. Topics in detail to be covered are:

Mathematical combinatorics;

Smarandache multi-spaces and Smarandache geometries with applications to other sciences;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

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Famous Words:

I want to bring out the secrets of nature and apply them for the happiness of man. I don't know of any better service to offer for the short time we are in the world.

By *Thomas Edison*, an American inventor.

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Mathematical Combinatorics

– My Philosophy Promoted on Science Internationally

Linfan MAO

1. Chinese Academy of Mathematics and System Science, Beijing 100190, P.R. China
2. Academy of Mathematical Combinatorics & Applications (AMCA), Colorado, USA

E-mail: maolinfan@163.com

Abstract: Mathematical science is the human recognition on the evolution laws of things that we can understand with the principle of logical consistency by mathematics, i.e., mathematical reality. So, *is the mathematical reality equal to the reality of thing?* The answer is not because there always exists contradiction between things in the eyes of human, which is only a local or conditional conclusion. Such a situation enables us to extend the mathematics further by combinatorics for the reality of thing from the local reality and then, to get a combinatorial reality of thing. This is the *combinatorial conjecture for mathematical science*, i.e., CC conjecture that I put forward in my postdoctoral report for *Chinese Academy of Sciences* in 2005, namely *any mathematical science can be reconstructed from or made by combinatorialization*. After discovering its relation with Smarandache multi-spaces, it is then be applied to generalize mathematics over 1-dimensional topological graphs, namely the *mathematical combinatorics* that I promoted on science internationally for more than 20 years. This paper surveys how I proposed this conjecture from combinatorial topology, how to use it for characterizing the non-uniform groups or contradictory systems and furthermore, why I introduce the continuity flow G^L as a mathematical element, i.e., vectors in Banach space over topological graphs with operations and then, how to apply it to generalize a few of important conclusions in functional analysis for providing the human recognition on the reality of things, including the subdivision of substance into elementary particles or quarks in theoretical physics with a mathematical supporting.

Key Words: Science's limitation, CC conjecture, Smarandachely denied axiom, Smarandache mutispace, non-harmonious group, non-solvable equation system, continuity flow, combinatorial notion, neutrosophic set, recognitive philosophy.

AMS(2010): 05C10, 05C21, 34A12, 34D06, 35A08, 46B25, 51D20,51H20, 51P05.

§1. Introduction

Science is the recognition of human on the law of things in the universe under conditions by the "*six sense organs*" of human, i.e., the eyes, ears, nose, tongue, body and mind, including their extension using by the scientific instruments or facilities, and mathematics is the formal

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system of symbols in accordance with the principle of logical consistency in order to describe the evolution of thing by abstract symbols, i.e, the mathematical reality $T_{\mathcal{M}}$ holds on thing T, which is a recognitive process from non-being to being of human and greatly depends on the human recognition from the known to the unknown, including the characteristics, indicators and methods of recognition on nature of thing. For example, let the observable characteristics of thing T be $\chi_1, \chi_2, \cdots$ such as the spatial location, geometry, color, state, odor, rate and direction of change, melting point, boiling point, hardness, density, structure, acidity, alkalinity, oxidation, reducibility, thermal stability, metabolism, growth, reproduction and development, heredity and variation, etc. Then, T is recognized on its characteristics of $\{\chi_1, \chi_2, \cdots\}$ by humans. Notice that the recognition on characteristics of $\chi_1, \chi_2, \cdots$ of thing T is a gradual process in general. That is why it needs to constantly improve, modify or extend our theory on thing T so as to approach to the reality of thing T infinitely. For example, the six blind men touched the elephant's teeth, trunk, ears, stomach, legs and the tail in fable of the blind men with an elephant and then, an elephant was characterized respectively by the blind men as a big radish, pipe, a leaf fan, a wall, a pillar or a rope, such as those shown in Figure 1.

(a) (b)

Figure 1 An elephant's shape recognized by blind men

In this case, *why did the blind men argue for the shape of an elephant?* The answer is because each of them touched different parts of the elephant's body, which results in the recognitions on the elephant different. Similarly, the human recognition on a thing T by its characteristics of $\chi_1, \chi_2, \cdots$ is similar to that of the case of a blind man. It is also in local recognition on T one by one characterizes of $\chi_1, \chi_2, \cdots$. However, *can such a recognition really equal to the reality of thing T and realize $T_{\mathcal{M}} = T$?* The answer is not because it is the human in recognizing things T and it mainly depends on the sense and reason of human, which has been asserted $T_{\mathcal{M}} \neq T$ in the discussion of sages. For example, "*Tao told is not the eternal Tao; Name named is not the eternal Name*" in Chapter 1 of Lao Zi's *Tao Te Ching*, "*Color is not different from the Empty, Empty is not different from the Color and the Color is the Empty, the Empty is the Color*" in *Heart Sutra* and also Kant's *Critique of Pure Reason* or "*what can I know?* " and so on. All of their discussions show that the human recognition is relative or conditional reality $T_{\mathcal{M}}$, not equivalent to the reality of thing T but only a gradual process, i.e., $T_{\mathcal{M}} \to T$. Furthermore, *can the mathematical reality of things be realized $T_{\mathcal{M}} \to T$ by human under the principle of logical consistency?* The answer is also not because the mathematical system follows the logical consistency but a contradiction exists everywhere in human recognition. It is impossible to completely describe the evolution of thing T with a logical consistency system of symbols. In this case, it is necessary to recognize that such a contradiction is caused by human's describing on the evolution, not the truth colour of thing T with mathematics. Thus, there are 3 questions need further to discuss at least in recognition of thing T by characteristics of $\chi_1, \chi_2, \cdots$ and then, we realize the thing T, including the blind men with the radish, pipe, fan, wall, pillar, rope and others for characterizing the shape of an elephant, respectively.

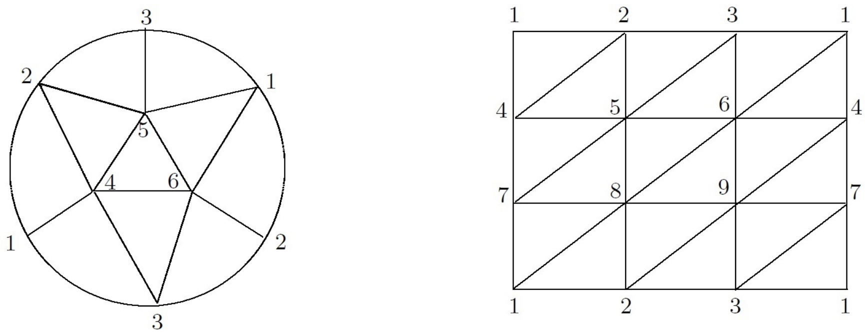
(Q1) *For an integer $i \geq 1$, is it complete for understanding thing T only by the characteristic χ_i?* The answer is certainly not because χ_i is only one characteristic of thing T, not the whole. In the fable of blind men with an elephant, although the sophist told the blind men that "*you are all right about the elephant*", he also said that "*the reason why you think the elephant's shape different is because each of you touches the different part of the elephant's body. In fact, an elephant has those all characteristics that you are talking about*", namely the sophist pointed out that each recognition of them is local also. Similarly, knowing thing T in terms of characteristic χ_i is necessarily incomplete but it is the normal case of human recognition. And so, all human activities led by the incomplete scientific recognitions are bound to be constrained by their application field, scope and achieving conditions.

(Q2) *How to recognize the characteristics of $\chi_1, \chi_2, \cdots$ of thing T?* In fact, there are many methods for the recognition on characteristics of $\chi_1, \chi_2, \cdots$ of thing T in science, including mathematical, physical, chemical and biological methods such as a radish can be eaten, a pillar can support others and a rope is soft but can tie others, etc. But as long as its characteristics are quantitatively described by data χ_i, $i \geq 1$, it must be assumed that the characteristic χ_i follows a mathematical system S, namely the characteristic χ_i is described in accordance with the principle of logical consistency in mathematics. Now, the question is *whether it is correct in assuming that the characteristic χ_i follows the rules of mathematical system S, and whether the change of characteristic χ_i of thing T can be fully described?*

(Q3) *Are any combination of characteristics of $\chi_1, \chi_2, \cdots$ necessarily the thing T?* For example, in fable of the blind men with an elephant, *is any combination of* 2 *big radishes,* 1 *pipe,* 2 *leaf fans,* 1 *wall,* 4 *pillars and* 1 *rope be the shape of an elephant recognized by the blind men?* The answer is not because these six known objects can be combined to create a variety of geometrical objects, they do not necessarily be the shape of an elephant. In other words, the shape of an elephant made from 2 big radishes, 1 pipe, 2 leaf fans, 1 wall, 4 pillars and 1 rope is combined on a 1-dimensional topology or topological graph G^L, and this 1-dimensional topology G^L is accompanied by human recognition of things T, which is inevitable.

Different understandings on the previous 3 questions will inevitably lead to different developing ways of science. Most researchers are at the first level, namely acknowledging tacitly that a local characteristic of thing T is equal to thing T and so, the thing T is subdivided into microscopic particles, including cells and genes in biology to reduce the *effect* of thing T to *cause* of the behavior of microscopic particles, which is to recognize the whole in a partial one. Unlike the ordinary scholars, Prof.Smarandache introduced the neutrosophic set for describing thing T with characteristics of $\chi_1, \chi_2, \cdots$, lead a lot of mathematicians researching it deeply and obtained many academic achievements. So, *what is a neutrosophic set?* A neutrosophic set is such a set that associates each element $x \in \chi_i$ of a recognitive set with a ternary array $(\mathcal{T}, \mathcal{I}, \mathcal{F})$, where $\mathcal{T}, \mathcal{I}, \mathcal{F} \subseteq [0,1]$, are respectively the confident set $\mathcal{T}$, indefinite set $\mathcal{I}$ and fail set $\mathcal{F}$, see [36] for details. However, I believe personally that the human recognition of thing T should follow the rule of extending the known to the unknown, from the local to the whole because humans are bound to be unable to give definite recognition for an uncertain or unrecognizable thing that appear in recognition. Thus, it is undoubtedly a useable or feasible way that extends mathematics over the topological structure G^L inherited in human recognition by reduction on thing T and then, apply it to the recognition of unknown things.

I was working on compact 2-dimensional manifolds without boundary, namely the partition of a closed surface into regular polygons and counted the non-isomorphic ways of partition during my doctoral and postdoctoral periods. For this work, there is a classical conclusion in algebraic topology, namely ([33]) *there exists a finite triangulation* $\{T_1, T_2, \cdots, T_n\}$ *on a closed surface* S, where for any integer $1 \leq i \leq n$, T_i is homeomorphic to a triangle $\triangle$, i.e., an open disk $\{(x,y)|x^2+y^2<1\}$ on Euclideian plane $\mathbb{R}^2$, called a 2-*cell*. That is, a closed surface can be obtained by adhering triangles. For example, the partitions (a) and (b) in Figure 2 are triangulation of the projective plane and the torus, respectively.



(a) Projective plane **(b) Torus**

Figure 2 Triangulation of surface

Notice that the 1-dimensional skeleton in a 2-cell partition of closed surface corresponds to a topological graph G^L. Conversely, a 2-cell embedded of graph G^L on surface S is nothing else but a 2-cell partition of surface S ([7]), called also a *combinatorial map* ([10],[11]), which is adhering a closed surface with regular polygons. Similarly, the assembling objects of space by tetrahedrons, hexahedrons, octahedrons, dodecahedrons and icosahedrons, and the algebraic systems such as those of rings, fields consisted of commutative groups is also such a combinatorial one. Working on combinatorial topology years motivated me realized suddenly that the essence of this way is a combinatorial notion which can be applied for generalizing mathematical science in general. After thinking for a long time, I proposed the notion that of applying combinatorics for generalizing mathematical sciences, namely the words of "*a good idea is pullback measures on combinatorial objects again, ignored by the classical combinatorics and reconstructed or make combinatorial generalization for the classical mathematics such as the algebra, differential geometry, Riemann geometry,* $\cdots$ *and the mechanics, theoretical physics,* $\cdots$" in Introduction of Chapter 5 of my post-doctoral report "*On the Automorphisms of Maps*

& Klein Surface" ([11] 1st edition) for *Chinese Academy of Sciences* in 2005. And then, I formally proposed the *combinatorial conjecture for mathematical science* in my report "*Combinatorial speculations and combinatorial conjecture for mathematics*" at the *2nd Conference on Combinatorics and Graph Theory of China* (August 16-19, 2006, Tianjin), namely

Combinatorial Conjecture for Mathematics([14]) *Any mathematical science can be reconstructed from or made by combinatorialization.*

The combinatorial conjecture for mathematics, abbreviated to CC conjecture is not so much as a mathematical conjecture but a generalization of mathematical science for extending the local recognition of human on thing by a combinatorial approach, which implies that one can select a limited number of combinatorial rules and axioms to reconstruct or generalize mathematics so that classical mathematics is its special or partial. And meanwhile, different branches of mathematics can be combined into a union one and then, applied to generalize other mathematics and sciences, which is the *mathematical combination*. Even so, *how to generalize mathematical science by mathematical combinations?* For this objective, an effective way is to establish the *Smarandache multi-space* or *continuity flow theory* ([31]) by vectors in a Banach space over 1-dimensional topological structures G^L for extending mathematics, including the contradiction avoided in mathematics for the recognition of reality of thing T. This is nothing else but the recognitive way explained by the sophist to the blind men in fable of the blind men with an elephant. In this way, the human recognition of reality of thing T should be a combined one or combinatorial reality. Essentially, the complex network obtained by *reduction* in the human recognition of thing T happens to be such a 1-dimensional topological structure G^L but we are short of a mathematical theory that regards it as an element, which is also the reason in the previous assertion that mathematical reality can not induce $T_{\mathcal{M}} \to T$.

The main purpose of this paper is to summarize the contribution of CC conjecture to the generalization of classical algebraic systems, topology and geometry, analyze its relationship with Smarandachely denied axiom, multi-spaces and the philosophy of mathematical combinational G^L for recognizing the combinatorial reality of thing, show its contribution to scientific recognition. All terminologies and notations not defined in this paper are standard such as those of the algebra, topology, complex systems, functional analysis and topological graph are respectively referred to [4]-[7], and terminologies in Smarandache geometry and multi-space are referred to [11]-[12],[15] and [36]-[38].

§2. Smarandachely Denied Axiom

Generally, it is believed that the application of mathematics to describe the reality of thing T depends on the closed algebraic operation system such as groups, rings and fields for describing the evolving rule of thing T, regular geometrical bodies approaching the appearance of thing T and the combinatorial relations between elements of thing T in mathematics. In the early of 2005, I completed my post-doctoral report "*On the Automorphisms of Maps & Klein Surface*" ([11] 1st edition). Also in the year earlier, I received an email from Dr.Preze Mihn, the editor of American Research Press. He told me that they would fund me to publish a book in USA if it contains Smarandache geometry. I personally appreciate the notion that recognizes things by combinatorics proposed in my post-doctoral report and thought I should let more ones know this notion for developing science. So, I reorganized my post-doctoral report and emailed it to this publishing company in USA, told them that a lots of classical mathematical problems such as Riemannian surfaces, Riemannian geometry and algebraic curves were discussed by combinatorics in my book. Dr.Preze Mihn emailed me a few of documents after reading it and told me that Smarandache geometry is more extensive than Riemannian geometry. He suggested me to increase the content of Smarandache geometry in my book, which motivated me to turn my research on Smarandachely denied axiom and Smarandache geometry.

2.1.Smarandache Geometry So, *what is Smarandachely denied axiom, what is Smarandache geometry, and what things that can be described by them?* Surprisingly, a Smarandache geometry no longer complies with the principle of logical consistency but includes contradictions.

Definition 2.1([37],[38]) *An axiom is said Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways. A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom.*

What really interests me on Smarandache geometry is that it is different from the classical one, namely it includes contradictions, and I always believe that this is the most important question in human recognition of thing. For example, the Euclidean geometry is a geometry without contradiction, based on five axioms but the fifth axiom, i.e., "*given a line and a point exterior this line, there is one line parallel to this line*" is always felt to be less obvious than the other axioms. And then, the Lobachevshy-Bolyai-Gauss geometry replaces it by "*there are infinitely many line parallels to a given line passing through an exterior point*" and the Riemannian geometry replaces it by "*there is no parallel to a given line passing through an exterior point*". All of them are complied with the principle of logical consistency, but a Smarandache geometry can be partly Euclidean geometry or Riemannian geometry and partly Lobachevshy-Bolyai-Gauss geometry ([9]), which is probably the natural state of thing because it is always evolving non-harmoniously.

Usually, the research object in Riemannian geometry is the n-dimensional Riemannian manifolds for integers $n \geq 2$, which is endowed on n-dimensional topological manifold with smooth nature, establish further its vector field, tensors and connections with geometrical behaviors. So, *what is an n-dimensional topological manifold?* By definition, an n-dimensional topological manifold is a Hausdoff space M holds with the separation axiom, namely for two distinct points $p_1, p_2 \in M$ there are neighborhoods $U(p_1), U(p_2) \in M$ of p_1, p_2 such that $U(p_1) \bigcap U(p_2) = \emptyset$ and for any point $p \in M$ there exists a neighborhood $U(p)$ homeomorphic to n-dimensional Euclidean space $\mathbb{R}^n$, called also the *locally Euclidean space M*. After my post-doctoral report ([11] 1st edition) published in USA, I further studied Iseri's book [8] on Smarandache manifolds and found an easier way for constructing 2-dimensional Smarandache manifolds than that of [8], i.e., by the 2-cell embeddings M of graphs on surfaces, namely endowed with a real number $\mu(u)$, $\mu(u)\rho_M(u)$ (mod2π) on any vertex $u \in V(M)$, $\rho_M(u) \geq 3$ of 2-cell embedding M to get a 2-dimensional Smarandache manifold (M, μ), is said to be *map* *geometry* and points of (M,μ) are classified into *elliptic*, *Euclidean* or *hyperbolic* if $\rho(u)\mu(u) < 2\pi$, $\rho(u)\mu(u) = 2\pi$ or $\rho(u)\mu(u) > 2\pi$, where $\rho_M(u)$ is the valency of vertex u in M and $\mu : u \in V(M) \to (0,\pi)$ is said to be an *angle factor*, see [12] for details.

Notice that the generalization of this way can be applied to Euclidean space $\mathbb{R}^n$ for constructing n-dimensional pseudo-Euclidean space and then, to get n-dimensional Smarandache manifolds ([15]). Generally, let $\mathbb{R}^n$ be an n-dimensional Euclidean space with normal basis $\boldsymbol{\epsilon}_1 = (1,0,\cdots,0)$, $\boldsymbol{\epsilon}_2 = (0,1,\cdots,0)$, $\cdots$, $\boldsymbol{\epsilon}_n = (0,0,\cdots,1)$. Then, an *$n$-dimensional pseudo-Euclidean space* is defined to be a 2-tuple $(\mathbf{R^n}, \omega|_{\overrightarrow{O}})$, where $\omega|_{\overrightarrow{O}} : \mathbf{R}^n \to \mathscr{O}$ is such a continuous function that a straight line of orientation $\overrightarrow{O}$ passing through point $\mathbf{u} \in \mathbb{R}^n$ will turn its orientation to $\overrightarrow{O} + \omega|_{\overrightarrow{O}}(\mathbf{u})$. Certainly, an n-dimensional pseudo-Euclidean space $(\mathbb{R}^n, \omega|_{\overrightarrow{O}}) = \mathbb{R}^n$ if and only if $\omega|_{\overrightarrow{O}}(\mathbf{u}) = \mathbf{0}$ for any point $\mathbf{u} \in \mathbb{R}^n$, i.e., a *flat space*. And then, an n-dimensional Smarandache manifold is defined to be a local n-dimensional pseudo-Euclidean space $(M^n, \mathcal{A}^\omega)$, namely for any point $p \in M^n$ there exists a neighborhood U_p homeomorphic to n-dimensional pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$, where (U_p, φ_p^ω) is a chart at point p with a homeomorphism $\varphi_p^\omega : U_p \to (\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ and $\mathcal{A} = \{(U_p, \varphi_p^\omega)|p \in M^n\}$ is an atlas on manifold M^n.

Generally, the six sense organs of human can not feel the distortion of space but thinks that the space is flat priorly. This is why the Euclidean spaces $\mathbb{R}^n$ of $n \geq 3$ are often used to be the reference of thing, but it is not necessarily the nature of thing in universe. For example, one of Einstein's contributions to the gravitational field is to show that the substance field of universe is not flat but a curved one under gravitation, not even with the light, namely the nature of substance field of universe is a 3-dimensional Smarandache manifold rather than a Euclidean space $\mathbb{R}^3$, which should be the proper contribution of rational thinking of human.

2.2.Smarandache Multi-Space. Unlike the classical geometry, an axiom of Smarandache geometry behaves simultaneously validated and invalided, or only invalided but in multiple ways, which is not easy to find by the geometrical intuition of human but it is easy to construct Smarandache systems on algebra such as the combination of two non-isomorphic groups or rings defined on a set and then, the resulting system must be Smarandachely denied, which is essentially the application of CC conjecture to that of algebraic systems. I emailed this thing to Dr.Preze Mihn and told him that I was going to generalize algebra systems along this way. He wrote back that this way had been proposed by Smarandache a few years ago, i.e., the *Smarandache multi-space* and encouraged me to follow this thinking on mathematics.

Definition 2.2([12],[36]) *For an integer $n \geq 1$, let $(\mathcal{S}_1, \mathcal{O}_1), (\mathcal{S}_2, \mathcal{O}_2), \cdots, (\mathcal{S}_n, \mathcal{O}_n)$ be n distinct mathematical systems or spaces, namely for any integer $1 \leq i \neq j \leq n$, $\mathcal{O}_i \subset \mathcal{S}_i \times \mathcal{S}_i$ and $\mathcal{S}_i \neq \mathcal{S}_j$ or $\mathcal{S}_i = \mathcal{S}_j$ but $\mathcal{O}_i \neq \mathcal{O}_j 1 \leq i,j \leq n$. Then, a Smarandache multi-space is defined to be*

$$(\mathscr{S};\mathscr{O}) = \bigcup_{i=1}^{n} (\mathcal{S}_i, \mathcal{O}_i), \tag{2.1}$$

i.e., $\mathscr{S} = \bigcup\limits_{i=1}^{n} \mathcal{S}_i$ and $\mathscr{O} = \bigcup\limits_{i=1}^{n} \mathcal{O}_i$, where $\mathcal{S}_i$ is a set and $\mathcal{O}_i$ is operations on $\mathcal{S}_i$ for integers $1 \leq i \leq n$.

Now, *how to understand each characteristic of $\chi_1, \chi_2, \cdots$ in the human recognition of* *thing T?* Usually, these characteristics of $\chi_1, \chi_2, \cdots$ in human recognition of thing T are not only numerical values or data but a family of sets $\mathcal{S}_1, \mathcal{S}_2, \cdots$ with respective characteristics of $\chi_1, \chi_2, \cdots$, where $\chi(T)$ is a kind of nature of thing T, i.e.,

$$T_{\mathcal{M}} = \bigcup_{i=1}^{\infty} (\mathcal{S}_i; \mathcal{O}_i), \tag{2.2}$$

where $\mathcal{O}_i$ is the evolving rule of elements in $\mathcal{S}_i$ and $T_{\mathcal{M}}$ is nothing else but a Smarandache multi-space (2.1). For example, the sophist told the blind men that the shape of elephant is

$$\text{An Elephant} = \{\text{ 2 Big Radish}\}\bigcup\{\text{1 Pipe}\}\bigcup\{\text{2 Leaf Fans}\}$$
$$\bigcup\{\text{1 Wall}\}\bigcup\{\text{4 Pillars}\}\bigcup\{\text{1 Rope}\}.$$

in fable of the blind men with an elephant, which is a Smarandache multi-space. Notice that different characteristics χ_i corresponds to different evolving systems $\mathcal{S}_i$ if $n \geq 2$, namely a Smarandache multi-space $(\mathscr{S}; \mathscr{O})$ holds with Smarandachely denied axiom. Conversely, all elements in a Smarandachely denied system $\mathscr{S}$ can be classified into systems by each of Smarandachely denied axiom $\mathcal{A}$ validated or invalided, or each invalided case to get a mathematical system or space $(\mathcal{S}_i; \mathcal{A})$, namely a Smarandachely denied system $\mathscr{S}$ is nothing else but a Smarandache multi-space $(\mathscr{S}; \mathscr{O})$. Whence, a Smarandachely denied system is equivalent to a Smarandache multi-space. Then, *how to determine evolving rules in set $\mathcal{S}_i$ for an integer $i \geq 1$?* Usually, we assume that all elements with characteristic χ_i comply with the mathematical operations in $\mathcal{O}_i$ and then, describe the evolution of things in $\mathcal{S}_i$ by mathematics.

Thus, if we do not consider the combinatorial structure G^L inherited in recognitive sets $\mathcal{S}_1, \mathcal{S}_2, \cdots$, we can apply CC conjecture to generalize algebraic multi-systems, including group, ring and field, geometrical multi-spaces, compact n-dimensional manifolds in topology, etc., which already appears for a few of simple cases in classical mathematics. For example, the continuous groups, Lie groups are both Smarandache multi-space of $n = 2$ in (2.1). By the Smarandache multi-space, Kandasamy, Smarandache and others extensively generalized algebraic systems such as those groups, rings and algebraic properties ([39]-[41]). For example, let $(G_1; \circ), (G_2; \bullet)$ be 2 different groups. Then, $(G_1 \bigcup G_2; \{\circ, \bullet\})$ is said to be a *bigroup*. Particularly, if $(G_1 \bigcup G_2; \circ)$ is an Abelian group with unit 0, $(G_1 \bigcup G_2 \setminus \{0\}; \bullet)$ is a group and for any elements $x, y, z \in G_1 \bigcup G_2$, there is

$$x \bullet (y \circ z) = (x \bullet y) \circ (x \bullet z), \quad (y \circ z) \bullet x = (y \bullet x) \circ (z \bullet x), \tag{2.3}$$

then the bigroup $(G_1 \bigcup G_2; \{\circ, \bullet\})$ is a skew field. Furthermore, if $(G_1 \bigcup G_2 \setminus \{0\}; \bullet)$ is also an Abelian group, then the bigroup $(G_1 \bigcup G_2; \{\circ, \bullet\})$ is nothing else but a field. Generally, for an integer $n \geq 1$, let $(\mathcal{S}_1, \mathcal{O}_1), (\mathcal{S}_2, \mathcal{O}_2), \cdots, (\mathcal{S}_n, \mathcal{O}_n)$ be n groups, rings or modules. Then, a Smarandache multi-space defined by (2.1) on $(\mathcal{S}_1, \mathcal{O}_1), (\mathcal{S}_2, \mathcal{O}_2), \cdots, (\mathcal{S}_n, \mathcal{O}_n)$ is said to be respectively the *n-group, n-ring* or *n-module*, and we can determine their multi-subgroups, multi-subrings, multi-subideals with a homomorphic theorem on associative systems following, and the decomposition structure of n-module, see [15] for details.

Theorem 2.3([15]) *Let ω be an onto homomorphism from an associative multi-system $(\mathscr{H}_1; \widetilde{O}_1)$*

to $(\mathscr{H}_2;\widetilde{O}_2)$ *and let* $(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)$ *be a multi-system with unit* $1_{\circ^-}$ *for* $\forall\circ^-\in\widetilde{O}_2$ *and inverse* x^{-1} *for* $\forall x\in\mathcal{I}(\widetilde{O}_2)$ *in* $((\mathcal{I}(\widetilde{O}_2);\circ^-)$. *Then there are representation pairs* $(R_1,\widetilde{P}_1)$ *and* $(R_2,\widetilde{P}_2)$ *with* $\widetilde{P}_1\subset\widetilde{O},\widetilde{P}_2\subset\widetilde{O}_2$ *such that*

$$\left.\frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\widetilde{O}_1)}\right|_{(R_1,\widetilde{P}_1)}\cong\left.\frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}\right|_{(R_2,\widetilde{P}_2)}\tag{2.4}$$

if each element of $\widetilde{\mathrm{Ker}\omega}$ *has an inverse in* $(\mathscr{H}_1;\circ)$ *for* $\circ\in\widetilde{O}_1$, *where* $\mathcal{I}(\widetilde{O}_2)$ *denotes the set consisting of all units* $1_\circ,\circ\in\widetilde{O}_2$ *in multi-system* $(\mathscr{H}_2;\widetilde{O}_2)$, $\widetilde{\mathrm{Ker}\omega}=\{a|\omega(a)=1_\circ,a\in\mathscr{H}_1,\circ\in\widetilde{O}_1\}$, *a multi-system* $(\mathscr{H};\widetilde{O})$ *is associative if for* $\forall a,b,c\in\mathscr{H}$, $\forall\circ_1,\circ_2\in\widetilde{O}$, $(a\circ_1 b)\circ_2 c=a\circ_1(b\circ_2 c)$ *and* $(R_1,\widetilde{P}_1)$, $(R_2,\widetilde{P}_2)$ *denotes the pairs of* $\mathscr{H}_1=\bigcup\limits_{a\in R_1,\circ\in\widetilde{P}_1}a\circ\widetilde{\mathrm{Ker}\omega}$, $\mathscr{H}_2=\bigcup\limits_{a\in R_2,\circ\in\widetilde{P}_2}a\circ\widetilde{\mathrm{Ker}\omega}$.

Particularly, let $(\mathscr{H}_1;\widetilde{O}_1)$ and $(\mathscr{H}_2;\widetilde{O}_2)$ be n-groups. Then,

Corollary 2.4([12]) *If homomorphism* $\omega:(\mathscr{H}_1;\widetilde{O}_1)\to(\mathscr{H}_2;\widetilde{O}_2)$ *is onto then* $\mathscr{H}_1/\mathrm{Ker}\omega\simeq\mathrm{Im}\omega$.

Now, if $(\mathcal{S}_1,\rho_1),(\mathcal{S}_2,\rho_2),\cdots,(\mathcal{S}_n,\rho_n)$ are n metric spaces, then a Smarandache multi-space determined by (2.1) is said to be a *n-metric space* and we can introduce also the Cauchy sequence, complete space and the contraction mapping on such a space to generalize Banach fixed-point theorem.

Theorem 2.5([12]) *Let* $\widetilde{M}=\bigcup\limits_{i=1}^{m}M_i$ *be a completed multi-metric space. For an* ϵ*-disk sequence* $\{B(\epsilon_n,x_n)\}$ *with* $\epsilon_n>0$ *for* $n=1,2,3,\cdots$, *if* $B(\epsilon_1,x_1)\supset B(\epsilon_2,x_2)\supset\cdots\supset B(\epsilon_n,x_n)\supset\cdots$ *and* $\lim\limits_{n\to+\infty}\epsilon_n=0$, *then* $\bigcap\limits_{n=1}^{+\infty}B(\epsilon_n,x_n)$ *has only one point.*

Theorem 2.6([12]) *If* $\widetilde{M}=\bigcup\limits_{i=1}^{m}M_i$ *is a completed multi-metric space and* T *a contraction on* $\widetilde{M}$ *then* $1\leq|T_{\mathrm{fix}}|\leq m$, *where* $|T_{\mathrm{fix}}|$ *is the cardinality of fixed point set of* T.

Notice that a Smarandache multi-space defined by (2.1) with $\mathscr{S}=\bigcup\limits_{i=1}^{n}\mathcal{S}_i$ and $\mathscr{O}=\bigcup\limits_{i=1}^{n}\mathcal{O}_i$ is the union of elements in $\mathcal{S}_i$ with operations in $\mathcal{O}_i, 1\leq i\leq n$. However, any thing does not exist in isolation. We can determine the combinatorial structure G^L inherited in systems or spaces $(\mathcal{S}_1,\mathcal{R}_1), 1\leq i\leq n$ by CC conjecture further.

Definition 2.7 *For an integer* $n\geq 1$, *let* $(\mathscr{S};\mathscr{O})=(\bigcup\limits_{i=1}^{n}\mathcal{S}_i;\bigcup\limits_{i=1}^{n}\mathcal{O}_i)$ *be a Smarandache multi-space with an inherited combinatorial structure or vertex-edge labeled graph* $G^L[\mathscr{S},\mathscr{O}]$ *determined by*

$$\begin{aligned}V(G^L[\mathscr{S},\mathscr{O}])&=\{\mathcal{S}_1,\mathcal{S}_2,\cdots,\mathcal{S}_n\},\\E(G^L[\mathscr{S},\mathscr{O}])&=\{(\mathcal{S}_i,\mathcal{S}_j)|\mathcal{S}_i\bigcap\mathcal{S}_j\neq\emptyset,\ 1\leq i\neq j\leq n\}\end{aligned}$$

and label mapping

$$L:\ \mathcal{S}_i\to L(\mathcal{S}_i)=\mathcal{S}_i,\quad(\mathcal{S}_i,\mathcal{S}_j)\to L(\mathcal{S}_i,\mathcal{S}_j)=\mathcal{S}_i\bigcap\mathcal{S}_j,\quad 1\leq i\neq j\leq n.$$

Thus, there is a bijection

$$(\mathscr{S};\mathscr{O})\ \stackrel{1-1}{\longleftrightarrow}\ G^L[\mathscr{S},\mathscr{O}]\tag{2.5}$$

between Smarandache multi-space $(\mathscr{S};\mathscr{O})$ and the labeled graph $G^L[\mathscr{S},\mathscr{O}]$. For example, let $\mathscr{G}_1 = \langle\alpha,\beta\rangle$, $\mathscr{G}_2 = \langle\alpha,\gamma,\theta\rangle$, $\mathscr{G}_3 = \langle\beta,\gamma\rangle$, $\mathscr{G}_4 = \langle\beta,\delta,\theta\rangle$ be freely Abelian groups generated by elements $\alpha,\beta,\gamma,\delta$ and θ with $\alpha\neq\beta\neq\gamma\neq\delta\neq\theta$. Calculation shows that $\mathscr{G}_1\bigcap\mathscr{G}_2 = \langle\alpha\rangle$, $\mathscr{G}_2\bigcap\mathscr{G}_3 = \langle\gamma\rangle$, $\mathscr{G}_3\bigcap\mathscr{G}_4 = \langle\delta\rangle$, $\mathscr{G}_1\bigcap\mathscr{G}_4 = \langle\beta\rangle$ and $\mathscr{G}_2\bigcap\mathscr{G}_4 = \langle\theta\rangle$. So, the vertex-edge labeled graph $G^L[\mathscr{S},\mathscr{O}]$ determined by Smarandache multi-group $(\mathscr{S};\mathscr{O})$ is shown in Figure 3.

2.3.Combinatorial Manifold. A generalization of manifolds in geometry by combinatorics is the combinatorial manifolds. By definition, a *combinatorial manifold* $\widetilde{M}$ is the combination of finite manifolds $M_1, M_2, \cdots, M_m$ over a topological graph G^L, namely the space $(\mathcal{S}_i,\mathcal{O}_i)$ is a manifold M_i for any integer $1\leq i\leq m$ in Definition 2.7 ([14],[15]). Such a Smarandache multi-space $(\mathscr{S};\mathscr{O})$ is denoted by $\widetilde{M}(n_1,n_2,\cdots,n_m)$. It should be noted that a topological graph G^L is inherited in the combinatorial manifold $\widetilde{M}$, namely

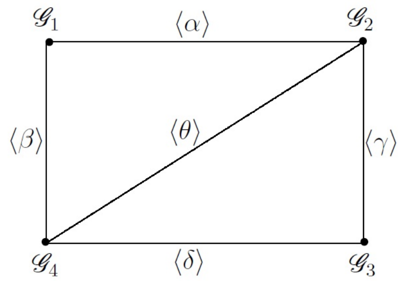


Figure 3. A labeled graph

$$V\left(G^L\right) = \{M_1, M_2, \cdots, M_m\},$$
$$E\left(G^L\right) = \left\{(M_i,M_j)|M_i\bigcap M_j\neq\emptyset, 1\leq i,j\leq m\right\}$$

and

$$L:\ M_i\to L(\mathcal{S}_i)=M_i,\quad (M_i,M_j)\to L(M_i,M_j)=M_i\bigcap M_j,\quad 1\leq i\neq j\leq m,$$

such as those shown in Figure 4, where M^3 is a 3-dimensional manifold, B^1 and T^2 are respectively a bouquet and a torus.

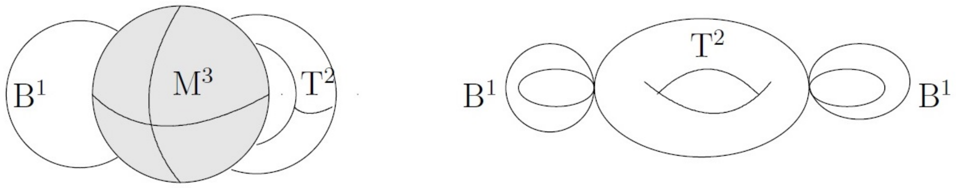


Figure 4. Examples of combinatorial manifolds

Locally, for any integer sequence $0<n_1<n_2<\cdots<n_m$, a combinatorial manifold can be geometrically defined also to be a Hausdoff space $\widetilde{M}$ holds with the separation axiom and there always is a neighborhood U_p with a homeomorphism $\varphi_p : U_p \to \widetilde{\mathbb{R}}(n_1(p), n_2(p),\cdots,n_{s(p)})$ for point $p\in\widetilde{M}$, where $\widetilde{\mathbb{R}}(n_1(p),n_2(p),\cdots,n_{s(p)})$ is a combinatorial Euclidean space by $s(p)$ Euclidean spaces $\mathbb{R}^{n_1},\mathbb{R}^{n_2},\cdots,\mathbb{R}^{n_{s(p)}}$, is said to be a *combinatorial Euclidean fan-space*, i.e., for integers $1\leq i\neq j\leq s(p)$, $\mathbb{R}^{n_i}\bigcap\mathbb{R}^{n_j} = \bigcap\limits_{k=1}^{s(p)}\mathbb{R}^{n_k}$ and

$$\bigcup_{p\in\widetilde{M}}\left\{n_1(p),n_2(p),\cdots,n_{s(p)}(p)\right\} = \{n_1,n_2,\cdots,n_m\}. \tag{2.6}$$

Now, *is a combinatorial manifold a topological manifold or Smarandache manifold?* By the definition in topology, the intersection $M_i \bigcap M_j$ of two n-dimensional manifolds M_i and M_j is also an n-dimensional manifold, which is consistent with the visual perception of human. Whence, if the intersection of manifolds $M_1, M_2, \cdots, M_m$ in same dimension complies with the intersection rule of topology in a combinatorial manifold $\widetilde{M}$, $\widetilde{M}$ is a manifold also. Otherwise, if the dimensions $\dim(M_1) = n_1, \dim(M_2) = n_2, \cdots, \dim(M_n) = n_m$ of manifolds $M_1, M_2, \cdots, M_m$ in combinatorial manifold $\widetilde{M}$ are not all the same, or not complies with the intersection rule of topology, i.e., for integers $1 \leq i \neq j \leq m$, the intersection $M_i \bigcap M_j$ satisfies

$$\dim(M_i \bigcap M_j) < \min\{\dim(M_i), \dim(M_j)\} \tag{2.7}$$

then, the combinatorial manifold $\widetilde{M}$ is no longer a topological manifold but a Smarandache manifold. In this way, the combinatorial manifold includes both the case of topological manifold and Smarandache manifold.

A typical nature of combinatorial manifold is that it simultaneously displays both the nature of manifold and topological graph, and so it can be characterized by the natures of manifolds and topological graphs. For example, let the combinatorial manifolds $\widetilde{M}_1$ and $\widetilde{M}_2$ be respectively consisted of manifolds $M_i^1, 1 \leq i \leq m$ and $M_k^2, 1 \leq k \leq s$. If there is such an isomorphism $\varphi : V(G^L[\widetilde{M}_2]) \to V(G^L[\widetilde{M}_2])$ between labeled graphs $G^L[\widetilde{M}_1]$ and $G^L[\widetilde{M}_2]$ that for any integers $1 \leq i, j \leq m$, if $\varphi : M_i^1 \to M_j^2$ then there is a homeomorphism $h_{M_i^1} : \varphi\left(M_i^1\right) \to M_j^2$ such that $\varphi(M_i^1)$ homeomorphic to M_j^2, then $h \circ \varphi$ is a homeomorphism between combinatorial manifold $\widetilde{M}_1$ and $\widetilde{M}_2$. We can therefore characterize the connectivity, d-dimensional fundamental group, homology group of combinatorial manifold $\widetilde{M}$ by topological graph $G^L[\widetilde{M}]$, and also the main objects in Riemannian geometry such as those of vector field, tensor field with local coordinates, Riemannian tenor with connection ([14]). Among them, what can reflect the most of CC conjecture is to establish the $|\Gamma|$-multiple covering space of combinatorial manifold $\widetilde{M}$ by voltage graph ([6]), where Γ is the finite group in voltage graph $G^L[\widetilde{M}]$.

(1)Regular covering space. Let G^L be a topological graph and let $(\Gamma; \circ)$ be a finite group. So, *what is a voltage graph and what is the lifting of a voltage graph?* Firstly, let $e = (v, u) \in E(G^L)$ be an edge of G^L. Its plus and minus orientations e^+, e^- on e are defined to be $v \to u$ and $u \to v$, respectively. And then, a voltage assignment $\alpha : E(G^L) \to \Gamma$ is a mapping from the plus-edges e^+ to Γ for $e \in E(G^L)$, i.e., $\alpha(e)$ is an element in group $(\Gamma; \circ)$ holding with $\alpha(e^-) = \alpha^{-1}(e)$. A topological graph G^L with voltage $\alpha(e)$ for any edge $e \in E(G^L)$ is said to be a *voltage graph* on group $(\Gamma; \circ)$, denoted by $(G^L; \alpha)$.

Notice that the voltage graph $(G^L; \alpha)$ is only a labeled graph with edge labels in a finite group $(\Gamma, \circ)$. Certainly, it is also a Smarandache multi-space of a topological graph G^L with a finite group $(\Gamma, \circ)$. However, the most interesting of voltage graph $(G^L; \alpha)$ is its *lifting* G^{L^α} defined by (see [7] for details)

$$\begin{aligned} V\left(G^{L^\alpha}\right) &= \left\{(v, a) = v_a \in V\left(G^L\right) \times \Gamma\right\}, \\ E\left(G^{L^\alpha}\right) &= \left\{(v_a, u_{a \circ b}) | e^+ = (v, u) \in E\left(G^L\right), \alpha\left(e^+\right) = b\right\} \end{aligned}$$

for regular covering of G^L. For example, let $G^L = K_3^L$, $\Gamma = \mathbb{Z}_2$ be respectively a topological graph and a finite group. Then, (a) is a voltage graph $(K_3^L;\alpha)$ and (b) is the lifting $K_3^{L^\alpha}$ of voltage graph $(K_3^L;\alpha)$ of (a) in Figure 5.

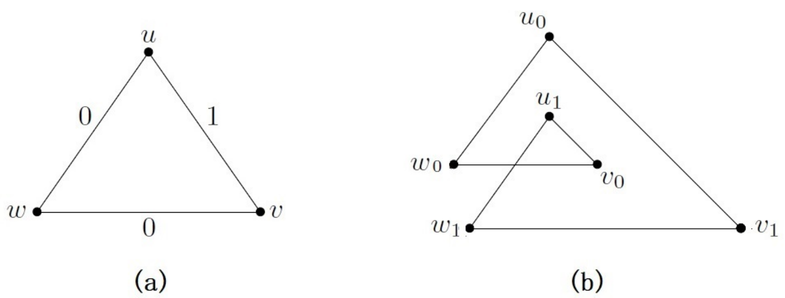


Figure 5. A voltage graph with lifting

There is a natural projection $\pi : G^{L^\alpha} \to G^L$ from the lifting G^{L^α} of voltage graph $(G^L;\alpha)$ on the topological graph G^L, namely for any vertices $v, u, \in V(G^L)$ and edge $(v,u) \in E(G^L)$ with $\alpha(v,u) = h$, define $\pi(v_g) = v$, $\pi(u_g) = u$ and $\pi(v_g, u_{g\circ h}) = (v,u)$. So, all vertices in the lifting G^{L^α} projected on $v \in V(G^L)$ consists of vertices $v_g, g \in \Gamma$ and edges that projected on edge (v,u) consists of edges $(v_g, u_{g\circ h})$, denoted by $\pi^{-1}(v)$ and $\pi^{-1}(v,u)$, respectively.

Notice that the lifting G^{L^α} of voltage graph $(G^L;\alpha)$ is a regular covering of topological graph G^L ([33]), i.e., a $|\Gamma|$-multiple covering on topological space G^L of dimension 1, and this method can be generalized for constructing regular covering space of combinatorial manifold $\widetilde{M}$ by the inherited topological structure $G^L[\widetilde{M}]$ of $\widetilde{M}$. That is, to assign voltages $\alpha : e \in E(G^L[\widetilde{M}])$ on edges of $G^L[\widetilde{M}]$ to get a voltage graph $(G^L;\alpha)$ with its lifting combinatorial manifold $\widetilde{M}^*$, where for any vertex $v_g \in V(G^L[\widetilde{M}^*])$, the labeling mapping $L^\alpha(v_g) = M^*$ is a covering space of M at vertex $v_M \in V(G^L[\widetilde{M}])$. In this case, for any manifold $M \in V(G^L[\widetilde{M}])$, let h_M be a covering mapping $h_M : M^* \to M$, $\varsigma_M : x \in M \to M$ and define $\pi^* = h_M \circ (\varsigma_M^{-1} \circ \pi \circ \varsigma_M)$. Then, the mapping $\pi^* : \widetilde{M}^* \to \widetilde{M}$ is a $|\Gamma|$-multiple covering mapping, which shows that the combinatorial manifold $\widetilde{M}^*$ is a regular covering space of combinatorial manifold $\widetilde{M}$.

