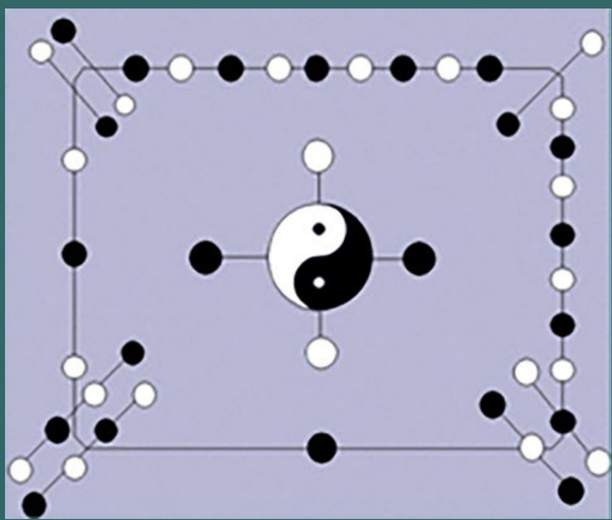
ISSN 1937 - 1055

VOLUME 2, 2023

INTERNATIONAL JOURNAL OF

MATHEMATICAL COMBINATORICS



EDITED BY

THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

ACADEMY OF MATHEMATICAL COMBINATORICS & APPLICATIONS, USA

June, 2023

Vol.2, 2023 ISSN 1937-1055

International Journal of

Mathematical Combinatorics

(www.mathcombin.com)

Edited By

The Madis of Chinese Academy of Sciences and

Academy of Mathematical Combinatorics & Applications, USA

June, 2023

Aims and Scope: The *mathematical combinatorics* is a subject that applying combinatorial notion to all mathematics and all sciences for understanding the reality of things in the universe, motivated by *CC Conjecture* of Dr.L.F. MAO on mathematical sciences. The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly, which publishes original research papers and survey articles in all aspects of mathematical combinatorics, Smarandache multi-spaces, Smarandache geometries, non-Euclidean geometry, topology and their applications to other sciences. Topics in detail to be covered are:

Mathematical combinatorics;

Smarandache multi-spaces and Smarandache geometries with applications to other sciences;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

Mathematical theory on gravitational fields and parallel universes;

Applications of Combinatorics to mathematics and theoretical physics.

Generally, papers on applications of combinatorics to other mathematics and other sciences are welcome by this journal.

It is also available from the below international databases:

Serials Group/Editorial Department of EBSCO Publishing
10 Estes St. Ipswich, MA 01938-2106, USA
Tel.: (978) 356-6500, Ext. 2262 Fax: (978) 356-9371
http://www.ebsco.com/home/printsubs/priceproj.asp

and

Gale Directory of Publications and Broadcast Media, Gale, a part of Cengage Learning
27500 Drake Rd. Farmington Hills, MI 48331-3535, USA
Tel.: (248) 699-4253, ext. 1326; 1-800-347-GALE Fax: (248) 699-8075
http://www.gale.com

Indexing and Reviews: Mathematical Reviews (USA), Zentralblatt Math (Germany), Index EuroPub (UK), Referativnyi Zhurnal (Russia), Mathematika (Russia), EBSCO (USA), Google Scholar, Baidu Scholar, Directory of Open Access (DoAJ), International Scientific Indexing (ISI, impact factor 2.012), Institute for Scientific Information (PA, USA), Library of Congress Subject Headings (USA), CNKI(China).

Subscription A subscription can be ordered by an email directly to

Linfan Mao

The Editor-in-Chief of *International Journal of Mathematical Combinatorics*
Chinese Academy of Mathematics and System Science Beijing, 100190, P.R.China, and also the President of Academy of Mathematical Combinatorics & Applications (AMCA), Colorado, USA

Email: maolinfan@163.com

Price: US$48.00

Famous Words:

Basically, I'm not interested in doing research and I never have been. I'm interested in understanding, which is quite a different thing. And often to understand something you have to work it out yourself because no one else has done it.

By David Blackwell, an American statistician and mathematician.

International J.Math. Combin. Vol.2(2023), 1-17

Some Common Fixed Point Theorems for Contractive Type Conditions in Complex Valued S-Metric Spaces

G. S. Saluja

(H.N. 3/1005, Geeta Nagar, Raipur, Raipur - 492001 (C.G.), India)

E-mail: saluja1963@gmail.com

Abstract: The aim of this paper is to establish some common fixed point theorems for four self-mappings satisfying $(E.A)$ and weak compatible properties and contractive type condition involving rational expressions in the setting of complex valued $\mathcal{S}$-metric spaces. The results presented in this paper extend and generalize several results from the existing literature.

Key Words: Common fixed point, $(E.A)$-property, weak compatibility, contractive type condition, complex valued $\mathcal{S}$-metric space.

AMS(2010): 47H10, 54H25.

§1. Introduction

In the development of nonlinear analysis, fixed point theory plays a very important role. Banach contraction principle [6] was the starting point for many researchers during the last decades in the field of nonlinear analysis. The Banach contraction principle with rational expressions have been expanded and some fixed point and common fixed point theorems have been obtained in [12], [13], [14], [