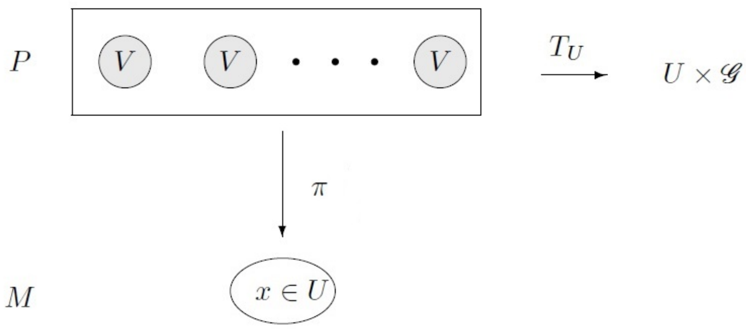


Figure 6. Principal fiber bundle on manifold

(2)Principal fiber bundle. Now, we turn our attention to differentiable manifolds. Firstly, a Lie group $(G;\circ)$ is a Smarandache multi-space that all elements in G both have algebraic and geometrical nature, namely for any $x, y \in G$, $x \circ y$ and x^{-1} both are C^∞-mapping, where G is a differentiable manifold in geometry and each point is an element in group

$(G;\circ)$ also. So, *what is a principal fiber bundle on a differentiable manifold M?* A principal fiber bundle is essentially a covering space on a differentiable manifold M. By definition, the *principal fiber bundle* on a differentiable manifold M is a 3-tuple $\{P, M; \mathscr{G}\}$ with covering space P, manifold M, a projection π and a mapping T_u such as those shown in Figure 6, where $\pi : P \to M$ is a projection, $\mathscr{G}$ is a Lie group acting on P such that $(g,h) \to g \circ h$ is C^∞ for any $g, h \in \mathscr{G}$ and holding with 3 conditions following:

(C_1) *The right action of Lie group $\mathscr{G}$ acting on P is free, namely for any $g \in \mathscr{G}$ there is a diffeomorphism $R_g : P \to P$ such that $R_g(p) = pg$ for any $p \in P$, $p(g_1g_2) = (pg_1)g_2$ for any $g_1, g_2 \in \mathscr{G}$, and $pe = p$ for any $e \in \mathscr{G}$ if and only if e is the identity of Lie group $\mathscr{G}$;*

(C_2) *The mapping $\pi : P \to M$ is onto, and $\pi^{-1}(\pi(p)) = \{pg | g \in \mathscr{G}\}$;*

(C_3) *For any point $x \in M$ there is an opened set U such that $x \in U$ and there is a diffeomorphism $T_U : \pi^{-1}(U) \to U \times \mathscr{G}$, i.e., $T_U(p) = (\pi(p), s_U(p))$, where $s_U : \pi^{-1}(U) \to \mathscr{G}$ holds with $s_U(pg) = s_U(p)g$ for any $g \in \mathscr{G}, p \in \pi^{-1}(U)$.*

Now, if a combinatorial manifold $\widetilde{M}$ consisted of differentiable manifolds $M_1, M_2, \cdots, M_m$ is differentiable and 3-tuples $\{P_1, M_1; \mathscr{G}_1\}$, $\{P_2, M_2; \mathscr{G}_2\}$, $\cdots$, $\{P_m, M_m; \mathscr{G}_m\}$ are respectively the principal fiber bundles on manifolds $M_1, M_2, \cdots, M_m$ with projections $\pi_i : P_i \to M_i$ for integers $1 \leq i \leq m$, then

$$\widetilde{P} = \bigcup_{i=1}^{m} P_i, \quad \left(\widetilde{\mathscr{G}}; \mathscr{O}\right) = \bigcup_{i=1}^{m} (\mathscr{G}_i; \circ_i) \tag{2.8}$$

are respectively the covering spaces and Lie multi-groups on combinatorial manifold $\widetilde{M}$, namely $\widetilde{\mathscr{G}}$ is a Lie multi-group in algebra and a differentially combinatorial manifold in geometry. Then, $\{\widetilde{P}, \widetilde{M}; \widetilde{\mathscr{G}}\}$ is a principal fiber bundle on $\widetilde{M}$.

Furthermore, let $(\Gamma;\circ)$ be a finite group, $\alpha : E(G^L[\widetilde{M}]) \to \Gamma$ is a voltage assignment on topological graph $G^L[\widetilde{M}]$. Then, by the lifting of voltage graph $(G^L[\widetilde{M}];\alpha)$, we can obtain the principal fiber bundles $\{\widetilde{P}^\alpha, \widetilde{M}; \widetilde{\mathscr{G}}\}$ on differentiable manifold $\widetilde{M}$ in general, namely let $L^\alpha(v_g) = P$ for any vertex $v_g \in V(G^{L^\alpha}[\widetilde{M}])$ for a projection $\pi : G^L[\widetilde{P}] \to G^L[\widetilde{M}]$, i.e., for any label $M \in V(G^L[\widetilde{M}])$ if $\pi(P_M) = M$ then $\pi^{-1}(M) = \{P_M^1, P_M^2, \cdots, P_M^m\}$, where P_M^i is differentially homeomorphic to P_M for integers $1 \leq i \leq m$ such as those shown in Figure 7.

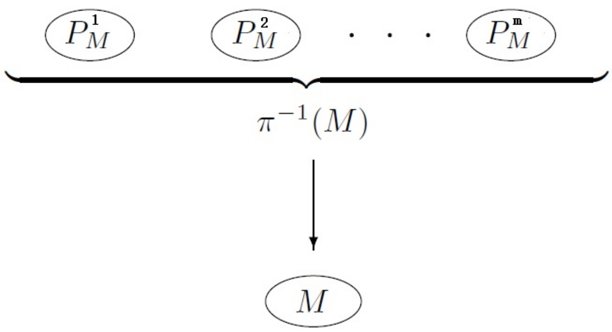


Figure 7. A principal fiber bundle on combinatorial manifold

Now, let $\{\widetilde{P}^\alpha, \widetilde{M}; \widetilde{\mathscr{G}}\}$ be a principal fiber bundles constructed in this way. Then, we can introduce the connection on differentially combinatorial manifold $\widetilde{M}$ and get the general form of curvature for characterizing differentially combinatorial manifolds. For example, a local and global connection on a principal fiber bundle $\{\widetilde{P}^{\alpha}, \widetilde{M}; \mathscr{G}\}$ are respectively a local linear mapping ${}^{i}\Gamma_u : T_x(\widetilde{M}) \to T_u(\widetilde{P}), u \in \Pi_i^{-1}(x) = {}^{i}F_x, x \in M_i$ for an integer $1 \leq i \leq l$ and a global linear mapping $\Gamma_u : T_x(\widetilde{M}) \to T_u(\widetilde{P})$ for $u \in \Pi^{-1}(x) = F_x$, $x \in \widetilde{M}$ holding with (i) $(d\Pi_i){}^{i}\Gamma_u$ =identity and $(d\Pi)\Gamma_u$ = identity mapping on $T_x(\widetilde{M})$; (ii) ${}^{i}\Gamma_{{}^{i}R_g \circ_i u} = d^{i}R_g \circ_i {}^{i}\Gamma_u$ and $\Gamma_{R_g \circ u} = dR_g \circ \Gamma_u$ for $\forall g \in \mathscr{L}_G$, $\forall \circ \in \mathscr{O}(\mathscr{L}_G)$, where ${}^{i}R_g$, R_g are the right translation respectively on P_{M_i} and $\widetilde{P}$; (iii) the mappings $u \to {}^{i}\Gamma_u$ and $u \to \Gamma_u$ both are C^{∞}, and a *curvature form* of a local or global connection is a $\mathfrak{Y}(\mathscr{H}_{\circ_i}, \circ_i)$ or $\mathfrak{Y}(\mathscr{L}_G)$-valued 2-form ${}^{i}\Omega = (d\ {}^{i}\omega)h$ or $\Omega = (d\omega)h$, where $(d^{i}\omega)h(X,Y) = d^{i}\omega(hX, hY)$ and $(d\omega)h(X,Y) = d\omega(hX,hY)$ for $X, Y \in \mathscr{X}(P_{M_i})$ or $X, Y \in \mathscr{X}(\widetilde{P})$. Then, the generalizations of Cartan's theorem and Bianchi identity on differentially combinatorial manifolds are obtained in the following.

Theorem 2.8([15]) *Let ${}^{i}\omega$, $1 \leq i \leq l$ and ω be respectively a local or global connection forms on a principal fiber bundle $\{\widetilde{P}^{\alpha}, \widetilde{M}; \mathscr{G}\}$. Then $(d^{i}\omega)(X,Y) = -[{}^{i}\omega(X), {}^{i}\omega(Y)] + {}^{i}\Omega(X,Y)$ and $d\omega(X,Y) = -[\omega(X), \omega(Y)] + \Omega(X,Y)$ for vector fields $X, Y \in \mathscr{X}(P_{M_i})$ and $\mathscr{X}(\widetilde{P})$, respectively.*

Theorem 2.9([15]) *Let ${}^{i}\omega$, $1 \leq i \leq l$ and ω be respectively a local or global connection forms on a principal fiber bundle $\{\widetilde{P}^{\alpha}, \widetilde{M}; \mathscr{G}\}$. Then, $(d^{i}\Omega)h = 0$ and $(d\Omega)h = 0$.*

§3. Non-Harmonious Groups

There is an implicit assumption in human recognition by the reduction when subdividing a thing T into microscopic particles, cells or genes, namely the evolving behavior of microscopic particles, cells or genes are all match in step and a solvable equation can be applied to describe its behavior, predict its evolution further. However, this assumption is not true in general unless the evolution of all microscopic particles, cells or genes that constitute thing T is synchronous. Otherwise, it is wrong even in the macroscopic world. For example, let $A = \{H_1, H_2, H_3, H_4\}$ and $B = \{H_1', H_2', H_3', H_4'\}$ be 2 families consisting of 4 horses that run respectively along 4 lines (LES^N) or (LES^S) on Euclidean plane $\mathbb{R}^2$ in Figure 8.

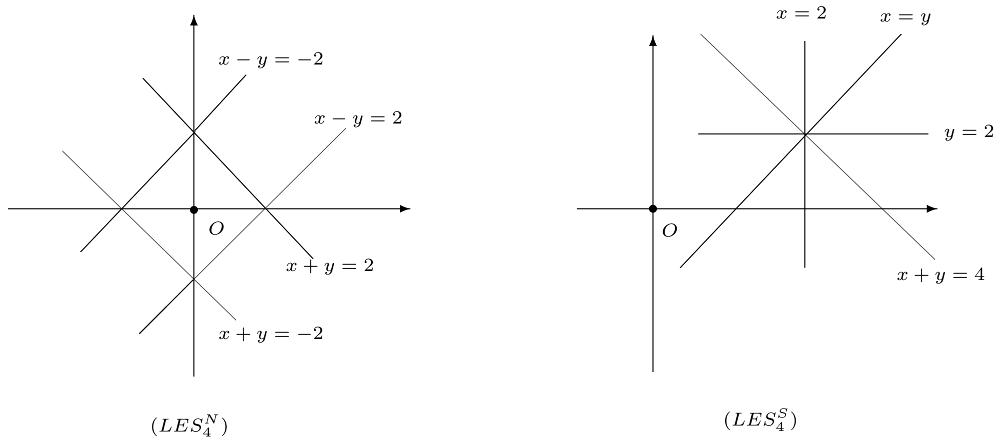


Figure 8. Horses running on 4 straight lines

So, we are easily know the systems of linear equations

$$(LES_4^N)\begin{cases} x+y &= 2 \\ x+y &= -2 \\ x-y &= -2 \\ x-y &= 2 \end{cases} \qquad (LES_4^S)\begin{cases} x=y \\ x+y=4 \\ x=2 \\ y=2 \end{cases}$$

that each horse in families A or B runs along 4 lines (LES^N) or (LES^S) such as those shown in Figure 8.

Now, *how to characterize the running behavior of horses in families A or B?* Generally, we use to the solution of equation systems (LES_4^N) or (LES_4^S) for characterizing the horse behaviors in families A or B. However, the equation system (LES_4^N) is non-solvable and the solution of equation system (LES_4^S) is $x=2, y=2$, which is only a point $(2,2)$ on Euclidean plan $\mathbb{R}^2$, and both of them can not be used to characterize the horse behavior in families A or B, even their running orbits. Here, a central question is the horses in families A or B are all conscious, not necessarily in synchronization or consistence in their respective running.

Similarly, there are also the biological populations, communities and the self-organizing or self-regulating system of cells, genes etc., whose evolution can not be characterized by a solvable equation. Whence, it is impossible to describe the evolution of groups in nature such as the self-organizing or self-regulating systems without an extending of mathematical elements, including the unified field by a solvable equation that of Einstein, which is the essence for discussing the non-harmonious groups.

The original thinking on non-harmonious groups came from my characterizing Smarandachely denied axiom on equations for reality of thing. In the first half of year 2012, I finished paper [16] on systems of linear equations. It is so happen that I went on a business trip to Guangzhou during the time that Prof.Smarandache's visiting Guangdong University of Technology in 2012. I visited him and introduced my thinking on Smarandachely denied axiom with the combinatorial characterizing of non-solvable systems of linear equations [16] that I just finished to him, which shows the necessity for the suitable form of Smarandachely denied axiom by non-solvable equations because a physical law is always described by differential equations on the evolution of thing. I got his greatly approval for this thinking.

Definition 3.1([29],[31]) *A non-harmonious group is such a group $\mathcal{T}$ consisting of elements P_i, $1\leq i\leq p, p\geq 2$ with internal relations that P_i is constrained on an equation $\mathscr{F}_i(\mathbf{x},\mathbf{y})=0$ on time t in space, namely its system state equation in n-dimensional Euclidean space $\mathbb{R}^n$ is*

$$\mathcal{T} \triangleq \begin{cases} \mathscr{F}_1(\mathbf{x},\mathbf{y})=0 \\ \mathscr{F}_2(\mathbf{x},\mathbf{y})=0 \\ \cdots\cdots\cdots\cdots \\ \mathscr{F}_m(\mathbf{x},\mathbf{y})=0 \end{cases} \tag{3.1}$$

where $\mathscr{F}_i(\mathbf{x}^0,\mathbf{y}^0)=0$ and $\mathscr{F}_i$ holds at a neighborhood U of point $(\mathbf{x}^0,\mathbf{y}^0)$ with the condition of *implicit function theorem, i.e., each equation $\mathscr{F}_i(\mathbf{x},\mathbf{y})=0$ is solvable for integers $1\leq i\leq m$.*

So, *how to characterize the evolution of a non-harmonious group $\mathcal{T}$?* Notice that each equation $\mathscr{F}_i(\mathbf{x},\mathbf{y})=0$ holds with the condition of implicit function theorem for any integer $1\leq i\leq m$. It must exist a solution manifold $S_{\mathscr{F}_i}\subset\mathbb{R}^n$ with $\mathscr{F}_i: S_{\mathscr{F}_i}\to 0$ for any integer $1\leq i\leq m$. Then, the condition for system (3.1) of equations having no solution or having a solution geometrically is

$$\bigcap_{i=1}^{p} S_{\mathscr{F}_i}=\emptyset \quad \text{or} \quad \bigcap_{i=1}^{p} S_{\mathscr{F}_i}\neq\emptyset. \tag{3.2}$$

In this case, *how to explain that system* (3.1) *has or has no solution?* Notice that the solution of system (3.1) represents only the overlap state of elements $P_1,P_2,\cdots,P_m$ at time t, not the state of elements $P_1,P_2,\cdots,P_m$ because the state of element P_i is the solution manifold $S_{\mathscr{F}_i}, 1\leq i\leq m$. Correspondingly, the non-solvable case of system (3.1) indicates only that there are no overlap state in system elements, not that the they do not exist because the states of elements $P_1,P_2,\cdots,P_p$ are described by the solution manifold $S_{\mathscr{F}_i}$ for integers $1\leq i\leq m$. And so, the system state of a non-harmonious group $\mathcal{T}$ should be described by Smarandache multi-space $\bigcup\limits_{i=1}^{m} S_{\mathscr{F}_i}$, not $\bigcap\limits_{i=1}^{m} S_{\mathscr{F}_i}$ as the usual, namely the system solution of equation (3.1) of a non-harmonious group $\mathcal{T}$ should be characterized by a combinatorial manifold $G^L[\widetilde{S}]$.

Theorem 3.2([29][31]) *For any integer $m\geq 1$, the combinatorial solution or G-solution to system* (3.1) *of a non-harmonious group $\mathcal{T}$ is a combinatorial manifold $\widetilde{S}$ inherited a topological graph $G^L[\widetilde{S}]$ with*

$$\begin{aligned} V\left(G^L[\widetilde{S}]\right) &= \{S_{\mathscr{F}_i},\ 1\leq i\leq m\};\\ E\left(G^L[\widetilde{S}]\right) &= \left\{(S_{\mathscr{F}_i},S_{\mathscr{F}_j})|S_{\mathscr{F}_i}\bigcap S_{\mathscr{F}_j}\neq\emptyset, 1\leq i,j\leq m\right\}\end{aligned}$$

with a labelling

$$L:\ S_{\mathscr{F}_i}\to S_{\mathscr{F}_i},\quad (S_{\mathscr{F}_i},S_{\mathscr{F}_j})\to S_{\mathscr{F}_i}\bigcap S_{\mathscr{F}_j},\quad 1\leq i,j\leq m.$$

For example, let the orbit of a horse in families A or B running on a straight line $ax+by=c$ be a point set $L_{a,b,c}=\{(x,y)|ax+by=c, ab\neq 0\}$. Then, the system states of horse families A and B can be characterized respectively by the combinatorial solutions $C_4^L[LES_4^N]$ and $K_4^L[LES_4^S]$ of equation systems (LES_4^N) and (LES_4^S), such as those shown in Figure 9,

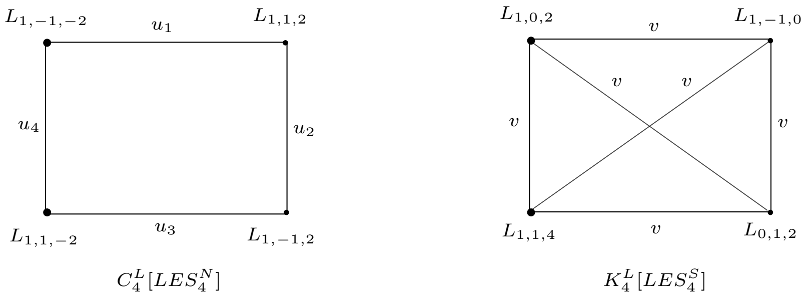


Figure 9. Combinatorial solutions of systems (LES_4^N) and (LES_4^S)

where

$$u_1 = L_{1,-1,-1}\bigcap L_{1,1,2}, \quad u_2 = L_{1,1,2}\bigcap L_{1,-1,2},$$
$$u_3 = L_{1,-1,2}\bigcap L_{1,1,-2}, \quad u_4 = L_{1,1,-2}\bigcap L_{1,-1,-2}.$$

Generally, we can apply Theorem 3.2, i.e., a combinatorial approach to discuss the non-solvable systems of algebraic equation, ordinary differential equations or partial differential equation for describing the system states of non-harmonious groups with the stability of systems, including biological systems that they correspond, respectively. Furthermore, the non-solvable systems of homogeneous algebraic equations in 3-variables can be used to determine the genus $g(\widetilde{S})$ of combinatorial surfaces $\widetilde{S}$ and normalization of complex non-singular curves, etc., see [17]-[18],[21],[23],[25] for details. Let us take the systems $(LDES_m^1)$ and (LDE_m^n) of non-solvable ordinary differential equations

$$(LDES_m^1)\begin{cases}\dot{X}=A_1X\\ \dot{X}=A_2X\\ \cdots\cdots\cdots\\ \dot{X}=A_mX\end{cases}, \qquad (LDE_m^n)\begin{cases}x^{(n)}+a_{11}^{[0]}x^{(n-1)}+\cdots+a_{1n}^{[0]}x=0\\ x^{(n)}+a_{21}^{[0]}x^{(n-1)}+\cdots+a_{2n}^{[0]}x=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x^{(n)}+a_{m1}^{[0]}x^{(n-1)}+\cdots+a_{mn}^{[0]}x=0\end{cases}$$

as examples, which are a system of 1-order of linear ordinary differential equations and an n-order of linear differential equations with constant coefficients respectively, where $a_{ij}^{[k]}$ is a real number for integers $0\leq k\leq m$, $1\leq i\leq n$, $1\leq j\leq s$, $A_k=\left(a_{ij}^k\right)_{n\times s}$ is a matrix and $X=(x_1,x_2,\cdots,x_n)^T$. Notice that the solution manifolds of $(LDES_m^1)$ and (LDE_m^n) are both linear spaces spanned by their basic solutions. We can therefore replace labels $\mathcal{S}_{\mathscr{F}_i}$, $\mathcal{S}_{\mathscr{F}_j}, 1\leq i,j\leq m$ in Theorem 3.2 by basic solutions and then, obtain their uniquely basic graphs $G^L[LDES_m^1]$ and $G^L[LDE_m^n]$ of systems $(LDES_m^1)$ and (LDE_m^n), respectively. For example, let $m=6$ with a system of linear ordinary differential equations

$$(LDES_6^1)\begin{cases}\ddot{x}-3\dot{x}+2x=0 & (1)\\ \ddot{x}-5\dot{x}+6x=0 & (2)\\ \ddot{x}-7\dot{x}+12x=0 & (3)\\ \ddot{x}-9\dot{x}+20x=0 & (4)\\ \ddot{x}-11\dot{x}+30x=0 & (5)\\ \ddot{x}-7\dot{x}+6x=0 & (6)\end{cases}$$

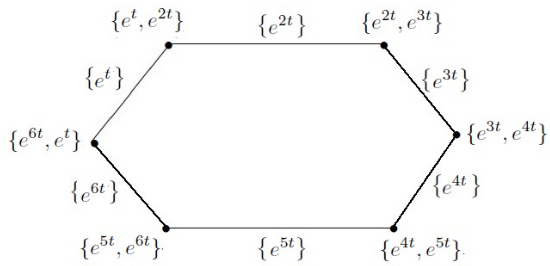


Figure 10. A basic graph

Then, the basic solutions of differential equations (1) – (6) are respectively

$$\{e^t,e^{2t}\},\{e^{2t},e^{3t}\},\{e^{3t},e^{4t}\},\{e^{4t},e^{5t}\},\{e^{5t},e^{6t}\},\{e^{6t},e^t\},$$

with a combinatorial solution or basic graph $G^L[LDES_6^1]$ shown in Figure 10.

Notice that there always exists a combinatorial solution $G^L[\widetilde{S}]$ of a non-harmonious group

(3.1) by Theorem 3.2, which enables us to introduce the system stability of non-harmonious group. For example, a combinatorial solution $G[LDES_m^1]$ or $G^L[LDE_m^n]$ is said to be *prod-stable* or *asymptotically prod-stable* if

$$\left\| \prod_{v\in V(H)} Y_v(t) - \prod_{v\in V(H)} X_v(t) \right\| < \varepsilon \quad \text{or} \quad \lim_{t\to 0} \left\| \prod_{v\in V(H)} Y_v(t) - \prod_{v\in V(H)} X_v(t) \right\| = 0. \tag{3.3}$$

holds with all solutions $Y_v(t)$, $v \in V(G^L)$ of $G[LDES_m^1]$ or $G^L[LDE_m^n]$ with $\|Y_v(0) - X_v(0)\| < \delta_v$ exists for all $t \geq 0$. In this case, we have

Theorem 3.3([18]) *A combinatorial zero-solution, i.e., all labels on basic graphs $G^L[LDES_m^1]$ and $G^L[LDE_m^n]$ are 0 of systems $(LDES_m^1)$ and $(LDES_m^n)$ of linear homogeneous differential equations is asymptotically prod-stable if and only if*

$$\sum_{v\in V(G^L[LDES_m^1])} \mathrm{Re}\alpha_v \ < \ 0 \quad \textit{or} \quad \sum_{v\in V(G^L[LDE_m^n])} \mathrm{Re}\lambda_v \ < \ 0 \tag{3.4}$$

for any basic solutions $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ of $(LDES_m^1)$ or $t^{l_v}e^{\lambda_v t} \in \mathscr{C}_v$ of (LDE_m^n).

In classical meaning, all systems correspondent to non-harmonious groups (3.1) are non-mathematical systems in general, namely they do not comply with the principle of logical consistency. So, *how to transform a non-mathematical system into a mathematical system and characterize the evolution of thing?* The answer is to transform a system of non-mathematics to mathematics by combinatorial approach discussed profitably in [19], including those of non-groups, non-rings, non-fields, non-solvable equations in algebra, non-solvable differential equations in calculus, non-spaces, non-manifolds, non-differentiable manifolds in geometry and others, which are all decomposing into mathematical systems over topological graphs G^L. This is exactly the application of my CC conjecture, i.e., mathematical combinatorics on non-mathematical groups.

§4. Continuity Flows

In the human recognition, the essence of subdividing a substance T into microscopic particles such as those of elementary particles, cells or genes by reduction is holding on the reality of thing T by the combinatorial solution $G^L[\widetilde{S}]$ on the behavior of microscopic particles. Notice that an edge $(S_{\mathscr{F}_i}, S_{\mathscr{F}_i}) \in V(G^L[\widetilde{S}])$ if and only if $S_{\mathscr{F}_i} \bigcap S_{\mathscr{F}_i} \neq \emptyset$ by definition, namely the action $S_{\mathscr{F}_i}$ and $S_{\mathscr{F}_i}$ in system $\widetilde{S}$ is symmetric. However, the interaction between particles or in general, the energy, information transmission are mostly not symmetric but a unidirectional one. For example, the transformation of energy with conservation. According to the uncertainty principle of microscopic particle, the random evolution of particle is introduced and then, the microscopic particle is described by complex networks ([5]). For example, Barabaśi and Albert described the growth of nodes of complex network by randomness in [2],[3]. But, *is the mechanism of natural evolution really random?* If so, *how can it be possible to describe the evolution of thing by random models established by humans?* Among them, a necessary way should be to understand the non-harmonious groups $\widetilde{S}$ by the recognizability of thing rather than infinitely subdividing a thing

T into microscopic particles or recognizing a thing T by randomness because the randomness is a by-product of the limitation of human recognition for lacking of information, may not the truth colour of thing T.

I reflected deeply the previous questions on the combinatorial solutions $G^L[\widetilde{S}]$ of non-harmonious groups in 2014. After finishing the paper [19], I began to think about how a combinational notion should correctly describe the evolution of thing T by reduction in human recognition and then, vectors $\mathbf{v}$ in a Banach space $\mathscr{B}$ were used for labelling the edge of topological graph or network G^L following with the conservation law at each vertex of G^L ([20]), i.e., the $\overrightarrow{G}$*-flow*, a generalization of network with operations. Subsequently, I proposed the *action flow* $\overrightarrow{G}^L$ in my plenary report at the "*National Conference on Emerging Trends in Mathematics and Mathematical Sciences*" (December 17-19, 2015, Kolkata) invited by the *Calcutta Mathematical Society* as an honorary guest, which is an extension from the conservation of vertices in $\overrightarrow{G}$-flow to the conservation of action flows by edge operators ([22]) and then, the *continuity flow* $\overrightarrow{G}^L$([24]) was put forward in my J.C.& K.L.Saha memorial lecture at the "*International Conference on Geometry and Mathematical Models in Complex Phenomena*" (December 5-7, 2017, Kolkata), which can be viewed also as a mathematical element with algebraic, differential and integral operations. In this way, the Banach and Hilbert flow spaces are established for providing theories of human recognition, especially for the reduction step by step.

So, *what is a continuity flow?* A continuity flow $(\overrightarrow{G}; L, \mathscr{A})$ is essentially a Banach space over a topological graph G^L, a generalization of mathematics by applying my CC conjecture.

Definition 4.1([24]) *A continuity flow* $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ *is an oriented topological graph* $\overrightarrow{G}^L$ *in space* $\mathscr{S}$ *associated with a mapping* $L: v \to L(v),\ (v,u) \to L(v,u)$, 2 *end-operators* $A^+_{vu}: L(v,u) \to L^{A^+_{vu}}(v,u)$ *and* $A^+_{uv}: L(u,v) \to L^{A^+_{uv}}(u,v)$ *on a Banach space* $\mathscr{B}$ *over a field* $\mathscr{F}$ *such as those shown in Figure* 11 *with* $L(v,u) = -L(u,v)$, $A^+_{vu}(-L(v,u)) = -L^{A^+_{vu}}(v,u)$ *for* $\forall (v,u) \in E\left(G^L\right)$ *and meanwhile, holding with the continuity equation*

$$\sum_{u \in N^-_G(v)} L^{A^+_{uv}}(u,v) - \sum_{u \in N^+_G(v)} L^{A^+_{uv}}(u,v) \ = \ L(v) \tag{4.1}$$

at any vertex $v \in V(G^L)$ *of topological graph* G^L, *where* $N^-_G(v), N^+_G(v)$ *are respectively the in-neighborhood and out-neighborhood of vertex* $v \in V(G^L)$, *namely all vertices in* $N^-_G(v) \subset N_G(v)$ *or* $N^+_G(v) \subset N_G(v)$ *flow into or out of the vertex* v *and* $N^-_G(v) \cup N^+_G(v) = N_G(v)$.

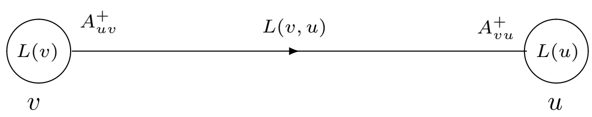


Figure 11. Flow with end-operators on an edge

Now, *why is the continuity flow important to human?* The answer is that the continuity flow provides us with a mathematical support for reduction on thing T, also answer the 3 questions in Section 1 because the result of human recognition by reduction to a thing T is such a continuity flow G^L. However, all existing sciences, including the mathematics can be only used for describing the evolution of a particle or particles of a system evolving all in synchronization, namely there are few evolutionary theory that regards a thing T as a self-organized or self-adjusted system by mathematics.

4.1.Continuity Flow Space. All operations on continuity flows are the composition of the union of topological graphs with composition of mappings. Generally, let $G^L, G'^{L'}$ be continuity flows on Banach space $\mathscr{B}$ over field $\mathscr{F}$, $\lambda \in \mathscr{F}$. Then, the addition, multiplication and scalar multiplication on continuity flows are defined by

$$\begin{aligned} G^L + G'^{L'} &= (G \setminus G')^L \bigcup \left(G \bigcap G'\right)^{L+L'} \bigcup (G' \setminus G)^{L'}, &(4.2)\\ G^L \cdot G'^{L'} &= (G \setminus G')^L \bigcup \left(G \bigcap G'\right)^{L\cdot L'} \bigcup (G' \setminus G)^{L'}, &(4.3)\\ \lambda \cdot G^L &= G^{\lambda\cdot L} &(4.4)\end{aligned}$$

where $L(v), L'(v), L(v,u), L'(v,u) \in \mathscr{B}$ for any vertex $v \in V(G)$ and edge $(v,u) \in E(G)$ with

$$\begin{aligned} L + L' &: v \to L(v) + L'(v),\ (v,u) \to L(v,u) + L'(v,u),\\ L \cdot L' &: v \to L(v) \cdot L'(v),\ (v,u) \to L(v,u) \cdot L'(v,u),\\ \lambda \cdot L &: v \to \lambda \cdot L(v),\ (v,u) \to \lambda \cdot L(v,u), \end{aligned}$$

and $L(v) \cdot L'(v)$, $L(v,u) \cdot L'(v,u)$ denote the operation of Hadamard product on vectors in Banach space $\mathscr{B}$, namely

$$(x_1, x_2, \cdots, x_n) \cdot (y_1, y_2, \cdots, y_n) = (x_1y_1, x_2y_2, \cdots, x_ny_n). \tag{4.5}$$

Generally, let $\mathscr{G} = \{G_1, G_2, \cdots, G_m\}$ be a closed family under the union operation of topological graphs and let $\mathscr{B}$ be a Banach space. All continuity flows with vectors in $\mathscr{B}$ over topological graph $G^L \in \mathscr{G}$ are denoted by $\mathscr{G}_{\mathscr{B}} = \{G^L | G \in \mathscr{G}, L : v \to L(v) \in \mathscr{B}, (v,u) \to L(v,u) \in \mathscr{B}, v \in V(G), (v,u) \in E(G)\}$. Then, *$(\mathscr{G}_{\mathscr{B}}; +, \cdot)$ is a bigroup under operations of addition "+" and Hadamard product "·"*. Furthermore, *if $(\mathscr{B}; +, \cdot)$ is a field and all end-operators are $1_{\mathscr{B}}$ on any continuity flow $G^L \in \mathscr{G}_{\mathscr{B}}$, then the bigroup $(\mathscr{G}_{\mathscr{B}}; +, \cdot)$ is also a field.* In addition, *$(\mathscr{G}_{\mathscr{B}}; \mathscr{F})$ is always a linear space under operations of addition and scalar multiplication.*

In this case, if each end-operator on a continuity flow $G^L \in \mathscr{G}_{\mathscr{B}}$ is linearly continuous on $\mathscr{B}$, then the norm of continuity flow G^L is defined by

$$\left\|G^L\right\| = \sum_{(v,u)\in E(G)} \left\|L^{A^+_{vu}}(v,u)\right\|, \tag{4.6}$$

where $\|\cdot\|$ denotes the norm on Banach space $\mathscr{B}$. And then, *the continuity flow space $(\mathscr{G}_{\mathscr{B}}; \mathscr{F})$ is a normed space.* Furthermore, for any $G^L, G'^{L'} \in \mathscr{G}_{\mathscr{B}}$ define the metric of continuity flows $G^L, G'^{L'}$ to be

$$\rho\left(G^L, G'^{L'}\right) = \left\|G^L - G'^{L'}\right\|. \tag{4.7}$$

Then, *each Cauchy sequence in continuity flow space $(\mathscr{G}_{\mathscr{B}}; \mathscr{F})$ is complete*, and *if the linear space $(\mathscr{B}; \mathscr{F})$ is a Banach or Hilbert space, then $(\mathscr{G}_{\mathscr{B}}; \mathscr{F})$ is a Banach or Hilbert space* ([24]).

4.2.G-Isomorphic Operator. By functional analysis, an operator on Banach space $\mathscr{B}$ maps one vector to another one. As a Banach space, an operator on continuity flow space $(\mathscr{G}_{\mathscr{B}};\mathscr{F})$ can be also defined to be a mapping that maps one continuity flow G^L to another $G'^{L'}$. However, such a definition does not reflect the topological nature in continuous flow G^L. So, it is of little significance. Whence, it is necessary to define a typical operator that leaves the topological structure of continuity flow G^L unchanged, which is nothing else but the G-isomorphic operator.

Definition 4.2([29],[31]) *Let $G_1^{L_1}, G_2^{L_2} \in \mathscr{G}_{\mathscr{B}}$ be continuity flows. A mapping $f : G_1^{L_1} \to G_2^{L_2}$ is said to be a G-isomorphic operator between continuity flows $G_1^{L_1}$, $G_2^{L_2}$ and the continuity flow $G_1^{L_1}$ is said to be G-isomorphic to $G_2^{L_2}$ if*

(1) *G_1, G_2 are isomorphic in graphs, i.e., there is an isomorphism $\varphi : G_1 \to G_2$ of graph;*

(2) *$L_2 = f\circ\varphi\circ L_1$ for $\forall(v,u)\in E(G_1)$.*

Notice that Definition 4.2 can be applicable only if $G_1^{L_1}, G_2L_2$ are isomorphic in labeled graphs, which should be extended to the general case. Usually, it is conventionalized that $\widehat{G}^{\widehat{L}} = G^L$ for a topological graph $\widehat{G} \supset G$ if $\widehat{L}(x) = L(x)$ for $x \in V(G)\cup E(G)$ and $\widehat{L}(x) = \mathbf{0}$ for $x \notin V(G)\cup E(G)$, which reflects the essence of continuity flow. And by this convention, a $\widehat{G}$-isomorphism between continuity flows $G_1^{L_1}, G_2^{L_2}$ can be generally defined even if $G_1^{L_1}, G_2^{L_2}$ are non-isomorphic but with a supergraph $\widehat{G}$ as $\widehat{G} \supseteq G_1 \bigcup G_2$.

Definition 4.3([29],[31]) *A mapping $f : G_1^{L_1} \to G_2^{L_2}$ is said to be a G-isomorphic operator between continuity flows $G_1^{L_1}$ and $G_2^{L_2}$ if*

(1) *there is an isomorphism $\varphi : \widehat{G} \to \widehat{G}$ with $\widehat{G} \supset G_1, G_2$ in graph;*

(2) *for $\forall(v,u)\in E(G_1)$ there is $L_2 = f\circ\varphi\circ L_1$ but for $\forall(v,u)\in E(G_2\setminus G_1)$, $f : \mathbf{0} \to L_2(v,u)$ and for $\forall(v,u)\in E(G_1\backslash G_2)$ and $\forall(v,u)\in E(\widehat{G}\backslash(G_1\bigcup G_2))$, $f : L(v,u) \to \mathbf{0}$.*

Particularly, let $\varphi = \mathrm{id}_G$, i.e., the identity mapping on topological graph G. Then, a G-isomorphic operator f is determined by the equation

$$L_2(v,u) \;=\; f\circ L_1(v,u), \quad \forall(v,u)\in E(G) \tag{4.8}$$

and the linearity of G-isomorphic operator with its nature such as the continuous, bounded, image and others can be introduced similar to the usual Banach space, and generalize a few of well-known results such as those of *a G-isomorphic linear operator $f : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}}$ is continuous if and only if it is bounded* and *if $f : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}}$ is closed then f is continuous*, etc. Furthermore, if the Banach space $\mathscr{B}$ is a function field on variable $\mathbf{x}$ then the G-isomorphic equation (4.8) is equivalent to

$$f\left(G^L[\mathbf{x}]\right) \;=\; G^{f(L[\mathbf{x}])}. \tag{4.9}$$

For example, let $\mathscr{B}$ be a real number field. We can construct the power $G^{aL}[\mathbf{x}]$ and exponent $a^{G^L[\mathbf{x}]}$ of a continuity flow $G^L[\mathbf{x}]$, define the limitation of continuity flow sequence, differential and integral operations, i.e., the theory of calculus on continuity flows $G^L[\mathbf{x}]$ and obtain the fundamental theorem ([29])

$$\int_a^b f\frac{d}{dt}\left(\overrightarrow{G}^L[t]\right)dt = f\left(\overrightarrow{G}^L[t]\right)\Big|_{t=b} - f\left(\overrightarrow{G}^L[t]\right)\Big|_{t=a} \tag{4.10}$$

similar to that of calculus.

In this case, assume the mapping $\mathscr{L} : (v,u) \in E(G) \to \mathscr{L}[\mathcal{L}(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))(v,u)]$ is differentiable and commutative with all end-operators A^+_{vu}, then the action $J\left[G^{\mathscr{L}}[t]\right]$ and variation $\delta J\left[G^{\mathscr{L}}[t]\right]$ on a continuity flow $G^{\mathscr{L}}[t]$ are respectively defined by

$$J\left[G^{\mathscr{L}}[t]\right] = \left|\int_{t_1}^{t_2} G^{\mathscr{L}[\mathcal{L}(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))]}dt\right|, \quad \delta J\left[G^{\mathscr{L}}[t]\right] = \left|\delta\int_{t_1}^{t_2} G^{\mathscr{L}[\mathcal{L}(t,\mathbf{q}(t),\dot{\mathbf{q}}(t))]}dt\right|, \tag{4.10}$$

where the variation $\delta : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}}$ is a G-isomorphic operator. Then, by the least action principle $\delta J\left[G^{\mathscr{L}}[t]\right](v,u) = 0$ for $\forall (v,u) \in E\left(G^{\mathscr{L}}[t]\right)$ and the norm property in Banach flow space $\mathscr{G}_{\mathscr{B}}$, we can induce Euler-Lagrange equations on continuity flow $G^{\mathscr{L}}[t]$ to be

$$\frac{\partial G^{\mathscr{L}}}{\partial q_i} - \frac{d}{dt}\frac{\partial G^{\mathscr{L}}}{\partial \dot{q}_i} \ = \ \mathbf{O}, \quad 1 \le i \le n. \tag{4.11}$$

and the 3 interesting exponential identities ([29]) following

$$e^x \ = \ 1 + \frac{x}{1!} + \frac{x^{2]}}{2!} + \cdots + \frac{x^n}{n!} + \cdots, \tag{4.12}$$

$$e^{tA} \ = \ \mathbf{I} + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \cdots + \frac{t^nA^n}{n!} + \cdots, \tag{4.13}$$

$$e^{G^L[\mathbf{x}]} \ = \ \mathbf{I} + \frac{G^L[\mathbf{x}]}{1!} + \frac{G^{2L}[\mathbf{x}]}{2!} + \cdots + \frac{G^{nL}[\mathbf{x}]}{n!} + \cdots, \tag{4.14}$$

where for an integer $n \ge 1$, A is an $n \times n$ matrix, $G^L[\mathbf{x}]$ is a continuity flow that continuous in variable $\mathbf{x}$, namely the formula (4.12) is the exponential identity in calculus, (4.13) is the exponential identity on matrix A which is a generalization of the exponential identity (4.12) and (4.14) is the exponential identity on continuity flow $G^L[\mathbf{x}]$, which is a generalization of the exponential identity (4.13) on matrix A.

Notice that a linear functional $f : \mathscr{B} \to \mathbb{R}$ or $\mathbb{C}$ is a linear operator on a Banach space by definition. So, *how to extend a linear functional on a Banach space to the Banach flow space $\mathscr{G}_{\mathscr{B}}$?* If there really exists such a G-isomorphic linear operator $f : \mathscr{G}_{\mathscr{B}} \to \mathbb{R}$ or $\mathbb{C}$, f is referred to a functional on $\mathscr{G}_{\mathscr{B}}$. And so, a fundamental question is to determine *whether there exists a continuously linear functional on the Banach flow space $\mathscr{G}_{\mathscr{B}}$?* The answer is certainly yes because we can generalize the functional extension theorem, i.e., Hahn-Banach theorem on Banach space to the Banach flow space $\mathscr{G}_{\mathscr{B}}$.

Theorem 4.4([26]) *Let $\mathscr{H}_{\mathscr{B}}$ be a subspace of Banach flow space $\mathscr{G}_{\mathscr{B}}$, $F : \mathscr{H}_{\mathscr{B}} \to \mathbb{C}$ is a continuously linear G-isomorphic functional on $\mathscr{H}_{\mathscr{B}}$. Then, there is a continuously linear G-isomorphic functional $\widetilde{F} : \mathscr{G}_{\mathscr{B}} \to \mathbb{C}$ holding with (i) if $G^L \in \mathscr{H}_{\mathscr{B}}$ then $\widetilde{F}\left(G^L\right) = F\left(G^L\right)$ and (ii) $\|\widetilde{F}\| = \|F\|$.*

Particularly, if $\mathbf{O} \ne G_0^{L_0} \in \mathscr{G}_{\mathscr{B}}$, there is a continuously linear G-isomorphic functional F such that $\|F\| = 1$ and $\|F(G_0^{L_0})\| = \|G_0^{L_0}\|$, where $G_0^{L_0}$ is the continuity flow $\mathbf{O}$, i.e., $L : v \to \mathbf{0}$ and $(v,u) \to \mathbf{0}$ for $\forall \in V(G_0^{L_0})$ and $\forall (v,u) \in E(G_0^{L_0})$.

Furthermore, by Theorem 4.4 of the functional extension theorem on $\mathscr{G}_{\mathscr{B}}$ we have

Corollary 4.5 *For a continuity flow $G^L \in \mathscr{G}_{\mathscr{B}}$ if $F\left(G^L\right) = 0$ holds with all linear functionals F, there must be $\overrightarrow{G}^L = \mathbf{O}$.*

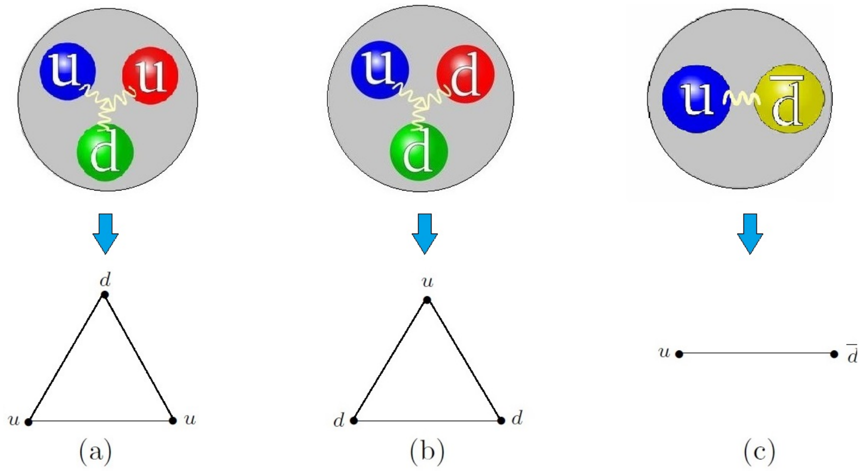


Figure 12. Hadron's quark model with continuity flow

Notice that there are 3 assumptions in quantum mechanics with a Hilbert space as the model of quantum states ([35]), i.e., (i) A pure state of quantum can be characterized in terms of a normalized vector $|\psi\rangle$ in Hilbert space $\mathcal{H}$ with $\langle\psi|\psi\rangle = 1$; (ii) For a physical quantity a of quantum, an observation on a in state $|\psi\rangle$ is the eigenvalue λ_j of an Hermitian operator A acting on $\mathcal{H}$ that exists, i.e., $A\,|\lambda_j\rangle = \lambda_j\,|\lambda_j\rangle$; (iii) The evolution of quantum state is governed by Schrödinger equation $i\hbar d\,|\psi\rangle/dt = H\,|\psi\rangle$, where $\hbar$ is the Planck's constant and H denotes a Hermitian operator corresponding to the energy of system. However, if we describe a microscopic particle such as a hadron, i.e., a proton, neutron or a meson by quarks, its model is no longer a particle but a continuity flow G^L ([31],[35]) shown in Figure 12.

In this case, *are there any reason to conclude there is a Hermitian operator A holding with the* 3 *assumptions in quantum mechanics?* Certainly, there are no such an affirmatively answer unless we priorly assume that all quarks $u, d, \overline{d}$ are evolving in synchronization. However, this assumption is incorrect on a self-organizing or self-regulating system such as a biological system consisting of cells. But its correctness can be verified by Theorem 4.4, i.e., let $f : \mathscr{G}_{\mathscr{B}} \to \mathbb{C}$ be a continuously linear G-isomorphism on continuity flows G^L. Then, there exists a uniquely continuity flow $\widehat{G}^{\widehat{L}} \in \mathscr{G}_{\mathscr{B}}$ holding with $f\left(G^L\right) = \left\langle G^L, \widehat{G}^{\widehat{L}}\right\rangle$ for a continuity flow $G^L \in \mathscr{G}_{\mathscr{B}}$ by Theorem 4.4, i.e., no matter how we subdivide a particle into a continuity flow G^L, there always exists a Hermitian operator A which holds with the 3 assumptions in quantum mechanics.

4.4.Example. A typical example of continuity flow G^L is the twelve meridians on human body ([31]), which consist of the lung meridian of Hand-Taiyin (LU) belongs to the lung and connects with the large intestine, the heart meridian of Hand-Shaoyin (HT) belongs to the heart and connects with the small intestine, the pericardium meridian of Hand-Jueyin (PC) is belongs to the pericardium and connects with the Sanjiao, the spleen meridian of Foot-Taiyin (SP) belongs to the spleen and connects with the stomach, the kidney meridian of Foot-Shaoyin (KI) belongs to the kidney and connects with the bladder, the liver meridian of Foot-Jueyin (LR) belongs to the liver and connects with gallbladder; the large intestine meridian of Hand-Yangming

(LI) belongs to the large intestine and connects with the lung, the small intestine meridian of Hand-Taiyang (SI) belongs to the small intestine and connects with the heart, the Sanjiao meridian of Hand-Shaoyang (SJ) belongs to the Sanjiao and connects with the pericardium, the stomach meridian of Foot-Yangming (ST) belongs to the stomach and connects with the spleen, the bladder meridian of Foot-Taiyang (BL) belongs to the bladder and connects with the kidney, and the gallbladder meridian of Foot-Shaoyang (GB) belongs to the gallbladder and connects with the liver of human body. All of the meridians on the way are shown in Figure 13 with 310 acupoints in the national standard of China, i.e., *Body Model for Both Meridian and Extraordinary Points of China* (GB 12346-90) for details.

Figure 13. Twelve meridians on human body

Now, *how to construct a continuity flow G^L on 12 meridians of human body?* Certainly, we are easily to construct a continuity flow G^L of human body, namely let all vertices v of continuity flow G^L be all acupoints on the twelve meridians of human body and let all edges (v,u) be paths between two successive acupoints v, u on one of the twelve meridians with a labelling $L : V \to$ *internal organs* of human body and $L : (v,u) \to$ *vital energy flow* $L(v,u)$ on (v,u) of human meridians. Notice that the flow of vital energy between two successive acupoints varies at different times of the day and depends on the state of human body. And so, the traditional Chinese medicine classifies $L(v,u)$ into two parts, i.e., *Yin* $\mathbf{Y}^-$ and *Yang* $\mathbf{Y}^+$, which can be regarded as a pair of interacting vectors and believes that the essence of normal operating of human body lies in the balance of Yin and Yang following the natural law, i.e., for $\forall v \in V(G^L)$ and any direction $\overrightarrow{O}$ on v in space, there must be

$$\mathbf{Y}^-(v) + \mathbf{Y}^+(v) \ = \ \mathbf{C}\left(v, \overrightarrow{O}\right) \tag{4.13}$$

which is a general ruler for determining whether the human body is operating normally, where $\mathbf{C}(v, \overrightarrow{O})$ is a constant vector at the point v in direction $\overrightarrow{O}$. Thus, an illness of human body is abstractly equivalent to the imbalance of Yin $\mathbf{Y}^-$ and Yang $\mathbf{Y}^+$ on some meridians of human body, and it is necessary to adjust the flows of vital energy on human meridians by "*reducing the excess with supply the insufficient*" to recover the balance of Yin $\mathbf{Y}^-$ and Yang $\mathbf{Y}^+$, i.e., the natural law of vital energy operating on human body.

§5. Conclusion

Clearly, the original motivation of CC conjecture is to apply the combinatorial approach for extending mathematical sciences in order to improve human's ability for recognizing the nature. But since it is a conjecture, it is necessary to give a proof on its correctness. So, *how to prove its correctness of CC conjecture?* In fact, the CC conjecture is not so much as a mathematical conjecture but a recognitive thought. Its correctness lies in the fact that humans extend their local recognitions of thing T to the whole, i.e., by characteristics of $\chi_1, \chi_2, \cdots$ of thing T showing up in front of human. Certainly, the recognitive conclusion must be a combination of local recognitive outcomes on characteristics of $\chi_1, \chi_2, \cdots$ over an inherited 1-dimensional topological structure G^L of thing T, namely the human recognition on reality of thing T can only be a combinatorial one ([31]), including the science and technology ([1],[32]). This is exactly what the sophist told the blind men in fable of the blind men with an elephant. Surely, the mathematics follows the principle of logical consistency in human recognition, which inevitably leads to the limitation of effect in human recognition by mathematics, namely the mathematical reality $T_{\mathcal{M}}$ is only a local recognitive or conditional conclusion. My CC conjecture is only consistent with the extension of human recognition from the local to the whole. That is, the combinatorial approach on reality $T_{\mathcal{M}}$ of thing T is prior to human recognition, including science and mathematics, which is a philosophy of human recognition, no proof further is required, namely the combinatorial notion implied by the human recognition of reduction is on the first and the recognition or science is followed, only on the second, i.e., the essence of mathematical combinatorics following.

Mathematical Combinatorics. *All mathematical sciences should be generalized or reconstructed over topological graph G^L which is consistent with the 1-dimensional topological graph G^L inherited in human recognition of thing by reduction.*

Notice that the "*reduction*" here is subdivided thing T into the recognizable elements that humans understand the reality of thing T, not an infinitely subdividing of thing T and there is essentially no need to subdivide any thing $T \to$ molecule $\to$ atom $\to$ elementary particle or any living $T \to$ biological macromolecule $\to$ cell $\to$ gene. Otherwise, it will artificially cause the complexity in the recognition with no benefits for recognizing the truth colour of thing. For example, the number of cells of an adult is about $4 \times 10^{14} - 6 \times 10^{14}$. So, *does an analysis on the behavior of an adult need to subdivide it into cells?* The answer is certainly not because the essence of reducing of thing T into elementary particles, cells and genes is to treat thing T as a complex network, which simultaneously increases the complexity in recognition. In contrast, the Chinese science established on the interaction of Yin $\mathbf{Y}^-$ and Yang $\mathbf{Y}^+$, the promoting and restraining potentials of five elements, i.e., the *metal, wood, water, fire, earth* is more suitable for leading the developing of humans, which is essentially a system science for the harmonious coexistence of humans with nature on the human recognition in today's terms.

In 2003, Prof.Tagmark of *Massachusetts Institute of Technology* proposed a *mathematical universe hypothesis* ([42]) which claims that *the natural reality outside of human is a mathematical structure, namely the universe can not only be characterized by mathematics but the universe itself is a mathematical structure.* Of course, this is an encouraging hypothesis that excites most researchers because it allows mathematics to describe and model the evolution of everything in the universe. However, *can the mathematics that follows the principle of logical consistency be applied already to describe the reality of everything in the universe?* The answer is certainly not because the mathematical reality is still a local or conditional one. It is essentially not equivalent to the reality of thing. And meanwhile, this hypothesis can not be verified because humans have not yet been able to arrive at each corner in the universe. In this case, it is necessary to extend mathematics including contradictions, i.e., Smarandache multi-spaces over the 1-dimensional topological graphs G^L inherited in things in order to understand the nature of things. This is my philosophy of mathematical combinatorics, which includes also the mathematical universe hypothesis of Prof.Tagmark as a deduction.

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Semi-Invariant Submanifolds of (k,μ)-Contact Manifold Admitting Semi-Symmetric Metric Connection

M.S. Siddesha

(Department of Mathematics, Jain Deemed-to-be-University, Global Campus-562112, Karnataka, India)

C.S. Bagewadi

(Department of Mathematics, Kuvempu University, Shankaraghatta, Shimoga-577451, Karnataka, India)

E-mail: mssiddesha@gmail.com, prof_bagewadi@yahoo.co.in

Abstract: In this paper, we define a semi-symmetric metric connection in a (k,μ)-contact manifold and study semi-invariant submanifolds of a (k,μ)-contact manifold endowed with a semi-symmetric metric connection. We determine the integrability conditions of distributions on semi-invariant submanifolds of a (k,μ)-contact manifold with a semi-symmetric metric connection.

Key Words: (k,μ)-contact manifold, semi-invariant submanifold, semi-symmetric metric connection.

AMS(2010): 53C25, 53C17, 53C40.

§1. Introduction

The torsion tensor T of a linear connection ∇ on a Riemannian manifold M is given by

$$T(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y],\ \ X,Y\in\chi(M). \tag{1.1}$$

The connection ∇ is symmetric if its torsion tensor T vanishes identically, otherwise it is non-symmetric. Again a linear connection ∇ is said to be a semi-symmetric connection if, the torsion T of the connection ∇ satisfies

$$T(X,Y)=\eta(Y)X-\eta(X)Y, \tag{1.2}$$

where η is a 1-form and ϕ is a tensor field of type $(1,1)$. Further, a connection ∇ is called metric connection on a Riemannian manifold M if

$$(\nabla_X g)(Y,Z)=0. \tag{1.3}$$

Otherwise, it is non-metric. The connection ∇ is said to be semi-symmetric metric connection

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[2]Corresponding author: M.S. Siddesha, E-mail: mssiddesha@gmail.com.

if it satisfies (1.1)-(1.3). In 1924, Friedmann and Schouten [8] were the pioneers who unveiled the notion of a semisymmetric linear connection on a Riemannian manifold. The subsequent introduction of the concept of a metric connection came about through the work of Hayden [9] in 1932.

In 1981, Bejancu and Papaghiuc introduced the idea of semi-invariant submanifolds, as a generalization of invariant and anti-invariant submanifolds of contact metric manifolds. On the other hand in 1995, Blair, Koufogiorgos and Papantoniou [5] introduced the new class of contact metric manifolds with ξ belonging to (k,μ)-nullity distributions which are known as (k,μ)-contact metric manifolds. In our previous work [12]-[17], we have investigated various categories of submanifolds, such as invariant, slant, and semi-slant submanifolds within these manifolds.

Recently, a noteworthy contribution was made by Avijit Sarkar et al. [10], who engaged in a study focused on semi-invariant submanifolds within the context of (k,μ)-contact manifolds. Moreover, an exploration of semi-invariant submanifolds extended to various classes of almost contact manifolds has attracted attention from several geometers such as [1, 2, 7, 11, 20] and other researchers.

Building upon the insights from the aforementioned research endeavors, this present paper embarks on an investigation into semi-invariant submanifolds within the realm of (k,μ)-contact manifolds with a semi-symmetric metric connection. The paper's structure unfolds as follows: In Section 2, a concise introduction to (k,μ)-contact manifolds sets the stage for the ensuing exploration. Moving into Section 3, the analysis demonstrates that the connection induced on semi-invariant submanifolds of a (k,μ)-contact manifold, endowed with a semi-symmetric metric connection, retains both the attributes of being semi-symmetric and metric. Section 4 is dedicated to laying the groundwork by presenting fundamental outcomes pertinent to semi-invariant submanifolds of (k,μ)-contact manifolds with a semi-symmetric metric connection. Concluding the paper, the final section delves into a comprehensive discussion regarding the integrability conditions governing distributions on semi-invariant submanifolds of (k,μ)-contact manifolds with semi-symmetric metric connection comes into play.

§2. (k,μ)-Contact Manifolds

A contact manifold is a $C^\infty-(2n+1)$ manifold $\tilde{M}^{2n+1}$ equipped with a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$ everywhere on $\tilde{M}^{2n+1}$. Given a contact form η it is well known that there exists a unique vector field ξ, called the characteristic vector field of η, such that $\eta(\xi)=1$ and $d\eta(X,\xi)=0$ for every vector field X on $\tilde{M}^{2n+1}$. A Riemannian metric is said to be associated metric if there exists a tensor field ϕ of type $(1,1)$ such that

$$\phi^2 = -I+\eta\otimes\xi,\quad \eta(\xi)=1,\quad \phi\xi=0,\quad \eta\cdot\phi=0, \tag{2.1}$$

$$g(\phi X,\phi Y) = g(X,Y)-\eta(X)\eta(Y),\quad g(X,\xi)=\eta(X), \tag{2.2}$$

for all vector fields $X,Y\in T\tilde{M}$. Then the structure (ϕ,ξ,η,g) on $\tilde{M}^{2n+1}$ is called a contact metric structure and the manifold $\tilde{M}^{2n+1}$ equipped with such a structure is called a contact metric manifold [4].

Given a contact metric manifold $\tilde{M}^{2n+1}(\phi, \xi, \eta, g)$, we define a $(1, 1)$ tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi \phi$, where $\mathcal{L}$ denotes Lie differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. Thus, if λ is an eigenvalue of h with eigenvector X, $-\lambda$ is also an eigenvalue with eigenvector ϕX. Also we have $Tr \cdot h = Tr \cdot \phi h = 0$ and $h\xi = 0$. Moreover, if $\tilde{\nabla}$ denotes the Riemannian connection of g, then the following relation holds:

$$\tilde{\nabla}_X \xi = -\phi X - \phi h X. \tag{2.3}$$

It is seen that the vector field ξ is a Killing vector with respect to g if and only if $h = 0$. In this case the manifold becomes a K-contact manifold. A K-contact structure on $\tilde{M}$ gives rise to an almost complex structure on the product $\tilde{M}^{2n+1} \times R$. If this almost complex structure is integrable, the contact metric manifold is said to be Sasakian. Equivalently, a contact metric manifold is Sasakian if and only if

$$\tilde{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

holds for all X, Y, where $\tilde{R}$ denotes the curvature tensor of the manifold $\tilde{M}$.

The (k, μ)-nullity distribution of a contact metric manifold $\tilde{M}^{2n+1}(\phi, \xi, \eta, g)$ is a distribution [5]

$$\begin{aligned} &N(k, \mu) : p \to N_p(k, \mu) = \{Z \in T_p M : \\ &R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY]\}. \end{aligned}$$

for any $X, Y \in T_p\tilde{M}$. Hence if the characteristic vector field ξ belongs to the (k, μ)-nullity distribution, then we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.4}$$

Thus a contact metric manifold satisfying the relation (2.4) is called a (k, μ)-contact metric manifold. In particular, if $\mu = 0$, then the notion of (k, μ)-nullity distribution reduces to the notion of k-nullity distribution, introduced by Tanno [19]. A (k, μ)-contact metric manifold is Sasakian if $k = 1$. In a (k, μ)-contact metric manifold the following relations hold [5]:

$$h^2 = (k - 1)\phi^2, \quad k \leq 1, \tag{2.5}$$

$$(\tilde{\nabla}_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX). \tag{2.6}$$

§3. Semi-Invariant Submanifolds

In this section, we introduce the notion of semi-invariant submanifold of a (k, μ)-contact manifold which generalizes the notion of both invariant and anti-invariant submanifolds.

A non degenerated submanifold M of a (k, μ)-contact manifold is called a semi-invariant submanifold, if there exists a pair of orthogonal distributions $\{D, D^{\perp}\}$ on M such that

(i) $TM = D \oplus D^{\perp} \oplus < \xi >$;
(ii) The distribution D is invariant under ϕ, that is $\phi D_x = D_x$, for each $x \in M$;
(iii) The distribution $D^{\perp}$ is anti-invariant under ϕ, that is $\phi D_x^{\perp} \subset T_x^{\perp} M$, for each $x \in M$.

A semi-invariant submanifold becomes invariant (resp. anti-invariant) submanifold if $D_x^{\perp} = 0$ (resp. $D_x = 0$) for all $x \in M$. Further, a submanifold which is neither invariant nor anti-invariant is called a proper semi-invariant submanifold.

We denote by $\tilde{\nabla}$ the Levi-Civita connection on $\tilde{M}$ with respect to induced metric g. Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{3.1}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{3.2}$$

for any tangent vector fields X, Y and the normal vector field N on M, where σ, A, ∇ and $\nabla^{\perp}$ are the second fundamental form, the shape operator, induced connection on M and the normal connection respectively. If the second fundamental form σ is identically zero, then the manifold is said to be totally geodesic. The second fundamental form σ and the shape operator A_N are related by

$$g(\sigma(X, Y), N) = g(A_N X, Y).$$

Now, we define a semi-symmetric metric connection $\bar{\bar{\tilde{\nabla}}}$ in a (k, μ)-contact manifold $\tilde{M}$ by

$$\bar{\bar{\tilde{\nabla}}}_X Y = \tilde{\nabla}_X Y + \eta(Y)X - g(X, Y)\xi. \tag{3.3}$$

for all $X, Y \in T\tilde{M}$.

Proposition 3.1 *Let M be a semi-invariant submanifold of a (k, μ)-contact manifold $\tilde{M}$ with a semi-symmetric metric connection. Then*

$$(\bar{\bar{\tilde{\nabla}}}_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) + g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{3.4}$$

$\forall\ X, Y \in \Gamma(TM)$.

Proof By virtue of (2.6) and (3.3), the proposition follows after having done similar computations as in the proof of Theorem 3 in [20]. □

Proposition 3.2 *Let M be a semi-invariant submanifold of a (k, μ)-contact manifold $\tilde{M}$ with a semi-symmetric metric connection. Then*

$$\bar{\bar{\tilde{\nabla}}}_X \xi = -\phi X - \phi hX + X - \eta(X)\xi, \tag{3.5}$$

$\forall\ X, Y \in \Gamma(TM)$.

Proof By virtue of (3.3) and (2.3), the proposition follows after having done similar computations as in the proof of Theorem 4 in [20]. □

Theorem 3.3 *The connection induced on a semi-invariant submanifold of a (k,μ)-contact manifold that admits a semi-symmetric metric connection also admits a semi-symmetric metric connection.*

Proof Let $\bar{\nabla}$ be the induced connection with respect to the unit normal N on semi-invariant submanifold M of a (k,μ)-contact manifold with semi-symmetric metric connection $\bar{\bar{\nabla}}$. Then,

$$\bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + m(X,Y), \tag{3.6}$$

where m is a tensor field of type $(0,2)$ on semi-invariant submanifold M.

Using (3.1) and (3.3), we have

$$\bar{\nabla}_X Y + m(X,Y) = \nabla_X Y + \sigma(X,Y) + \eta(Y)X - g(X,Y)\xi.$$

Equating tangential and normal components of the above equation, we get

$$\begin{aligned} m(X,Y) &= \sigma(X,Y). \\ \nabla_X Y &= \nabla^*_X Y + \eta(Y)X - g(X,Y)\xi. \end{aligned} \tag{3.7}$$

Thus, $\bar{\nabla}$ is also semi-symmetric metric connection. □

Now, the Gauss and Weingarten formulae for semi-invariant submanifolds of a (k,μ)-contact manifold with a semi-symmetric metric connection are given by

$$\bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \sigma(X,Y), \tag{3.8}$$

$$\bar{\bar{\nabla}}_X N = (-A_N + \eta(N))X + \bar{\nabla}^{\perp}_X N, \tag{3.9}$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where σ and A_N are the second fundamental form and Weingarten endomorphism associated with N, and are related by

$$g(\sigma(X,Y),N) = g((-A_N + \eta(N))X, Y).$$

For $X \in \Gamma(TM)$, $N \in \Gamma(T^{\perp}M)$, we can write

$$X = PX + QX + \eta(X)\xi, \tag{3.10}$$

$$\phi N = BN + CN, \tag{3.11}$$

where P and Q are the projection operators of TM to D and $D^{\perp}$ respectively, and BN (resp. CN) denote the tangential (resp. normal) component of ϕN.

§4. Basic Results

Lemma 4.1 *Let M be a semi-invariant submanifold of a (k,μ)-contact manifold $\tilde{M}$ with a* *semi-symmetric metric connection, then we have*

$$\begin{aligned}(\bar{\bar{\nabla}}_X\phi)Y &= (\bar{\nabla}_X P)Y + \sigma(X,PY) - A_{QY}X + \eta(N)X \\ &\quad +(\bar{\nabla}_X Q)Y - B\sigma(X,Y) - C\sigma(X,Y), \qquad (4.1)\\ (\bar{\bar{\nabla}}_X\phi)N &= (\bar{\nabla}_X B)N + \sigma(X,BN) - A_{CN}X + P(-A_N)X + Q(-A_N)X \\ &\quad +(\bar{\nabla}_X C)N + P\eta(N)X + Q\eta(N)X. \qquad (4.2)\end{aligned}$$

Proof Using (3.10), (3.11), the Gauss and Weingarten formulae, necessary arrangements are made to obtain the desired. □

We state the next Lemma whose proof is straightforwardly deduced by applying (3.4) in (4.1) and (4.2), hence omitted.

Lemma 4.2 *Let M be a semi-invariant submanifold of a (k,μ)-contact manifold $\tilde{M}$ with a semi-symmetric metric connection, then we have*

$$\begin{aligned}(\bar{\nabla}_X P)Y - A_{QY}X + \eta(N)X - B\sigma(X,Y) &= g(X+hX,Y)\xi \\ &\quad -\eta(Y)(X+hX) - \eta(Y)PX, \qquad (4.3)\\ (\bar{\nabla}_X Q)Y + \sigma(X,PY) - C\sigma(X,Y) &= -\eta(Y)QX, \qquad (4.4)\\ (\bar{\nabla}_X B)N - A_{CN}X\sigma(X,PY) - C\sigma(X,Y) &= 0, \qquad (4.5)\\ (\bar{\nabla}_X C)N + \sigma(X,BN) + Q(-A_N X + \eta(N)X) &= 0, \qquad (4.6)\\ g(PX,Y) &= 0, \qquad (4.7)\\ g(QX,Y) &= 0, \qquad (4.8)\end{aligned}$$

for all $X,Y\in\Gamma(TM)$, $N\in\Gamma(T^{\perp}M)$.

Lemma 4.3 *Let M be a semi-invariant submanifold of a (k,μ)-contact manifold $\tilde{M}$ with a semi-symmetric metric connection such that $\xi\in\Gamma(TM)$, we have*

$$\begin{aligned}\bar{\nabla}_X\xi &= -\phi X - \phi hX + X - \eta(X)\xi, \quad \sigma(X,\xi) = 0. \qquad (4.9)\\ \bar{\nabla}_\xi\xi &= 0, \ \ \sigma(\xi,\xi) = 0, \quad A_N\xi = 0. \qquad (4.10)\end{aligned}$$

Proof Using (3.3) and (3.5) we have (4.10). In addition, we get

$$o = g(\sigma(X,\xi),N) = g(\sigma(\xi,X),N) = g(A_N\xi,X).$$

This completes the proof. □

§5. Integrability of Distributions

In this section, we study the integrability of all the distributions involved in the definition of semi-invariant submanifolds.

For all $X, Y \in \Gamma(D)$, we have

$$g([X,Y],\xi) = g(\nabla_X Y,\xi) - g(\nabla_Y X,\xi).$$

Using (3.1) in above, we get

$$g([X,Y],\xi) = g(\tilde{\nabla}_X Y,\xi) - g(\tilde{\nabla}_Y X,\xi). \tag{5.1}$$

Taking account of (3.3) in (5.1) and using (3.5), we obtain

$$g([X,Y],\xi) = g(Y,\phi X + \phi hX) - g(X,\phi Y + \phi hY)$$

and so, $g([X,Y],\xi) \neq 0$. This leads to the following:

Theorem 5.4 *Let M be a semi-invariant submanifold of a (k,μ)-contact manifold $\tilde{M}$ with a semi-symmetric metric connection such that dim$(D) \neq 0$. Then the distribution D is not integrable.*

Theorem 5.5 *Let M be a semi-invariant submanifold of a (k,μ)-contact manifold $\tilde{M}$ with a semi-symmetric metric connection. Then the distribution $D\oplus <\xi>$ is integrable if and only if*

$$\sigma(X,\phi Y) = \sigma(\phi X, Y),$$

for all $X, Y \in \Gamma(D\oplus <\xi>)$.

Proof For $X, Y \in \Gamma(D)$, we have

$$\phi([X,Y]) = \phi(\nabla_X Y - \nabla_Y X).$$

Using (3.1) in the above equation, we get

$$\phi([X,Y]) = \phi(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X). \tag{5.2}$$

Making use of relation (3.3) in (5.2), we get

$$\phi([X,Y]) = \bar{\tilde{\nabla}}_X \phi Y - (\bar{\tilde{\nabla}}_X \phi)Y - \eta(Y)\phi X - \bar{\tilde{\nabla}}\phi X + (\bar{\tilde{\nabla}}_Y \phi)X - \eta(X)\phi Y. \tag{5.3}$$

Taking account of (3.8) in (5.3) and using (3.4), we obtain

$$\begin{aligned}\phi([X,Y]) &= \bar{\nabla}_X \phi Y + \sigma(X,\phi Y) + \bar{\nabla}_Y \phi X + \sigma(Y,\phi X)\\ &\quad -g(hX,Y)\xi + \eta(Y)(X+hX) + g(hY,X)\xi - \eta(X)(Y+hY) + 2g(\phi X,Y),\end{aligned}$$

where $\phi([X,Y])$ shows the component of $\nabla_X Y$ from the orthogonal complementary distribution of $D\oplus <\xi>$ in M. Then, $[X,Y]\in\Gamma(D\oplus<\xi>)$ if and only if $\sigma(X,\phi Y)=\sigma(Y,\phi X)$. □

Theorem 5.6 *Let M be a semi-invariant submanifold of a (k,μ)-contact manifold $\tilde{M}$ with a semi-symmetric metric connection. Then the distribution $D^{\perp}$ is integrable.*

Proof For all $X,Y\in\Gamma(D^{\perp})$, we have

$$g([X,Y],\xi)=g(\nabla_X Y,\xi)-g(\nabla_Y X,\xi).$$

Using (3.1) in the above equation, we get

$$g([X,Y],\xi)=g(\tilde{\bar{\nabla}}_X Y,\xi)-g(\tilde{\bar{\nabla}}_Y X,\xi). \tag{5.4}$$

Taking account of (3.3) in (5.4) and using (3.5), we obtain

$$g([X,Y],\xi)=g(Y,\phi X+\phi hX)-g(X,\phi Y+\phi hY),$$

which implies that $g([X,Y],\xi)=0$. So $\eta([X,Y])=0$. Then, we have $[X,Y]\in\Gamma(D^{\perp})$. □

Theorem 5.7 *Let M be a semi-invariant submanifold of a (k,μ)-contact manifold $\tilde{M}$ with a semi-symmetric metric connection. Then the distribution $D^{\perp}\oplus<\xi>$ is integrable if and only if*

$$A_{\phi X}Y=A_{\phi Y}X,$$

for all $X,Y\in\Gamma(D^{\perp}\oplus<\xi>)$.

Proof For $X,Y\in\Gamma(D^{\perp})$, we have

$$\phi([X,Y])=\phi(\nabla_X Y-\nabla_Y X).$$

Using (3.1) in above, we get

$$\phi([X,Y])=\phi(\tilde{\bar{\nabla}}_X Y-\tilde{\bar{\nabla}}_Y X). \tag{5.5}$$

Making use of relation (3.3) in (5.5), we get

$$\phi([X,Y])=\bar{\tilde{\bar{\nabla}}}_X\phi Y-(\bar{\tilde{\bar{\nabla}}}_X\phi)Y-\eta(Y)\phi X-\bar{\tilde{\bar{\nabla}}}\phi X+(\bar{\tilde{\bar{\nabla}}}_Y\phi)X-\eta(X)\phi Y. \tag{5.6}$$

Taking account of (3.9) in (5.6) and using (3.4), we obtain

$$\begin{aligned}\phi([X,Y]) \;\; = \;\; & A_{\phi X}Y-A_{\phi Y}X+\bar{\nabla}^{\perp}_X\phi Y-\bar{\nabla}^{\perp}_Y\phi X+g(hY,X)\xi-g(hX,Y)\xi\\ & +\eta(Y)(X+hX)-\eta(X)(Y+hY).\end{aligned}$$

Then ,we get

$$[X,Y]\in\Gamma(D^{\perp}\oplus<\xi>)\Rightarrow A_{\phi X}Y=A_{\phi Y}X.$$

Conversely

$$\begin{aligned}\phi^2([X,Y]) &= \phi\{A_{\phi X}Y - A_{\phi Y}X + \bar{\nabla}^{\perp}_{X}\phi Y - \bar{\nabla}^{\perp}_{Y}\phi X + g(hY,X)\xi - g(hX,Y)\xi \\ &\quad +\eta(Y)(X+hX) - \eta(X)(Y+hY)\}.\end{aligned}$$

Making use of (2.1) and using the equality $A_{\phi X}Y = A_{\phi Y}X$, the above equation can be written as

$$[X,Y] = \phi(\bar{\nabla}^{\perp}_{X}\phi Y) - \phi(\bar{\nabla}^{\perp}_{Y}\phi X) + \eta(Y)\phi(X+hX) - \eta(X)\phi(Y+hY).$$

Thus, $[X,Y] \in \Gamma(D^{\perp} \oplus <\xi>)$. Hence the distribution $D^{\perp} \oplus <\xi>$ is integrable. □

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On the M-polynomial and Degree-Based Topological Indices of Dandelion Graph

H. M. Nagesh

(Department of Science and Humanities, PES University, Bangalore, India)

M. C. Mahesh Kumar

(Department of Mathematics, Government First Grade College, K. R. Puram, Bangalore, India)

E-mail: nageshhm@pes.edu, softmahe15@gmail.com

Abstract: For a graph G, the M-polynomial is defined to be

$$M(G;x,y) = \sum_{\delta \le \alpha \le \beta \le \Delta} m_{\alpha\beta}(G) x^{\alpha} y^{\beta},$$

where $m_{\alpha\beta}(\alpha, \beta \ge 1)$ is the number of edges ab of G such that $deg_G(a) = \alpha$ and $deg_G(b) = \beta$; and δ is the minimum degree and Δ is the maximum degree of G. The physicochemical properties of chemical graphs are found by topological indices, in particular, the degree-based topological indices, which can be determined from an algebraic formula called M-polynomial. In this paper, we first compute the M-polynomial of the Dandelion graph and the line graph of Dandelion graph. Further, we derive some degree-based topological indices of these graphs from their respective M-polynomial.

Key Words: M-polynomial, Smarandachely M-polynomial, degree-based topological indices, Dandelion graph, line graph.

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§1. Introduction

Throughout this paper, by a graph $G = (V, E)$, we mean a simple, undirected, and finite graph of order n and size m. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. A *chemical graph* is a labeled graph where the atoms correspond to the vertices and the chemical bonds of the compound corresponds to the edges. A numerical quantity which is used to analyse both the physical and chemical properties of compounds is termed as a *topological index*. A topological index is also called a *graph invariant*. In general, the physicochemical properties and boiling activities of a chemical graph are investigated using topological indices.

The number of vertices of G adjacent to a given vertex v is the *degree* of the vertex v and is denoted by $deg_G(v)$. In a chemical graph, the degree of any vertex is at most 4. For all terms and definitions, not defined specifically in this paper, we refer to [5].

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The study of topological indices was first initiated by H. Wiener [16] in the year 1947. He introduced Weiner index in order to understand the correlation of the measured properties of molecules in a compound with their structural properties. In the year 1972, the Weiner index was interpreted by Hosoya [6] using distances between vertices in a graph. Over the last decade, various topological indices were introduced and studied by different authors [1,3,4].

Historically, various topological indices have been computed based on their mathematical definitions. Efforts have been made to explore a more streamlined approach capable of recovering multiple topological indices within a specific category. In this pursuit, the concept of a general polynomial was introduced, designed to yield the values of necessary topological indices through its derivatives and integrals. For instance, the Hosoya polynomial [7] is employed for calculating distance-based topological indices, while the NM-polynomial [15] is used to derive neighborhood degree sum-based indices. In 2015, Deutsch and Klažar [2] introduced the concept of M-polynomial to address the computation of degree-based topological indices. For more details on degree-based topological indices using the M-polynomial, we refer the readers to the references [11,12,13,14].

Definition 1.1 *For a graph G, the M-polynomial is defined to be*

$$M(G;x,y) = \sum_{\delta \leq \alpha \leq \beta \leq \Delta} m_{\alpha\beta}(G) x^{\alpha} y^{\beta},$$

where $m_{\alpha\beta}(\alpha, \beta \geq 1)$ is the number of edges ab of G such that $deg_G(a) = \alpha$ and $deg_G(b) = \beta$.

Particularly, if $deg_G(a) \neq deg_G(b)$ for all edges $ab \in E(G)$ such a M-polynomial is called a *Smarandachely M-polynomial* because it posses the character of Smarandachely denied axiom. For example, a star S_{n-1}. The authors in [8] introduced the concept of Dandelion graph while studying the Wiener inverse interval problem.

Definition 1.2 *A star graph, written S_{n-1}, is a graph on n vertices, consisting of some vertex, connected to $n-1$ leaves.*

Definition 1.3 *A Dandelion graph, written $D(n,l)$, is a graph on n vertices, consisted of a copy of the star S_{n-1} and copy of a path p_l on vertices $p_0, p_1, p_2, \ldots, p_{l-1}$, where p_0 is identified with a star center,*

Figure 1 shows an example of Dandelion graph $D(17,8)$.

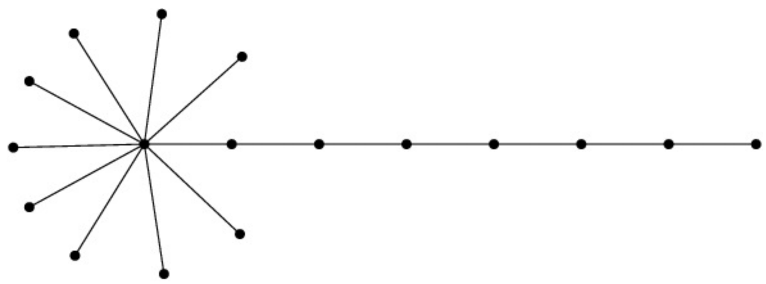

Figure 1. Dandelion graph $D(17,8)$

Siros Ghobadi et al. [9] computed the F-polynomial of Dandelion graphs. Again in [10], Siros Ghobadi et al. computed the first Zagreb index, F-index, and F-coindex of the line graph of Dandelion graph using subdivision concept. Motivated by this, we aim to calculate several algebraic polynomials and degree-based topological indices of Dandelion graph and the line graph of Dandelion graph.

§2. Methodology

We first divide the edge set of Dandelion graph and the line graph of Dandelion graph into different classes based on the degree of end vertices. With the help of this edge division, we compute the M-polynomial of Dandelion graph and the line graph of Dandelion graph. Further, by using M-polynomial, we compute the degree-based topological indices as listed in Table 1.

Table 1. Operations to derive degree-based topological indices form M-polynomial

Notation	Topological Index	Derivation from $M(G;x,y)$
$M_1(G)$	First Zagreb index	$(D_x+D_y)(M(G;x,y))\|_{x=y=1}$
$M_2(G)$	Second Zagreb index	$(D_xD_y)(M(G;x,y))\|_{x=y=1}$
${}^mM_2(G)$	Second modified Zagreb index	$(S_xS_y)(M(G;x,y))\|_{x=y=1}$
$R_\alpha(G)$	Randić index	$(D_x^\alpha D_y^\alpha)(M(G;x,y))\|_{x=y=1}$
$RR_\alpha(G)$	Inverse Randić index	$(S_x^\alpha S_y^\alpha)(M(G;x,y))\|_{x=y=1}$
$SSD(G)$	Symmetric division index	$(D_xS_y+D_yS_x)(M(G;x,y))\|_{x=y=1}$
$H(G)$	Harmonic index	$2S_xJ(M(G;x,y))\|_{x=1}$

where

$$\begin{aligned} M(G;x,y) &= f(x,y),\quad D_x = x\frac{\partial f(x,y)}{\partial x},\quad D_y = y\frac{\partial f(x,y)}{\partial y},\\ S_x &= \int_o^x \frac{f(t,y)}{t}dt,\quad S_y = \int_o^y \frac{f(x,t)}{t}dt,\quad J(f(x,y)) = f(x,x)\end{aligned}$$

are the operators. As discussed in [2], each of these topological indices can be found using M-polynomial as given in Table 1.

§3. M-polynomial of Dandelion Graph

In this section we find the M-polynomial of Dandelion graph.

Theorem 3.1 *Let $G=D(n,l)$ be the Dandelion graph. Then the M-polynomial of G is*

$$M(G;x,y) = xy^2 + (l-3)x^2y^2 + x^2y^{n-l+1} + (n-l)xy^{n-l+1}.$$

Proof Let $G=D(n,l)$ be the Dandelion graph. From Figure 1, $|V(G)|=n$ and $|E(G)|=$

$n-1$. Since each of the vertices of G is of degree either 1 or 2 or $n-l+1$, the vertex set of G has three partitions with respect to degree

$V_1(G) = \{u \in V(G) : deg_G(u) = 1\}$;

$V_2(G) = \{u \in V(G) : deg_G(u) = 2\}$;

$V_3(G) = \{u \in V(G) : deg_G(u) = n-l+1\}$ such that $|V_1(G)| = n-l+1$, $|V_2(G)| = l-2$, $|V_3(G)| = 1$. Further, the edge set of G has four partitions based on the degree of the end vertices.

$E_1(G) = \{e = uv \in E(G) : deg_G(u) = 1, deg_G(v) = 2\}$,

$E_2(G) = \{e = uv \in E(G) : deg_G(u) = 2, deg_G(v) = 2\}$,

$E_3(G) = \{e = uv \in E(G) : deg_G(u) = 2, deg_G(v) = n-l+1\}$,

$E_4(G) = \{e = uv \in E(G) : deg_G(u) = 1, deg_G(v) = n-l+1\}$ such that $|E_1(G)| = 1; |E_2(G)| = l-3$, $|E_3(G)| = 1$, $|E_4(G)| = n-l$.

Now, from the definition of the M-polynomial,

$$\begin{aligned} M(G;x,y) &= \sum_{\alpha\le\beta} m_{\alpha\beta}(G)x^\alpha y^\beta \\ &= m_{12}(G)xy^2 + m_{22}(G)x^2y^2 + m_{2(n-l+1)}(G)x^2y^{n-l+1} + m_{1(n-l+1)}(G)xy^{n-l+1} \\ &= xy^2 + (l-3)x^2y^2 + x^2y^{n-l+1} + (n-l)xy^{n-l+1} \end{aligned}$$

This completes the proof. $\square$

Now, we compute some degree-based topological indices of the Dandelion graph from this M-polynomial.

Theorem 3.2 *Let $G = D(n,l)$ be the Dandelion graph. Then*

$$\begin{aligned} M_1(G) &= n^2 + (3-2l)n + l^2 + l - 6, \\ M_2(G) &= n^2 + (3-2l)n + l^2 + l - 8, \\ {}^mM_2(G) &= \frac{(l+3)n - l^2 - 2l + 1}{4n - 4l + 4}, \\ R_\alpha(G) &= 2^\alpha(n-l+1) + (n-l)(n-l+1) + 4^\alpha(l-3) + 2^\alpha, \\ RR_\alpha(G) &= \frac{2^\alpha + 4^\alpha(n-l) + ((l-3)n - l^2 + 4l - 3) + 2^\alpha(n-l+1)}{4^\alpha(n-l+1)}, \\ SSD(G) &= \frac{2n^3 + (5-6l)n^2 + (6l^2 - 6l - 1)n - 2l^3 + l^2 + 5l - 2}{2n - 2l + 2}, \\ H(G) &= \frac{(3l+7)n^2 + (-6l^2 + l + 23)n + 3l^3 - 8l^2 - 5l - 6}{6n^2 + (30 - 12l)n + 6l^2 - 30l + 36}. \end{aligned}$$

Proof From Theorem 3.1, we have

$$M(G;x,y) = f(x,y) = xy^2 + (l-3)x^2y^2 + x^2y^{n-l+1} + (n-l)xy^{n-l+1}.$$

Then, we get the following:

$$\begin{aligned}
D_x f(x,y) &= 2x^2y^{n-l+1} + (n-l)xy^{n-l+1} + 2(l-3)x^2y^2 + xy^2,\\
D_y f(x,y) &= (n-l+1)x^2y^n + (n-l)(n-l+1)xy^n + 2(l-3)x^2y^2 + 2xy^2,\\
(D_yD_x)(f(x,y)) &= 2(n-l+1)x^2y^n + (n-l)(n-l+1)xy^n + 4(l-3)x^2y^2 + 2xy^2,\\
S_x(f(x,y)) &= \frac{1}{2}x^2y^{n-l+1} + (n-l)xy^{n-l+1} + \frac{1}{2}(l-3)x^2y^2 + xy^2,\\
S_y(f(x,y)) &= \frac{1}{n-l+1}x^2y^{n-l+1} + \frac{n-l}{n-l+1}xy^{n-l+1}\\
&\quad + \frac{\left((l-3)n-l^2+4l-3\right)x^2y^2}{2(n-l+1)} + \frac{1}{2}xy^2,\\
S_xS_y(f(x,y)) &= \frac{1}{2(n-l+1)}x^2y^{n-l+1} + \frac{n-l}{n-l+1}xy^{n-l+1}\\
&\quad + \frac{\left((l-3)n-l^2+4l-3\right)x^2y^2}{4(n-l+1)} + \frac{1}{2}xy^2,\\
S_yD_x(f(x,y)) &= \frac{2x+n-l}{n-l+1}xy^{n-l+1} + \frac{\left((2l-6)n-2l^2+8l-6\right)}{2(n-l+1)}x^2y^2 + \frac{1}{2}xy^2,\\
S_xD_y(f(x,y)) &= \frac{n-l+1}{2}x^2y^{n-l+1} + \frac{\left(2n^2+(2-4l)n+2l^2-2l\right)}{2}xy^{n-l+1}\\
&\quad +(l-3)x^2y^2 + 2xy^2,\\
2S_xJ(f(x,y)) &= \frac{\left((3l-9)n^2+(-6l^2+33l-45)n+3l^3-24l^2+63l-54\right)x^4}{6n^2+(30-12l)n+6l^2-30l+36}\\
&\quad + \frac{\left(4n^2+(20-8l)n+4l^2-20l+24\right)x^3}{6n^2+(30-12l)n+6l^2-30l+36}\\
&\quad + \frac{\left((12n-12l+24)x^3+(12n^2+(36-24l)n+12l^2-36l)x^2\right)x^{n+l}}{6n^2+(30-12l)n+6l^2-30l+36}
\end{aligned}$$

Now, we have the following from Table 1:

(1) The first Zagreb index

$$M_1(G) = (D_x + D_y)(f(x,y))|_{x=y=1} = n^2 + (3-2l)n + l^2 + l - 6.$$

(2) The second Zagreb index

$$M_2(G) = (D_xD_y)(f(x,y))|_{x=y=1} = n^2 + (3-2l)n + l^2 + l - 8.$$

(3) The second modified Zagreb index

$${}^mM_2(G) = (S_xS_y)(f(x,y))|_{x=y=1} = \frac{(l+3)n-l^2-2l+1}{4n-4l+4}.$$

(4) The Randić index

$$R_\alpha(G) = (D_x^\alpha D_y^\alpha)(f(x,y))|_{x=y=1} = 2^\alpha(n-l+1) + (n-l)(n-l+1) + 4^\alpha(l-3) + 2^\alpha.$$

(5) The inverse Randić index

$$RR_\alpha(G) = (S_x^\alpha S_y^\alpha)(f(x,y))|_{x=y=1} = \frac{2^\alpha + 4^\alpha(n-l) + ((l-3)n - l^2 + 4l - 3) + 2^\alpha(n-l+1)}{4^\alpha(n-l+1)}.$$

(6) The symmetric division index

$$SSD(G) = (D_xS_y + D_yS_x)(f(x,y))|_{x=y=1} = \frac{2n^3 + (5-6l)n^2 + (6l^2 - 6l - 1)n - 2l^3 + l^2 + 5l - 2}{2n - 2l + 2}.$$

(7) The harmonic index

$$H(G) = 2S_xJ(f(x,y))|_{x=1} = \frac{(3l+7)n^2 + (-6l^2 + l + 23)n + 3l^3 - 8l^2 - 5l - 6}{6n^2 + (30-12l)n + 6l^2 - 30l + 36}.$$

This completes the proof. □

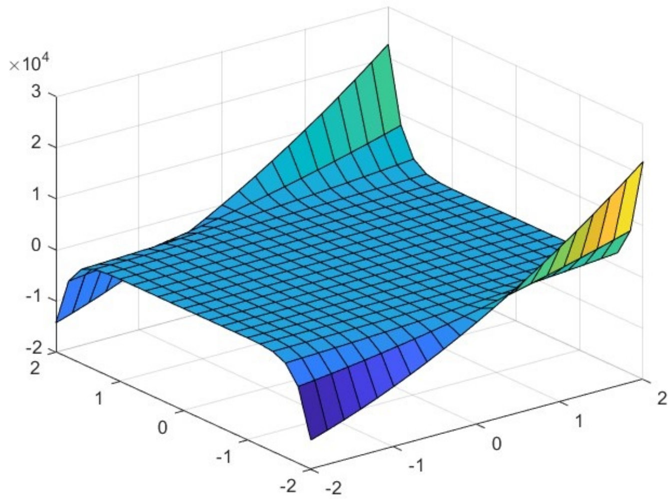

Figure 2 Plot of M-polynomial of Dandelion graph $D(17,8)$

§4. M-polynomial of the Line Graph of Dandelion Graph

There are many graph operators (or graph valued functions) with which one can construct a new graph from a given graph, such as the line graphs, line cut-vertex graphs; total graphs; middle graphs; and their generalizations. A *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common.

In the next theorem, we find the M-polynomial of the line graph of Dandelion graph.

Theorem 4.1 *Let $G = D(n, l), l > 4$, be a Dandelion graph. Then, the M-polynomial of $L(G)$ is*

$$M(L(G); x, y) = xy^2 + (l-4)x^2y^2 + x^2y^{n-l+1} + (n-l)x^{n-l}y^{n-l+1} + \frac{n^2 + l^2 - 2nl - n + l}{2}x^{n-l}y^{n-l}.$$

Proof Let $G = D(n, l), l > 4$, be the Dandelion graph. Then $|V(L(G))| = n - 1$. Since each of the vertices of $L(G)$ is of degree either 1 or 2 or $n - l$ or $n - l + 1$, the vertex set of $L(G)$ has four partitions with respect to degree:

$V_1(L(G)) = \{u \in V(L(G)) : deg_{L(G)}(u) = 1\}$,
$V_2(L(G)) = \{u \in V(L(G)) : deg_{L(G)}(u) = 2\}$,
$V_3(L(G)) = \{u \in V(L(G)) : deg_{L(G)}(u) = n - l\}$ and
$V_4(L(G)) = \{u \in V(L(G)) : deg_{L(G)}(u) = n - l + 1\}$ such that

$$|V_1(L(G))| = 1, \quad |V_2(L(G))| = l - 3, \quad |V_3(L(G))| = n - l, \quad |V_4(L(G))| = 1.$$

Further, the edge set of G has five partitions based on the degree of the end vertices.

$E_1(L(G)) = \{e = uv \in E(L(G)) : deg_{L(G)}(u) = 1, deg_{L(G)}(v) = 2\}$,
$E_2(L(G)) = \{e = uv \in E(L(G)) : deg_{L(G)}(u) = 2, deg_{L(G)}(v) = 2\}$,
$E_3(L(G)) = \{e = uv \in E(L(G)) : deg_{L(G)}(u) = 2, deg_{L(G)}(v) = n - l + 1\}$,
$E_4(L(G)) = \{e = uv \in E(L(G)) : deg_{L(G)}(u) = n - l, deg_{L(G)}(v) = n - l + 1\}$,
$E_5(L(G)) = \{e = uv \in E(L(G)) : deg_{L(G)}(u) = n - l, deg_{L(G)}(v) = n - l\}$ such that

$$|E_1(L(G))| = 1, \quad |E_2(L(G))| = l - 4, \quad |E_3(L(G))| = 1, \quad |E_4(L(G))| = n - l,$$

and
$$|E_5(L(G))| = \frac{n^2 + l^2 - 2nl - n + l}{2}.$$

Now, from the definition of the M-polynomial,

$$M(L(G); x, y) = xy^2 + (l-4)x^2y^2 + x^2y^{n-l+1} + (n-l)x^{n-l}y^{n-l+1} + \frac{n^2 + l^2 - 2nl - n + l}{2}x^{n-l}y^{n-l}.$$

This completes the proof. □

Now, we compute some degree-based topological indices of the line graph of Dandelion graph from this M-polynomial.

Theorem 4.2 *Let $G = D(n, l), l > 4$, be the Dandelion graph. Then,*

$$M_1(L(G)) = 2n^2 + (-4l + 2k + 2)n + 2l^2 + (2 - 2k)l - 10,$$

$$\begin{aligned}
M_2(L(G)) &= n^3+(-3l+k+1)n^2+(3l^2+(-2k-2)l+2)n-l^3+(k+1)l^2+2l-12,\\
{}^mM_2(L(G)) &= \frac{(l-2)n^3+(-3l^2+7l+4)n^2+(3l^3-8l^2-8l+4k)n-l^4+3l^3+4l^2-4kl+4k}{4n^3+(4-12l)n^2+(12l^2-8l)n-4l^3+4l^2},\\
R_\alpha(L(G)) &= 2^\alpha(n-l+1)+(n-l)(n-l+1)+4^\alpha(l-3)+2^\alpha,\\
RR_\alpha(L(G)) &= \frac{2^\alpha+4^\alpha(n-l)+((l-3)n-l^2+4l-3)+2^\alpha(n-l+1)}{4^\alpha(n-l+1)},\\
SSD(L(G)) &= \frac{5n^2+(-6l+4k-5)n+l^2+(9-4k)l+4k-4}{2n-2l+2},\\
H(L(G)) &= \frac{A}{B}.
\end{aligned}$$

where

$$\begin{aligned}
A &= (6l-4)n^3+(-18l^2+33l+12k+4)n^2+(18l^3-54l^2\\
&\quad +(1-24k)l+42k-12)n-6l^4+25l^3+(12k-5)l^2+(12-42k)l+18k,\\
B &= 12n^3+(42-36l)n^2+(36l^2-84l+18)n-12l^3+42l^2-18l.
\end{aligned}$$

Proof From Theorem 4.1, we have

$$M(L(G);x,y)=xy^2+(l-4)x^2y^2+x^2y^{n-l+1}+(n-l)x^{n-l}y^{n-l+1}+kx^{n-l}y^{n-l},$$

where

$$k=\frac{n^2+l^2-2nl-n+l}{2}.$$

Then, we have the following

$$\begin{aligned}
D_xf(x,y) &= (n-l)^2x^{n-l}y^{n-l+1}+2x^2y^{n-l+1}+k(n-l)x^{n-l}y^{n-l}+2(l-4)x^2y^2+xy^2,\\
D_yf(x,y) &= (n-l)(n-l+1)x^{n-l}y^{n-l+1}+(n-l+1)x^2y^{n-l+1}+k(n-l)x^{n-l}y^{n-l}\\
&\quad +2(l-4)x^2y^2+2xy^2,\\
(D_yD_x)(f(x,y)) &= (n-l)^2(n-l+1)x^{n-l}y^{n-l+1}+2(n-l+1)x^2y^{n-l+1}+k(n-l)^2x^{n-l}y^{n-l}\\
&\quad +4(l-4)x^2y^2+2xy^2.
\end{aligned}$$

$$\begin{aligned}
S_x(f(x,y)) &= \frac{((l-4)n-l^2+4l)x^{l+2}y^{l+2}+(2n-2l)x^{l+1}y^{l+2}+(2n-2l)x^ny^{n+1}+(n-l)x^{l+2}y^{n+1}}{(2n-2l)x^ly^l}\\
&\quad +\frac{2kx^ny^n}{(2n-2l)x^ly^l}
\end{aligned}$$

$$\begin{aligned}
S_y(f(x,y)) &= \frac{((l-4)n^2+(-2l^2+9l-4)n+l^3-5l^2+4l)x^{l+2}+(n^2+(1-2l)n+l^2-l)x^{l+2}y^{l+2}}{(2n^2+(2-4l)n+2l^2-2l)x^ly^l}\\
&\quad +\frac{(2n^2-4nl+2l^2)x^ny^{n+1}+(2n-2l)x^{l+2}y^{n+1}+(2kn-2kl+2k)x^ny^n}{(2n^2+(2-4l)n+2l^2-2l)x^ly^l}
\end{aligned}$$

$$S_xS_y(f(x,y)) = \frac{((l-4)n^3+(-3l^2+13l-4)n^2+(3l^3-14l^2+8l)n-l^4-5l^3-4l^2)x^{l+2}y^{l+2}}{(2n-2l)(2n^2+(2-4l)n+2l^2-2l)x^ly^l}$$
$$+\frac{(2n^3+(2-6l)n^2+(6l^2-4l)n-2l^3+2l^2)x^{l+2}y^{l+2}}{(2n-2l)(2n^2+(2-4l)n+2l^2-2l)x^ly^l}$$
$$+\frac{(4n^2-8ln+4l^2)x^ny^{n+1}+(2n^2-4nl+2l^2)x^{l+2}y^{n+1}+(4kn-4kl+4k)x^ny^n}{(2n-2l)(2n^2+(2-4l)n+2l^2-2l)x^ly^l},$$

$$S_yD_x(f(x,y)) = \frac{\left((2l-8)n-2l^2+10l-8)x^2+(n-l+1)x\right)x^ly^{l+2}}{(2n-2l+2)x^ly^l}$$
$$+\frac{\left((2n^2-4ln+2l^2)x^n+4x^{l+2}\right)y^{n+1}+(2kn-2kl+2k)x^ny^n}{(2n-2l+2)x^ly^l},$$

$$S_yD_x(f(x,y)) = \frac{((2l-8)x^2+4x)x^ly^{l+2}+(2n-2l+2)x^ny^{n+1}+(n-l+1)x^{l+2}y^{n+1}+2kx^ny^n}{2x^ly^l},$$

$$2S_xJ(f(x,y)) = \frac{(24n^2+(12-48l)n+24l^2-12l)x^{n+l+3}}{(12n^3+(42-36l)n^2+(36l^2-84l+18)n-12l^3+42l^2-18l)\,x^{2l}}$$
$$+\frac{\left((6l-24)n^3+(-18l^2+93l-84)n^2+(18l^3-114l^2+177l-36)n\right)x^4}{(12n^3+(42-36l)n^2+(36l^2-84l+18)n-12l^3+42l^2-18l)}$$
$$+\frac{\left(-6l^4+45l^3-93l^2+36l)\right)x^4}{(12n^3+(42-36l)n^2+(36l^2-84l+18)n-12l^3+42l^2-18l)}$$
$$+\frac{\left((8n^3+(28-24l)n^2+(24l^2-56l+12)n-8l^3+28l^2-12l)\right)x^3}{(12n^3+(42-36l)n^2+(36l^2-84l+18)n-12l^3+42l^2-18l)}$$
$$+\frac{\left(12n^3+(36-36l)n^2+(36l^2-72l)n-12l^3+36l^2\right)x^{2n+1}}{(12n^3+(42-36l)n^2+(36l^2-84l+18)n-12l^3+42l^2-18l)\,x^{2l}}$$
$$+\frac{\left(12kn^2+(42k-24kl)n+12kl^2-42kl+18k\right)x^{2n}}{(12n^3+(42-36l)n^2+(36l^2-84l+18)n-12l^3+42l^2-18l)\,x^{2l}}$$

We finally get the results by applying the appropriate operations given in Table 1. □

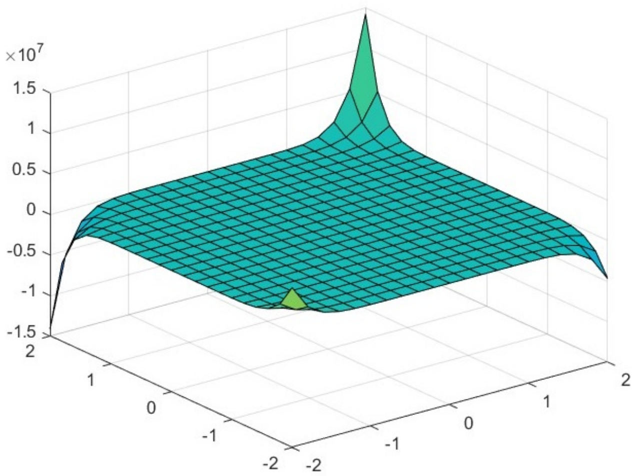

Figure 3 Plot of M-polynomial of the line graph of Dandelion graph $D(17, 8)$

§5. Conclusion

Topological indices play an important role in understanding many physical and chemical properties of a chemical compound. Some of the degree-based topological indices can be found by means of the M-polynomial of the corresponding chemical graph. In this paper, we have determined some of these topological indices using the closed form of the M-polynomial of the Dandelion graph and the line graph of Dandelion graph. The M-polynomial can be determined for graph operations, graph products, and graph powers also. The study on M-polynomials with respect to different types of graph operators also seem to be much promising.

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Some Inequalities for the Entire Sombor Index

N. Harish and B. Chaluvaraju

(Department of Mathematics, Bangalore University, Jnana Bharathi Campus, Bangalore -560 056, India)

E-mail: harish.jaga16@gmail.com, bcrbub@gmail.com

Abstract: The entire Sombor index of a graph G was introduced by Movahedi and Akhbari [6]. By motivated their work, we obtained some properties, inequalities and characterization in terms of order, size, degree and other degree based graphical indices. Also, we present the computed values of certain families of graphs. In addition to that, we compare the statistical behaviour of Sombor based graphical indices such as KG-Sombor index, Reformulated Sombor index and Entire Sombor index of molecular graph of linear $[k]$- alkanes.

Key Words: Entire Sombor index, Sombor index, Kulli-Gutman Sombor index, topological indices.

AMS(2010): 05C05, 05C07, 05C09, 05C92.

§1. Introduction

A simple undirected graph $G = (V, E)$ is the set of all ordered pairs, such as the set of all vertices $V(G)$ are related to atoms and the set of all edges $E(G)$ are related to chemical bonds among atoms with $\mid V \mid= n$ and $\mid E \mid= m$. The vertices u and v are adjacent vertices if and only if they end vertices of a common edge $e = uv \in E(G)$. Let $(\delta(G), \Delta(G))$ be the set of all ordered pairs represented by the minimum and maximum number of adjacent edges incident on the vertex. Also, edge degree can be written as

$$d_G(e) = d_G(uv) = d_G(u) + d_G(v) - 2.$$

The entire Sombor index of a graph G is defined as the sum of the square root of the terms x and y are the two member of the set $B(G)$. Where $B(G)$ is the set of all subsets of two members $\{x, y\} \subseteq V(G) \cup E(G)$ such that x and y are adjacent or incident to each other. For more details on graph theoretical terminologies, we refer to [14,17].

Topological descriptors have made sure their essential importance, because of their easy formation and speed with which those tests may be accomplished. There are numerous graph associated numerical descriptors, that have proven their value in one-of-a-kind areas. Thereby, the system of finding the topological descriptors has emerge as a fascinating and appealing direction of research. We discussed in this paper are depicted in Table 1. For more details refer

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to [9, 10, 11, 23, 26, 27, 28, 29].

Graphical Indices	**Mathematical Representation**
First Zagreb index (Gutman and Trinajstic [12])	$M_1(G) = \sum_{uv\in E(G)} d_G(u) + d_G(v)$
Second Zagreb index (Gutman and Trinajstic [12])	$M_2(G) = \sum_{uv\in E(G)} d_G(u)\, d_G(v)$
Re-defined Zagreb index (Ranjini et al., [22])	$ReZG_3(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)\big[d_G(u) + d_G(v)\big]$
First general Zagreb index (X.Li et al.,[20])	$M_4(G) = \sum_{u\in V(G)} (d_G(u))^4$
Forgotten index (Fortula and Gutman [7])	$F(G) = \sum_{uv\in E(G)} \big[d_G(u)^2 + d_G(v)^2\big]$
Sombor index (Gutman [8])	$SO(G) = \sum_{uv\in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}$
Reformulated Sombor index (Harish et al.,[15])	$RS(G) = \sum_{e\sim f} \sqrt{d_G(e)^2 + d_G(f)^2}$
KG Sombor index (Kulli et al.,[18])	$KG(G) = \sum_{ue} \sqrt{d_G(u)^2 + d_G(e)^2}$
First Entire Zagreb index (Alwardi et al.,[2])	$EM_1(G) = \sum_{\{x,y\}\subseteq B(G)} [d_G(x) + d_G(y)]$
Second Entire Zagreb index (Alwardi et al.,[2])	$EM_2(G) = \sum_{\{x,y\}\subseteq B(G)} [d_G(x)\, d_G(y)]$
Entire Forgotten index (Bharall et al.,[3])	$EF(G) = \sum_{\{x,y\}\subseteq B(G)} \big[d_G(x)^2 + d_G(y)^2\big]$
Entire Randic index (Saleh et al.,[25])	$ER(G) = \sum_{\{x,y\}\subseteq B(G)} \frac{1}{\sqrt{d_G(x)d_G(y)}}$
Entire ABC index (Saleh et al., [24])	$EABC(G) = \sum_{\{x,y\}\subseteq B(G)} \sqrt{\frac{d_G(x)+d_G(y)-2}{d_G(x)d_G(y)}}$
Platt index (Platt [21])	$Pl(G) = \sum_{u\in V(G)} d_G(u)(d_G(u) - 1)$

Table 1. Degree-based graphical indices and its representation

§2. Entire Sombor Index

The entire Sombor index of a graph G and is defined as

$$ES(G) = \sum_{\{x,y\}\subseteq B(G)} \sqrt{d_G(x)^2 + d_G(y)^2}.$$

2.1. Existing and Preliminary Results

Observation 2.1([6]) Let G be a connected graph with $n \geqslant 3$. Then

$$ES(G) = SO(G) + RS(G) + \sum_{u \text{ is incident to } e} \sqrt{d_G(u)^2 + d_G(e)^2}. \tag{2.1}$$

For edge $e = uv \in E(G)$, equation (2.1) can be expressed as

$$\begin{aligned} ES(G) &= SO(G) + RS(G) + \sum_{uv\in E(G)} \Big[\sqrt{d_G(u)^2 + (d_G(u) + d_G(v) - 2)^2} \\ &\quad + \sqrt{d_G(v)^2 + (d_G(u) + d_G(v) - 2)^2}\Big]. \end{aligned} \tag{2.2}$$

Observation 2.2([1, 13]) For any graph G with $n \geq 2$,

$$Pl(G) = M_1(G) - 2m.$$

Observation 2.3 Let G be a connected graph with $n \geqslant 3$. Then,

(*i*) $|B(G)| = 2m + \frac{M_1(G)}{2}$;
(*ii*) $|B(G)| = 3m + \frac{Pl(G)}{2}$.

Observation 2.4([5]) For any connected graph G with $n \geq 2$,

(*i*) $2m\,(\delta - 1) \leq Pl(G) \leq 2m\,(\Delta - 1)$;
(*ii*) $m \leq Pl(G) \leq 2m\,(n - 2)$.

Proposition 2.1([6]) *Let G be a r-regular graph with $n \geqslant 3$ and $r \geqslant 1$. Then*

$$ES(G) = nr\left[\frac{r}{\sqrt{2}} + \sqrt{2}(r-1)^2 + \sqrt{5r^2 - 8r + 4}\right].$$

Proposition 2.2([6]) *Let G be a complete graph K_n, cycle C_n, a complete bipartite graph $K_{r,s}$ or a path P_n.*

(*i*) *If $n \geq 2$ in K_n then*

$$ES(K_n) = n(n-1)\left[\frac{n-1}{\sqrt{2}} + \sqrt{2}(n-2)^2 + \sqrt{5n^2 - 18n + 17}\right].$$

(*ii*) *If $n \geqslant 3$ in C_n then*

$$ES(C_n) = 8\sqrt{2}\,n.$$

(iii) If $1 \leqslant r \leqslant s$ *in* $K_{r,s}$ *then*

$$\begin{aligned} ES(K_{r,s}) &= rs\left[\sqrt{r^2+s^2}+\frac{\sqrt{2}}{2}(r+s-2)^2+\sqrt{r^2+(r+s-2)^2}\right. \\ &\quad \left. + \sqrt{s^2+(r+s-2)^2}\right]. \end{aligned}$$

(iv) If $n \geqslant 4$ *in* P_n *then*

$$ES(P_n) = 6\sqrt{5} + 8\sqrt{2}\,(n-3).$$

Proposition 2.3 *For any Wheel* W_n *with* $n \geqslant 4$*,*

$$\begin{aligned} ES(W_n) &= n\left[7\sqrt{2}+10\right]+n\sqrt{n^2+9}+2n\sqrt{n^2+2n+17} \\ &\quad +n\left[\sqrt{n^2+2n+10}+\sqrt{2n^2+2n+1}\right]. \end{aligned}$$

Proof Let W_n be a Wheel with $n \geqslant 4$. Then $|V_{3,3}(G)| = n$, $|V_{3,n}(G)| = n$, $|E_{4,4}(G)| = n$, $|E_{4,n+1}(G)| = 2n$, $|E_{n+1,n+1}(G)| = n(n-1)/2$, $A_{3,4}(G) = 2n$, and $A_{3,n+1}(G) = A_{n,n+1} = n$. By Observation 2.1, we have

$$\begin{aligned} ES(W_n) &= n\sqrt{18}+n\sqrt{n^2+9}+n\sqrt{32}+2n\sqrt{16+(n+1)^2}+\frac{n(n-1)}{2} \\ &\quad \times \sqrt{2(n+1)^2}+2n\sqrt{25}+n\sqrt{9+(n+1)^2}+n\sqrt{n^2+(n+1)^2}. \\ ES(W_n) &= n[7\sqrt{2}+10]+n\sqrt{n^2+9}+2n\sqrt{n^2+2n+17}+n[\sqrt{n^2+2n+10} \\ &\quad + \sqrt{2n^2+2n+1}]. \end{aligned}$$

On simplification, we have the required result. □

Now, we obtain the relation between the Sombor index, reformulated Sombor index and KG Sombor index of a graph G as follows:

Theorem 2.1 *Let* G *be a connected graph with* $n \geqslant 3$*. Then*

$$ES(G) = SO(G) + RS(G) + KG(G).$$

Proof Let G be a connected graph with $n \geqslant 3$. Then

$$ES(G) = \sum_{\substack{x \text{ is either adjacent} \\ \text{or} \\ \text{incident to } y}} \sqrt{d_G(x)^2+d_G(y)^2} = \sum_{xy \in E(G)} \sqrt{d_G(x)^2+d_G(y)^2}$$

$$+\sum_{e,f\in E(G),\, e\sim f}\sqrt{d_G(e)^2+d_G(f)^2}+\sum_{x\ is\ incident\ to\ e}\sqrt{d_G(x)^2+d_G(e)^2}$$
$$=SO(G)+RS(G)+KG(G).$$

Thus, the result follows. □

2.2. Inequalities in Terms of Order, Size and Minimum/Maximum Degree

Theorem 2.2 *Let G be a connected graph with $n\geqslant 3$. Then*

$$\frac{\sqrt{2}m}{n}\left[n\delta+4(\delta-1)(2m-n)+\sqrt{2}n\sqrt{5\delta^2-8\delta+4}\right]\leq ES(G)$$
$$\leq\sqrt{2}\left[m\Delta+2(\Delta-1)(n\Delta^2-2m)+\sqrt{2}m\sqrt{5\Delta^2-8\Delta+4}\right].$$

The both left and right inequalities holds if and only if G is regular.

Proof Let G be a connected graph with $n\geqslant 3$. Then

$$\begin{aligned}
ES(G)&=\sum_{xy\in E(G)}\sqrt{d_G(x)^2+d_G(y)^2}+\sum_{e,f\in E(G),\, e\sim f}\sqrt{d_G(e)^2+d_G(f)^2}\\
&\quad+\sum_{x\ (e)}\sqrt{d_G(x)^2+d_G(e)^2}\\
&\geqslant\sum_{xy\in E(G)}\sqrt{2}\delta+\sum_{e,f\in E(G),\, e\sim f}2\sqrt{2}(\delta-1)\\
&\quad+\sum_{x\in V(G)}\sum_{e\in E(G)}\sqrt{5\delta^2-8\delta+4}\\
&\geq\sqrt{2}m\delta+2\sqrt{2}(\delta-1)\sum_{x\in V(G)}\binom{d_G(x)}{2}+\sum_{x\in V(G)}d_G(x)\sqrt{5\delta^2-8\delta+4}\\
&\geq\sqrt{2}m\delta+2\sqrt{2}(\delta-1)(M_1(G)-2m)+2m\sqrt{5\delta^2-8\delta+4}\\
&\geq\sqrt{2}m\delta+2\sqrt{2}(\delta-1)(\frac{4m^2}{n}-2m)+2m\sqrt{5\delta^2-8\delta+4}\\
&\geq\sqrt{2}m\delta+4\frac{\sqrt{2}}{n}(\delta-1)(2m^2-mn)+2m\sqrt{5\delta^2-8\delta+4}
\end{aligned}$$

i.e.,

$$ES(G)\geq\frac{\sqrt{2}m}{n}\left[n\delta+4(\delta-1)(2m-n)+\sqrt{2}n\sqrt{5\delta^2-8\delta+4}\right].$$

Similarly, we have to prove the right inequality.

The both left and right inequalities holds if and only if G is regular. □

Theorem 2.3 *Let G be a connected graph with $n \geqslant 3$. Then*

$$\sqrt{2}m\big[\sqrt{2}\sqrt{5\delta^2-8\delta+4}+4(\delta-1)^2+\delta\big] \leqslant ES(G)$$
$$\leq \sqrt{2}m\left[\sqrt{2}\sqrt{5\Delta^2-8\Delta+4}+4(\Delta-1)^2+\Delta\right].$$

The both left and right inequalities holds if and only if G is regular.

Proof Let G be a connected graph with $n \geqslant 3$. By Theorem 2.2, Observations 2.2 and 2.4(i), we have

$$\begin{aligned} ES(G) &\geq \sqrt{2}m\delta + 2\sqrt{2}(\delta-1)(M_1(G)-2m) + 2m\sqrt{5\delta^2-8\delta+4} \\ &\geq \sqrt{2}m\delta + 2\sqrt{2}(\delta-1)2m(\delta-1) + 2m\sqrt{5\delta^2-8\delta+4} \\ &= \sqrt{2}m\delta + 4m\sqrt{2}(\delta-1)^2 + 2m\sqrt{5\delta^2-8\delta+4}. \end{aligned}$$

i.e.,

$$ES(G) \geqslant \sqrt{2}m\left[\sqrt{2}\sqrt{5\delta^2-8\delta+4}+4(\delta-1)^2+\delta\right].$$

Similarly, we have to prove the right inequality.

The both left and right inequalities holds if and only if G is regular. □

Theorem 2.4 *Let G be a connected graph with $n \geqslant 3$. Then*

$$\sqrt{2}m\left[\delta+2(\delta-1)+\sqrt{10\delta^2-16\delta+8}\right] \leqslant ES(G)$$
$$\leq \sqrt{2}m\left[\Delta+4(n-2)(\Delta-1)+\sqrt{10\Delta^2-16\Delta+8}\right].$$

The both left and right inequalities holds if and only if G is regular.

Proof Let G be a connected graph with $n \geqslant 3$. By Theorems 2.1 and 2.2, Observations 2.2 and 2.4(ii), we have

$$\begin{aligned} ES(G) &\leqslant \sqrt{2}m\Delta + 2\sqrt{2}(\Delta-1)(M_1(G)-2m) + 2m\sqrt{5\Delta^2-8\Delta+4} \\ &\leqslant \sqrt{2}m\Delta + 2\sqrt{2}(\Delta-1)2m(n-2) + 2m\sqrt{5\Delta^2-8\Delta+4} \end{aligned}$$
$$ES(G) \leqslant \sqrt{2}m\big[\Delta+4(n-2)(\Delta-1)+\sqrt{10\Delta^2-16\Delta+8}\big].$$

Similarly, we have to prove left inequality.

The both left and right inequalities holds if and only if G is regular. □

2.3. Inequalities in Terms of Other Degree Based Graphical Indices

Theorem 2.5 *Let G be a connected graph with $n \geqslant 3$. Then*

$$\frac{EM_1(G)}{\sqrt{2}} \leqslant ES(G) \leqslant EM_1(G).$$

The both left and right inequalities holds if and only if G is regular.

Proof Let G be a connected graph with $n \geqslant 3$. We have

$$\frac{d_G(x)+d_G(y)}{\sqrt{2}} \leqslant \sqrt{d_G(x)^2+d_G(y)^2} \leqslant d_G(x)+d_G(y).$$

Therefore, the above inequalities which satisfies for each members and also taking the summation of all the above inequalities, we have

$$\begin{aligned}\frac{1}{\sqrt{2}}\sum_{\{x,y\}\subseteq B(G)}[d_G(x)+d_G(y)] &\leqslant \sum_{\{x,y\}\subseteq B(G)}\sqrt{d_G(x)^2+d_G(y)^2}\\ &\leqslant \sum_{\{x,y\}\subseteq B(G)} d_G(x)+d_G(y).\end{aligned}$$

Thus,

$$\frac{EM_1(G)}{\sqrt{2}} \leqslant ES(G) \leqslant EM_1(G).$$

The both left and right inequalities holds if and only if G is regular. □

Theorem 2.6 *Let G be a connected graph with $n \geqslant 3$. Then*

$$EM_2(G)\frac{\sqrt{2}}{\Delta} \leqslant ES(G) \leqslant EM_2(G)\frac{\sqrt{2}}{\delta}.$$

The both left and right inequalities holds if and only if G is regular.

Proof Let G be a connected graph with $n \geqslant 3$. Then

$$\begin{aligned}ES(G) &= \sum_{\{x,y\}\subseteq B(G)}\sqrt{d_G(x)^2+d_G(y)^2}\\ &= \sum_{\{x,y\}\subseteq B(G)} d_G(x).d_G(y)\sqrt{\frac{1}{d_G(x)^2}+\frac{1}{d_G(y)^2}}\\ &\leq \sum_{\{x,y\}\subseteq B(G)}[d_G(x).d_G(y)]\left(\sqrt{\frac{1}{\delta^2}+\frac{1}{\delta^2}}\right)\\ &\leq EM_2(G)\frac{\sqrt{2}}{\delta}.\end{aligned}$$

Similarly, we have to prove the left inequality. Therefore,

$$EM_2(G)\frac{\sqrt{2}}{\Delta} \leqslant ES(G) \leqslant EM_2(G)\frac{\sqrt{2}}{\delta}.$$

The both left and right inequalities holds if and only if G is regular. □

To prove our next result, we make use of the definition of line graph following.

Any two vertices of a line graph $L(G)$ are adjacent if and only if the corresponding edges of a graph G are incident with the same vertex of G. The line graph $L(G)$ of a graph G is a graph whose vertices equal to the edges of G.

Theorem 2.7 *Let G be a connected graph with $n \geqslant 3$. Then,*

$$8\sqrt{2}m - 3\sqrt{2}|B(G)| + \sqrt{2}M_2(G) + \frac{F(G)}{\sqrt{2}}$$
$$\leqslant ES(G) \leqslant 16m - 6|B(G)| + 2M_2(G) + F(G).$$

The both left and right inequalities holds if and only if G is regular.

Proof Let G be a connected graph with $n \geqslant 3$. By Theorem 2.5, we have

$$\begin{aligned}
EM_1(G) &= M_1(G) + M_1(L(G)) \\
&= \sum_{uv\in E(G)} \Big[d_G(u) + d_G(v) + (d_G(uv))^2\Big] \\
&\leq 4m - 3M_1(G) + 2M_2(G) + \sum_{uv\in E(G)} [d_G(u)^2 + d_G(v)^2] \\
&\leq 4m - 3\big((2|B(G)| - 4m)\big) + 2M_2(G) + F(G) \\
&\leq 16m - 6|B(G)| + 2M_2(G) + F(G).
\end{aligned}$$

Similarly, we have to prove the left inequality.

The both left and right inequalities holds if and only if G is regular. □

Theorem 2.8 *Let G be a connected graph with $n \geqslant 3$. Then*

$$\frac{EF(G)}{\sqrt{2}\,\Delta} \leqslant ES(G) \leq \frac{EF(G)}{\sqrt{2}\,\delta}.$$

The both left and right inequalities holds if and only if G is regular.

Proof Let G be a connected graph with $n \geqslant 3$. Then

$$\begin{aligned}
ES(G) &= \sum_{\{x,y\}\subseteq B(G)} \sqrt{d_G(x)^2 + d_G(y)^2} \\
&= \sum_{\{x,y\}\subseteq B(G)} \frac{d_G(x)^2 + d_G(y)^2}{\sqrt{d_G(x)^2 + d_G(y)^2}} \\
&\leq \sum_{\{x,y\}\subseteq B(G)} [d_G(x)^2 + d_G(y)^2]\Big(\frac{1}{\sqrt{\delta^2+\delta^2}}\Big) \\
&\leq \frac{EF(G)}{\sqrt{2}\,\delta}.
\end{aligned}$$

Similarly, we have to prove the left inequality. Therefore,

$$\frac{EF(G)}{\sqrt{2}\,\Delta} \leqslant ES(G) \leqslant \frac{EF(G)}{\sqrt{2}\,\delta}.$$

The both left and right inequalities holds if and only if G is regular. $\square$

Theorem 2.9 *Let G be a connected graph with $n \geqslant 3$. Then*

$$ER(G)\sqrt{2}\,\delta^2 \leqslant ES(G) \leqslant ER(G)\sqrt{2}\,\Delta^2.$$

The left and right inequalities holds if and only if G is regular.

Proof Let G be a connected graph with $n \geqslant 3$. Then,

$$\begin{aligned}
ES(G) &= \sum_{\{x,y\}\subseteq B(G)} \sqrt{d_G(x)^2 + d_G(y)^2} \\
&= \sum_{\{x,y\}\subseteq B(G)} \frac{1}{\sqrt{d_G(x).d_G(y)}}\left[\sqrt{d_G(x).d_G(y)\big(d_G(x)^2 + d_G(y)^2\big)}\right] \\
&\leq \sum_{\{x,y\}\subseteq B(G)} \frac{1}{\sqrt{d_G(x).d_G(y)}}\left[\sqrt{\Delta^2\ 2(\Delta)^2}\right] \\
&\leq ER(G)\ \sqrt{2}\,\Delta^2.
\end{aligned}$$

Similarly, we have to prove the left inequality. Therefore,

$$ER(G)\sqrt{2}\,\delta^2 \leqslant ES(G) \leqslant ER(G)\sqrt{2}\,\Delta^2.$$

The both left and right inequalities holds if and only if G is regular. $\square$

Theorem 2.10 *Let G be a connected graph with $n \geqslant 3$. Then*

$$ES(G) \leqslant \sqrt{\frac{\Delta^2 + \delta^2}{\Delta\delta}}ER(G).$$

The inequality holds if and only if G is regular.

Proof Let G be a connected graph with $n \geqslant 3$. Then,

$$\begin{aligned}
ES(G) &= \sum_{\{x,y\}\subseteq B(G)} \sqrt{d_G(x)^2 + d_G(y)^2} \\
&= \sum_{\{x,y\}\subseteq B(G)} \sqrt{\left[\frac{d_G(x)}{d_G(y)} + \frac{d_G(y)}{d_G(x)}\right] d_G(x)\ d_G(y)} \\
&\leq \sqrt{\left[\frac{\Delta^2 + \delta^2}{\Delta\ \delta}\right]} \sum_{\{x,y\}\subseteq B(G)} \sqrt{d_G(x)\ d_G(y)} \\
&\leq \sqrt{\left[\frac{\Delta^2 + \delta^2}{\Delta\ \delta}\right]} ER(G).
\end{aligned}$$

The inequality holds if and only if G is regular. $\square$

Corollary 2.1 *Let G be an r-regular connected graph with $n \geqslant 3$. Then*

$$ER(G) \leq ES(G) \leq \sqrt{2}ER(G).$$

Theorem 2.11 *Let G be a connected graph with $n \geqslant 3$. Then,*

(i) $ES(G) \leqslant \sqrt{\Big(|B(G)|\Big)\ EF(G)}$ *and*

(ii) $ES(G) \leqslant \sqrt{\Big(|B(G)|\Big)\ MEF^{*}(G)\ (EF(G))^2}$,

where,

$$MEF^{*}(G) = \sum_{\{x,y\}\subseteq B(G)} \frac{1}{d_G(x)^2 + d_G(y)^2}$$

is the modified forgotten index.

Proof Let G be a connected graph with $n \geq 3$.

(i) Consider

$$\begin{aligned} ES(G) &= \sum_{\{x,y\}\subseteq B(G)} \sqrt{d_G(x)^2 + d_G(y)^2} \\ &\leq \sqrt{\Big(|B(G)|\Big) \sum_{\{x,y\}\subseteq B(G)} d_G(x)^2 + d_G(y)^2} \leq \sqrt{\Big(|B(G)|\Big)\ EF(G)}. \end{aligned}$$

(ii) Consider

$$\begin{aligned} ES(G) &= \sum_{\{x,y\}\subseteq B(G)} \sqrt{d_G(x)^2 + d_G(y)^2} \\ &\leq \sqrt{(|B(G)|) \sum_{\{x,y\}\subseteq B(G)} \frac{\left[d_G(x)^2 + d_G(y)^2\right]^2}{d_G(x)^2 + d_G(y)^2}} \\ &\leq \sqrt{(|B(G)|) \sum_{\{x,y\}\subseteq B(G)} \frac{1}{d_G(x)^2 + d_G(y)^2} \sum_{\{x,y\}\subseteq B(G)} \left[d_G(x)^2 + d_G(y)^2\right]^2} \\ &\leq \sqrt{(|B(G)|)\ MEF^{*}(G)\ (EF(G))^2}. \end{aligned}$$

We obtained the desired results. □

Theorem 2.12 *Let G be a connected graph with $n \geqslant 3$. Then*

$$2\delta^2\sqrt{\Big(|B(G)|\Big)\ MEF^{*}(G)} \leqslant ES(G) \leqslant 2\Delta^2\sqrt{\Big(|B(G)|\Big)\ MEF^{*}(G)}.$$

The both left and right inequalities holds if and only if G is regular.

Proof Let G be a connected graph with $n \geqslant 3$. Then,

$$\begin{aligned} ES(G) &= \sum_{\{x,y\}\subseteq B(G)} \sqrt{d_G(x)^2 + d_G(y)^2} \\ &\leq \sqrt{(|B(G)|) \sum_{\{x,y\}\subseteq B(G)} \frac{1}{d_G(x)^2 + d_G(y)^2}(d_G(x)^2 + d_G(y)^2)^2} \\ &\leq \sqrt{(|B(G)|)\, 4\Delta^4 MEF^*(G)} \\ &\leq 2\Delta^2 \sqrt{(|B(G)|)\ MEF^*(G)}. \end{aligned}$$

Similarly, we have to prove the left inequality. Therefore,

$$2\delta^2 \sqrt{\Big(|B(G)|\Big)\ MEF^*(G)} \leqslant ES(G) \leqslant 2\Delta^2 \sqrt{\Big(|B(G)|\Big)\ MEF^*(G)}.$$

The both left and right inequalities holds if and only if G is regular. □

Corollary 2.2 *Let G be a connected graph with $n \geqslant 3$. Then,*

(*i*) $ES(G) \leqslant \sqrt{\left(\frac{Pl(G)}{2} + 3\,m\right) EF(G)}$ *and*

(*ii*) $ES(G) \leqslant \sqrt{\left(\frac{Pl(G)}{2} + 3\,m\right) MEF^*(G)\ (EF(G))^2}$.

Corollary 2.3 *Let G be a connected graph with $n \geqslant 3$. Then,*

(*i*) $ES(G) \leqslant \sqrt{\left(2m + \frac{M_1(G)}{2}\right) EF(G)}$ *and*

(*ii*) $ES(G) \leqslant \sqrt{\left(2m + \frac{M_1(G)}{2}\right) MEF^*(G)\ (EF(G))^2}$.

§3. Chemical Applicability of Linear $[k]$ Alkanes

Hydrocarbons are one of the major part of chemical graph theory. The hydrocarbons are the organic compounds containing carbon and hydrogen. For example alkane, alkene and alkynes. Alkanes are saturated, open chain hydrocarbons containing carbon-carbon single bonds. For example, methane (CH_4), ethane (C_2H_6), propane (C_3H_8), etc. These hydrocarbons are inert under normal conditions (i.e do not react with acids, bases and other reagents). Hence, they were earlier known as *paraffins*. The uses of alkanes depends on the quantity of carbon atoms. The first four alkanes are used largely for heating and culinary purposes. For more details, we refer to [4, 16, 19].

The molecular graph of alkane is a tree in which vertices corresoponds to atoms and edges to carbon-carbon or hydrogen-carbon bonds in a chemical alkane. The molecular formula for alkane C_nH_{2n+2} which contains $(3n+2)$ - vertices and $(3n+1)$ - edges and the linear $[k]$ alkanes can be represented as $A[k]$, see Figure 1. For partitioned of $A[k]$, the value of k represents the stages of alkanes $A[k]$. If $k = 1$ the number of pair of adjacent edges in $(3, 3)$ is 6. For $k = 2$

the number of pair of adjacent edges in $(3,6)$ and $(6,6)$ are shown in Table 2.

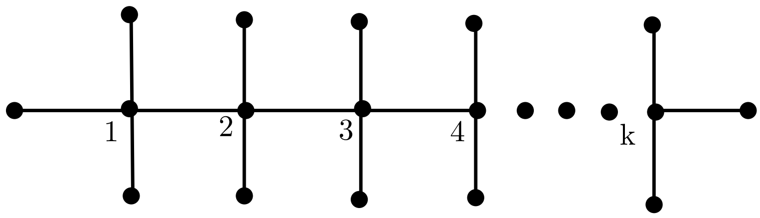

Figure 1. Linear [k] Alkanes

$\big(d_G(u), d_G(v)\big) : uv \in E(G)$	$(1,4)$	$(4,4)$	
Number of edges	$(2k+2)$	$(k-1)$	
$\big(d_G(e), d_G(f)\big) : e, f \in E(G)$	$(3,3)$	$(3,6)$	$(6,6)$
Number of pair of adjacent edges	$\big[(k-2)+6\big]$	$\big[4(k-1)+2\big]$	$(k-2)$
$\big(d_G(u), d_G(v)\big) : uv \in E(G)$	$(1,4)$	$(4,4)$	
$d_G(e)$	3	6	
Number of edges	$(2k+2)$	$(k-1)$	

Table 2. Vertex-edges set partitions and their values.

Mathematically, the computed values of Sombor related indices of a graph can represents as, $SO(G) < KG(G) \leqslant RS(G) < ES(G)$. The computed values of graphical indices of different stages k for $1 \leq k \leq 4$ as shown in Table 3 and its comparative analysis as shown in Figure 2. This shows the highest and least value of Sombor related topological indices.

Indices	**Computed values**
$SO(G)$	$[2\sqrt{17}+4\sqrt{2}]k + [2\sqrt{17}-4\sqrt{2}]$
$RS(G)$	$[5+6\sqrt{2}]k + [24\sqrt{2}+6\sqrt{5}]$
$KG(G)$	$[2\sqrt{10}+4\sqrt{13}+10]k + [2\sqrt{10}-4\sqrt{13}+10]$
$ES(G)$	$[10\sqrt{2}+2\sqrt{17}+2\sqrt{10}+4\sqrt{13}+15]k$ $+[20\sqrt{2}+2\sqrt{17}+6\sqrt{5}+2\sqrt{10}-4\sqrt{13}+4]$

Table 3. The computed values of Sombor related indices

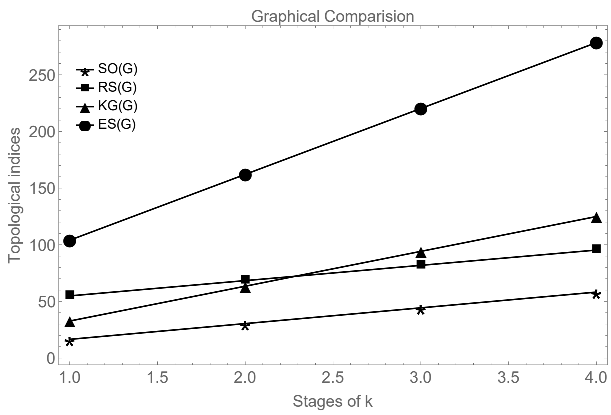

Figure 2. The computed values of Sombor related indices

Also, the computed values of graphical indices with respected to the particular values k for $1 \leq k \leq 4$ as shown in Table 4 and its comparative analysis as shown in Figure 3 as follows:

Stages of k	**Sombor related indices**			
	$SO(G)$	$RS(G)$	$KG(G)$	$ES(G)$
k=1	16.492	54.842	32.649	103.984
k=2	30.395	68.328	63.395	162.119
k=3	44.298	81.813	94.142	220.254
k=4	58.201	95.298	124.889	278.389

Table 4. The particular values of Sombor related indices

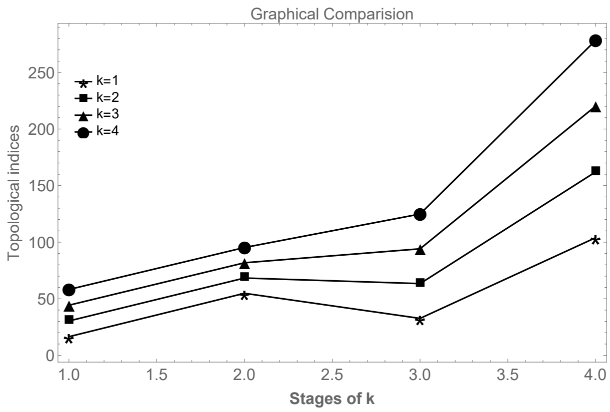

Figure 3. The particular values of Sombor related indices

§4. Conclusion

A topological descriptor can be assumed to be a function that provides the information in numerical form about any underline molecular structure. Topological descriptors capture the symmetry of chemical compounds and present the facts in the numerical form including the presence of heteroatoms, molecular size, multiple bonds, shape and branching.

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Decomposition of Tensor Product of Complete Graphs into Connected Unicyclic Bipartite Graphs with Eight Edges

S. Duraimurugan and A. Muthusamy

(Department of Mathematics, Periyar University, Salem, Tamil Nadu, India)

E-mail: duraipu93@gmail.com, appumuthusamy@gmail.com

Abstract: In this paper, we obtain necessary and sufficient conditions for decomposing tensor product of complete graphs into some connected unicyclic bipartite graphs with eight edges.

Key Words: Decomposition, Smarandache decomposition, wreath product, tensor product, unicyclic graph.

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§1. Introduction

All the graphs considered here are loopless and finite. For a given graph G and an integer $\lambda \geq 1$, we use the notation $G(\lambda)$ to represent the multigraph obtained from G by replacing each of its edges with λ parallel edges. Similarly, λG denotes the graph consisting of λ edge-disjoint copies of G. The notations P_t, C_t, K_t, and $\overline{K}_t$ represents the path, cycle, complete graph, and complement of the complete graph, each with t vertices, respectively. Also, we denote the induced subgraph H of G induced by S as $\langle S \rangle$. Consider a *complete bipartite graph* $K_{t,t}$ with bipartition (X, Y), where $X = \{x_0, x_1, \cdots, x_{t-1}\}$ and $Y = \{y_0, y_1, \cdots, y_{t-1}\}$. We define the spanning subgraph $F_i(X, Y)$ of $K_{t,t}$ as $\langle \{x_j y_{j+i} : 0 \leq j \leq t-1\} \rangle$, where addition in the subscripts are taken modulo t. It is clear that $F_i(X, Y)$ is a 1-factor of $K_{t,t}$ with a distance i from X to Y. Moreover, $K_{t,t} = \bigoplus_{i=0}^{t-1} F_i(X, Y)$, where $\oplus$ denotes the edge-disjoint union of graphs, also called a *Smarandache decomposition* if $K_{t,t}$ is labeled.

For two graphs G and H, their *lexicographic product* $G \otimes H$ has the vertex set $V(G \otimes H) = V(G) \times V(H)$ and the edge set $E(G \otimes H) = \{(g_1, h_1)(g_2, h_2) : g_1 g_2 \in E(G)$ or $g_1 = g_2$ and $h_1 h_2 \in E(H)\}$. Similarly, the *tensor product* $G \times H$ of two graphs G and H has the vertex set $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H) = \{(g_1, h_1)(g_2, h_2) : g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)\}$. Note that, the tensor product is commutative and distributive over edge-disjoint union of graphs, that is, if $G = G_1 \oplus G_2 \oplus \cdots \oplus G_u$, then $G \times H = (G_1 \times H) \oplus \cdots \oplus (G_u \times H)$. One can easily observe that $(K_u \otimes \overline{K}_g) - gK_u \cong K_u \times K_g$, where gK_u denotes g disjoint copies of K_u.

For some integer $r \geq 1$, we say that the graph G has a *decomposition* into the subgraphs

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$G_1, G_2, \cdots, G_r$ if $G = \oplus_{i=1}^{r} G_i$, and $G_1, G_2, \cdots, G_r$ are pairwise edge-disjoint subgraphs of G. For each $i,\ 1 \leq i \leq r$, if $G_i \cong H$, then we say that G has an *H-decomposition* and we denote such decomposition by $H|G$. A graph G is said to be *unicyclic* if it has exactly one cycle.

Decomposition of graphs into subgraphs has been an interesting research area in graph theory since 1950s. Adams et al. [1] published an excellent survey on decomposing complete graphs into subgraphs containing up to six vertices. Tian et al. [17] established the decomposition of complete graphs into unicyclic graphs with six vertices and seven edges, while Froncek et al. [10] proved the decomposition of complete graphs into connected unicyclic bipartite graphs with seven edges. In recent studies, Froncek et al. [11,12] proved the decomposition of complete graphs into tri-cyclic and bi-cyclic graphs, each with eight edges. Furthermore, Fahnenstiel et al. [5] established the necessary and sufficient conditions for the existence of a decomposition of complete graphs into connected unicyclic bipartite graphs with eight edges. Huang et al. [13] proved the decomposition of complete equipartite graphs into connected unicyclic graphs, each having a size of five vertices. Similarly, Paulraja et al. [14] established the decomposition of certain regular graphs into unicyclic graphs of order five. Sowndhariya et al. [15] proved the decomposition of product graphs into sunlet graphs of order eight. Aspenson et al. [3], proved the decomposition K_{18n} and K_{18n+1} into connected unicyclic graphs with nine edges. Similarly, Bonhert et al. [4], proved the decompositions of complete graphs into unicyclic disconnected bipartite graphs with nine edges. Recently, we have proved the existence of decomposition of λ-fold complete equipartite graphs into connected unicyclic bipartite graphs with eight edges in [6] and the general problem is open for other classes of product of graphs. In this paper, we show the existence of such decomposition in tensor product of complete graphs.

Let G_1, G_2, G_3, G_4 and G_5 be the graphs shown in Figure 1. We assume that these graphs have the vertex set $\{v_1, v_2, \cdots, v_8\}$. The edge set of the unicyclic graphs G_1, G_2, G_3, G_4, and G_5 are denoted by $(v_1v_2v_3v_4)$ $[v_1v_5v_6v_7v_8]$, $(v_1v_2v_3v_4)$ $[v_1v_5v_7v_8]$ $[v_5v_6]$, $(v_1v_2v_3v_4)$ $[v_2v_6v_7v_8]$ $[v_1v_5]$, $(v_1v_2v_3v_4)$ $[v_1v_5v_6]$ $[v_3v_7v_8]$, and $(v_1v_2v_3v_4)[v_1v_5v_6]$ $[v_4v_7]$ $[v_3v_8]$, respectively. Clearly, each $G_i,\ 1 \leq i \leq 5$, is a connected unicyclic bipartite graph with eight edges.

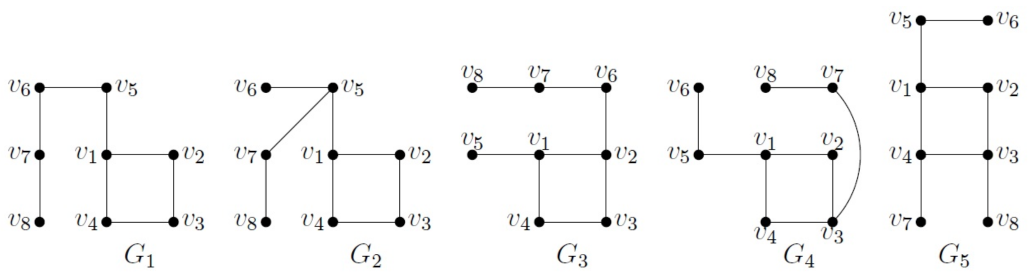

Figure 1. Connected unicyclic bipartite graphs with eight edges

To prove our results we state the following:

Theorem 1.1([16]) *There exists a P_{m+1}-decomposition of $K_u(\lambda)$ if and only if $\lambda u(u-1) \equiv 0 \pmod{2m}$, $u \geq m+1$.*

Theorem 1.2([2]) *For all positive odd integers m and n with $3 \leq m \leq n$, there exists a C_m-decomposition of K_n if and only if $n(n-1) \equiv 0 \pmod{2m}$.*

Theorem 1.3([6]) *There exists a G_i-decomposition of $K_{4x,4y}$, $1 \le i \le 5$.*

§2. G_i-Decomposition of Base Graphs

In this part, we have established some crucial lemmas to prove our main results.

Lemma 2.1 *The graphs $K_{4,2}$, $K_{4,4}$ and $K_{4,6}$ admits a P_3-decomposition.*

Proof Our proof is divided into two cases.

Case 1. $P_3|K_{4,4}$

Let $V(K_{4,4}) = (U, V)$, where $U = \{u_0, u_1, u_2, u_3\}$ and $V = \{v_0, v_1, v_2, v_3\}$. Let $P_3^{j,1} = [v_j u_j v_{j+1}]$ and $P_3^{j,2} = [u_j v_{j+2} u_{j+3}]$, $j \in \mathbb{Z}_4$ and additions in the subscripts of u and v are taken modulo 4. When j varies, $\{P_3^{j,1}, P_3^{j,2}\}$ gives a required P_3-decomposition of $K_{4,4}$.

Case 2. $P_3|K_{4,6}$

Let $V(K_{4,4}) = (U, V)$, where $U = \{u_0, u_1, u_2, u_3\}$ and $V = \{v_0, v_1, \cdots, v_5\}$. Let $P_3^{j,1} = [v_j u_j v_{j+1}]$, $P_3^{j,2} = [u_j v_{j+2} u_{j+3}]$, and $P_3^{j,3} = [v_4 u_j v_5]$, $j \in \mathbb{Z}_4$ and additions in the subscripts of u and v are taken modulo 4. When j varies, $\{P_3^{j,1}, P_3^{j,2}, P_3^{j,3}\}$ gives a required P_3-decomposition of $K_{4,6}$. □

Lemma 2.2 *There exists a G_i-decomposition of $P_3 \times K_5$, $1 \le i \le 5$.*

Proof Let $V(P_3 \times K_5) = \cup_{i \in \mathbb{Z}_3} X_i$, where $X_0 = \{u_0, u_1, \cdots, u_4\}$, $X_1 = \{v_0, v_1, \cdots, v_4\}$ and $X_2 = \{w_0, w_1, \cdots, w_4\}$. The required G_i-decomposition of $P_3 \times K_5$ is shown below.

Let $G_1^j = (u_{j+1} v_{j+3} u_{j+4} v_{j+2})[v_{j+2} w_{j+3} v_{j+4} w_{j+2} v_j]$,
$G_2^j = (u_{j+1} v_{j+3} u_{j+4} v_{j+2})[v_{j+2} w_{j+3} v_{j+4} w_{j+2}][w_{j+3} v_{j+1}]$,
$G_3^j = (u_{j+1} v_{j+3} w_{j+1} v_{j+2})[u_{j+1} v_j w_{j+2} v_{j+1}][v_{j+3} u_j]$,
$G_4^j = (u_{j+1} v_{j+3} u_{j+4} v_{j+2})[v_{j+3} w_j v_{j+1}][v_{j+2} w_{j+3} v_j]$, and
$G_5^j = (u_{j+1} v_{j+3} w_{j+1} v_{j+2})[w_{j+1} v_{j+4} w_j][v_{j+2} u_{j+4}][u_{j+1} v_j], j \in \mathbb{Z}_5$, where the additions

in the subscripts of u, v, and w are taken modulo 5. Clearly, $G_i^j \cong G_i, i = 1, 2, 3, 4, 5,$ $j \in \mathbb{Z}_5$ shown in Figure 1. When j varies we get the required decomposition of $P_3 \times K_5$. □

Lemma 2.3 *There exists a G_i-decomposition of $P_3 \times K_8$, $1 \le i \le 5$.*

Proof Let $V(P_3 \times K_8) = \cup_{i \in \mathbb{Z}_3} X_i$, where $X_0 = \{u_0, u_1, \cdots, u_7\}$, $X_1 = \{v_0, v_1, \cdots, v_7\}$ and $X_2 = \{w_0, w_1, \cdots, w_7\}$. The required G_i-decomposition of $P_3 \times K_8$ is shown below.

Let $G_1^{j,1} = (u_{j+5} v_7 w_{j+6} v_{j+4})[w_{j+6} v_{j+3} u_{j+2} v_j w_7]$,
$G_1^{j,2} = (u_j v_{j+2} w_{j+1} v_{j+3})[u_j v_{j+4} w_{j+5} v_{j+1} u_7]$,
$G_2^{j,1} = (u_{j+6} v_7 w_{j+6} v_{j+5})[v_{j+5} w_{j+4} v_{j+2} u_7][w_{j+4} v_{j+1}]$,
$G_2^{j,2} = (u_{j+5} v_j w_{j+5} v_{j+1})[v_{j+1} u_j v_{j+5} w_7][u_j v_{j+4}]$,

$G_3^{j,1} = (u_{j+6}v_7w_{j+6}v_{j+5})[u_{j+6}v_{j+4}w_{j+3}v_{j+1}][v_{j+5}u_7]$,
$G_3^{j,2} = (u_{j+5}v_jw_{j+5}v_{j+1})[v_{j+1}u_jv_{j+4}w_7][w_{j+5}v_{j+2}]$,
$G_4^{j,1} = (u_{j+6}v_7w_{j+6}v_{j+5})[u_{j+6}v_{j+4}u_7][w_{j+6}v_{j+3}w_7]$,
$G_4^{j,2} = (u_jv_{j+2}w_jv_{j+1})[u_jv_{j+3}u_{j+6}][w_jv_{j+5}w_{j+2}]$,
$G_5^{j,1} = (u_{j+6}v_7w_{j+6}v_{j+5})[w_{j+6}v_{j+4}u_7][v_{j+5}w_7][u_{j+6}v_{j+3}]$ and
$G_5^{j,2} = (u_jv_{j+2}w_jv_{j+1})[u_jv_{j+3}u_{j+5}][v_{j+1}w_{j+5}][w_jv_{j+4}]$, $j \in \mathbb{Z}_7$, where the additions in

the subscripts of u, v, and w are taken modulo 7. Clearly, $G_i^{j,l} \cong G_i, i = 1,2,3,4,5,\ j \in \mathbb{Z}_7, l \in \{1,2\}$.When j and l varies, we get the required decomposition of $P_3 \times K_8$. □

Lemma 2.4 *There exists a G_1-decomposition of $P_3 \times K_{12}$.*

Proof Let $V(P_3 \times K_{12}) = \cup_{i\in\mathbb{Z}_3} X_i$, where $X_0 = \{u_0, u_1, \cdots, u_{11}\}$, $X_1 = \{v_0, v_1, \cdots, v_{11}\}$ and $X_2 = \{w_0, w_1, \cdots, w_{11}\}$. The required G_1-decomposition of $P_3 \times K_{12}$ is given below.

Let $G_1^{j,1} = (u_{j+10}v_{11}w_{j+10}v_{j+9})[w_{j+10}v_{j+8}u_{j+7}v_{j+5}w_{11}]$,
$G_1^{j,2} = (u_{j+10}v_{j+6}w_{j+10}v_{j+7})[u_{j+10}v_{j+5}w_{j+4}v_{j+10}u_{11}]$ and
$G_1^{j,3} = (u_jv_{j+2}w_{j+9}v_{j+3})[u_jv_{j+5}w_{j+3}v_{j+6}u_{j+2}]$, $j \in \mathbb{Z}_{11}$, where the additions in the

subscripts of u, v, and w are taken modulo 11. Clearly, $G_1^{j,l} \cong G_1, j \in \mathbb{Z}_{11}, l \in \{1,2,3\}$. When j and l varies, we get the required decomposition of $P_3 \times K_{12}$. □

Lemma 2.5 *There exists a G_i-decomposition of $P_5 \times K_6$, $1 \le i \le 5$.*

Proof Let $V(P_5 \times K_6) = \cup_{i\in\mathbb{Z}_5} X_i$, where $X_0 = \{u_0, u_1, \cdots, u_5\}$, $X_1 = \{v_0, v_1, \cdots, v_5\}$, $X_2 = \{w_0, w_1, \cdots, w_5\}$, $X_3 = \{x_0, x_1, \cdots, x_5\}$, and $X_4 = \{y_0, y_1, \cdots, y_5\}$. The required G_i-decomposition of $P_5 \times K_6$ is given below.

Let $G_1^{j,1} = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_5w_{j+4}v_{j+3}u_5]$,
$G_1^{j,2} = (v_jw_{j+3}x_{j+1}w_{j+2})[w_{j+3}v_{j+4}w_5x_{j+4}y_5]$,
$G_1^{j,3} = (w_{j+4}x_5y_{j+4}x_{j+1})[y_{j+4}x_jy_{j+1}x_{j+4}w_{j+3}]$,
$G_2^{j,1} = (u_jv_{j+1}u_{j+3}v_{j+2})[v_{j+2}w_{j+3}v_5u_{j+4}][w_{j+3}x_{j+4}]$,
$G_2^{j,2} = (v_jw_{j+3}x_{j+1}w_{j+2})[w_{j+3}v_{j+4}w_5x_{j+4}][v_{j+4}u_5]$,
$G_2^{j,3} = (w_{j+4}x_5y_{j+4}x_{j+1})[y_{j+4}x_jy_{j+1}x_{j+4}][x_jy_5]$,
$G_3^{j,1} = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_5w_{j+3}x_{j+4}][v_{j+1}w_{j+2}]$,
$G_3^{j,2} = (v_jw_{j+3}x_{j+1}w_{j+2})[w_{j+3}v_{j+4}w_5x_{j+4}][v_ju_5]$,
$G_3^{j,3} = (w_{j+4}x_5y_{j+4}x_{j+1})[y_{j+4}x_jy_{j+1}x_{j+4}][x_{j+1}y_5]$,
$G_4^{j,1} = (u_{j+4}v_5w_{j+4}v_{j+3})[u_{j+4}v_{j+2}u_5][w_{j+4}v_{j+1}w_{j+3}]$,
$G_4^{j,2} = (v_{j+4}w_5x_{j+4}w_{j+3})[v_{j+4}u_{j+3}v_j][x_{j+4}y_{j+3}x_j]$,
$G_4^{j,3} = (w_{j+4}x_5y_{j+4}x_{j+3})[w_{j+4}x_{j+1}w_{j+3}][y_{j+4}x_{j+2}y_5]$,
$G_5^{j,1} = (u_{j+4}v_5w_{j+4}v_{j+3})[u_{j+4}v_{j+2}u_5][v_{j+3}w_j][w_{j+4}v_{j+1}]$,

$G_5^{j,2} = (v_{j+4}w_5x_{j+4}w_{j+3})[v_{j+4}u_{j+3}v_j][w_{j+3}x_{j+1}][x_{j+4}y_{j+3}]$ and
$G_5^{j,3} = (w_{j+4}x_5y_{j+4}x_{j+3})[y_{j+4}x_{j+2}y_5][x_{j+3}y_{j+1}][w_{j+4}x_{j+1}]$, $j \in \mathbb{Z}_5$, where the

additions in the subscripts of u, v, w, x, and y are taken modulo 5. Clearly, $G_i^{j,l} \cong G_i, i = 1,2,3,4,5,\ j \in \mathbb{Z}_5, l \in \{1,2,3\}$. When j and l varies, we get the required decomposition of $P_5 \times K_6$. □

Lemma 2.6 *There exists a G_i-decomposition of $K_9 \times K_2$, $1 \le i \le 5$.*

Proof Let $V(K_9 \times K_2) = (U, V)$, where $U = \{u_0, u_1, \cdots, u_8\}$ and $V = \{v_0, v_1, \cdots, v_8\}$. The required G_i-decomposition of $K_9 \times K_2$ is given below.

Let $G_1^j = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_{j+6}u_{j+2}v_{j+7}u_{j+1}]$,
$G_2^j = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_{j+6}u_{j+2}v_{j+8}][v_{j+6}u_{j+1}]$,
$G_3^j = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_{j+6}u_{j+2}v_{j+8}][v_{j+2}u_{j+6}]$,
$G_4^j = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_{j+6}u_{j+2}][u_jv_{j+5}u_{j+8}]$, and
$G_5^j = (u_jv_{j+1}u_{j+3}v_{j+2})[u_{j+3}v_{j+6}u_{j+2}][v_{j+2}u_{j+5}][u_jv_{j+5}]$, $j \in \mathbb{Z}_9$, where the additions

in the subscripts of u and v are taken modulo 9. Clearly, $G_i^j \cong G_i, i = 1,2,3,4,5,\ j \in \mathbb{Z}_9$. When j varies, we get the required decomposition of $K_9 \times K_2$. □

Lemma 2.7 *There exists a G_i-decomposition of $C_3 \times K_8$, $1 \le i \le 5$.*

Proof Let $V(C_3 \times K_8) = \cup_{i \in \mathbb{Z}_3} X_i$, where $X_0 = \{u_0, u_1, \cdots, u_7\}$, $X_1 = \{v_0, v_1, \cdots, v_7\}$ and $X_2 = \{w_0, w_1, \cdots, w_7\}$. The required G_i-decomposition of $C_3 \times K_8$ is given below.

Let $G_1^{j,1} = (u_{j+5}v_{j+6}u_7w_{j+6})[u_{j+5}v_ju_{j+4}v_{j+1}u_{j+3}]$,
$G_1^{j,2} = (u_{j+6}v_7w_{j+6}v_{j+5})[v_{j+5}w_jv_{j+4}w_{j+1}v_{j+3}]$,
$G_1^{j,3} = (u_{j+6}w_7v_{j+6}w_{j+5})[w_{j+5}u_jw_{j+4}u_{j+1}w_{j+3}]$,
$G_2^{j,1} = (u_{j+5}v_{j+6}u_7w_{j+6})[u_{j+5}v_ju_{j+4}v_{j+1}][v_ju_{j+2}]$,
$G_2^{j,2} = (u_{j+6}v_7w_{j+6}v_{j+5})[v_{j+5}w_jv_{j+4}w_{j+1}][w_jv_{j+2}]$,
$G_2^{j,3} = (u_{j+6}w_7v_{j+6}w_{j+5})[w_{j+5}u_jw_{j+4}u_{j+1}][u_jw_{j+2}]$,
$G_3^{j,1} = (u_{j+5}v_{j+6}u_7w_{j+6})[u_{j+5}v_ju_{j+4}v_{j+1}][v_{j+6}w_{j+4}]$,
$G_3^{j,2} = (u_{j+6}v_7w_{j+6}v_{j+5})[v_{j+5}w_jv_{j+4}w_{j+1}][w_{j+6}u_{j+4}]$,
$G_3^{j,3} = (u_{j+6}w_7v_{j+6}w_{j+5})[w_{j+5}u_jw_{j+4}u_{j+1}][u_{j+6}v_{j+4}]$,
$G_4^{j,1} = (u_{j+5}v_{j+6}u_7w_{j+6})[v_{j+6}w_{j+1}v_{j+5}][w_{j+6}u_{j+1}w_{j+5}]$,
$G_4^{j,2} = (u_{j+6}v_7w_{j+6}v_{j+5})[u_{j+6}v_{j+1}u_{j+5}][w_{j+6}u_{j+4}w_j]$,
$G_4^{j,3} = (u_{j+6}w_7v_{j+6}w_{j+5})[u_{j+6}v_{j+3}u_{j+5}][v_{j+6}w_{j+3}v_{j+5}]$,
$G_5^{j,1} = (u_{j+5}v_{j+6}u_7w_{j+6})[w_{j+6}u_{j+3}w_{j+5}][u_{j+5}v_j][v_{j+6}w_{j+1}]$,
$G_5^{j,2} = (u_{j+6}v_7w_{j+6}v_{j+5})[u_{j+6}v_{j+2}u_{j+5}][v_{j+5}w_{j+1}][w_{j+6}u_{j+2}]$ and
$G_5^{j,3} = (u_{j+6}w_7v_{j+6}w_{j+5})[v_{j+6}w_{j+2}v_{j+5}][w_{j+5}u_j][u_{j+6}v_{j+4}]$, $j \in \mathbb{Z}_7$,

where the additions in the subscripts of u, v, and w are taken modulo 7. Clearly, $G_i^{j,l} \cong G_i, i = 1,2,3,4,5,\ j \in \mathbb{Z}_7, l \in \{1,2,3\}$. When j and l varies, we get the required decomposition of $C_3 \times K_8$. □

Lemma 2.8 *There exists a G_i-decomposition of $K_4 \times K_4,\ 2 \le i \le 5$.*

Proof Let $V(K_4 \times K_4) = \cup_{i\in\mathbb{Z}_4} X_i$, where $X_0 = \{u_0,u_1,u_2,u_3\},\ X_1 = \{v_0,v_1,v_2,v_3\},\ X_2 = \{w_0,w_1,w_2,w_3\}$, and $X_3 = \{x_0,x_1,x_2,x_3\}$. The required G_i-decomposition of $K_4 \times K_4$ is given below.

$$\begin{aligned}
\text{Let}\quad G_2^{j,1} &= (u_3v_{j+2}w_3x_{j+2})[u_3w_{j+1}x_3u_{j+1}][w_{j+1}u_j],\\
G_2^{j,2} &= (u_jv_{j+1}w_jx_{j+1})[w_ju_{j+1}v_3w_{j+2}][u_{j+1}w_3],\\
G_2^{j,3} &= (u_{j+1}v_jw_{j+1}x_j)[v_jx_{j+1}v_{j+2}x_3][x_{j+1}v_3],\\
G_3^{j,1} &= (u_3v_{j+2}w_3x_{j+2})[v_{j+2}x_3w_{j+2}v_3][u_3w_{j+1}],\\
G_3^{j,2} &= (u_jv_{j+1}w_jx_{j+1})[v_{j+1}x_{j+2}v_3u_{j+2}][u_jw_3],\\
G_3^{j,3} &= (u_{j+1}v_jw_{j+1}x_j)[u_{j+1}w_ju_{j+2}x_3][v_jx_{j+2}],\\
G_4^{j,1} &= (u_3v_{j+2}w_3x_{j+2})[u_3w_{j+1}u_j][w_3u_{j+1}x_3],\\
G_4^{j,2} &= (u_jv_{j+1}w_jx_{j+1})[u_jw_{j+2}x_3][w_jv_3u_{j+2}],\\
G_4^{j,3} &= (u_{j+1}v_jw_{j+1}x_j)[v_jx_{j+1}v_3][x_jv_{j+1}x_3],\\
G_5^{j,1} &= (u_3v_{j+2}w_3x_{j+2})[u_3w_{j+1}u_j][v_{j+2}x_{j+1}][w_3u_{j+2}],\\
G_5^{j,2} &= (u_jv_{j+1}w_jx_{j+1})[w_jx_3u_{j+2}][v_{j+1}x_{j+2}][u_jv_3] \text{ and}\\
G_5^{j,3} &= (u_{j+1}v_jw_{j+1}x_j)[w_{j+1}v_3x_{j+2}][v_jx_3][u_{j+1}w_j],\ j \in \mathbb{Z}_3,
\end{aligned}$$

where the additions in the subscripts of u, v, w, and x are taken modulo 3. Clearly, $G_i^{j,l} \cong G_i, i = 1,2,3,4,5,\ j \in \mathbb{Z}_3, l \in \{1,2,3\}$. When j and l varies, we get the required decomposition of $K_4 \times K_4$. □

Lemma 2.9 *For $g \equiv 0 \pmod 8$, there exists a G_i-decomposition of $K_6 \times K_g,\ 1 \le i \le 5$.*

Proof Let $g = 8x,\ x \ge 1$. We can write $K_{8x} = (K_x \otimes \overline{K}_8) \oplus xK_8 = \binom{x}{2}(K_2 \otimes \overline{K}_8) \oplus xK_8 \cong \binom{x}{2}K_{8,8} \oplus xK_8$ and hence $K_{8x} \times K_6 = \binom{x}{2}(K_{8,8} \times K_6) \oplus x(K_8 \times K_6) = 15\binom{x}{2}(K_{8,8} \times K_2) \oplus x(K_8 \times K_6)$. By Theorem 1.3, $G_i|K_{8,8}$, since G_i is bipartite, $G_i \times K_2 = 2G_i$. By Theorem 1.1, $P_5|K_8$ and hence $G_i|P_5 \times K_6$ by Lemma 2.5. Therefore, the graph $K_6 \times K_{8x}$ has a required G_i-decomposition. □

Lemma 2.10 *For $g \equiv 0 \pmod 8$, there exists a G_i-decomposition of $P_3 \times K_g,\ 1 \le i \le 5$.*

Proof Let $g = 8x,\ x \ge 1$. We can write $P_3 \times K_{8x} = ((P_3 \times K_x) \otimes \overline{K}_8) \oplus x(P_3 \times K_8) = ((P_3 \times \binom{x}{2}K_2) \otimes \overline{K}_8) \oplus x(P_3 \times K_8) = \left(\binom{x}{2}(P_3 \times K_2) \otimes \overline{K}_8\right) \oplus x(P_3 \times K_8) = 4\binom{x}{2}(K_2 \otimes \overline{K}_8) \oplus x(P_3 \times K_8) = 4\binom{x}{2}K_{8,8} \oplus x(P_3 \times K_8)$. By Theorem 1.3 and Lemma 2.3, the graph $P_3 \times K_{8x}$ has a required G_i-decomposition. □

Lemma 2.11 *For $u \equiv 0, 4 \pmod 8$ and $g \equiv 0 \pmod 4$, G_1-decomposition of $K_u \times K_g$ exists.*

Proof Let $u = 8x + t,\ x \ge 1$ and $t \in \{0, 4\}$. We can write $K_{8x+t} = K_{8+t} \oplus (x-1)K_8 \oplus$

$(x-1)K_{8,8+t} \oplus (K_{x-1} \otimes \overline{K}_8) = K_{8+t} \oplus (x-1)K_8 \oplus (x-1)K_{8,8+t} \oplus (\binom{x-1}{2}(K_2 \otimes \overline{K}_8)) = K_{8+t} \oplus (x-1)K_8 \oplus (x-1)K_{8,8+t} \oplus \binom{x-1}{2}K_{8,8}$. By Theorem 1.1, $P_3|K_g$ and $G_1|K_{8+t} \times P_3$, by Lemmas 2.3 and 2.4. By Theorem 1.3, $G_1|K_{8,8+t}$ and hence $G_1 \times K_g = G_1 \times (\binom{g}{2}K_2) = \binom{g}{2}(G_1 \times K_2)$, since G_1 is bipartite, $G_1 \times K_2 = 2G_1$. Therefore, G_1-decomposition of $K_u \times K_g$ exists. □

§3. G_i-Decomposition of $K_u \times K_g$

Theorem 3.1 *Let $u, g \geq 4$. For $1 \leq i \leq 5$, $G_i|K_u \times K_g$ if and only if $ug(u-1)(g-1) \equiv 0 \pmod{16}$, except possibly $(u, g, G_i) = (4, 4, G_1)$.*

Proof *Necessity:* The number of edges in $K_u \times K_g$ are $\binom{u}{2}(g^2 - g)$ and G_i has 8 edges. If $G_i|K_u \times K_g$, then $8|\binom{u}{2}(g^2 - g)$. Hence $ug(u-1)(g-1) \equiv 0 \pmod{16}$.

Sufficiency: To prove the sufficiency, from the edge divisibility condition, it is enough to discuss the following cases.

- $u \equiv 0 \pmod 4$ and $g \equiv 0 \pmod 4$;
- $u \equiv 0 \pmod 4$ and $g \equiv 1 \pmod 4$;
- $u \equiv 2 \pmod 4$ and $g \equiv 0 \pmod 8$;
- $u \equiv 2 \pmod 4$ and $g \equiv 1 \pmod 8$;
- $u \equiv 3 \pmod 4$ and $g \equiv 0 \pmod 8$;
- $u \equiv 3 \pmod 4$ and $g \equiv 1 \pmod 8$;
- $u \equiv 1 \pmod 4$ and $g \equiv 1 \pmod 4$.

Case 1. $u \equiv 0 \pmod 4$ and $g \equiv 0 \pmod 4$

By Lemma 2.11, $G_1|K_u \times K_g$ exists and hence it is enough to prove $G_i|K_u \times K_g$, $2 \leq i \leq 5$. Let $u = 4x$ and $g = 4y$, $x, y \geq 1$. We can write $K_{4x} = (K_x \otimes \overline{K}_4) \oplus xK_4 = \binom{x}{2}(K_2 \otimes \overline{K}_4) \oplus xK_4 = \binom{x}{2}K_{4,4} \oplus xK_4$ and $K_{4y} = \binom{y}{2}K_{4,4} \oplus yK_4$. Then $K_{4x} \times K_{4y} = \big(\binom{x}{2}K_{4,4} \oplus xK_4\big) \times \big(\binom{y}{2}K_{4,4} \oplus yK_4\big) = \binom{x}{2}\binom{y}{2}(K_{4,4} \times K_{4,4}) \oplus y\binom{x}{2}(K_{4,4} \times K_4) \oplus x\binom{y}{2}(K_{4,4} \times K_4) \oplus xy(K_4 \times K_4) = 16\binom{x}{2}\binom{y}{2}(K_{4,4} \times K_2) \oplus 6y\binom{x}{2}(K_{4,4} \times K_2) \oplus 6x\binom{y}{2} \oplus (K_{4,4} \times K_2) \oplus xy(K_4 \times K_4)$. By Theorem 1.3, $G_i|K_{4,4}$ since G_i is bipartite, $G_i \times K_2 = 2G_i$. By Lemma 2.8, $G_i|K_4 \times K_4$, $2 \leq i \leq 5$. Therefore, the graph $K_{4x} \times K_{4y}$ has a required G_i-decomposition.

Case 2. $u \equiv 0, 1 \pmod 4$ and $g \equiv 1 \pmod 4$

Let $g = 4x + 1$, $x \geq 1$. We can write $K_{4x+1} = (K_x \otimes \overline{K}_4) \oplus xK_5 = \binom{x}{2}K_{4,4} \oplus xK_5$ and hence $K_u \times K_{4x+1} = \binom{x}{2}(K_u \times K_{4,4}) \oplus x(K_u \times K_5) = \binom{u}{2}\binom{x}{2}(K_2 \times K_{4,4}) \oplus x(K_u \times K_5)$. By Theorem 1.3, $G_i|K_{4,4}$, since G_i is bipartite, $G_i \times K_2 = 2G_i$. By Theorem 1.1, $P_3|K_u$ and hence $G_i|P_3 \times K_5$ by Lemma 2.2. Therefore, the graph $K_u \times K_{4x+1}$ has a required G_i-decomposition.

Case 3. $u \equiv 2 \pmod 4$ and $g \equiv 0 \pmod 8$

Let $u = 4x+2$, $x \geq 1$. We can write $K_{4x+2} = K_6 \oplus (x-1)K_4 \oplus (x-1)K_{4,6} \oplus (K_{x-1} \otimes \overline{K}_4) = K_6 \oplus (x-1)K_4 \oplus (x-1)K_{4,6} \oplus \binom{x-1}{2}K_{4,4}$ and hence $K_{4x+2} \times K_g = (K_6 \times K_g) \oplus (x-1)(K_4 \times K_g) \oplus (x-1)(K_{4,6} \times K_g) \oplus \binom{x-1}{2}(K_{4,4} \times K_g)$. By Lemma 2.9, the graph $K_6 \times K_g$ has a G_i-decomposition. By Theorem 1.1 and Lemma 2.1, $P_3|K_4$, $P_3|K_{4,4}$, and $P_3|K_{4,6}$ and hence $G_i|P_3 \times K_g$ by Lemma 2.10. Therefore, the graph $K_{4x+2} \times K_g$ has a required G_i-decomposition.

Case 4. $u \equiv 3 \pmod 4$ and $g \equiv 0 \pmod 8$

Let $u = 4x+3,\ x \geq 1$. We can write $K_{4x+3} = K_7 \oplus (x-1)K_5 \oplus (x-1)K_{4,6} \oplus (K_{x-1} \otimes \overline{K}_4) = K_7 \oplus (x-1)K_5 \oplus (x-1)K_{4,6} \oplus \binom{x-1}{2}K_{4,4}$ and hence $K_{4x+3} \times K_g = (K_7 \times K_g) \oplus (x-1)(K_5 \times K_g) \oplus (x-1)(K_{4,6} \times K_g) \oplus \binom{x-1}{2}(K_{4,4} \times K_g)$. By Theorem 1.2, $C_3|K_7$ and the graphs $K_5,\ K_{4,2},\ K_{4,4}$ has P_3-decomposition by Theorem 1.1 and Lemma 2.1. Then by Lemmas 2.7 and 2.3, the graphs $C_3 \times K_g$ and $P_3 \times K_g$ has G_i-decomposition. Therefore, the graph $K_{4x+3} \times K_g$ has a required G_i-decomposition.

Case 5. $u \equiv 2, 3 \pmod 4$ and $g \equiv 1 \pmod 8$

Let $g = 8x + 1,\ x \geq 1$. We can write $K_{8x+1} = (K_x \otimes \overline{K}_8) \oplus xK_9 = \binom{x}{2}K_{8,8} \oplus xK_9$ and hence $K_u \times K_{8x+1} = \binom{x}{2}(K_u \times K_{4,4}) \oplus x(K_u \times K_9) = \binom{u}{2}\binom{x}{2}(K_2 \times K_{8,8}) \oplus x\binom{u}{2}(K_2 \times K_9)$. By Theorem 1.3, $G_i|K_{8,8}$, since G_i bipartite, $G_i \times K_2 = 2G_i$ and by Lemma 2.6, $G_i|K_9 \times K_2$. Therefore, the graph $K_u \times K_{8x+1}$ has a required G_i-decomposition. □

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Second Order Connectivity Indices of Some Chemical Trees

Jingling Fang, Jie Li, Li Li and Zhen Lin

(School of Mathematics and Statistics, Qinghai Normal University, Xining, 810008, Qinghai, China)

E-mail: lnlinzhen@163.com

Abstract: Based on the higher order Randić index, we propose three other high order connectivity indices, and obtain the calculation formulas for the new connectivity indices of some chemical trees.

Key Words: Connectivity indices, higher order Randić index, chemical trees.

AMS(2010): 05C05, 05C07, 05C09, 05C92.

§1. Introduction

In 1975, the connectivity index (now also called the Randić index or the branching index) of a graph G, denoted by $R(G)$, introduced by the chemist Milan Randić [6], is the degree-based topological index that is most frequently applied in quantitative structure-property and structure-activity studies. For a simple undirected graph $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$, its Randić index is defined as

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}},$$

where d_u denotes the degree of the vertex u in G.

The sum-connectivity index [14], the atom-bond connectivity index [3] and the atom-bond sum-connectivity index [1] are the class of successful variants of the connectivity index, and defined as

$$\begin{aligned} SCI(G) &= \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}, \\ ABC(G) &= \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}, \\ ABS(G) &= \sum_{uv \in E(G)} \sqrt{1 - \frac{2}{d_u + d_v}}. \end{aligned}$$

In 1976, Kier et al. [4] modified the Randić index, and proposed the higher order Randić

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[3]Corresponding author: Zhen Lin, E-mail: lnlinzhen@163.com.

index ${}^hR(G)$ of a graph G, that is,

$$ {}^hR(G) = \sum_{v_1,v_2,\ldots,v_{h+1}\in E_h(G)} \frac{1}{\sqrt{d_{v_1}d_{v_2}\cdots d_{v_{h+1}}}}, $$

where $E_h(G)$ is all paths of length h in G. Clearly, $E_1(G)$ is the edge of G. Thus the higher order Randić index is a natural extension of the Randić index. The higher order Randić index is of great interest in mathematics [2,7,9,11] and the theory of molecular topology [10]. In particular, the lower order Randić index has attracted widespread attention from scholars, and extensively studied [5,8,12,13]. However, these investigations all focused on benzenoid systems. The results of the chemical trees (alkanes) have not been reported.

Naturally, we need to study other high order connectivity indices, especially the second order connectivity indices. By definition, the second sum-connectivity index, second atom-bond connectivity index and the second atom-bond sum-connectivity index are respectively

$$ \begin{aligned} {}^2SCI(G) &= \sum_{uvw\in E_2(G)} \frac{1}{\sqrt{d_u+d_v+d_w}}, \\ {}^2ABC(G) &= \sum_{uvw\in E_2(G)} \sqrt{\frac{d_u+d_v+d_w-3}{d_ud_vd_w}}, \\ {}^2ABS(G) &= \sum_{uvw\in E_2(G)} \sqrt{1-\frac{3}{d_u+d_v+d_w}}. \end{aligned} $$

In this paper, the expression of the second order connectivity indices of some chemical trees is found.

§2. Chemical Trees of Module 2

The chemical trees of T_2^0 and T_2^1 are shown in Figure 1.

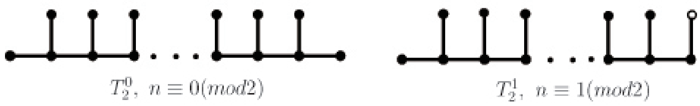

Figure 1. T_2^0 and T_2^1

Theorem 2.1 *The second order connectivity indices of T_2^0 with n vertices are given by*

$$ \begin{aligned} {}^2R(T_2^0) &= \frac{(6+\sqrt{3})n}{18} + \frac{\sqrt{3}-2}{3}, \\ {}^2SCI(T_2^0) &= \frac{(7+6\sqrt{7})n}{42} + \frac{14\sqrt{5}-10\sqrt{7}-35}{35}, \\ {}^2ABC(T_2^0) &= \frac{(4+\sqrt{2})n}{6} + \frac{2\sqrt{6}-3\sqrt{2}-4}{3}, \\ {}^2ABS(T_2^0) &= \frac{(12\sqrt{7}+7\sqrt{6})n}{42} + \frac{14\sqrt{10}-20\sqrt{7}-35\sqrt{6}}{35}. \end{aligned} $$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_2^0 in the following table.

$m(1,3,1)$	$m(1,3,3)$	$m(3,3,3)$
2	$n-2$	$\frac{n-6}{2}$

Thus, we have

$$\begin{aligned}
{}^2R(T_2^0) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_ud_vd_w}} \\
&= 2\times\frac{1}{\sqrt{3}}+(n-2)\times\frac{1}{\sqrt{9}}+\frac{n-6}{2}\times\frac{1}{\sqrt{27}} \\
&= \frac{(6+\sqrt{3})n}{18}+\frac{\sqrt{3}-2}{3}, \\
{}^2SCI(T_2^0) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}} \\
&= 2\times\frac{1}{\sqrt{5}}+(n-2)\times\frac{1}{\sqrt{7}}+\frac{n-6}{2}\times\frac{1}{\sqrt{9}} \\
&= \frac{(7+6\sqrt{7})n}{42}+\frac{14\sqrt{5}-10\sqrt{7}-35}{35}, \\
{}^2ABC(T_2^0) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_ud_vd_w}} \\
&= 2\times\frac{\sqrt{6}}{3}+\frac{2n-4}{3}+\frac{n-6}{2}\times\frac{\sqrt{2}}{3} \\
&= \frac{(4+\sqrt{2})n}{6}+\frac{2\sqrt{6}-3\sqrt{2}-4}{3}, \\
{}^2ABS(T_2^0) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}} \\
&= 2\times\frac{\sqrt{2}}{\sqrt{5}}+(n-2)\times\frac{2}{\sqrt{7}}+\frac{n-6}{2}\times\frac{\sqrt{6}}{3} \\
&= \frac{(12\sqrt{7}+7\sqrt{6})n}{42}+\frac{14\sqrt{10}-20\sqrt{7}-35\sqrt{6}}{35}.
\end{aligned}$$

This completes the proof. □

Theorem 2.2 *The second order connectivity indices of T_2^1 with n vertices are given by*

$$\begin{aligned}
{}^2R(T_2^1) &= \frac{(6+\sqrt{3})n}{18}+\frac{6\sqrt{6}-\sqrt{3}+3\sqrt{2}-24}{18}, \\
{}^2SCI(T_2^1) &= \frac{(7+6\sqrt{7})n}{42}+\frac{84\sqrt{5}-240\sqrt{7}+140\sqrt{6}-490+105\sqrt{2}}{420}, \\
{}^2ABC(T_2^1) &= \frac{(4+\sqrt{2})n}{6}+\frac{2\sqrt{6}-16-\sqrt{2}+\sqrt{10}}{6}, \\
{}^2ABS(T_2^1) &= \frac{(7\sqrt{6}+12\sqrt{7})n}{42}+\frac{189\sqrt{10}-490\sqrt{6}-480\sqrt{7}+420\sqrt{2}}{420}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_2^1 in the following table.

$m(1,3,1)$	$m(1,3,3)$	$m(3,3,3)$	$m(1,2,3)$	$m(1,3,2)$	$m(3,3,2)$
1	$n-4$	$\frac{n-7}{2}$	1	1	1

Thus we have

$$\begin{aligned}
{}^2R(T_2^1) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_ud_vd_w}}\\
&= 1\times\frac{1}{\sqrt{3}}+(n-4)\times\frac{1}{\sqrt{9}}+\frac{n-7}{2}\times\frac{1}{\sqrt{27}}+1\times\frac{1}{\sqrt{6}}+1\times\frac{1}{\sqrt{6}}+1\times\frac{\sqrt{2}}{6}\\
&= \frac{1}{\sqrt{3}}+\frac{n-4}{3}+\frac{n-7}{6}\times\frac{1}{\sqrt{3}}+\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}+\frac{\sqrt{2}}{6}\\
&= \frac{(6+\sqrt{3})n}{18}+\frac{6\sqrt{6}-\sqrt{3}+3\sqrt{2}-24}{18},\\
{}^2SCI(T_2^1) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}}\\
&= 1\times\frac{1}{\sqrt{5}}+(n-4)\times\frac{1}{\sqrt{7}}+\frac{n-7}{2}\times\frac{1}{\sqrt{9}}+1\times\frac{1}{\sqrt{6}}+1\times\frac{1}{\sqrt{6}}+1\times\frac{1}{\sqrt{8}}\\
&= \frac{\sqrt{5}}{5}+\frac{\sqrt{7}n-4\sqrt{7}}{7}+\frac{n-7}{6}+\frac{2\sqrt{6}}{6}+\frac{\sqrt{2}}{4}\\
&= \frac{(7+6\sqrt{7})n}{42}+\frac{84\sqrt{5}-240\sqrt{7}+140\sqrt{6}-490+105\sqrt{2}}{420},\\
{}^2ABC(T_2^1) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_ud_vd_w}}\\
&= 1\times\frac{\sqrt{6}}{3}+(n-4)\times\frac{2}{3}+\frac{n-7}{2}\times\frac{\sqrt{2}}{3}+1\times\frac{\sqrt{2}}{2}+1\times\frac{\sqrt{2}}{2}+1\times\frac{\sqrt{10}}{6}\\
&= \frac{\sqrt{6}}{3}+\frac{2n-8}{3}+\frac{n-7}{2}\times\frac{\sqrt{2}}{3}+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}+\frac{\sqrt{10}}{6}\\
&= \frac{(4+\sqrt{2})n}{6}+\frac{2\sqrt{6}-16-\sqrt{2}+\sqrt{10}}{6},\\
{}^2ABS(T_2^1) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}}\\
&= \frac{\sqrt{10}}{5}+(n-4)\times\frac{2}{\sqrt{7}}+\frac{n-7}{2}\times\frac{\sqrt{6}}{3}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{\sqrt{10}}{4}\\
&= \frac{(7\sqrt{6}+12\sqrt{7})n}{42}+\frac{189\sqrt{10}-490\sqrt{6}-480\sqrt{7}+420\sqrt{2}}{420}.
\end{aligned}$$

This completes the proof. $\square$

§3. Chemical Trees of Module 3

The chemical trees of T_3^0, T_3^1 and T_3^2 are shown in Figure 2.

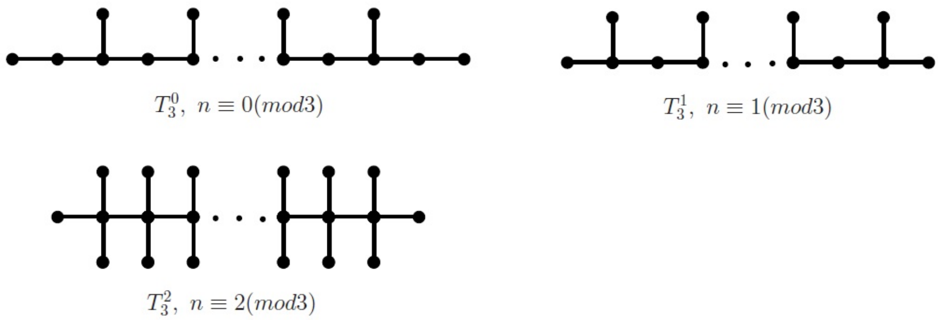

Figure 2. T_3^0, T_3^1 and T_3^2

Theorem 3.1 *The second order connectivity indices of T_3^0 with n vertices are given by*

$$\begin{aligned}
{}^2R(T_3^0) &= \frac{(2\sqrt{6}+\sqrt{3}+\sqrt{2})n}{18}-\frac{2\sqrt{2}+\sqrt{3}}{6},\\
{}^2SCI(T_3^0) &= \frac{(28\sqrt{6}+12\sqrt{7}+21\sqrt{2})n}{252}-\frac{2\sqrt{7}+7\sqrt{2}}{14},\\
{}^2ABC(T_3^0) &= \frac{(6\sqrt{2}+2\sqrt{3}+\sqrt{10})n}{18}-\frac{\sqrt{3}+\sqrt{10}}{3},\\
{}^2ABS(T_3^0) &= \frac{(28\sqrt{2}+8\sqrt{7}+7\sqrt{10})n}{84}-\frac{4\sqrt{7}+7\sqrt{10}}{14}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_3^0 in the following table.

$m(1,2,3)$	$m(2,3,1)$	$m(2,3,2)$	$m(3,2,3)$
2	$\frac{2n-6}{3}$	$\frac{n-3}{3}$	$\frac{n-6}{3}$

Thus, we have

$$\begin{aligned}
{}^2R(T_3^0) &= \sum_{uvw\in E_2(T)}\frac{1}{\sqrt{d_ud_vd_w}}\\
&= 2\times\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}\times\frac{2n-6}{3}+\frac{\sqrt{3}}{6}\times\frac{n-3}{3}+\frac{\sqrt{2}}{6}\times\frac{n-6}{3}\\
&= \frac{\sqrt{6}}{3}+\frac{\sqrt{6}n-3\sqrt{6}}{9}+\frac{\sqrt{3}n-3\sqrt{3}+\sqrt{2}n-6\sqrt{2}}{18}\\
&= \frac{(2\sqrt{6}+\sqrt{3}+\sqrt{2})n}{18}-\frac{2\sqrt{2}+\sqrt{3}}{6},
\end{aligned}$$

$$
\begin{aligned}
{}^2SCI(T_3^0) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}}\\
&= 2\times\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}\times\frac{2n-6}{3}+\frac{\sqrt{7}}{7}\times\frac{n-3}{3}+\frac{\sqrt{2}}{4}\times\frac{n-6}{3}\\
&= \frac{(28\sqrt{6}+12\sqrt{7}+21\sqrt{2})n}{252}-\frac{2\sqrt{7}+7\sqrt{2}}{14},\\
{}^2ABC(T_3^0) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_ud_vd_w}}\\
&= 2\times\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\times\frac{2n-6}{3}+\frac{\sqrt{3}}{3}\times\frac{n-3}{3}+\frac{\sqrt{10}}{6}\times\frac{n-6}{3}\\
&= \frac{(6\sqrt{2}+2\sqrt{3}+\sqrt{10})n-6\sqrt{3}-6\sqrt{10}}{18}\\
&= \frac{(6\sqrt{2}+2\sqrt{3}+\sqrt{10})n}{18}-\frac{\sqrt{3}+\sqrt{10}}{3},\\
{}^2ABS(T_3^0) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}}\\
&= 2\times\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\times\frac{2n-6}{3}+\frac{2}{\sqrt{7}}\times\frac{n-3}{3}+\frac{\sqrt{5}}{2\sqrt{2}}\times\frac{n-6}{3}\\
&= \frac{(28\sqrt{2}+8\sqrt{7}+7\sqrt{10})n}{84}-\frac{4\sqrt{7}+7\sqrt{10}}{14}.
\end{aligned}
$$

This completes the proof. □

Theorem 3.2 *The second order connectivity indices of T_3^1 with n vertices are given by*

$$
\begin{aligned}
{}^2R(T_3^1) &= \frac{(2\sqrt{6}+\sqrt{3}+\sqrt{2})n}{18}+\frac{5\sqrt{3}-2\sqrt{6}-4\sqrt{2}}{18},\\
{}^2SCI(T_3^1) &= \frac{(28\sqrt{6}+12\sqrt{7}+21\sqrt{2})n}{252}-\frac{15\sqrt{2}+5\sqrt{6}+15\sqrt{7}-18\sqrt{5}}{45},\\
{}^2ABC(T_3^1) &= \frac{(6\sqrt{2}+2\sqrt{3}+\sqrt{10})n}{18}+\frac{6\sqrt{6}-3\sqrt{2}-7\sqrt{3}-2\sqrt{10}}{9},\\
{}^2ABS(T_3^1) &= \frac{(28\sqrt{2}+8\sqrt{7}+7\sqrt{10})n}{84}+\frac{\sqrt{10}-5\sqrt{2}-10\sqrt{7}}{15}.
\end{aligned}
$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_3^1 in the following table.

$m(1,3,1)$	$m(1,3,2)$	$m(3,2,3)$	$m(2,3,2)$
2	$\frac{2n-2}{3}$	$\frac{n-4}{3}$	$\frac{n-7}{3}$

Thus, we have

$$\begin{aligned}
{}^2R(T_3^1) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u d_v d_w}} \\
&= 2\times\frac{\sqrt{3}}{3}+\frac{\sqrt{6}}{6}\times\frac{2n-2}{3}+\frac{\sqrt{2}}{6}\times\frac{n-4}{3}+\frac{\sqrt{3}}{6}\times\frac{n-7}{3} \\
&= \frac{(2\sqrt{6}+\sqrt{3}+\sqrt{2})n}{18}+\frac{5\sqrt{3}-2\sqrt{6}-4\sqrt{2}}{18}, \\
{}^2SCI(T_3^1) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}} \\
&= 2\times\frac{\sqrt{5}}{5}+\frac{\sqrt{6}}{6}\times\frac{2n-2}{3}+\frac{\sqrt{2}}{4}\times\frac{n-4}{3}+\frac{\sqrt{7}}{7}\times\frac{n-7}{3} \\
&= \frac{(28\sqrt{6}+12\sqrt{7}+21\sqrt{2})n}{252}-\frac{15\sqrt{2}+5\sqrt{6}+15\sqrt{7}-18\sqrt{5}}{45}, \\
{}^2ABC(T_3^1) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_u d_v d_w}} \\
&= 2\times\frac{\sqrt{6}}{3}+\frac{\sqrt{2}}{2}\times\frac{2n-2}{3}+\frac{\sqrt{10}}{6}\times\frac{n-4}{3}+\frac{\sqrt{3}}{3}\times\frac{n-7}{3} \\
&= \frac{(6\sqrt{2}+2\sqrt{3}+\sqrt{10})n}{18}+\frac{6\sqrt{6}-3\sqrt{2}-7\sqrt{3}-2\sqrt{10}}{9}, \\
{}^2ABS(T_3^1) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}} \\
&= 2\times\frac{\sqrt{10}}{5}+\frac{2n-2}{3}\times\frac{\sqrt{2}}{2}+\frac{n-4}{3}\times\frac{\sqrt{10}}{4}+\frac{n-7}{3}\times\frac{2\sqrt{7}}{7} \\
&= \frac{(28\sqrt{2}+8\sqrt{7}+7\sqrt{10})n}{84}+\frac{\sqrt{10}-5\sqrt{2}-10\sqrt{7}}{15}.
\end{aligned}$$

This completes the proof. □

Theorem 3.3 *The second order connectivity indices of T_3^2 with n vertices are given by*

$$\begin{aligned}
{}^2R(T_3^2) &= \frac{4+13n}{24}, \\
{}^2SCI(T_3^2) &= \frac{(\sqrt{6}+8+\sqrt{3})n}{18}+\frac{5\sqrt{6}-14-4\sqrt{3}}{9}, \\
{}^2ABC(T_3^2) &= \frac{(3+4\sqrt{3}+8\sqrt{6})n}{24}+\frac{10\sqrt{3}-6-7\sqrt{6}}{6}, \\
{}^2ABS(T_3^2) &= \frac{(3\sqrt{2}+3\sqrt{3}+8\sqrt{6})n}{18}+\frac{15\sqrt{2}-12\sqrt{3}-14\sqrt{6}}{9}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_3^2 in the following table.

$m(1,4,1)$	$m(1,4,4)$	$m(4,4,4)$
$\frac{n+10}{3}$	$\frac{4n-14}{3}$	$\frac{n-8}{3}$

Thus, we have

$$\begin{aligned}
{}^2R(T_3^2) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u d_v d_w}} \\
&= \frac{n+10}{3}\times\frac{1}{2}+\frac{4n-14}{3}\times\frac{1}{4}+\frac{n-8}{3}\times\frac{1}{8} \\
&= \frac{4+13n}{24},
\end{aligned}$$

$$\begin{aligned}
{}^2SCI(T_3^2) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}} \\
&= \frac{n+10}{3}\times\frac{1}{\sqrt{6}}+\frac{4n-14}{3}\times\frac{1}{\sqrt{9}}+\frac{n-8}{3}\times\frac{1}{\sqrt{12}} \\
&= \frac{4n-14}{9}+\frac{10\sqrt{6}+\sqrt{6}n+(n-8)\sqrt{3}}{18} \\
&= \frac{(\sqrt{6}+8+\sqrt{3})n}{18}+\frac{5\sqrt{6}-14-4\sqrt{3}}{9},
\end{aligned}$$

$$\begin{aligned}
{}^2ABC(T_3^2) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_u d_v d_w}} \\
&= \frac{n+10}{3}\times\frac{\sqrt{3}}{2}+\frac{4n-14}{3}\times\frac{\sqrt{6}}{4}+\frac{n-8}{3}\times\frac{3}{8} \\
&= \frac{(3+4\sqrt{3}+8\sqrt{6})n}{24}+\frac{10\sqrt{3}-6-7\sqrt{6}}{6},
\end{aligned}$$

$$\begin{aligned}
{}^2ABS(T_3^2) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}} \\
&= \frac{n+10}{3}\times\frac{\sqrt{2}}{2}+\frac{4n-14}{3}\times\frac{\sqrt{6}}{3}+\frac{n-8}{3}\times\frac{\sqrt{3}}{2} \\
&= \frac{(3\sqrt{2}+3\sqrt{3}+8\sqrt{6})n}{18}+\frac{15\sqrt{2}-12\sqrt{3}-14\sqrt{6}}{9}.
\end{aligned}$$

This completes the proof. □

§4. Chemical Trees of Module 4

The chemical trees of T_4^0, T_4^1, T_4^2 and T_4^3 are shown in Figure 2.

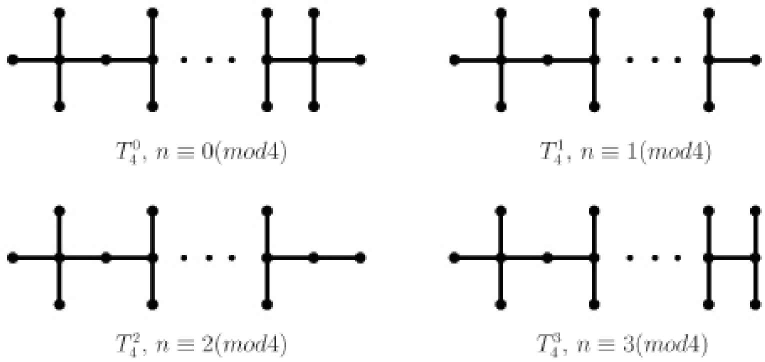

Figure 3. T_4^0, T_4^1, T_4^2 and T_4^3

Theorem 4.1 *The second order connectivity indices of T_4^0 with n vertices are given by*

$$\begin{aligned}
{}^2R(T_4^0) &= \frac{(6+9\sqrt{2})n}{32} + \frac{20-15\sqrt{2}}{8},\\
{}^2SCI(T_4^0) &= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}\\
&\quad + \frac{40\sqrt{6}-6\sqrt{10}-45\sqrt{2}-60\sqrt{7}+100}{60},\\
{}^2ABC(T_4^0) &= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32} + \frac{10\sqrt{6}+16\sqrt{3}-28\sqrt{2}+\sqrt{14}-6\sqrt{5}}{8},\\
{}^2ABS(T_4^0) &= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}\\
&\quad + \frac{120\sqrt{2}-6\sqrt{70}-45\sqrt{10}-120\sqrt{7}+100\sqrt{6}}{60}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then, we can obtain the basic information on T_4^0 in the following table.

$m(1,4,1)$	$m(1,4,2)$	$m(1,4,4)$	$m(2,4,4)$	$m(4,2,4)$	$m(2,4,2)$
$\frac{n+16}{4}$	$n-7$	5	1	$\frac{n-8}{4}$	$\frac{n-12}{4}$

Thus, we have

$$\begin{aligned}
{}^2R(T_4^0) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u d_v d_w}}\\
&= \frac{n+16}{4}\times\frac{1}{2} + \frac{n-7}{2\sqrt{2}} + \frac{5}{4} + \frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}\times\frac{n-8}{4} + \frac{1}{4}\times\frac{n-12}{4}\\
&= \frac{(6+9\sqrt{2})n}{32} + \frac{20-15\sqrt{2}}{8}.
\end{aligned}$$

$$\begin{aligned}
{}^2SCI(T_4^0) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}} \\
&= \frac{n+16}{4}\times\frac{1}{\sqrt{6}}+\frac{n-7}{\sqrt{7}}+\frac{5}{3}+\frac{1}{\sqrt{10}}+\frac{1}{\sqrt{10}}\times\frac{n-8}{4}+\frac{1}{2\sqrt{2}}\times\frac{n-12}{4} \\
&= \frac{\sqrt{6}n-18\sqrt{2}}{24}+\frac{\sqrt{7}n}{7}+\frac{\sqrt{10}n-4\sqrt{10}}{40}+\frac{\sqrt{2}n}{16}+\frac{2\sqrt{6}+5}{3}-\sqrt{7} \\
&= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{40\sqrt{6}-6\sqrt{10}-45\sqrt{2}-60\sqrt{7}+100}{60},
\end{aligned}$$

$$\begin{aligned}
{}^2ABC(T_4^0) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_ud_vd_w}} \\
&= \frac{n+16}{4}\times\frac{\sqrt{3}}{2}+(n-7)\frac{4\sqrt{2}}{8}+\frac{5\sqrt{6}}{4}+\frac{\sqrt{14}}{8}+\frac{\sqrt{14}}{8}\times\frac{n-8}{4}+\frac{\sqrt{5}}{4}\times\frac{n-12}{4} \\
&= \frac{\sqrt{3}n-\sqrt{14}}{8}+\frac{\sqrt{2}n-7\sqrt{2}}{2}+\frac{5\sqrt{6}-3\sqrt{5}-\sqrt{14}}{4}+\frac{\sqrt{14}n+2\sqrt{5}n}{32} \\
&= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{10\sqrt{6}+16\sqrt{3}-28\sqrt{2}-\sqrt{14}-6\sqrt{5}}{8},
\end{aligned}$$

$$\begin{aligned}
{}^2ABS(T_4^0) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}} \\
&= \frac{n+16}{4}\times\frac{\sqrt{2}}{2}+(n-7)\frac{2\sqrt{7}}{7}+\frac{5\sqrt{6}}{3}+\frac{\sqrt{70}}{10}\times\frac{n-4}{4}+\frac{\sqrt{10}}{4}\times\frac{n-12}{4} \\
&= \frac{5\sqrt{2}n-4\sqrt{70}+\sqrt{70}n}{40}+\frac{6\sqrt{7}n+35\sqrt{6}}{21}+\frac{\sqrt{10}n-12\sqrt{10}}{16}+2\sqrt{2}-2\sqrt{7} \\
&= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}+\frac{120\sqrt{2}-6\sqrt{70}-45\sqrt{10}-120\sqrt{7}+100\sqrt{6}}{60}.
\end{aligned}$$

This completes the proof. □

Theorem 4.2 *The second order connectivity indices of T_4^1 with n vertices are given by*

$$\begin{aligned}
{}^2R(T_4^1) &= \frac{(6+9\sqrt{2})n}{32}+\frac{42-29\sqrt{2}}{32}, \\
{}^2SCI(T_4^1) &= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{210\sqrt{6}-189\sqrt{2}-144\sqrt{7}-42\sqrt{10}}{336}, \\
{}^2ABC(T_4^1) &= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{60\sqrt{3}-48\sqrt{2}-5\sqrt{14}-18\sqrt{5}}{32}, \\
{}^2ABS(T_4^1) &= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}+\frac{210\sqrt{2}-96\sqrt{7}-63\sqrt{10}-14\sqrt{70}}{112}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then, we can obtain the basic information on T_4^1 in the following table.

$m(1,4,1)$	$m(1,4,2)$	$m(4,2,4)$	$m(2,4,2)$
$\frac{n+15}{4}$	$n-3$	$\frac{n-5}{4}$	$\frac{n-9}{4}$

Thus, we have

$$\begin{aligned}
{}^2R(T_4^1) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u d_v d_w}} \\
&= \frac{n+15}{4}\times\frac{1}{2}+\frac{n-3}{2\sqrt{2}}+\frac{n-5}{4}\times\frac{1}{4\sqrt{2}}+\frac{n-9}{16} \\
&= \frac{(6+9\sqrt{2})n}{32}+\frac{42-29\sqrt{2}}{32}, \\
{}^2SCI(T_4^1) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}} \\
&= \frac{n+15}{4}\times\frac{\sqrt{6}}{6}+\frac{n-3}{\sqrt{7}}+\frac{1}{\sqrt{10}}\times\frac{n-5}{4}+\frac{n-9}{8\sqrt{2}} \\
&= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{210\sqrt{6}-189\sqrt{2}-144\sqrt{7}-42\sqrt{10}}{336}, \\
{}^2ABC(T_4^1) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_u d_v d_w}} \\
&= \frac{n+15}{4}\times\frac{\sqrt{3}}{2}+\frac{n-3}{\sqrt{2}}+\frac{\sqrt{14}}{8}\times\frac{n-5}{4}+\frac{n-9}{4}\times\frac{\sqrt{5}}{4} \\
&= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{60\sqrt{3}-48\sqrt{2}-5\sqrt{14}-18\sqrt{5}}{32}, \\
{}^2ABS(T_4^1) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}} \\
&= \frac{n+15}{4}\times\frac{\sqrt{2}}{2}+\frac{2n-6}{\sqrt{7}}+\frac{7}{\sqrt{70}}\times\frac{n-5}{4}+\frac{\sqrt{10}n-9\sqrt{10}}{16} \\
&= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}+\frac{210\sqrt{2}-96\sqrt{7}-63\sqrt{10}-14\sqrt{70}}{112}.
\end{aligned}$$

This completes the proof. □

Theorem 4.3 *The second order connectivity indices of T_4^2 with n vertices are given by*

$$\begin{aligned}
{}^2R(T_4^2) &= \frac{(6+9\sqrt{2})n}{32}+\frac{6-11\sqrt{2}}{16}, \\
{}^2SCI(T_4^2) &= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{140\sqrt{6}-240\sqrt{7}-126\sqrt{10}-315\sqrt{2}}{560}, \\
{}^2ABC(T_4^2) &= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{12\sqrt{3}-16\sqrt{2}-6\sqrt{5}-3\sqrt{14}}{16}, \\
{}^2ABS(T_4^2) &= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}+\frac{210\sqrt{2}-42\sqrt{70}-105\sqrt{10}-160\sqrt{7}}{280}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_4^2 in the following table.

$m(1,4,1)$	$m(1,4,2)$	$m(4,2,4)$	$m(4,2,1)$	$m(2,4,2)$
$\frac{n+6}{4}$	$n-3$	$\frac{n-6}{4}$	1	$\frac{n-6}{4}$

Thus, we have

$$\begin{aligned}
{}^2R(T_4^2) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_ud_vd_w}} \\
&= \frac{n+6}{4}\times\frac{1}{2}+\frac{n-3}{2\sqrt{2}}+\frac{1}{4\sqrt{2}}\times\frac{n-6}{4}+1\times\frac{1}{2\sqrt{2}}+\frac{1}{4}\times\frac{n-6}{4} \\
&= \frac{n+6}{4}\times\frac{1}{2}+\frac{n-6}{4}\times(\frac{1}{4\sqrt{2}}+\frac{1}{4})+\frac{n-2}{2\sqrt{2}} \\
&= \frac{(6+9\sqrt{2})n}{32}+\frac{6-11\sqrt{2}}{16}, \\
{}^2SCI(T_4^2) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}} \\
&= \frac{n+6}{4}\times\frac{1}{\sqrt{6}}+\frac{n-3}{\sqrt{7}}+\frac{1}{\sqrt{10}}\times\frac{n-6}{4}+\frac{1}{\sqrt{7}}+\frac{1}{2\sqrt{2}}\times\frac{n-6}{4} \\
&= \frac{n+6}{4}\times\frac{1}{\sqrt{6}}+\frac{n-6}{4}\times(\frac{1}{\sqrt{10}}+\frac{1}{2\sqrt{2}})+\frac{n-2}{\sqrt{7}} \\
&= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{140\sqrt{6}-160\sqrt{7}-84\sqrt{10}-210\sqrt{2}}{560}, \\
{}^2ABC(T_4^2) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_ud_vd_w}} \\
&= \frac{n+6}{4}\times\frac{\sqrt{3}}{2}+\frac{n-3}{\sqrt{2}}+\frac{\sqrt{5}}{4}\times\frac{n-6}{4}+\frac{1}{\sqrt{2}}+\frac{\sqrt{7}}{\sqrt{32}}\times\frac{n-6}{4} \\
&= \frac{n+6}{4}\times\frac{\sqrt{3}}{2}+\frac{n-6}{4}\times(\frac{\sqrt{5}}{4}+\frac{\sqrt{7}}{4\sqrt{2}})+\frac{n-2}{\sqrt{2}} \\
&= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{12\sqrt{3}-16\sqrt{2}-6\sqrt{5}-3\sqrt{14}}{16}, \\
{}^2ABS(T_4^2) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}} \\
&= \frac{n+6}{4}\times\frac{\sqrt{3}}{\sqrt{6}}+\frac{2n-6}{\sqrt{7}}+\frac{\sqrt{7}}{\sqrt{10}}\times\frac{n-6}{4}+\frac{2}{\sqrt{7}}+\frac{\sqrt{5}}{2\sqrt{2}}\times\frac{n-6}{4} \\
&= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}+\frac{210\sqrt{2}-42\sqrt{70}-105\sqrt{10}-160\sqrt{7}}{280}.
\end{aligned}$$

This completes the proof. $\square$

Theorem 4.4 *The second order connectivity indices of* T_4^3 *with* n *vertices are given by*

$$\begin{aligned}
{}^2R(T_4^3) &= \frac{(6+9\sqrt{2})n}{32}+\frac{8\sqrt{6}+96\sqrt{3}-165\sqrt{2}-6}{96},\\
{}^2SCI(T_4^3) &= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{35\sqrt{2}}{112}+\frac{8\sqrt{5}-7\sqrt{10}}{40}\\
&\quad -\frac{6\sqrt{7}}{7}+\frac{35\sqrt{6}+56}{168},\\
{}^2ABC(T_4^3) &= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{8\sqrt{6}-10\sqrt{3}-48\sqrt{2}-11\sqrt{5}+8}{16}\\
&\quad -\frac{7\sqrt{14}}{32}+\frac{35\sqrt{15}}{96},\\
{}^2ABS(T_4^3) &= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}\\
&\quad +\frac{350\sqrt{2}-980\sqrt{70}+287\sqrt{10}-960\sqrt{7}}{560}+\frac{\sqrt{6}}{3}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_4^3 in the following table.

$m(1,4,1)$	$m(1,4,2)$	$m(4,2,4)$	$m(2,4,2)$	$m(1,3,1)$	$m(1,4,3)$	$m(1,3,4)$	$m(2,4,3)$
$\frac{n+5}{4}$	$n-6$	$\frac{n-7}{4}$	$\frac{n-11}{4}$	1	2	2	1

Thus, we have

$$\begin{aligned}
{}^2R(T_4^3) &= \sum_{uvw\in E_2(T)}\frac{1}{\sqrt{d_ud_vd_w}}\\
&= \frac{n+5}{4}\times\frac{1}{2}+\frac{n-6}{2\sqrt{2}}+\frac{n-7}{4}\times\frac{1}{4\sqrt{2}}+\frac{n-11}{4}\times\frac{1}{4}+\frac{1}{2\sqrt{6}}+\frac{1}{\sqrt{3}}+4\times\frac{1}{2\sqrt{3}}\\
&= \frac{(6+9\sqrt{2})n}{32}+\frac{8\sqrt{6}+96\sqrt{3}-165\sqrt{2}-6}{96},\\
{}^2SCI(T_4^3) &= \sum_{uvw\in E_2(T)}\frac{1}{\sqrt{d_u+d_v+d_w}}\\
&= \frac{n+5}{4}\times\frac{\sqrt{6}}{6}+\frac{n-6}{\sqrt{7}}+\frac{1}{\sqrt{10}}\times\frac{n-7}{4}+\frac{1}{\sqrt{8}}\times\frac{n-11}{4}\\
&\quad +\frac{1}{3}+\frac{1}{\sqrt{5}}+2\times\frac{1}{\sqrt{8}}\\
&= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{35\sqrt{2}}{112}+\frac{8\sqrt{5}-7\sqrt{10}}{40}\\
&\quad -\frac{6\sqrt{7}}{7}+\frac{35\sqrt{6}+56}{168}.
\end{aligned}$$

$$
\begin{aligned}
{}^2ABC(T_4^3) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_ud_vd_w}} \\
&= \frac{n+5}{4}\times\frac{\sqrt{3}}{2}+(n-6)\times\frac{\sqrt{2}}{2}+\frac{n-7}{4}\times\frac{\sqrt{14}}{8}+\frac{\sqrt{5}}{4}\times\frac{n-11}{4} \\
&\quad +\frac{1}{2}+\frac{\sqrt{6}}{3}+2\times\frac{\sqrt{5}}{\sqrt{3}} \\
&= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{8\sqrt{6}-10\sqrt{3}-48\sqrt{2}-11\sqrt{5}+8}{16}-\frac{7\sqrt{14}}{32}+\frac{35\sqrt{15}}{96}, \\
{}^2ABS(T_4^3) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+dw}} \\
&= \frac{n+5}{4}\times\frac{\sqrt{2}}{2}+(n-6)\times\frac{2\sqrt{7}}{7}+\frac{\sqrt{7}}{\sqrt{10}}\times\frac{n-7}{4}+\frac{\sqrt{5}}{\sqrt{8}}\times\frac{n-11}{4}+\frac{\sqrt{8}}{\sqrt{5}} \\
&\quad +\frac{\sqrt{2}}{\sqrt{5}}+4\times\frac{\sqrt{5}}{\sqrt{8}} \\
&= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}+\frac{350\sqrt{2}-980\sqrt{70}+287\sqrt{10}-960\sqrt{7}}{560}+\frac{\sqrt{6}}{3}.
\end{aligned}
$$

This completes the proof. □

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Several Fundamental Findings on Intuitionistic Fuzzy Strong ∅-b-Normed Linear Spaces

Ram Milan Singh

(Institute for Excellence in Higher Education, Bhopal)

E-mail: rammilansinghlig@gmail.com

Abstract: Following the concept of fuzzy normed linear space that Bag and Samanta provided in general t-norm settings, a definition of fuzzy strong-b-normed linear space is provided in this study. In this case, a general function $\varnothing(c)$ that satisfies certain requirements is used in place of the scalar function $|c|$. We study some fundamental results on finite dimensional fuzzy strong b-normed linear space.

Key Words: Intuitionistic fuzzy norm, t-norm, intuitionistic fuzzy normed linear space, neutrosophic set, intuitionistic fuzzy strong ϕ-b-normed linear space.

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§1. Introduction

Zadeh [19] was the first to develop the idea of a fuzzy set in 1965. The theory of fuzzy sets has since been extensively expanded by other authors. Fuzzy metric spaces were first introduced by Osmo Kaleva [10], Kramosil and Michalek [14], Georage and Veeramani [9], et al. in various ways. On the other hand, the concept of fuzzy normed linear spaces has been provided in several ways by Katsaras [11], Felbin [6], Cheng and Mordeson [4] and Bag and Samanta [1].

Different generalised metric and norm types, such as the 2-metric [7], b-metric [5], strong-b-metric [13], G-metric [15], 2-norm [13], G-norm [12], etc., as well as generalised fuzzy metric and fuzzy norm types, such as the fuzzy b-metric [16], strong-fuzzy b-metric [18], fuzzy cone metric [17], fuzzy cone norm [2], G-fuzzy norm [3], etc.

Oner proposed fuzzy strong b-metric spaces and produced some topological findings on these spaces in [18].

§2. Preliminaries

In this section, some definitions and results are collected which are used in this paper.

Definition 2.1 *A binary operation* $*: [0,1]\times[0,1]\to[0,1]$ *is called a t-norm if it satisfies the following conditions:*

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$(a) *$ *is commutative and associative;*
$(b) *$ *is continuous;*
(c) $a * 1 = a$ *for all* $a \in [0,1]$*;*
(d) $a *\ b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0,1]$.

If $*$ *is continuous, then it is called continuous t-norm.*

The following are examples of some t-norms.

(i) Standard intersection: $a * b = \min\{a, b\}$.
(ii) Algebraic product: $a * b = ab$.
(iii) Bounded difference: $a * b = \max\{0, a + b - 1\}$.

Definition 2.2 *A binary operation* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is a continuous tconorm if it satisfies the following conditions:*

(a) *is commutative and associative;*
(b) *is continuous;*
(c) $\diamond =\ a$ *for all* $a \in [0,1]$*;*
(d) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each of* $a, b, c, d \in [0,1]$.
If $*$ *is continuous, then it is called continuous t-norm.*

The following are examples of some t-norms.

(i) Standard intersection: $abb = \max\{a, b\}$.
(ii) Algebraic product: $a \downarrow b = ab$.
(iii) Bounded difference: $a \diamond b = \min\{0, a + b - 1\}$.

Definition 2.3 *A three tuple* $(X, M, *)$ *is said to be a fuzzy metric space, a case of neutrosophic set if* X *is an arbitrary set,* $*$ *a continuous t-norm and* M *a fuzzy set on* $X^2 \times [0, \infty)$ *satisfying the following condition, for all* $\mathrm{x, y, z \in X}$ *and* $\mathrm{t, s} > 0$ *:*

(a) $M(x, y, 0) = 0$*;*
(b) $M(x, y, t) = 1$ *for all* $t > 0$ *iff* $x = y$*;*
(c) $M(x, y, t) = M(y, x, t)$*;*
(d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$*;*
(e) $M(x, y,) : .[0, \infty) \to [0, 1]$ *is left continuous;*
(f) $\lim_{\mathrm{n} \to \infty} M(x, y, t) = 1$.

Definition 2.4 *A 5-tuple* $(X, M, N, *, \diamond)$*issaidtobeanintuitionisticfuzzymetric space (shortly IFM-Space) if* X *is an arbitrary set,* $*$ *is a continuous t-norm,* $\forall$ *is a continuous t-conorm and* M, N *are fuzzy sets on* $X^2 \times [0, \infty)$ *satisfying the following conditions:*

(a) $M(x, y, t) + N(x, y, t) \leq 1$ *for all* $x, y \in X$ *and* $t > 0$*;*
(b) $M(x, y, 0) = 0$ *for all* $x, y \in X$*;*
(c) $M(x, y, t) = 1$ *for all* $x, y \in X$ *and* $t > 0$ *if and only if* $x = y$*;*
(d) $M(x, y, t) = M(y, x, t)$ *for all* $x, y \in X$ *and* $t > 0$*;*
(e) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ *for all* $x, y, z \in X$ *and* $s, t > 0$*;*
(f) $M(x, y,) : .[0, \infty) \to [0, 1]$ *is left continuous for all* $x, y \in X$*;*
(g) $\lim_{\mathrm{n} \to \infty} M(x, y, t) = 1$*;*

(*h*) $N(x, y, 0) = 1$ *for all* $x, y \in X$*;*
(*i*) $N(x, y, t) = 0$ *for all* $x, y \in X$ *and* $t > 0$ *if and only if* $x = y$*;*
(*j*) $N(x, y, t) = N(y, x, t)$ *for all* $x, y \in X$ *and* $t > 0$*;*
(*k*) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ *for all* $x, y, z \in X$ *and* $s, t > 0$*;*
(*l*) $M(x, y,) : .[0, \infty) \to [0, 1]$ *is right continuous for all* $x, y \in X$*;*
(*m*) $\lim\limits_{\mathrm{n}\to\infty} N(x, y, t) = 0$ *for all* $x, y \in X$*,*

then, (M, N) *is called an intuitionistic fuzzy metric on* X*. The functions* $M(x, y, t)$ *and* $N(x, y, t)$ *denote the degree of nearness and degree of non nearness between* x *and* y *with respect to* t*, respectively.*

Definition 2.5 *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric space. Then,*

(*a*) *A sequence* $\{x_n\}$ *is said to be convergent* x *in* X *if for each* $\epsilon > 0$ *and* $t > 0$*, there exist* $\mathrm{n}_0 \in N$ *such that* $M(x_n, x, t) > 1 - \epsilon$ *and* $N(x_n, x, t) < 0 - \epsilon$ *for all* $n \geq n_0$*;*

(*b*) *A sequence* $\{\mathrm{x_n}\}$ *is said to be Cauchy if for each* $\epsilon > 0$ *and* $\mathrm{t} > 0$*, there exist* $\mathrm{n}_0 \in \mathrm{N}$ *such that* $M(\mathrm{x_n}, \mathrm{x_m}, \mathrm{t}) > 1 - \epsilon$ *and* $N(\mathrm{x_n}, \mathrm{x_m}, \mathrm{t}) < 0 - \epsilon$ *for all* $\mathrm{n}, \mathrm{m} \geq \mathrm{n}_0$*;*

(*c*) *An intuitionistic fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*

Definition 2.6 *A sequence* $\{\mathrm{Si}\}$ *of self maps on a complete intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$ *is said to be intuitionistic mutually contractive if for* $t > 0$ *and* $\mathrm{i}\epsilon\mathrm{N}$

$$M(S_{\mathrm{i}}x, S_jy, t) \geq M\left(x, y, \frac{t}{p}\right) \quad \text{and} \quad N(S_ix, S_jy, t) \leq N\left(x, y, \frac{t}{p}\right),$$

where $x, y \in X, p \in (0, 1), i \neq j$ *and* $x \neq y$*.*

Definition 2.7 *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy normed linear space.*

(*i*) *A sequence* $\{x_n\}$ *is said to be convergent if there exists* $x \in X$ *such that*

$$\lim_{n\to\infty} \mathrm{M}(x_n - x, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(x_n - x, t) = 0$$

for all $t > 0$*. Then* x *is called the limit of the sequence* $\{x_n\}$ *and denoted by* $\lim\limits_{n\to\infty} x_n$*;*

(*ii*) *A sequence* $\{x_n\}$ *in an intuitionistic fuzzy normed linear space* (X, N) *is said to be Cauchy if*

$$\lim_{n\to\infty} \mathrm{M}(x_{n+p} - x_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(x_{n+p} - x_n, t) = 0$$

for all $t > 0$ *and* $p = 1, 2, \cdots$*;*

(*iii*) $A \subseteq X$ *is said to be closed if for any sequence* $\{x_n\}$ *in* A *converges to* $x \in A$*;*

(*iv*) $A \subseteq X$ *is said to be the closure of* A*, denoted by* $\bar{A}$ *if for any* $x \in \bar{A}$*, if there is a sequence* $\{x_n\} \subseteq A$ *such that* $\{x_n\}$ *converges to* x*.* (*v*) $A \subseteq X$ *is said to be compact if any sequence* $\{x_n\} \subseteq A$ *has a subsequence converging to an element of* A*.*

Lemma 2.6 *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy normed linear space and let* $M(x, .)$*,* $N(x, \cdot)$ *be with* $x \neq 0)$*. If the set* $A = \{x : M(x, 1) > 0$ *and* $\mathrm{N(x, 1)} < 0\}$ *is compact, then* X *is finite dimensional.*

§3. Intuitionistic Fuzzy Strong ϕ-b-Normed Linear Space

In this section, we give the definition of intuitionistic fuzzy normed linear space in a new approach.

Definition 3.1 *Let ϕ be a function defined on $\mathbb{R}$ to $\mathbb{R}^+$ with the following properties*

($\phi 1$) $\phi(-t) = \phi(t)$ *for all* $t \in \mathbb{R}$;
($\phi 2$) $\phi(1) = 1$;
($\phi 3$) ϕ *is strictly increasing and continuous on* $(0, \infty)$;
($\phi 4$) $\lim\limits_{\alpha \to 0} \phi(\beta) = 0$ *and* $\lim_{\alpha \to \infty} \phi(\beta) = \infty$.

The followings are examples of such functions.

(i) $\phi(\beta) = |\beta|$ for all $\beta \in \mathbb{R}$.
(ii) $\phi(\beta) = |\beta|^p$ for all $\beta \in \mathbb{R}, p \in \mathbb{R}^+$.
(iii) $\phi(\beta) = \dfrac{2\beta^{2n}}{|\beta| + 1}$ for all $\beta \in \mathbb{R}, n \in \mathbb{N}$.

Definition 3.2 *Let X be a linear space over the field $\mathbb{R}$ and $b \geq 1$ be a given real number. A fuzzy subset N of $X \times \mathbb{R}$ is called intuitionistic fuzzy strong ϕ-b-norm on X if for all $x, y \in X$ the following conditions hold:*

(i) $\forall t \in \mathbb{R}$ *with* $t \leq 0, M(x,t) = 0$;
(ii) $(\forall t \in \mathbb{R}, t > 0, M(x,t) = 1)$ *iff* $x = \theta$;
(iii) $\forall t \in \mathbb{R}, t > 0, M(cx,t) = M\left(x, \frac{t}{\phi(c)}\right)$ *if* $\phi(c) \neq 0$;
(iv) $\forall s, t \in \mathbb{R}, M(x+y, s+bt) \geq M(x,s) * N(y,t)$;
(v) $M(x,\cdot)$ *is a non-decreasing function of t and* $\lim_{t\to\infty} M(x,t) = 1$;
(vi) $\forall t \in \mathbb{R}$ *with* $t \geq 0, N(x,t) = 0$;
(vii) $(\forall t \in \mathbb{R}, t = 0, N(x,t) = 0)$ *iff* $x = \theta$;
$(viii)$ $\forall t \in \mathbb{R}, t < 0, N(cx,t) = N\left(x, \frac{t}{\phi(c)}\right)$ *if* $\phi(c) \neq 0$;
(ix) $\forall s, t \in \mathbb{R}, N(x+y, s+bt) \leq N(x,s) \diamond N(y,t)$;
(x) $N(x,\cdot)$ *is a non-increasing function of t and* $\lim_{t\to\infty} N(x,t) = 0$.

Then $(X, \mathrm{M}, N, \phi, b, *)$ is called intuitionistic fuzzy strong ϕ-b-normed linear space.

§4. Finite Dimensional Intuitionistic Fuzzy Strong ϕ-b-Normed Linear Spaces

In this section, some basic results on finite dimensional intuitionistic fuzzy strong ϕ-b-normed linear spaces are established.

Lemma 4.1 *Let $(X, \mathrm{M}, N, \phi, b, *, \diamond)$ be a Intuitionistic fuzzy strong ϕ-b-normed linear space with the underlying t-norm $*$ continuous and t-co norm at $(1,1)$ and $\{x_1, x_2, \cdots x_n\}$ be a linearly independent set of vectors in X. Then there exists $c > 0$ and $\delta \in (0,1)$ such that for any set of* *scalars* $\{\beta_1, \beta_2, \cdots, \beta_n\}$ *with* $\sum_{i=1}^{n} |\beta_i| \neq 0$

$$M\left(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{n}|\beta_i|}\right)}\right) < 1 - \delta. \tag{4.1}$$

and

$$N\left(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{n}|\beta_i|}\right)}\right) > 0 - \delta. \tag{4.2}$$

Proof Notice that the equations

$$M\left(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{n}|\beta_i|}\right)}\right) < 1 - \delta.$$

and

$$N\left(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{n}|\beta_i|}\right)}\right) > 0 - \delta.$$

are equivalent to the relations

$$M\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, bc\right) < 1 - \delta$$

and

$$N\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, bc\right) > 0 - \delta$$

for some $c > 0$, $\delta \in (0,1)$ and for all set of scalars $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ with $\sum_{i=1}^{n} |\alpha_i| = 1$ If possible, suppose that (4.1) does not hold. Thus, for each $c > 0$ and $\delta \in (0,1)$, there exists a set of scalars $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ with $\sum_{i=1}^{n} |\alpha_i| = 1$ for which

$$M\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, bc\right) \geq 1 - \delta$$

and

$$N\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, bc\right) \leq 0 - \delta.$$

Then, for $c = \delta = \frac{1}{m}, m = 1, 2, \ldots$, there exists a set of scalars $\left\{\alpha_1^{(m)}, \alpha_2^{(m)}, \ldots, \alpha_n^{(m)}\right\}$ with $\sum_{i=1}^{n} \left|\alpha_i^{(m)}\right| = 1$ such that

$$M\left(y_m, \frac{b}{m}\right) \geq 1 - \frac{1}{m}$$

and

$$N\left(y_m, \frac{b}{m}\right) \leq 0 - \frac{1}{m},$$

where $y_m = \alpha\beta_1^{(m)} x_1 + \beta_2^{(m)} x_2 + \cdots + \beta_n^{(m)} x_n$. Since $\sum_{i=1}^{n} \left|\alpha_i^{(m)}\right| = 1$, we have $0 \leq \left|\alpha_i^{(m)}\right| \leq 1$ for $i = 1, 2, \ldots, n$. So for each fixed i, the sequence $\left\{\alpha_i^{(m)}\right\}$ is bounded and hence $\left\{\alpha_i^{(m)}\right\}$ has a convergent subsequence. Let α_1 denotes the limit of that subsequence and let $\{y_{1,m}\}$ denotes the corresponding subsequence of $\{y_m\}$. By the same argument $\{y_{1,m}\}$ has a subsequence $\{y_{2,m}\}$ for which the corresponding subsequence of scalars $\left\{\alpha_2^{(m)}\right\}$ converges to α_2. Continuing in this way, after n steps we obtain a subsequence $\{y_{n,m}\}$ where

$$y_{n,m} = \sum_{i=1}^{n} \gamma_i^{(m)} x_i \quad \text{with} \quad \sum_{i=1}^{n} \left|\gamma_i^{(m)}\right| = 1$$

and $\gamma_i^{(m)} \to \alpha_i$ as $m \to \infty$ for each $i = 1, 2, \cdots, n$.

Let $y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$. Now,

$$\begin{aligned}
M(y_{n,m} - y, t) &= M\left(\sum_{j=1}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, t\right) \\
&= M\left(\left(\gamma_1^{(m)} - \alpha_1\right) x_1 + \sum_{j=2}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, \frac{t}{n} + b(n-1)\frac{t}{nb}\right) \\
&\geq M\left(\left(\gamma_1^{(m)} - \alpha_1\right) x_1, \frac{t}{n}\right) * M\left(\sum_{j=2}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, (n-1)\frac{t}{nb}\right) \\
&= M\left(\left(\gamma_1^{(m)} - \alpha_1\right) x_1, \frac{t}{n}\right) \\
&\quad * M\left(\left(\gamma_2^{(m)} - \alpha_2\right) x_2 + \sum_{j=3}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, \frac{t}{nb} + b\left(1 - \frac{2}{n}\right)\frac{t}{b^2}\right) \\
&\geq M\left(\left(\gamma_1^{(m)} - \alpha_1\right) x_1, \frac{t}{n}\right) * M\left(\left(\gamma_2^{(m)} - \alpha_2\right) x_2, \frac{t}{nb}\right) \\
&\quad * M\left(\sum_{j=3}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, \left(1 - \frac{2}{n}\right)\frac{t}{b^2}\right) \\
&\geq M\left(\left(\gamma_1^{(m)} - \alpha_1\right) x_1, \frac{t}{n}\right) * M\left(\left(\gamma_2^{(m)} - \alpha_2\right) x_2, \frac{t}{nb}\right) \\
&\quad * \cdots * M\left(\left(\gamma_n^{(m)} - \alpha_n\right) x_n, \frac{t}{nb^{n-1}}\right) \\
&= M\left(x_1, \frac{t}{n\phi\left(\left(\gamma_1^{(m)} - \alpha_1\right)\right)}\right) * \cdots * M\left(x_n, \frac{t}{nb^{n-1}\phi\left(\left(\gamma_n^{(m)} - \alpha_n\right)\right)}\right)
\end{aligned}$$

and

$$\begin{aligned}
N(y_{n,m} - y, t) &= N\left(\sum_{j=1}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, t\right) \\
&= N\left(\left(\gamma_1^{(m)} - \alpha_1\right) x_1 + \sum_{j=2}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, \frac{t}{n} + b(n-1)\frac{t}{nb}\right)
\end{aligned}$$

$$
\begin{aligned}
&= N\left(\left(\gamma_1^{(m)}-\alpha_1\right)x_1,\frac{t}{n}\right)\\
&\leq N\left(\gamma_1^{(m)}-\alpha_1\right)x_1,\frac{t}{n}\right)\diamond N\left(\sum_{j=2}^{n}\left(\gamma_j^{(m)}-\alpha_j\right)x_j,(n-1)\frac{t}{nb}\right)\\
&\quad\diamond N\left(\left(\gamma_2^{(m)}-\alpha_2\right)x_2+\sum_{j=3}^{n}\left(\gamma_j^{(m)}-\alpha_j\right)x_j,\frac{t}{nb}+b\left(1-\frac{2}{n}\right)\frac{t}{b^2}\right)\\
&\leq N\left(\left(\gamma_1^{(m)}-\alpha_1\right)x_1,\frac{t}{n}\right)\diamond N\left(\left(\gamma_2^{(m)}-\alpha_2\right)x_2,\frac{t}{nb}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&N\left(\sum_{j=3}^{n}\left(\gamma_j^{(m)}-\alpha_j\right)x_j,\left(1-\frac{2}{n}\right)\frac{t}{b^2}\right)\\
&\geq N\left(\left(\gamma_1^{(m)}-\alpha_1\right)x_1,\frac{t}{n}\right)\diamond N\left(\left(\gamma_2^{(m)}-\alpha_2\right)x_2,\frac{t}{nb}\right)\diamond\cdots\nabla N\left(\left(\gamma_n^{(m)}-\alpha_n\right)x_n,\frac{t}{nb^{n-1}}\right)\\
&= N\left(x_1,\frac{t}{n\phi\left(\left(\gamma_1^{(m)}-\alpha_1\right)\right)}\right)\diamond\cdots\nabla N\left(x_n,\frac{t}{nb^{n-1}\phi\left(\left(\gamma_n^{(m)}-\alpha_n\right)\right)}\right).
\end{aligned}
$$

Now taking limit as $m\to\infty$ on both sides, we have

$$\lim_{m\to\infty} M\left(y_{n,m}-y,t\right)\geq 1*1*\cdots*1,\quad \forall t>0$$

and

$$\lim_{m\to\infty} N\left(y_{n,m}-y,t\right)\leq 0>0\diamond\cdots\vee 0,\quad \forall t>0$$

i.e

$$\lim_{m\to\infty} M\left(y_{n,m}-y,t\right)=1,\quad \forall t>0$$

and

$$\lim_{m\to\infty} N\left(y_{n,m}-y,t\right)=0,\quad \forall t>0$$

Now, for $r>0$, choose m such that $\frac{1}{m}<\frac{r}{b^2}$. We have

$$
\begin{aligned}
M\left(y_{n,m},\frac{r}{b}\right)&=M\left(y_{n,m}+\theta,\frac{b}{m}+b\left(\frac{r}{b^2}-\frac{1}{m}\right)\right)\\
&\geq\left(y_{n,m},\frac{b}{m}\right)*M\left(\theta,\frac{r}{b^2}-\frac{1}{m}\right)\geq\left(1-\frac{b}{m}\right)*1
\end{aligned}
$$

and

$$
\begin{aligned}
N\left(y_{n,m},\frac{r}{b}\right)&=N\left(y_{n,m}+\theta,\frac{b}{m}+b\left(\frac{r}{b^2}-\frac{1}{m}\right)\right)\\
&\leq N\left(y_{n,m},\frac{b}{m}\right)\diamond N\left(\theta,\frac{r}{b^2}-\frac{1}{m}\right)\leq\left(1-\frac{b}{m}\right)\diamond 0
\end{aligned}
$$

which implies

$$\lim_{m\to\infty} M\left(y_{n,m},\frac{r}{b}\right)\geq 1 \quad \text{i.e.,} \quad \lim_{m\to\infty} M\left(y_{n,m},\frac{r}{b}\right)=1$$

and

$$\lim_{m\to\infty} N\left(y_{n,m},\frac{r}{b}\right)\leq 0 \quad \text{i.e.,} \lim_{m\to\infty} N\left(y_{n,m},\frac{r}{b}\right)=0.$$

Again,

$$\begin{aligned} M(y,2r) &= M\left(y-y_{n,m}+y_{n,m}, r+b\cdot\frac{r}{b}\right)\\ &\geq M\left(y-y_{n,m},r\right)\cdot N\left(y_{n,m},\frac{r}{b}\right) \end{aligned}$$

and

$$\begin{aligned} N(y,2r) &= N\left(y-y_{n,m}+y_{n,m}, r+b\cdot\frac{r}{b}\right)\\ &\leq N\left(y-y_{n,m},r\right)\diamond N\left(y_{n,m},\frac{r}{b}\right). \end{aligned}$$

Thus,

$$\begin{aligned} M(y,2r) &\geq \lim_{m\to\infty} M\left(y-y_{n,m},r\right) * \lim_{m\to\infty} M\left(y_{n,m},\frac{r}{b}\right)\\ &\Rightarrow M(y,2r)\geq 1\cdot 1=1 \Rightarrow M(y,2r)=1 \end{aligned}$$

and

$$\begin{aligned} N(y,2r) &\leq \lim_{m\to\infty} N\left(y-y_{n,m},r\right) \diamond \lim_{m\to\infty} N\left(y_{n,m},\frac{r}{b}\right)\\ &\Rightarrow N(y,2r)\leq 0\diamond 0=0 \Rightarrow N(y,2r)=0. \end{aligned}$$

Since $r>0$ is arbitrary, so $y=\theta$. Again since $\sum_{i=1}^{n}\left|\alpha_i^{(m)}\right|=1$ and $\{x_1,x_2,\ldots,x_n\}$ is a linearly independent set of vectors so $y=\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n\neq\theta$, thus we arrive at a contradiction and Lemma is proved. □

Theorem 4.2 *Every finite dimensional Intuitionistic fuzzy strong ϕ-b-normed linear space with the underlying t-norm * continuous and t-co norm $\diamond$ Continuous at $(1,1)$ is complete.*

Proof Let $(X,\mathrm{M},N,\phi,b,*,\diamond)$ be a Intuitionistic fuzzy strong ϕ-b-normed linear space where $b(>1)$ is a real constant. Let $\dim X=r$ and $\{e_1,e_2,\cdots,e_r\}$ be a basis for X. Let $\{x_p\}$ be a Cauchy sequence in X. Then, $x_n=\sum_{k=1}^{r}\alpha_k^{(n)}e_k$ for suitable scalars $\alpha_1^{(n)},\alpha_2^{(n)},\cdots,\alpha_r^{(n)}\cdot$ So

$$\lim_{m,n\to\infty}\mathrm{M}\left(x_m-x_n,t\right)=1, \quad \forall t>0$$

and

$$\lim_{m,n\to\infty} N\left(x_m-x_n,t\right)=0, \quad \forall t>0.$$

Now, by Lemma 4.1 it follows that $\exists c>0$ and $\delta\in(0,1)$ such that

$$M\left(\sum_{i=1}^{r}\left(a_i^{(m)}-\alpha_i^{(n)}\right)e_i, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|}\right)}\right)<1-\delta$$

and

$$N\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_i, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|}\right)}\right)>0-\delta. \tag{4.3}$$

If

$$\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|=0$$

then $\alpha_i^{(m)}=\alpha_i^{(n)}$ for any integer i implies that $\{x_n\}$ is a constant sequence and hence follows the theorem. So we may assume

$$\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|\neq 0.$$

Again, for $0<\delta<1$ from (4-3) it follows that there exists a positive integer $n_0(\delta,t)$ such that

$$M\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_i, t\right)>1-\delta, \quad \forall m,n\geq n_0(\delta,t) \tag{4.4}$$

and

$$N\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_i, t\right)<0-\delta, \quad \forall m,n\geq n_0(\delta,t). \tag{4.5}$$

Now, from (4.4) and (4.5), $\forall m,n\geq n_0(\delta,t)$ we have

$$M\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_i\frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|}\right)}\right)<M\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_{i,t}\right)$$

and

$$N\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_i\frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|}\right)}\right)>N\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_i, t\right).$$

Thus,

$$\frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{\tau}\left|\alpha_t^{(m)}-\alpha_i^{(m)}\right|}\right)}<t$$

since $M(x,t)$ is non-decreasing with respect to t and

$$\frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{\tau}\left|\alpha_t^{(m)}-\alpha_i^{(m)}\right|}\right)}>t$$

since $N(x,t)$ is non-increasing with respect to t. Hence, since $t>0$ is arbitrary, namely

$$\lim_{m,n\to\infty} \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|}\right.}\Bigg) = 0$$

then

$$\lim_{m,n\to\infty} \phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|}\right) = \infty.$$

Thus,

$$\phi\left(\frac{1}{\lim_{m\to\infty}\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|}\right) = \infty$$

since ϕ is continuous. Then

$$\lim_{m,n\to\infty} \sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right| = 0$$

since $\lim_{\alpha\to\infty}\phi(\beta)=\infty$. Therefore, $\left\{\alpha_i^{(m)}\right\}$ is a Cauchy sequence of scalars for each $i=1,2,\cdots,r$. So each sequence $\left\{\alpha_i^{(m)}\right\}$ converges. Let $\lim_{n\to\infty}\alpha_i^{(n)}=\alpha_i$ for $i=1,2,\ldots,r$. Define $x=\sum_{i=1}\alpha e_i$. Then clearly $x\in X$. By similar calculation as in Lemma 4.1 , it can be shown that $\lim_{n\to\infty} M(x_n-x,t)=1, \lim_{n\to\infty} N(x_n-x,t)=0, \forall t>0$. Hence X is complete.

§5. Conclusion

Recently, various writers have constructed various kinds of generalised fuzzy metric spaces as well as generalised fuzzy normed linear spaces. The concept of fuzzy strong b-normed linear spaces was presented after the introduction of fuzzy strong b-metric spaces, and various findings in finite finite dimensions fuzzy strong b-normed linear spaces were examined. We believe there is a vast area of research to be done in order to create fuzzy strong b-normed linear spaces. Open issues in such spaces include results on completeness and compactness, operator standards, etc.

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Some New Results on 4-Total Mean Cordial Graphs

R. Ponraj[1], S.Subbulakshmi[2] and M.Sivakumar[3]

1. Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627412, Tamilnadu, India
2. Department of Mathematics, Manonmaniam sundarnar university, Abishekapatti, Tirunelveli-627012, Tamilnadu, India
3. Department of Mathematics, Government Arts and Science College, Tittagudi-606106,Tamilnadu, India

E-mail: ponrajmaths@gmail.com, ssubbulakshmis@gmail.com, sivamaths1975@gmail.com

Abstract: Let G be a graph and let $f : V(G) \to \{0, 1, 2, \cdots, k-1\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign a label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ and f is called a k-total mean cordial labeling of G if $|t_{mf}(i) - t_{mf}(j)| \leq 1$ for all $i, j \in \{0, 1, \cdots, k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0, 1, 2, \cdots, k-1\}$. A graph with admit a k-total mean cordial labeling is called k-total mean cordial graph. In this paper we investigate the 4-total mean cordial labeling behavior of some graphs which are obtained from stars.

Key Words: k-Total mean cordial labeling, Smarandachely k-total mean cordial labeling, k-total mean cordial labeling graph, Smarandachely k-total mean cordial labeling graph, star, cycle, comb, ladder.

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§1. Introduction

All graphs in this paper are finite, simple and undirected graphs only. Graceful labeling was introduced by Rosa in [15]. Subsequently Graham and Sloan have introduced the notion of harmonious labeling [2]. Motivated by these works several author introduce varies types of graph labeling. The concept of k-total mean cordial labeling has been introduced in [4]. The 4-total mean cordial labeling behavior of several graphs like cycle, complete graph, star, bistar, comb and crown have been investigated in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. In this paper we investigate the 4- total mean cordial labeling behavior of some graphs which are obtained from stars. Let x be any real number. Then $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Terms are not defined here follow from Harary [3] and Gallian [1].

§2. k-Total Mean Cordial Graph

Definition 2.1 *Let G be a graph. Let $f : V(G) \to \{0, 1, 2, \cdots, k-1\}$ be a function where*

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$k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a k-total mean cordial labeling of G if $|t_{mf}(i) - t_{mf}(j)| \leq 1$ for all $i, j \in \{0, 1, 2, \cdots, k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0, 1, 2, \cdots, k-1\}$. Otherwise, if there exist integers $i, j \in \{0, 1, 2, \cdots, k-1\}$ such that $|t_{mf}(i) - t_{mf}(j)| \geq 2$, such a labeling f is called a Smarandachely k-total mean cordial labeling.

A graph with an admit a k-total mean cordial labeling or a Smarandachely k-total mean cordial labeling is called a k-total mean cordial graph or a Smarandachely k-total mean cordial graph.

§3. Preliminaries

Definition 3.1 *A complete bipartite graph $K_{1,n}$ is called a star. Let $V(K_{1,n}) = \{w, w_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{ww_i : 1 \leq i \leq n\}$, w is called the central vertex of the star $K_{1,n}$.*

Definition 3.2 *A graph obtained from the cycle C_n and star $K_{1,n}$ by identifying the vertex of C_n with these central vertex of $K_{1,n}$ is denoted by $C_n \oplus K_{1,n}$.*

Definition 3.3 *Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. A corona of G_1 with G_2 is the graph $G_1 \odot G_2$ obtained by taking one copy of G_1 , p_1 copies of G_2 and joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2 where $1 \leq i \leq p_1$.*

Definition 3.4 *A graph $P_n \odot K_1$ is called a comb. Let P_n be the path $u_1 u_2 \cdots u_n$. Let $V(P_n \odot K_1) = V(P_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1) = E(P_n) \cup \{u_i v_i : 1 \leq i \leq n\}$.*

Definition 3.5 *A graph $L_n = P_n + K_2$ is called a ladder. Let the vertex set be $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and the edge set $E(L_n) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$.*

§4. Main Results

Theorem 4.1 *A graph $C_n \oplus K_{1,n}$ is 4-total mean cordial labeling for all $n \geq 3$.*

Proof Let C_n be the cycle $u_1\ u_2 \cdots u_n\ u_1$, $V(C_n \oplus K_{1,n})) = V(C_n) \cup \{w, w_i : 1 \leq i \leq n,\ u_1 = w\}$ and $E(C_n \oplus K_{1,n})) = E(C_n) \cup \{u_1 w_i : 1 \leq i \leq n\}$. Obviously $|V(C_n \oplus K_{1,n})| + |E(C_n \oplus K_{1,n})| = 4n$.

Case 1. $n \equiv 1 \pmod 2$.

Let $n = 2r+1$, $r \in \mathbb{N}$. Consider the cycle $C_n : u_1\ u_2 \cdots u_n\ u_1$. Assign the label 0 to the $r+1$ vertices u_1, u_2, $\cdots$, u_{r+1}. Now we assign the label 1 to the r vertices u_{r+2}, u_{r+3}, $\cdots$, u_{2r+1}. Next move to the pendent vertices. We now assign the label 3 to the $2r+1$ vertices w_1, w_2, $\cdots$, w_{2r+1}.

Case 2. $n \equiv 0 \pmod 2$.

Let $n = 2r$, $r \in \mathbb{N}$. Assign the label 2 to the vertex u_1. Next we assign the label 0 to the r vertices u_2, u_3, $\cdots$, u_{r+1}. We now assign the label 1 to the $r-1$ vertices u_{r+2}, u_{r+3}, $\cdots$, u_{2r}. Now we assign the label 0 to the vertex w_1. Next we assign the label 2 to the $r-1$ vertices w_2, w_3, $\cdots$, w_r. Finally we assign the label 3 to the r vertices w_{r+1}, w_{r+2}, $\cdots$, w_{2r}.

This shows that f is a 4-total mean cordial labeling follows from Table 1.

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2r + 1$	$2r + 1$	$2r + 1$	$2r + 1$	$2r + 1$
$n = 2r$	$2r$	$2r$	$2r$	$2r$

Table 1

This completes the proof. □

Theorem 4.2 *A graph obtained from the comb $P_n \odot K_1$ and star $K_{1,n}$ by identifying the central vertex w of the star with the vertex u_1 of the comb is 4-total mean cordial.*

Proof Let G be the resulting graph. Take the vertex set and edge set of the comb is as in Definition 3.4. Let $V(G) = V(P_n \odot K_1) \cup \{w_i : 1 \leq i \leq n\}$. $E(G) = E(P_n \odot K_1) \cup \{u_1 w_i : 1 \leq i \leq n\}$. Clearly $|V(G)| + |E(G)| = 6n - 1$.

Case 1. $n \equiv 0 \pmod 2$.

Let $n = 2r$, $r \in \mathbb{N}$. Assign the label 2 to the r vertices $u_1, u_2, \cdots, u_r$. Now we assign the label 3 to the r vertices $u_{r+1}, u_{r+2}, \cdots, u_{2r}$. Next we assign the label 0 to the r vertices $v_1, v_2, \cdots, v_r$. We now assign the label 2 to the r vertices $v_{r+1}, v_{r+2}, \cdots, v_{2r}$. Now we assign the label 0 to the $2r$ vertices $w_1, w_2, \cdots, w_{2r}$.

Case 2. $n \equiv 1 \pmod 2$.

Let $n = 2r + 1$, $r \in \mathbb{N}$. Assign the label 2 to the r vertices $u_1, u_2, \cdots, u_r$. Next we assign the label 3 to the $r + 1$ vertices $u_{r+1}, u_{r+2}, \cdots, u_{2r+1}$. Now we assign the label 0 to the $r + 1$ vertices $v_1, v_2, \cdots, v_{r+1}$. We now assign the label 2 to the r vertices $v_{r+2}, v_{r+3}, \cdots, v_{2r+1}$. Next we assign the label 0 to the $2r$ vertices $w_1, w_2, \cdots, w_{2r}$. Finally we assign the label 1 to the vertex w_{2r+1}.

Thus, this vertex labeling f is a 4-total mean cordial labeling follows from Table 2.

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2r$	$3r$	$3r$	$3r - 1$	$3r$
$n = 2r + 1$	$3r + 1$	$3r + 1$	$3r + 1$	$3r + 2$

Table 2

This completes the proof. □

Theorem 4.3 *A graph G obtained from the comb $P_n \odot K_1$ and two stars by identifying the vertex u_1 of the comb with the central vertex of one star and u_n with the central vertex of another star is 4-total men cordial.*

Proof Take the vertex set and edge set of the comb is as in Definition 3.4. Let $V(G) = V(P_n \odot K_1) \cup \{x_i, y_i : 1 \leq i \leq n\}$ and $E(G) = E(P_n \odot K_1) \cup \{u_1 x_i, u_n y_i : 1 \leq i \leq n\}$. Note that $|V(G)| + |E(G)| = 8n - 1$.

Assign the label 0 to the n vertices $u_1, u_2, \cdots, u_n$. Next we assign the label 1 to the n vertices $v_1, v_2, \cdots, v_n$. Now we assign the label 3 to the n vertices $x_1, x_2, \cdots, x_n$. Finally we assign the label 3 to the n vertices $y_1, y_2, \cdots, y_n$.

Clearly $t_{mf}(0) = 2n - 1$, $t_{mf}(1) = t_{mf}(2) = t_{mf}(3) = 2n$. $\square$

Theorem 4.4 *A graph obtained from the comb $P_n \odot K_1$ and three stars by identifying the vertex u_1 of the comb with the central vertex of one star, u_n with the central vertex of second star and v_1 with the central vertex of third star is 4-total mean cordial.*

Proof Let G be the resulting graph. Take the vertex set and edge set of the comb is as in Definition 3.4. Let $V(G) = V(P_n \odot K_1) \cup \{x_i, y_i, z_i : 1 \leq i \leq n\}$ and $E(G) = E(P_n \odot K_1) \cup \{u_1x_i, u_ny_i, v_1z_i : 1 \leq i \leq n\}$. Clearly $|V(G)| + |E(G)| = 10n - 1$.

Case 1. $n \equiv 0 \pmod 2$.

Let $n = 2r$, $r \in \mathbb{N}$. Assign the label 0 to the $2r$ vertices $u_1, u_2, \cdots, u_{2r}$. Next we assign the label 2 to the $2r$ vertices $v_1, v_2, \cdots, v_{2r}$. Now we assign the label 3 to the $2r$ vertices $x_1, x_2, \cdots, x_{2r}$. Then we assign the label 1 to the r vertices $y_1, y_2, \cdots, y_r$. We now assign the label 3 to the r vertices $y_{r+1}, y_{r+2}, \cdots, y_{2r}$. Now we assign the label 0 to the r vertices $z_1, z_2, \cdots, z_r$. Finally we assign the label 3 to the r vertices $z_{r+1}, z_{r+2}, \cdots, z_{2r}$.

Case 2. $n \equiv 1 \pmod 2$.

Let $n = 2r + 1$, $r \in \mathbb{N}$. Assign the label 0 to the $2r + 1$ vertices $u_1, u_2, \cdots, u_{2r+1}$. Now we assign the label 2 to the $2r + 1$ vertices $v_1, v_2, \cdots, v_{2r+1}$. Next we assign the label 3 to the $2r + 1$ vertices $x_1, x_2, \cdots, x_{2r+1}$. We now assign the label 1 to the r vertices $y_1, y_2, \cdots, y_r$. Then we assign the label 3 to the $r + 1$ vertices $y_{r+1}, y_{r+2}, \cdots, y_{2r+1}$. Now we assign the label 3 to the r vertices $z_1, z_2, \cdots, z_r$. Finally we assign the label 0 to the $r + 1$ vertices $z_{r+1}, z_{r+2}, \cdots, z_{2r+1}$.

Thus, the vertex labeling f is a 4-total mean cordial labeling follows from Table 3.

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2r$	$5r - 1$	$5r$	$5r$	$5r$
$n = 2r + 1$	$5r + 2$	$5r + 2$	$5r + 3$	$5r + 2$

Table 3

This completes the proof. $\square$

Theorem 4.5 *A graph G obtained from the comb $P_n \odot K_1$ and four stars by identifying the vertex u_1 of the comb with the central vertex of one star, u_n with the central vertex of second star, v_1 with the central vertex of third star and v_n with the central vertex of fourth star is 4-total mean cordial.*

Proof Take the vertex set and edge set of the comb is as in Definition 3.4. Let $V(G) = V(P_n \odot K_1) \cup \{w_i, x_i, y_i, z_i : 1 \leq i \leq n\}$ and $E(G) = E(P_n \odot K_1) \cup \{u_1w_i, u_nx_i, v_1y_i, v_nz_i : 1 \leq i \leq n\}$. Note that $|V(G)| + |E(G)| = 12n - 1$.

Case 1. $n \equiv 0 \pmod 2$.

Let $n = 2r$, $r \in \mathbb{N}$. Assign the label 0 to the $2r$ vertices $u_1, u_2, \cdots, u_{2r}$. Next we assign the label 0 to the r vertices v_1, v_2, $\cdots$, v_r. We now assign the label 1 to the r vertices v_{r+1}, v_{r+2}, $\cdots$, v_{2r}. Next we assign the label 3 to the $2r$ vertices w_1, w_2, $\cdots$, w_{2r}. Now we assign the label 3 to the $2r$ vertices x_1, x_2, $\ldots$, x_{2r}. Then we assign the label 1 to the $2r$ vertices y_1, y_2, $\cdots$, y_{2r}. Finally we assign the label 3 to the $2r$ vertices z_1, z_2, $\cdots$, z_{2r}.

Case 2. $n \equiv 1 \pmod 2$.

Let $n = 2r+1$, $r \in \mathbb{N}$. We now assign the label 0 to the $2r$ vertices u_1, u_2, $\cdots$, u_{2r}. Next we assign the label 0 to the $r+1$ vertices v_1, v_2, $\cdots$, v_{r+1}. We now assign the label 1 to the r vertices v_{r+2}, v_{r+3}, $\cdots$, v_{2r+1}. Next we assign the label 3 to the $2r+1$ vertices w_1, w_2, $\cdots$, w_{2r+1}. Now we assign the label 3 to the $2r+1$ vertices x_1, x_2, $\cdots$, x_{2r+1}. Then we assign the label 1 to the $2r+1$ vertices y_1, y_2, $\cdots$, y_{2r+1}. Finally we assign the label 3 to the $2r+1$ vertices z_1, z_2, $\cdots$, z_{2r+1}.

Thus, this vertex labeling f is a 4-total mean cordial labeling follows from Table 4.

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2r$	$6r-1$	$6r$	$6r$	$6r$
$n = 2r+1$	$6r+3$	$6r+2$	$6r+3$	$6r+3$

Table 4

This completes the proof. □

Theorem 4.6 *A graph obtained from the ladder L_n and star $K_{1,n}$ by identifying the vertex u_1 of the ladder with the central vertex of star is* 4*-total mean cordial.*

Proof Let G be the resulting graph. Take the vertex set and edge set of the ladder is as in Definition 3.5. Let $V(G) = V(L_n) \cup \{w_i : 1 \le i \le n\}$ and $E(G) = E(L_n) \cup \{u_1 w_i : 1 \le i \le n\}$. Obviously $|V(G)| + |E(G)| = 7n - 2$.

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \in \mathbb{N}$. Assign the label 0 to the r vertices u_1, u_2, $\cdots$, u_r. Next we assign the label 1 to the r vertices u_{r+1}, u_{r+2}, $\cdots$, u_{2r}. Now we assign the label 2 to the r vertices u_{2r+1}, u_{2r+2}, $\cdots$, u_{3r}. We now assign the label 3 to the r vertices u_{3r+1}, u_{3r+2}, $\cdots$, u_{4r}. Now we assign the label 0 to the r vertices v_1, v_2, $\cdots$, v_r. We now assign the label 1 to the r vertices v_{r+1}, v_{r+2}, $\cdots$, v_{2r}. Next we assign the label 2 to the r vertices v_{2r+1}, v_{2r+2}, $\cdots$, v_{3r}. We now assign the label 3 to the r vertices v_{3r+1}, v_{3r+2}, $\cdots$, v_{4r}. Now we assign the label 0 to the $r+1$ vertices w_1, w_2, $\cdots$, w_{r+1}. Then we assign the label 1 to the r vertices w_{r+2}, w_{r+3}, $\cdots$, w_{2r+1}. We now assign the label 3 to the $2r-1$ vertices w_{2r+2}, w_{2r+3}, $\cdots$, w_{4r}.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r+1$, $r \in \mathbb{N}$. As in Case 1 assign the label to the vertices u_i, v_i, w_i $(1 \le i \le 4r)$. Now we assign the labels 3, 0, 1 to the vertices u_{4r+1}, v_{4r+1}, w_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r+2$, $r \in \mathbb{N}$. Label the vertices u_i, v_i, w_i $(1 \le i \le 4r+1)$ as in Case 2. Next we assign the labels 3, 0, 2 to the vertices u_{4r+2}, v_{4r+2}, w_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4r+3$, $r \in \mathbb{N}$. In this case assign the label for the vertices u_i, v_i, w_i $(1 \le i \le 4r+2)$ as in Case 3. We now assign the labels 3, 0, 2 to the vertices u_{4r+3}, v_{4r+3}, w_{4r+3}.

This vertex labeling f is a 4-total mean cordial labeling follows from Table 5.

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$7r$	$7r$	$7r-1$	$7r-1$
$n = 4r+1$	$7r+1$	$7r+2$	$7r+1$	$7r+1$
$n = 4r+2$	$7r+3$	$7r+3$	$7r+3$	$7r+3$
$n = 4r+3$	$7r+5$	$7r+4$	$7r+5$	$7r+5$

Table 5

This completes the proof. □

Theorem 4.7 *A graph G obtained from the ladder L_n and two stars $K_{1,n}$ by identifying the vertex u_1 with the central vertex of one star and u_n with the central vertex of another star is 4-total mean cordial.*

Proof Take the vertex set and edge set of the ladder is as in Definition 3.5. Let $V(G) = V(L_n) \cup \{x_i, y_i : 1 \le i \le n\}$ and $E(G) = E(L_n) \cup \{u_1x_i, u_ny_i : 1 \le i \le n\}$. Note that $|V(G)| + |E(G)| = 9n-2$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4r$, $r \in \mathbb{N}$. Assign the label 0 to the r vertices $u_1, u_2, \cdots, u_r$. Now we assign the label 1 to the r vertices $u_{r+1}, u_{r+2}, \cdots, u_{2r}$. We now assign the label 2 to the r vertices u_{2r+1}, $u_{2r+2}, \cdots, u_{3r}$. Next assign the label 3 to the r vertices $u_{3r+1}, u_{3r+2}, \cdots, u_{4r}$. We now assign the label 0 to the r vertices $v_1, v_2, \cdots, v_r$. Next we assign the label 1 to the r vertices v_{r+1}, $v_{r+2}, \cdots, v_{2r}$. Now we assign the label 2 to the r vertices $v_{2r+1}, v_{2r+2}, \cdots, v_{3r}$. We now assign the label 3 to the r vertices $v_{3r+1}, v_{3r+2}, \cdots, v_{4r}$. Next we assign the label 0 to the $2r+1$ vertices $x_1, x_2, \cdots, x_{2r+1}$. Now we assign the label 2 to the $2r-1$ vertices $x_{2r+2}, x_{2r+3}, \cdots$, x_{4r}. Then we assign the label 1 to the $2r$ vertices $y_1, y_2, \cdots, y_{2r}$. Finally we assign the label 3 to the $2r$ vertices $y_{2r+1}, y_{2r+2}, \cdots, y_{4r}$.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r+1$, $r \in \mathbb{N}$. Label the vertices u_i, v_i, x_i, y_i $(1 \le i \le 4r)$ as in Case 1. Next we assign the labels 3, 1, 0, 1 to the vertices u_{4r+1}, v_{4r+1}, x_{4r+1} y_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r+2$, $r \in \mathbb{N}$. In this case assign the label for the vertices u_i, v_i, x_i, y_i $(1 \le i \le 4r+1)$ as in Case 2. We now assign the labels 3, 1, 0, 1 to the vertices u_{4r+2}, v_{4r+2}, x_{4r+2} y_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4r+3$, $r \in \mathbb{N}$. As in Case 3, we assign the label to the vertices u_i, v_i, x_i, y_i

$(1 \leq i \leq 4r+2)$. Finally we assign the labels 3, 1, 0, 1 to the vertices u_{4r+3}, v_{4r+3}, x_{4r+3} y_{4r+3}.

Thus, the vertex labeling f is a 4-total mean cordial labeling follows from Table 6.

Size of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$9r$	$9r-1$	$9r-1$	$9r$
$n = 4r+1$	$9r+2$	$9r+1$	$9r+2$	$9r+2$
$n = 4r+2$	$9r+4$	$9r+4$	$9r+4$	$9r+4$
$n = 4r+3$	$9r+6$	$9r+7$	$9r+6$	$9r+6$

Table 6

This completes the proof. □

Theorem 4.8 *A graph obtained from the ladder L_n and three stars $K_{1,n}$ by identifying the vertex u_1 with the central vertex of one star, u_n with the central vertex of second star and v_1 with the central vertex of third star is 4-total mean cordial.*

Proof We denote G as the resulting graph. Take the vertex set and edge set of the ladder is as in Definition 3.5. Let $V(G) = V(L_n) \cup \{x_i, y_i, z_i : 1 \leq i \leq n\}$ and $E(G) = E(L_n) \cup \{u_1x_i, u_ny_i, v_1z_i : 1 \leq i \leq n\}$. Clearly $|V(G)| + |E(G)| = 11n - 2$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4r$, $r \in \mathbb{N}$. Assign the label 0 to the r vertices u_1, u_2, $\cdots$, u_r. Next we assign the label 1 to the r vertices u_{r+1}, u_{r+2}, $\cdots$, u_{2r}. Now we assign the label 2 to the r vertices u_{2r+1}, u_{2r+2}, $\cdots$, u_{3r}. We now assign the label 3 to the r vertices u_{3r+1}, u_{3r+2}, $\cdots$, u_{4r}. Now we assign the label 0 to the r vertices v_1, v_2, $\cdots$, v_r. We now assign the label 1 to the r vertices v_{r+1}, v_{r+2}, $\cdots$, v_{2r}. Next we assign the label 2 to the r vertices v_{2r+1}, v_{2r+2}, $\cdots$, v_{3r}. We now assign the label 3 to the r vertices v_{3r+1}, v_{3r+2}, $\cdots$, v_{4r}. Next we assign the label 0 to the $3r+1$ vertices x_1, x_2, $\cdots$, x_{3r+1}. Now we assign the label 2 to the $r-1$ vertices x_{3r+2}, x_{3r+3}, $\cdots$, x_{4r}. Then we assign the label 1 to the r vertices y_1, y_2, $\cdots$, y_r. We now assign the label 3 to the $3r$ vertices y_{r+1}, y_{r+2}, $\cdots$, y_{4r}. Finally we assign the label 2 to the $4r$ vertices z_1, z_2, $\cdots$, z_{4r}.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r+1$, $r \in \mathbb{N}$. In this case we assign the label to the vertices u_i, v_i, x_i, y_i, z_i $(1 \leq i \leq 4r)$ as in Case 1. We now assign the labels 3, 0, 0, 1, 1 to the vertices u_{4r+1}, v_{4r+1}, x_{4r+1} y_{4r+1}, z_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r+2$, $r \in \mathbb{N}$. As in Case 2, we assign the label to the vertices u_i, v_i, x_i, y_i z_i $(1 \leq i \leq 4r+1)$. Finally we assign the labels 3, 1, 0, 2, 2 to the vertices u_{4r+2}, v_{4r+2}, x_{4r+2}, y_{4r+2}, z_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4r + 3$, $r \in \mathbb{N}$. Label the vertices u_i, v_i, x_i, y_i, z_i $(1 \le i \le 4r + 2)$ as in Case 3. Next we assign the labels 3, 0, 0, 2, 1 to the vertices u_{4r+3}, v_{4r+3}, x_{4r+3}, y_{4r+3}, z_{4r+3}.

This vertex labeling f is a 4-total mean cordial labeling follows from Table 7.

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$11r$	$11r - 1$	$11r - 1$	$11r$
$n = 4r + 1$	$11r + 3$	$11r + 2$	$11r + 2$	$11r + 2$
$n = 4r + 2$	$11r + 5$	$11r + 5$	$11r + 5$	$11r + 5$
$n = 4r + 3$	$11r + 8$	$11r + 8$	$11r + 7$	$11r + 8$

Table 7

This completes the proof. □

Theorem 4.9 *A graph G obtained from the ladder L_n and four stars by identifying the vertex u_1 with the central vertex of one star, u_n with the central vertex of second star, v_1 with the central vertex of third star and v_n with the central vertex of fourth star is 4-total mean cordial.*

Proof Take the vertex set and edge set of the ladder is as in Definition 3.5. Let $V(G) = V(L_n) \cup \{w_i, x_i, y_i, z_i : 1 \le i \le n\}$ and $E(G) = E(L_n) \cup \{u_1 w_i, u_n x_i, v_1 y_i, v_n z_i : 1 \le i \le n\}$. Obviously $|V(G)| + |E(G)| = 13n - 2$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4r$, $r \in \mathbb{N}$. Assign the label 0 to the $4r$ vertices u_1, u_2, $\cdots$, u_{4r}. Next we assign the label 2 to the $4r$ vertices v_1, v_2, $\cdots$, v_{4r}. We now assign the label 0 to the $2r + 1$ vertices w_1, w_2, $\cdots$, w_{2r+1}. Now we assign the label 1 to the $2r - 1$ vertices w_{2r+2}, w_{2r+3}, $\cdots$, w_{4r}. Next we assign the label 1 to the $r + 1$ vertices x_1, x_2, $\cdots$, x_{r+1}. Then we assign the label 3 to the $3r - 1$ vertices x_{r+2}, x_{r+3}, $\cdots$, x_{4r}. We now assign the label 0 to the $r - 1$ vertices y_1, y_2, $\cdots$, y_{r-1}. Next we assign the label 1 to the $2r + 1$ vertices y_r, y_{r+1}, $\cdots$, y_{3r}. Now we assign the label 3 to the r vertices y_{3r+1}, y_{3r+2}, $\cdots$, y_{4r}. Finally we assign the label 3 to the $4r$ vertices z_1, z_2, $\cdots$, z_{4r}.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r + 1$, $r \in \mathbb{N}$. Label the vertices u_i, v_i, w_i, x_i, y_i, z_i $(1 \le i \le 4r)$ as in Case 1. Next we assign the labels 0, 2, 2, 2, 3, 3 to the vertices u_{4r+1}, v_{4r+1}, w_{4r+1}, x_{4r+1}, y_{4r+1}, z_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r + 2$, $r \in \mathbb{N}$. As in Case 1, we assign the label to the vertices u_i, v_i, w_i, x_i, y_i z_i $(1 \le i \le 4r)$. Finally we assign the labels 0, 0, 2, 2, 1, 0, 2, 3, 3, 1, 3, 3 to the vertices u_{4r+1}, u_{4r+2}, v_{4r+1}, v_{4r+2}, w_{4r+1}, w_{4r+2}, x_{4r+1}, x_{4r+2}, y_{4r+1}, y_{4r+2}, z_{4r+1}, z_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4r + 3$, $r \in \mathbb{N}$. In this case we assign the label to the vertices u_i, v_i, w_i, x_i, y_i, z_i $(1 \le i \le 4r + 2)$ as in Case 3. Finally we assign the labels 0, 2, 1, 3, 0, 3 to the vertices u_{4r+3}, v_{4r+3}, w_{4r+3}, x_{4r+3}, y_{4r+3}, z_{4r+3}.

This vertex labeling f is a 4-total mean cordial labeling follows from Table 8.

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$13r$	$13r$	$13r - 1$	$13r - 1$
$n = 4r + 1$	$13r + 2$	$13r + 3$	$13r + 3$	$13r + 3$
$n = 4r + 2$	$13r + 6$	$13r + 6$	$13r + 6$	$13r + 6$
$n = 4r + 3$	$13r + 9$	$13r + 10$	$13r + 9$	$13r + 9$

Table 8

This completes the proof. □

Example 4.1 A 4-total mean cordial labeling of the graph G obtained from Theorem 4.9 with $n = 5$ is given in Figure 1.

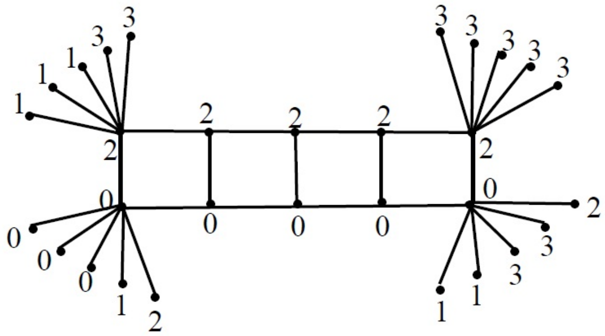

Figure 1

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Graph Coloring, Types and Applications: A Survey

Sunil B. Thakare

Department of Mathematics, School of Engineering and Sciences
MIT Art, Design and Technology University, Pune-412201, (MS) India

Haribhau R. Bhapkar

Department of Mathematics, Central University of Kashmir, Ganderbal-191201, Kashmir, India

E-mail: sunilthakare@gmail.com, hrbhapkar@gmail.com

Abstract: In recent decade, Graph Theory has many applications in problems like security key generation, brain MRI segmentation and tumor detection by cut sets, virus graph and its application during COVID-19 pandemic. The color assignment to various graph's elements is a significantly important topic for research in graph theory. It has a wide-ranging applications in sciences, medical sciences, computer engineering, electronics and telecommunication, electrical engineering, network theory, artificial intelligence and machine learning, psychology and economics, to name a few. Many conjectures are remains open problems and many researchers and mathematicians from around the world are working on it. In this paper, we review the graph's coloring, the types of coloring, theorems and axioms related to the graph-coloring, and applications.

Key Words: Graph coloring, Smarandachely Λ-coloring, vertex Smarandachely Λ-coloring, edge Smarandachely Λ-coloring, Smarandachely total coloring, total coloring, face Smarandachely Λ-coloring, perfect coloring, list coloring, strong edge coloring, acyclic coloring.

AMS(2010): 05C15, 05C69, 05C78.

§1. Introduction

The famous Konigsberg seven-bridge problem launched graph theory [85, 86]. The task was to begin at any point, walk through all seven bridges on the Pregel river just one time, and then came back to the initial point. In 1736, Euler [54] used a graph to resolve this issue. He represented lands with vertices and a bridge connecting two lands is an edge between them. In this manner, the problem was represented in a graph. Euler found that there is no such closed walk exists for this problem. As a result in graph theory, the Eulerian circuit concept was introduced and "*A connected graph is Eulerian if the degree of all vertices is an even number and vice versa.*" This was the first paper considered, and so the evolution of graph theory began.

Later, L.Euler [50,54,120] developed the planer graph formula based on the invariant of polyhedron in algebraic topology. If polyhedron P has n vertices, f faces and m edges then

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[2]Corresponding author: Haribhau R. Bhapkar, Email: hrbhapkar@gmail.com.

$n + f - m$ is invariant and $n + f - m = 2$. L.Euler made significant contributions to the field of mathematics and physics; especially for the advancement in graph theory.

In 1850, legendary four-color map brought graph theory to the forefront. For nearly 127 years, this was an unsolved difficult problem. Many mathematicians and researchers attempted to solve this problem but were unsuccessful.

§2. **Graph Coloring**

Coloring a graph began in the mid-nineteenth century with the legendary four-color conjecture (4CC). Francis Guthrie discovered that all the nations on the administrative map of England in 1850 were painted in only four colors, with every two adjacent nation-states painted in a different color. He talked about it with his brother, Frederick Guthrie. Later, Frederick discussed this problem with his professor, Augustus De-Morgan but he was unable to answer. Morgan questioned William Hamilton concerning 4CC in 1852. Following that, in 1878, Arthur Cayley worked on the problem and posted a question in the London mathematical society. May [105] quotes in an article given by Harary [69] that "*Any map on a plane or the surface of a sphere can be colored with on four colors so that no two adjacent nations have the same color?*" The 4CC was first proved by Kemp [91] in 1879. But there was an error in the proof shown by Heawood [80] and demonstrated that conjecture was correct for five colors. Many mathematicians are worked to prove 4CC for more than 100 years. In 1969, Ore and Stemple [111] revealed proof of 4CC with the numerical method for all maps with less than 40 countries. Meanwhile the work on coloring of graph elements started. Finally, in 1977, Apple and Haken [84,90,94] demonstrated 4CC using a computer with 1200 hours of computer time. First time in history, a famous mathematics problem was solved extensively by using the computer.

After this many mathematicians verified the proof of 4CC in various ways. Robertson, Sanders, Seymour and Thomas [117] proved 4CC with 633 unavoidable reducible configurations. H. R. Bhapkar [14] proved this by PNR of a graph. Birkhoff [17] proposed a Chromatic polynomial in 2012, which is based on Gauss [55] fundamental algebraic theorem, which states that each n-degree polynomial with a complex coefficient has precisely n zeros.

Graphs have the ability to exemplify a wide variety of practical problems. The solutions to these problems are given by graph theory. For example, the road network problems, electrical networks consisting of resisters, capacitors and the inductors, maximum flow problems, optimal path and minimized cost for transportation problems, the communication network, social media networks, time-table scheduling of flights, trains and buses, signal flow problems in signal transmission, representation of the structure of an organic molecule in chemistry, etc. These structures can be represented as graphs, which are collections of points and lines connecting some or all pairs of points and are known as vertices and edges, respectively.

2.1. Graph

Definition 3.1([48,69,144,145]) *A graph H made up of two sets, $V(H)$-nonempty set of elements called vertices of graph H represented by point, $E(H)$-set of unordered pairs of vertices joined by an arc or a line called an edge-set of graph H. It is symbolized with $H(V,E)$. Please* *refer to books for a basic understanding of graphs.*

2.2. Graph's Coloring

Graph coloring, it is procedure of giving colors to graph components like vertices, edges, regions in a manner that separates the colors of nearby elements. This term described as a proper coloring in graph. The significant work is done on vertex coloring because the graph's edge and region coloring is identical to its line and dual graph respectively. But many problems of coloring are studied in their original form in order for getting better results and applications.

Generally, a *Smarandachely* Λ-*coloring* of a graph G on a surface $\mathcal{S}$ by colors in $\mathscr{C}$ is a mapping $\varphi_\Lambda : \mathscr{C} \to V(G) \cup E(G) \cup F(G)$ such that $\varphi(u) \neq \varphi(v)$ if u and v are elements of a subgraph isomorphic to $\Lambda \prec G$, where $F(G)$ is the face set of 2-cell embedding of G on $\mathcal{S}$ ([95, 154]).

2.3. Proper Coloring

Definition 2.2([48,144]) *A proper coloring is the process of color allocating to graph's element so that neighboring elements colored differently. If k different colors are required, it is known as k-proper coloring or k-colorable.*

The primary categories of graph coloring and some other special types of coloring are surveyed in the follows sections.

§3. Vertex Coloring

Definition 3.1([48,95,144,154]) *A vertex coloring means adjoining vertices of a graph colored differently. If we needed k colors, then it is known as k-proper vertex coloring.*

Generally, a Smarandachely Λ-coloring $\varphi_\Lambda|_{V(G)} : \mathscr{C} \to V(G)$ is called a vertex Smarandachely Λ-coloring.

Definition 3.2 *A chromatic number is the number k of least set of distinct colors required for a graph to be k-properly vertex colorable. It is represented by χ.*

The cycle graph C_4 is 2-colorable, 3-colorable and 4-colorable. The bare minimum, though, is 2-colorable. Consequently, the chromatic number $\chi(C_4)$ is 2.

The following are some standard graphs and their chromatic numbers.

(1) A complete graph K_n is n-vertex colorable;
(2) A null graph is 1-vertex colorable;
(3) A bipartite graph ($K_{m,n}$) is 2-vertex colorable;
(4) A cycle C_n is 3 or 2-vertex colorable if n is odd or even, respectively;
(5) The chromatic number of tree is 2;
(6) The chromatic number of star graph $S_{1,n}$ is 2;
(7) The chromatic number of path P_n is 2.

Many researchers contributed to the understanding of vertex coloring. In 1936, Konig [85] characterized two-colorable graphs as below.

Theorem 3.1 *If a graph G is 2-vertex colorable then G is without an odd cycle and vice versa.*

Theorem 3.2 *A graph is bipartite if and only if it is without an odd cycle and vice versa.*

He also demonstrated how to divide any k-regular bipartite graph into one factor. In 1941, Brooks [5] characterized vertex coloring for connected graphs as below.

Theorem 3.3 *Let H be a connected graph with a largest degree Δ. Then,*

(A) *$\chi(H)$ is at most Δ, excluding a complete graph or/and an odd cycle;*
(B) *$\chi(H)$ is $\Delta+1$ for a complete graph and an odd cycle.*

In other words, Brook's theorem is equivalent to this: *An odd cycles and complete graph are both $(k-1)$-regular, k-critical graphs.* There is no way to compute chromatic number of any random graph. Greedy color algorithm is one of the most important graph color algorithm for getting this. The vertex coloring has many bounds. Thus, a clique number is one among them. Szekeres and Wilf [129] provide the upper bound.

Theorem 3.4 $\chi(H) \leq 1 + max[\ \delta(K)]$, *for a graph H and for all induced subgraphs $K \subseteq H$.*

And then, Berge C. [8, pp. 37] and Ore O. [110, pp. 225] provide a lower bound. Harary, Hedetniemi [70] provide an upper bound in terms of independent number, as below.

Theorem 3.5 *For graph H on $n-$ vertices, $\frac{n}{\beta} \leq \chi \leq n - \beta + 1$, where β is the cardinality of maximal independent subset of H.*

So, *is there any graph that has no triangles but a very high Chromatic number?* Dirac [49] posed this question, and Descartes [46] and Mycielski [104] responded positively. The result was proved for $n \geq 2$ by Kelly and Kelly [87], i.e., *there exists n-chromatic graph with a girth is more than 5.* Later, Erdos [51] and Lavasz [96] established the high Chromatic-number result as shown below.

Theorem 3.6 *For integers $p > 0$ and $q > 0$, there is a q-chromatic graph with a girth greater than p.*

In 1912, Birkhoff [17] introduced the Chromatic polynomial. It is an $n-$degree polynomial that give us count of vertex coloring for set of m colors $1, 2, \cdots, m$, where $m \geq \chi(H)$ is a positive integer. It is clear that it has integer roots, namely $1, 2, \cdots, m-1$ if $\chi = m$. There are many properties of chromatic polynomials that are explained by Birkhoff [17,6], Whitney [147], Rota [121], Read [118], and many other authors, as follows:

Theorem 3.7 *If a graph H has p vertices, q edges and k components with chromatic number χ, then the chromatic polynomial $f(H,x)$ of graph H yields the following results.*

(1) *The coefficients are alternate in sign;*
(2) *The polynomial $f(H,x)$ has degree p;*
(3) *The coefficient of x^n is 1;*
(4) *The coefficient of x^{n-1} is $-q$;*
(5) *The polynomial's constant term is zero;*

(6) *The chromatic polynomial with* k *components is*

$$f(H,x) = \prod_{i=1}^{k} f(H_i,x);$$

(7) *The smallest exponent of* x *is* q*;*

(8) *The numbers* $1, 2, 3, \cdots, \chi - 1$ *are zeros of chromatic polynomial.*

Theorem 3.8 *4CC is equivalent to* $f(H,x) = 0$ *for m=1,2,3 and 4.*

As a result of Birkhoff's attempt to solve the graph theory coloring problem using an algebraic method, more than 600 papers have been published since the advent of the chromatic polynomial until today. Benzer [7] identified the linear structure of the DNA molecule in 1955; as a result, Hajnal and Suranyi [72] introduced and studied interval graphs, a subclass of chordal graphs, in 1958. Chordal graphs has all of roots from a set $\{1, 2\cdots, (\chi-1)\}$. Dmitriev [47] discovered the characterization of chordal graphs and chromatic polynomials, i.e., "*A graph H is chordal if and only if all the roots of the chromatic polynomial for every induced subgraph* H' *are integers from the set* $\{1, 2, \cdots, (\chi-1)\}$*, which are roots of chromatic polynomial of a graph H.* There was lots of research going on chordal graphs and chromatic polynomials.

The complement graph's chromatic number is given as $\overline{\chi(H)} = \chi(\overline{H})$. It is clear that $\chi(\overline{H}) = \beta$. Nordhaus and Gaddum [107] provide bounds in terms of sum and product.

Theorem 3.9 *The chromatic numbers* χ*, satisfy inequalities for* n *vertices graph,*

(*A*) $2\sqrt{n} \le \chi + \overline{\chi} \le n+1$;

(*B*) $n \le \chi\overline{\chi} \le \left(\dfrac{n+1}{2}\right)^2$.

In general, the disjoint union (addition) of two graphs's chromatic number is largest among both chromatic numbers. The results of the graph operations on two or more graphs are discussed by Vizing [134] in 1963 and Aberth [3] in 1964, as below.

Theorem 3.10 *Let* P *and* Q *be two graphs. Then, the chromatic number of their Cartesian product is maximum from* $\chi(P)$ *or* $\chi(Q)$*.*

Theorem 3.11 *Let* P *and* Q *be two graphs. Then, the chromatic number of their join is a sum of* $\chi(P)$ *and* $\chi(Q)$*.*

Chvatal [44] verified the below result for graphs without triangles in 1970.

Theorem 3.12 *If a graph has no triangles and it is four regular, then it is a* 4*-chromatic graph.*

In 1970, Thatcher et al. [132] proved the complexity of vertex coloring as below.

Theorem 3.13 *The vertex coloring is NP-complete problem.*

In 2007, W. Klotz and T. Sander [93] gave a result on a unitary Cayley graph.

Theorem 3.14 *If* X_n *is unitary Cayley graph such that* p *is the minimum prime divisor of* n*, then its vertex coloring number of* X_n *is* p *and the vertex coloring number of its complementary graph is* n/p*.*

§4. Edge Coloring

Definition 4.1([48,69,95,144,145,154]) *The edge coloring means proper-coloring of edges of a graph. If we needed k colors, then it is known as k-proper edge coloring.*

Generally, a Smarandachely Λ-coloring $\varphi_\Lambda|_{E(G)}$: $\mathscr{C} \to E(G)$ is called an edge Smarandachely Λ-coloring.

Definition 4.2 *The chromatic Index means least k different colors needed such that graph is k-proper edge colorable. We symbolized this by χ'.*

Here are some examples of standard graphs and their edge chromatic numbers.

(1) For a complete graph K_n, $\chi'(K_n) = (n-1)$, where n is a positive integer;
(2) For a complete-bipartite graph $(K_{m,n})$, $\chi'(K_{m,n})$ = maximum of $\{m,n\}$;
(3) For a cycle C_n, $\chi'(C_n) = 2$ or 3 if n is even or odd positive integer, respectively;
(4) For a tree T, $\chi'(T) = 2$;
(5) For a star graph $S_{1,n}$, $\chi'(S_{1,n}) = 2$;
(6) For a path P_n, $\chi'(P_n) = 2$.

In 1890, Peter Tait [130] proved the result of edge coloring for a planar cubic map.

Theorem 4.1 *A cubic planar map with four colors is equivalent to 3 edge coloring and vice versa.*

Claude Shannon [122] published results on the tight bounds of lines in any electrical network colored differently for identification in 1949.

Theorem 4.2 *A multipartite graph H having highest degree Δ, satisfies $\Delta \leq \chi'(H) \leq \lfloor \frac{3}{2}\Delta \rfloor$.*

In 1964, Vizing [139] characterized tight bonds for edge coloring of simple connected graphs as below.

Theorem 4.3 *If graph H is a simple connected with highest degree is Δ, then $\Delta \leq \chi'(H) \leq \Delta + 1$.*

If $\chi'(H) = \Delta$, the graph H is classified as Class-I and if $\chi'(H) = \Delta + 1$, it is classified as Class-II.

Example 4.1 A complete K_{2n} is of class-I and K_{2n+1} is of class II.

After this, Vizing [133] worked on a simple planar cubic graph and its edge coloring in 1965 and gave the following characterization.

Theorem 4.4 *If a simple cubic planar graph H with an extreme degree is $\Delta \geq 8$, then $\chi'(H) = \Delta$.*

The above result was enhanced for $\Delta \geq 7$ by Grunewald [61], Zhang [151], and Sanders [123] independently in the years 2000 and 2001. The problem is still open for $\Delta \geq 6$. In general, for multigraph with a largest vertex of degree Δ having multiplicity μ, Vizing and

Gupta [58] proved the following theorem for edge coloring. This is known as the Vizing and Gupta Conjecture.

Theorem 4.5 *A connected multigraph H with the largest degree is Δ and having multiplicity μ, then $\chi'(H)$ is Δ or $\Delta+\mu$.*

Vizing [133] introduced critical graph concept in terms of an edge coloring of a graph and its deleted subgraph, if $\chi'(H-e)<\chi'(H)$, every edge e and proved the result.

Theorem 4.6 *Every critical graph has 3 and more vertices of maximum degree.*

In 1970, there was lots of research on critical graphs and the main critical graph conjecture is as below.

Theorem 4.7 *There is no critical graph of class-II exists for an even number of vertices.*

This result was proved for$n = 4, 6, 8,$ and 10 by Jakobsen [82,83]. Later, Fiorini and Beineke [21] extended to $n = 14$ and Lars Andersen [1,2,18] to $n = 16$ (with Fiorini).

In 1973, Mark Goldberg [62] observed that for graph $H, \omega(H)$ is a density function where

$$\omega(H) = \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil .$$

Then, $\chi'(H) = \omega$ is possibly the best lower bound for edge coloring, and he published the following conjecture.

Theorem 4.8(Goldberg conjecture) *A simple connected graph H has an edge coloring number χ' is Max $\{\Delta,\omega\}$ or $\Delta+1$.*

In 1974, Paul Seymour worked on the graph density function and edge coloring of graphs and derived the same result as Goldberg's conjecture. Later, it was known as the Seymour-Goldberg conjecture. In 1977, Seymour [124] published the following results from his work with planar multigraphs and edge coloring.

Theorem 4.9 *For a planar multigraph H, $\chi'(H) = \omega$ or Δ, where Δ is largest degree and ω is density of graph H.*

R. P. Gupta [63] also worked on Goldberg's conjecture in 1978 and published a result known as Gupta's conjecture, which is an equivalent form of Goldberg's conjecture.

Theorem 4.10 *For a planar multigraph H, the chromatic index*

$$\chi' > \Delta + 1 + \left(\frac{\Delta - 2}{2t} \right)$$

with $t \geq 1$ is a fixed number and encloses a sub-multigraph K of H having $2t'+1$ vertices such that

$$\chi' = \left\lceil \frac{|E(K)|}{t'} \right\rceil$$

if $1 \leq t' < t$.

For $t = 1$, it is Claude Shannon's bound [122]. For $t = 2$, it is the Goldberg conjecture [62]. Many researchers worked on this conjecture, and finally, in 1990, the famous 1.1-theorem was proved by Nishizeki and Kashiwagi [108] as below.

Theorem 4.11 *For a graph H, if $\chi' > 1.1\Delta(H) + 0.8$, then $\chi' = \omega$.*

Goldberg's conjecture is published in parameterized form by using Gupta's conjecture [63] as below.

Theorem 4.12 *For each graph H, it's chromatic index is $\chi' > \Delta(H) + 1 + \frac{(\Delta(H)-2)}{(m-1)}$, odd integer number $m > 4$ is an elementary-graph.*

Several researchers have demonstrated this result for m, which is from 5 to 38. For $m = 39$ the result was solved by Chen and Jing [34] in 2017.

In 1977, Erdos and Wilson [52] discussed the chromatic index in Combinatorial Theory Journal.

Theorem 4.13 *Almost all graphs have a distinctive vertex of maximum degree, and hence almost all graphs are of class-I.*

Garey [59] and Holyer [73] discussed about NP completeness of edge coloring of graphs. In 1981, Holyer demonstrated that it is NP-hard problem and proved result as below.

Theorem 4.14 *The chromatic index is NP-hard problem to decide for any arbitrary graph. Cubic graph is NP-complete to conclude whether chromatic index is 3 or 4.*

Chudnovsky [38,39] discussed about r-regular planar graphs and edge coloring in 2011.

Theorem 4.15 *All 7-regular planar graphs with oddly seven-edge connected are seven edge-colorable.*

Theorem 4.16 *Eight-regular planar graph is 8-edge-colorable if and only if graph is oddly 8-edge connected.*

In 2012, Huang and Wang [76] proved the result for a planar graph.

Theorem 4.17 *A planar graph not having seven-cycle with largest degree six is class-I graph.*

In 2013, Machadoa et al. [102] worked on chordless graphs coloring, their time complexity in polynomial time.

Theorem 4.18 *The chordless graph H having maximum degree $d > 2$, graph H is d-edge colorable and its time complexity is $O(|V(H)|^3|E(H)|)$.*

§5. Face Coloring

Definition 5.1([95,130,154]) *A face coloring means proper face F coloring of a planar graph. If it requires k colors, so-called k-proper region coloring. This is also known as face coloring or map coloring.*

Generally, a Smarandachely Λ*-coloring* $\varphi_\Lambda|_{F(G)} : \mathscr{C} \to F(G)$ *is called a face Smarandachely* Λ*-coloring.*

We know that graph coloring was started with four map color and hence four colorable; so region chromatic number is $1, 2, 3$ or 4.

Theorem 5.1 *All planar graphs are four-colorable.*

The result was proved for the first time by Kemp [91] in 1879. However, there was an error in the proof shown by Heawood [80] who proved the conjecture for five colors in 1890.

Theorem 5.2 *Every planar graph is five-colorable.*

Ore and Stemple [111] demonstrated 4CC for maps with fewer than 40 countries using a numerical method in 1969.

Theorem 5.3 *All planar graphs upto thirty nine faces are four-colorable.*

After 87 years, Appel Kenneth and Haken Wolfgang [90,94] presented a proof of 4CC by verifying more than 1900 unavoidable reducible configurations of a planar graph with the help of 1200 computer hours in 1977 and proved that each planar graph can be colored with 4 or less colors. Robertson et al. [117] gave revised proof with less than 650 unavoidable reducible configurations. In 2014, Bhapkar [14] proved 4CC by using PRN (Pivot Region Number) of graph. There are various characterizations of 4CC demonstrated by many mathematicians, as below. In 1931, Whitney [146,147] proved the result on Hamiltonian planar graphs as below:

Theorem 5.4 *The 4CC holds iff all hamiltonian planar graphs are four-colorable.*

Vizing [139] described 4CC in the form of a chromatic index, as below.

Theorem 5.5 *The 4CC is true iff all cubic planar graphs not comprising bridge are 3 edge colorable.*

In 1943, Hadwiger [71] introduced the concept of contraction in graph theory and gave the famous conjecture below.

Theorem 5.6 *All* n*-chromatic connected graphs are contractible to complete graph* K_n*.*

The converse of this was proved by Wagner [140] in 1960.

Theorem 5.7 *4CC is the same to Hadwiger's conjecture for* $n = 5$*.*

Grötzsch's [56] characterized 3-colorable graphs in 1958 as below.

Theorem 5.8(Three color problem) *All triangle free planar graphs are three-colorable.*

Grünbaum [57] characterized 3-colorable graphs in 1963 as below.

Theorem 5.9 *All planar graphs having less than 4 triangles are three-colorable.*

Another Characterization of plane graph is given by Ore and Stemple [111] in 1969, as shown below.

Theorem 5.10 *The 4CC holds iff all bridgeless cubic plane graph are 4-colorable.*

In 1976, Steinberg raised the question below, which was proved by Gimbel [126] in 1993.

Theorem 5.11 *Every planar graph not comprising four-cycle and five-cycle is three-colorable.*

Later in 2005, Borodin et al. [19] improved this result as follows.

Theorem 5.12 *A planar graph not consists of cycles of* $4-7$ *length are three-colorable.*

After this, Borodin et al. [20] extended this result up to cycle $3-9$ length in 2006.

§6. **Total Coloring**

Definition 6.1([10,95,154]) *A total coloring means coloring vertices and edges both together properly. It we use k-colors, so-called k-total coloring of a graph.*

Generally, a Smarandachely total coloring of a graph G by colors in $\mathscr{C}$ *is a mapping* $\varphi_\Lambda : \mathscr{C} \to V(G) \cup E(G)$ *such that* $\varphi(u) \neq \varphi(v)$ *if u and v are elements of a subgraph isomorphic to* $\Lambda \prec G$.

Definition 6.2 *If we required least k different colors for coloring of vertices and edges then graph is known as k-total colorable or total chromatic number, denoted by* χ''.

In 1965, Mehdi Behzad [9] introduced the idea of total coloring. One of most the important results was total coloring conjecture. Mehdi Behzad [9,10] and Vizing [139] was discussed separately this result which is listed below.

Theorem 6.1 *For any graph H, with extreme degree is* Δ*, total chromatic number* $\chi''(H)$ *holds inequality* $\Delta + 1 \leq \chi''(H) \leq \Delta + 2$.

Therefore, graphs are characterized in two types according to their total coloring number. A graph H is called Type-I if $\chi''(H) = \Delta + 1$ and Type-II if $\chi''(H) = \Delta + 2$.

Example 6.1 A cycle C_{2n} is Type-I and C_{2n+1} is Type-II.

For graphs having very large maximum degree Δ, Reed and Molloy [115] proved by the probabilistic approach that its Total Chromatic numbers is at most $\Delta + 10^{26}$. To determine the TCC f is NP-hard problem for any arbitrary graph, which was proved by Sanchez-Arroyo [127] in 1989.

In 1971, Rosenfeld [116] discussed the results of TCC for cubic graphs.

Theorem 6.2 *The total coloring number of cubic graph is four or five.*

In 1996, Kostochka [88] proved TCC for the largest degree of a graph being fewer than 6, as below.

Theorem 6.3 *The maximum five degree multigraph is at most 7-total colorable.*

Later, this result was improved to 6-total colorable, as shown below.

Theorem 6.4 *A multigraph having maximum degree four is at most 6-total colorable.*

Theorem 6.5 *A five-regular multigraph with perfect matching is at most seven-total colorable.*

In 1992, Seoud [128] established results for the Cartesian product of path graphs.

Theorem 6.6 *A graph* $(P_m \times P_n)$ *is of Type-I,* $m, n > 2$.

In 1999, Sanders, Daniel P., and Y. Zhao [125] proved that the TCC for the maximum degree of a planar graph is less than 8, as below.

Theorem 6.7 *If a planar graph has at the most degree seven, then it is nine-total colorable.*

In 2001, Bojarshinov [33] showed that TCC holds for an interval-graph. This is NP-hard problem with its polynomial time complexity as below.

Theorem 6.8 *An interval-graph* H *with odd maximum degree is of Type-I and its time complexity is* $O(|V(H)| + |E(H)| + (\Delta(H))^2)$.

Theorem 6.9 *An interval-graph* H *with even maximum degree is of Type-II and its time complexity is* $O(|V(H)| + |E(H)| + (\Delta(H))^2)$.

In 2003 and 2007, Campos and Mello [35,36], proved the results on a circulant graph that is a power of cycle graph, as below.

Theorem 6.10 *The power two of cycle graph* C_n *(circulant graph-*$C_n(1, 2)$*) is Type-I excluding* $n = 7$.

Theorem 6.11 *If* $C_n(1, 2, \cdots, k)$ *is a circulant graph where* $2 \leq k \leq \lfloor n/2 \rfloor$ *then it is type-II iff* k *is odd integer and* $k > \frac{(n-3)}{3}$.

In 2003, Hilton et al. [81], G. Li and L. Zhang [98] published results on total chromatic numbers of join graphs.

Theorem 6.12 *The graph* $H = H_1 + H_2$ *is a join graph, where* H_1 *and* H_2 *are bipartite graphs with maximum degree at most 2, then* H *is of Type-I if and only it if not isomorphic to* $K_{n,n}$ *or* K_4.

Theorem 6.13 *The graph* $H = H_1 + H_2$ *is a regular graph, where* H_1 *and* H_2 *are graphs having odd number of vertices, then* H *is of Type-II.*

Theorem 6.14 *The graph* $H = C_m + C_n$ *is a join graph. Then,* H *is of Type-II if and only* m *and* n *are odd integers with* $m = n$.

Theorem 6.15 *The graph* $H = K_{p,q} + C_n$ *is of Type-I for positive integers* n *and* $p > q$.

In 2005, Campos and Mello [37] proved some result on bipartite graph families as below.

Theorem 6.16 *A grid graph* $G_{m\times n}$, $m, n > 1$, *is of Type-I, a near ladder graph* B_k *is of Type-I and II for* k *is even and odd, respectively and a* k*-dimensional cube graph* Q_k *is of Type-I for integers* $k > 2$.

In 2008, Kowalik et al. [89] proved following result for maximum degree of planar graph is more than 8 as below.

Theorem 6.17 *If the largest degree of a planar graph is more than eight then it is Type-I.*

Khennoufa and Togni [92] discussed about fractional total coloring number for cubic circulant and four-regular graphs in 2008.

Theorem 6.18 *For a circulant H having n vertices and $(p, p, \cdots, p)$-stable for positive rational number p, the fractional total coloring number is $\leq (n/p)$.*

Theorem 6.19 *All cubic circulant graph $H = C_{2n}(1, n)$ with fractional $(p, p, 0)$-stable, then fractional total coloring number is $\leq (2n/p) + 1$.*

Theorem 6.20 *A four regular circulant graph $C_{5p}(1, k)$ is of Type-I for integers $p > 0$ and $(k-2)$, $(k-3)$ being multiple of 5, where $k < 3p/2$.*

Theorem 6.21 *A four regular circulant graph $C_{6p}(1, k)$ is of Type-I for integers $p > 3$ and $(k-1)$, $(k-2)$ being multiple of 3, where $k < 3p$.*

In 2010, Prnaver and Zmazek [113] results on direct product graph's total colorings.

Theorem 6.22 *The direct product of cycles C_m and C_n is 5-total colorable. Also, the direct product of cycle C_m and path P_n is 5-total colorable.*

In 2011, Campos et al. [40] published result for some snarks families graph's total colorings.

Theorem 6.23 *The total coloring number of infinite snarks families namely flower, Goldberg and twisted Goldberg snark is 4. Hence these graphs are of Type-I.*

In 2012, Campos et al. [41] published result of total-coloring of split indifference graph.

Theorem 6.24 *The total-coloring of split indifference graph with largest even and odd degree is Type-I and II respectively iff Hilton's condition satisfied.*

In 2013, Machadoa et al. [102] discussed TCC for chordless graph with its time complexity in polynomial time as below.

Theorem 6.25 *The chordless graph H having maximum degree is three and more is of Type-I and its time complexity is $O(|V(H)|^3|E(H)|)$.*

In 2015, Geetha and Somasundaram [65] published the total coloring for generalized sierpinski graph of hypergraph and cycle graph.

Theorem 6.26 *For $n > 1$, if a graph H is Type-I, so the Sierpinski graph $S(n, H)$ is Type-I.*

Theorem 6.27 *The Sierpinski graph $S(n, C_k)$ of cycle graph C_k is Type-I for positive integers $n > 1, k > 2$.*

Theorem 6.28 *The Sierpinski graph $S(n, Q_k)$ of hypercube graph Q_k is of Type-I for positive integer $n > 1$.*

In 3D topology, WK-recursive topology of graph G is constructed as l layers of 2D recursive topology of graph G. $K(l,n,G)$ for $l = 1$ is snark family of graph G that is Sierpinski graph $S(n,G)$. For WK-recursive topology of complete graph, Geetha and Somasundaram [65] proved following result of total coloring.

Theorem 6.29 *The graph $K(l,n,K_k)$-WK-recursive topology of complete graph K_k is of Type-I for positive integers $n,k > 1$ and $l > 0$.*

Theorem 6.30 *The graph $K(l,n,C_k)$-WK-recursive topology of cycle C_k is of Type-I for positive integers $n > 1, k > 1$ and $l > 0$.*

In 2016, Mohan et al. [100], discussed results of total coloring for compounded graph and rooted graph.

Theorem 6.31 *The compounded graph $H[G]$ for any two total colorable graphs H and G is of Type-I.*

Theorem 6.32 *A rooted graph $H \circ P_n$ for total colorable graph H and path P_n is Type-I.*

In 2017, Mohan, Geetha and Somasundaram [99], proved results of total colorings for corona product of two graphs.

Theorem 6.33 *For the path, cycle, complete and complete bipartite graphs, the corona product with any graph H is Type-I.*

In 2018, Geetha and Somasundaram [66] published the results on total coloring numbers of graph's product.

Theorem 6.34 *A graph $(K_n \times K_n)$ is Type-I for even positive integer n.*

Theorem 6.35 *A graph $(C_m \times C_n)$ is Type-I for positive integer $n \geq 3$ and m is multiple of $3, 5$ and 8.*

In 2018, Golumbic [64] discussed total coloring of rooted path graph and its polynomial time complexity. He also gave algorithm by using greedy algorithm to find total coloring number.

Theorem 6.36 *A rooted path graph having even maximum degree is Type-I. Otherwise it is Type-II. Its time complexity is $O(|V(H)| + |E(H)|)$.*

In 2018, Vignesh [136] discussed total coloring numbers for double graph.

Theorem 6.37 *A double graph of Type-I graph is Type-I. Otherwise, it is Type-II.*

Theorem 6.38 *For two Type-I graph's deleted lexicon product is Type-I.*

Theorem 6.39 *The deleted lexicon product of any graph H with path P_m for $m > 2$ is Type-I.*

Theorem 6.40 *Let K_n be a complete graph. Then, its line graph is Type-I.*

In 2020, Vignesh et al. [135] explained total coloring numbers of Cocktail Party, Core-Satellite, Shrikhande and Modular Product of Graphs.

Theorem 6.41 *A core-satellite graph is total colorable and it is Type-I if the core and satellite cliques are of Type-I.*

Theorem 6.42 *A cocktail party graph of order n is Type-I for $n > 2$.*

Theorem 6.43 *The modular product of P_3 graph with Cycle C_n and path P_n are total colorable graph.*

Theorem 6.44 *The Shrikhande graph is Type-I.*

In 2021, Mauro et al. [109] proved 5-total coloring of four regular circulant graphs that are Type-I as follow.

Theorem 6.45 *A circulant graphs $C_{3kp}(1,p)$ is Type-I for integer $k > 0$, p is a multiple of 3. Also proved if k is even then $C_{3nk}(1,k)$ is Type-I and $C_{3n}(1,3)$ is Type-I except that $C_{12}(1,3)$ is Type-II.*

In 2022, Prajnanaswaroopa et al. [114] described result on total coloring of Caley graph.

Theorem 6.46 *A Caley graph is of Type-II.*

Theorem 6.47 *The TCC holds for odd and mock threshold graph.*

For the vertex, edge or total coloring of a graph G, there are open problems following.

Problem 6.1 *Let the complete graph K_n be decomposed into 3 subgraphs G_1, G_2, G_3 such that $\chi(G_1) = n_1$, $\chi'(G_2) = n_2$ and $\chi''(G_3) = n_3$ for integers $n_1, n_2, n_3 \geq 1$.*

(1) *Determine all possible subgraphs G_1, G_2 and G_3;*
(2) *Determine all possible integers n_1, n_2, n_3.*

Problem 6.2 *For any connected graph G, can it be decomposed into 3 subgraphs G_1, G_2, G_3 such that $\chi(G_1) = n_1$, $\chi'(G_2) = n_2$ and $\chi''(G_3) = n_3$ for integers $n_1, n_2, n_3 \geq 1$, particularly, with some special numbers such as $n_1 = 0, 1$, $n_2 = 0, 1$ or $n_3 = 0, 1$ or other integers?*

§7. Perfect Coloring

Definition 7.1 *A perfect coloring means proper coloring of all components of planar graph. If it needs least k colors, then it is called k-proper perfect coloring.*

In 2018, Bhapkar [15] introduced perfect coloring of graphs. The following results proved.

Theorem 7.1 *A star graph is perfectly $(n + 2)$-colorable.*

Theorem 7.2 *A rose graph is perfectly $(m + 2)$-colorable.*

Theorem 7.3 *A chain graph is perfectly 4-colorable.*

Theorem 7.4 *A tree with largest degree d is perfectly $(d + 2)$-colorable.*

Theorem 7.5 *A cycle C_n is perfectly 5-colorable if the integer n is multiple of 3. Otherwise, it is perfectly 6-colorable.*

In 2019, Archana Bhange [30] proved result of perfect coloring of corona product of cycle graph with cycle, path and null graphs as below.

Theorem 7.6 *A perfect coloring of corona product of cycle graph C_n with C_m is $m+3$ for $n>4, m>4$.*

Theorem 7.7 *A perfect coloring of corona product of cycle graph C_n with path graph P_m is $m+3$ for $n>4, m>3$.*

Theorem 7.8 *A perfect coloring of corona product of cycle graph C_n with null graph N_m is $m+4$ for $n>4, m\geq 14$.*

In 2020, Bhange [16] collaborated with Bhapkar to define upper and lower bound and kinds of perfect coloring. They also worked on some standard families and their Perfect coloring.

Theorem 7.19 *The bound for perfect coloring is $\chi'' \leq \chi^p \leq \chi''+4$, where χ'' is total coloring number.*

Theorem 7.10 *There are following kinds of perfect coloring of graphs.*

(1) *Kind 0 perfect coloring if $\chi^p(H)=\chi''(H)$;*
(2) *Kind 1 perfect coloring if $\chi^p(H)=\chi''(H)+1$;*
(3) *Kind 2 perfect coloring if $\chi^p(H)=\chi''(H)+2$;*
(4) *Kind 3 perfect coloring if $\chi^p(H)=\chi''(H)+3$;*
(5) *Kind 4 perfect coloring if $\chi^p(H)=\chi''(H)+4$.*

Theorem 7.11 *There are no graph with*

$$\chi^p(H)=\chi''(H)+5.$$

Theorem 7.12 *The diamond graph is kind 0.*

Theorem 7.13 *The null graph, trees, friendship graph, ladder rung graph are kind 1.*

Theorem 7.14 *A prism and circular ladder graph are kind 2.*

Theorem 7.15 *A ladder graph is kind 3.*

Theorem 7.16 *A helm graph is kind 4.*

In 2022, Archana Bhange and Bhapkar [31] worked on perfect coloring of corona product of Fan graphs with sunlet graph, tadpole graph and proved following results.

Theorem 7.17 *A perfect coloring of corona product of sunlet graph and fan graph is $\Delta+1$.*

Theorem 7.18 *Perfect coloring of corona product of Tadpole graph and Fan graph is $\Delta+1$.*

We observed that the concept of vertex coloring was extended to edge and face coloring as well as the combination of these elements. After this many researchers in the field of graph theory have defined various types of graph coloring by enforcing some different conditions while coloring graphs. Now, let we discuss some other special types of coloring and their results.

§8. **Strong Edge Coloring of Graph**

Definition 8.1 *A properly-edge colored graph fulfills the condition $C(u_1) \neq C(u_2)$ for every edge u_1u_2 with color set $C(u)$ and $C(v)$ is known as strong edge coloring of graph.*

In 1997, Burris [26] introduced this coloring notion. It is also named vertex distinguishing proper edge coloring. It has the following properties

(1) Adjacent edges have the different color;

(2) If two vertices u_1 and u_2 are neighbors, their color sets are separate. i.e. $C(u_1) \neq C(u_2)$.

Notice that a color set $C(u)$ for a vertex u means set of colors of all edges incident at a vertex u after proper coloring of edge and the strong edge chromatic number symbolized by $\chi'_s(H)$ means minimum colors needed for this coloring. Burris, Schelp [26] validated the following result for cycle graph, bipartite, and complete bipartite.

Theorem 8.1 *For a cycle C_n,*

(1) $\chi'_s(C_n) = 5$, *for $n = 5$;*

(2) $\chi'_s(C_n) = 3$, *for n is multiple of 3;*

(3) $\chi'_s(C_n) = 4$, *else.*

Theorem 8.2 *For a complete bipartite graph $K_{m,n}, 1 \leq m \leq n$,*

(1) $\chi'_s(K_{m,n}) = (n+1)$, *if $m < n$ and*

(2) $\chi'_s(K_{m,n}) = (n+2)$, *if $m = n > 1$.*

Theorem 8.3 *For a complete graph $K_n, n \geq 3, \chi'_s(K_n)$ is n and $n+1$ for n is odd and even, respectively.*

Theorem 8.4 *For a star graph $K_{1,n}, n \geq 3$, $\chi'_s(K_{1,n}) = n$.*

For a graph H with n_k at least k colors as there are vertices having degree k. Thus lower bound is $\chi'_s(H) \geq max\{(k!n_k)^{(1/k)} + (k-1)/2 : for\ 1 \leq k \leq d\}$. We can improve this lower bound upto additive 1 as below.

Theorem 8.5 *If graph H has largest degree is d and for smallest integer j such that ${}^jC_k \geq n_k$ for $1 \leq k \leq d$. Then strong edge coloring $\chi'_s(H) = j$ or $j+1$.*

The upper bond for strong edge coloring was proved as below.

Theorem 8.6 *If a graph H is strong edge coloring and $n_i^{1/i} = max\{n_j^{1/j}$: for $j = 1$ to maximum degree d of graph $H\}$. Then $\chi'_s(H) \leq (\Delta+1)(2n_i^{1/i} + 5)$. This is upper bound.*

In 1997, Bazgan et al. [27] verified following result on strong edge coloring.

Theorem 8.7 *If H is any graph with n vertices, consists of no more than one isolated vertex and no isolated edges, then $\chi'_s(H) \leq n+1$.*

In 2002, Zhongfu Zhang et al. [152] published following results.

Theorem 8.8 *A graph H is formed with n connected components H_i then its strong edge coloring is $\chi'_s(H) = max\{\chi'_s(H_i) : \ for \ all \ i\}$.*

Theorem 8.9 *If T is tree graph with 3 and more vertices, then $\chi'_s(T) = d$ for two maximum degree vertices are not neighbors, otherwise $\chi'_s(T) = d + 1$.*

Theorem 8.10 *If two highest degree d vertices are neighbors in any graph, then $\chi'_s \geq d + 1$.*

Theorem 8.11 *A graph with highest degree d vertices are not neighbors and two neighbor vertices of different degree, then $\chi'_s = d$.*

In 2007 Balister et al. [32] proved following results for strong edge-colorings.

Theorem 8.12 *If H be any connected graph with 6 and more vertices, then $\chi'_s(H) \leq \Delta + 2$.*

Theorem 8.13 *If a graph H has non-isolated edges and the largest degree is 3, then $\chi'_s(H) \leq 6$.*

Theorem 8.14 *If a bipartite graph B has no isolated edge, then $\chi'_s(B) \leq \Delta + 2$.*

Theorem 8.15 *If a graph H is non-isolated and it is k-chromatic, then $\chi'_s(H) \leq \Delta + O(log\ k)$.*

In 2010, Wang et al. [142] proved strong edge coloring for maximum degree of graph more than 4 with the condition on maximum average degree (mad) as follows.

Theorem 8.16 *If a connected graph H has highest degree Δ with mad(H), then*

(1) *If $\Delta = 3$; $mad(H) < 7/2$, then $\chi'_s(H) \leq \Delta + 1$;*
(2) *If $\Delta \geq 3$; $mad(H) < 3$, then $\chi'_s(H) \leq \Delta + 2$;*
(3) *If $\Delta \geq 4$; $mad(H) < 5/2$, then $\chi'_s(H) \leq \Delta + 1$;*
(4) *If $\Delta \geq 5$; $mad(H) < 5/2$, then $\chi'_s(H) \leq \Delta + 1$ iff graph H has adjacent vertices of highest degree.*

In 2013, Hocquard, Montassier [78] generalized these results and proven the below results for the largest degree $\Delta \geq 5$ with a condition on mad.

Theorem 8.17 *For every graph H with $\Delta \geq 5$, $mad(H) < 3 - (2/\Delta)$, then $\chi'_s(H) \leq \Delta + 1$.*

In 2021, Borut et al [97] proved strong edge coloring of regular graph.

Theorem 8.18 *If a graph H is r-regular, then strong-edge-chromatic number $\chi'_s(H)$ is equal to $(2r - 1)$ if and only if it covers the Kneser graph $K(2r - 1,\ r - 1)$.*

Theorem 8.19 *A cubic graph is 5 strong-edge-chromatic iff it covers Petersen graph.*

§9. Vertex Distinguishing Total Coloring

Definition 9.1 *There is an additional constraint joined in total coloring, if the color sets of any two neighboring vertices should be different, such a coloring is known as AVD total coloring.*

In 2005, Zhang et al. [153] introduced this coloring type after adding one more restriction in the definition of total coloring. It has the following properties.

(1) Adjacent-vertices colored differently;

(2) Adjacent-edges colored differently;

(3) An edge with its end vertices are colored differently;

(4) For each two neighbor vertices u_1, u_2 of a graph H, both vertices color sets are different. That means $C(u_1) \neq C(u_2)$.

The AVD-total-chromatic number symbolized by $\chi_{at}(H)$. It is the least colors needed for AVD-total-coloring of a graph. Many researchers provided results on AVD-Total Coloring. The lower and upper bounds were discussed by Zhang et al. [153] in 2005.

Theorem 9.1(Lower bound) *If two maximum degree vertices are adjacent in a simple graph H, then $\chi_{at}(H) \geq \Delta + 2$; otherwise, $\chi_{at}(H) \geq \Delta + 1$.*

Theorem 9.2 *If graph H is simple and connected with minimum order 2, then $\chi_{at}(H) \leq \Delta + 3$.*

Thus, by the AVD-total-coloring conjecture we conclude that $\Delta + 1 \leq \chi_{at}(H) \leq \Delta + 3$. This is true for various graph families such as the graphs with $\Delta = 3$, bipartite graphs, complete graphs.

In 2007, Wang [143] and in 2008, Chen [42] separately verified result of AVD-total-coloring for maximum degree three graphs as below.

Theorem 9.3 *If the largest degree of graph H is at most three, then $\chi_{at}(H) \leq \Delta + 3$.*

In 2012, Huang [45] proved following theorem.

Theorem 9.4 *If the largest degree of a simple graph H is more than 2, then $\chi_{at}(H) \leq 2\Delta$.*

An algorithmic procedure described for four regular graph's AVD-total-coloring by Papaioannou and Raftopoulou [112] in 2014.

Theorem 9.5 $\chi_{at}(H) \leq \Delta + 3$ *for any four regular graphs having its maximum degree is Δ.*

In 2017 Yang [150] proved the result for planar graph.

Theorem 9.6 *If the largest degree of a planar graph is more than 10, then $\chi_{at}(H) \leq \Delta + 2$.*

In 2019, Wang [141] and Hu [77] independently demonstrated result for planar graph having extreme degree nine.

Theorem 9.7 *If the largest degree of a planar graph H is more than 8, then $\chi_{at}(H) \leq \Delta + 3$.*

In 2020, Yulin Chang et al. [43] revised this result for maximum degree is more than 7.

Theorem 9.8 *If the largest degree of a planar graph H is more than 7, then $\chi_{at}(H) \leq \Delta + 3$.*

§10. Acyclic Coloring

Definition 10.1 *If every two-chromatic subgraph is acyclic after graph's vertex coloring, then it is known as acyclic coloring. In other words, each cycle in a graph uses minimum three* *colors for proper vertex coloring. The least colors needed for such coloring is known as acyclic chromatic number.*

In 1973, Grunbaum [60] started work on acyclic coloring and proved result for graph with largest degree is 3 as below.

Theorem 10.1 *For any graph H with $\Delta = 3$, $A(H) \leq 4$.*

In 1979, Burstein [12] verified result for largest 4 degree graph.

Theorem 10.2 *Any graph H with $\Delta = 4$ is acyclic 5-coloring.*

In 1979, Borodin [11] proved result of acyclic coloring for planar graph.

Theorem 10.3 *Any planar graph H is at most acyclic 5-coloring.*

In 2011, Varagani et al. [138] verified result for largest 6 degree graph.

Theorem 10.4 *Any graph H with $\Delta = 6$ is at most acyclic 12-coloring.*

In 2011, Kostochka and Stocker [4] proved result for graph with maximum degree is 5.

Theorem 10.5 *Any graph H with $\Delta = 5$ is at most acyclic 7-coloring.*

§11. **List Coloring**

Definition 11.1 *List vertex coloring means proper vertex coloring of graph with color every vertex from available list of color only. The minimum colors necessary for this coloring is called List vertex chromatic number or vertex-choosability.*

Definition 11.2 *If we color edges of graph from an available list of colors for each edge, then it is called List edge-coloring. Thus choose a color for every edge from a list of colors only. The minimum colors necessary for this coloring is called List edge chromatic number or edge-choosability.*

Definition 11.13 *The total coloring from available list of colors for each vertex and edge is termed as list total-coloring.*

This type of coloring was introduced for the first time by Erdos et al. [53] in 1980.

Theorem 11.1 *For any graph, its lower bound of list vertex coloring is chromatic number.*

Theorem 11.2 *For any graph, the lower bound of list edge coloring is chromatic index of a graph.*

In 1976 Vizing [137] proved result for list edge coloring.

Theorem 11.3 *The vertex-choosability of every graph is at most $\Delta + 1$.*

Borodin [25] proved above result is true for planar graphs for $\Delta = 8$ in 1990. Woodall et al. [24] verified list chromatic index is Δ for planar graph for maximum 11 degree. In 1994, Borodin [23] proved the result for list coloring.

Theorem 11.4 *In a planar graph, if largest degree is more than 9; then total coloring $\leq$ list total coloring $\leq$ maximum degree $+2$.*

In 1995, Woodall [148] proved the result of list total coloring on planar graph.

Theorem 11.5 *A planar graph with largest degree is more than 5 and also girth is more than 5; then total coloring $=$ list total coloring $=$ maximum degree $+1$. Thus it is of Type-I graph.*

Above result proved for maximum degree 7 and its girth more than 3 by Borodin et al. [24] in 1997. Hou [74] in 2006 proved above result if there is no 4-cycle in graph.

In 2006 and 2007, Hou, Liu, Cai [74,75] proved the result below.

Theorem 11.6 *If a planar graph has largest degree is more than five and does not contains 4-8 length cycles; then List total coloring $=$ total coloring $=$ maximum degree $+1$. Thus it is of Type-I graph.*

The following are some equivalent results.

Theorem 11.7 *A planar graph with largest degree is more than eight, then list total coloring $=$ total coloring $=$ maximum degree $+1$, if the graph satisfies below conditions:*

(1) *There is no intersecting 3-cycle. (Wu-Wang [149] proved in 2008);*
(2) *Does not contain 5-cycle or 6-cycle. (Ma, Wu, Yu [101] proved in 2009).*

In 2006, Borowiecki et al. [22] discussed list coloring of product graphs.

Theorem 11.8 *The vertex-choosability of product of two graphs P and Q has tight upper bound, $\chi_l(P \times Q) \leq min\{\chi_l(P) + col(Q), \chi_l(Q) + col(P)\} - 1$.*

In 2006, Hou, Liu, Cai [74,75] proved the result of List edge coloring for graph without 14-cycle.

Theorem 11.9 *The edge-choosability of graph is Δ, if largest degree is more than 3 and without 14-cycle.*

In 2008, Wu-Wang [149] verified result for the maximum degree was more than 8 for planar graphs.

Theorem 11.10 *For a more than 8 largest degree planar graph, the List total-coloring $=$ total coloring $=$ maximum degree $+1$, if there is no 3 cycle.*

Certainly, all the previous colorings can be determined on a graph G, which enables us to generalize Problem 6.2 as follows.

Problem 11.1 *How to decompose a connected graph G, particularly, the complete graph K_n of order n into subgraphs $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8$, i.e., $G = \bigcup\limits_{i=1}^{8} G_i$ such that $\chi(G_1) = n_1$,*

$\chi'(G_2) = n_2$, $\chi''(G_3) = n_3$, $\chi^p(G_4) = n_4$, $\chi'_s(G_5) = n_5$ $\chi_{at}(G_6) = n_6$, $\chi_l(G_7) = n_7$ and $A(G_8) = n_8$ for chosen integers $n_i, 1 \leq i \leq 8$?

Clearly, Problem 11.1 is Problems 6.1 and 6.2 if $n_4 = n_5 = n_6 = n_7 = n_8 = 0$.

§12. π-Coloring and Incident Vertex π-Coloring

Bhapkar and Thakare [155, describe the π-coloring idea, which is based on properly coloring graph components with distinctive color patterns.

Definition 12.1(π-Coloring, [155]) *Let $H = (V, E)$ be a simple connected graph where V is vertex set, E is edge set, and $X = \{X_1, X_2, \cdots, X_r\}$ is a collection of distinct subsets of elements of graph H having some common properties. If there exists a function $f : X \longrightarrow P(C)$, where C is a set of colors and $P(C)$ is its power set, such that $f(X_p) \neq f(X_q)$, for all $p \neq q$ with some conditions, then it is called π-coloring of graph H. The least number of colors of set C is called π-chromatic number of a graph H corresponding to function $f(X)$. It is denoted by $\pi_f(H)$ or $\pi(H)$.*

Assigning distinct colors to each incidence vertex on the edges in set X, which is the collection of all incident vertices pairs of each edge in the graph is known as incident vertex π-coloring and it is defined as below [155].

Definition 12.2(Incident Vertex π-Coloring) *Let $H = (V, E)$ be a simple connected graph where V is vertex set, E is edge set, and $H = \{H_1, H_2, \cdots, H_r\}$, where $H_i = \{e_i = (u, v)|$ for all $u, v \in E\}$, that is a collection of order pair incident vertices of every single edge e in $E(H)$. Define a function $f : X \longrightarrow P(C)$, where C is set of colors and $P(C)$ is its power set, such that $f(X_i) \neq f(X_j)$, for all $i \neq j$, then it is called incident vertex π-coloring (IVPI) of graph H. The least number of colors of set C called Incident Vertex π chromatic number of graph H corresponding to function $f(X)$, and it is represented by $IV\Pi_f(H)$ or $IVPI(H)$.*

Bhapkar and Thakare [155] discussed the incident vertex π coloring of graphs namely star graph, double star graph, complete graph, wheel graph, fan graph, double fan graph and complete bipartite graph.

Theorem 12.1 *The incident vertex π chromatic number of $K_{1,n}$ is $n + 1$.*

Theorem 12.2 *The incident vertex π chromatic number of $K_{1,n,n}$ is $n + 1$.*

Theorem 12.3 *The incident vertex π chromatic number of a complete graph is n.*

Theorem 12.4 *The incident vertex π chromatic number of wheel graph W_{n+1} is $n + 1$.*

Theorem 12.5 *The incident vertex π chromatic number of fan graph $F_{1,n}$ is $\Delta + 1$.*

Theorem 12.6 *The incident vertex π chromatic number of double fan graph $F_{2,n}$ is $\Delta + 2$.*

Theorem 12.7 *The incident vertex π chromatic number of complete bipartite graph $B_{m,n}$ is $m + n$.*

§13. Applications of Graphs Coloring

The graph theory has a wide-ranging applications because it deals with real-world problems and their solutions (for more details, see Narsingh Deo [48], Roberts [119], and Berge [8]). Certainly, the graph theory is used in mathematics to solve problems involving linear systems such as signal flow problems. The Markov process is one of the most important methods in statistics and probability theory for solving problems in various areas such as statistical information, analysis of various computer programs, control theory, problems in genetics and inventory theory. As a result, the graph theory is used to solve Markov processes. In chemistry, the graph theory is used to represent and match the chemical structure of molecules. By using graph enumeration techniques, we can identify or characterize new chemical composites. Designing computer programs and analyzing them are the two most crucial aspects of computer engineering. In computer programming, the graph theory is used for running time estimation and storage requirements, identifying errors, segmenting and flow of a program, and creating a stochastic model for a program. It is also used for programme optimization, automatic flow charts, data structure as graph, and determining the equivalence and validity of various programmes by transforming their diagraph into canonical form.

There are numerous real-world applications for graph coloring (see [119] for the latest study), so it has received renewed interest in recent years. We can solve lots of real-world problems in sciences by using the graph coloring concept, which includes computer network problems, artificial intelligence problems, machine learning problems from computer engineering, electrical circuit problems from electrical engineering, and communication network problems from electronics and communication engineering. One of the most well-known of these applications is frequency allocation. Each radio transmitter in a radio transmitter network has its own set of operating frequencies. Once two adjacent transmitters utilize the same frequency, they can cause interference. The frequency bands assigned to these transmitter pairs should have been distinct in the simplest model. The aim is to reduce the overall frequency number used. The graph coloring solves this problem. Vertices are emitters in this case, an edge is added in among pair of emitters (vertices) that may interfere. As a result, the frequencies match the colors allocated to the vertices, and nearby vertices must have different colors. The required frequency separation in larger designs may be greater for closer transmitter pairs or for number of frequency bands assigned to the same transmitter; and hence, the goal is to generally reduce the variation between the lowest and highest frequency used (allocation range). It was widely assumed that assigned frequencies should be distinct and regularly spaced points on the spectrum (see Hale [79]). As a result, colors are commonly considered as numbers. Later from Hale's paper, two frequency assignment models are developed, one T-coloring and another is channel assignment. After this, Tesman [131] developed list T-coloring model for frequencies assignment which is with the restriction of frequencies available (using the concept of list coloring) for the transmitter.

In 2018, Bhapkar [29], explained how to generate security key with the help of perfect weighted planar graph. This paper also describes algorithm for public key and secret key generation.

In 1920, the virus graphs and its use were explained by Bhapkar et al. [28] in details. There are four types of virus graphs, i.e., the virus graph type I to IV, where types I and II are not death-defying but types III and IV are extremely harmful for human beings. In this paper, they also discussed about the importance of graph modeling during pandemic conditions and its rate spreading. To control the spreading of COVID-19, the cut set concept is used as an isolation of people.

In 2022, Ghorpade and Bhapkar [67, 68] worked together on brain MRI separation and used the cut-set concept to find the exact infected area that helps for medical treatment. They discussed brain MRI segmentation by cut and watershed model.

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Famous Words

Only people's social practice is the standard of the truth that people know about the outside world. The standard of truth can only be social practice.

By *Mao Zedong*, a great Marxist, proletarian revolutionary, strategist and theorist, and the main founder and leader of the communist party of China, the Chinese people's liberation army and the people's republic of China.

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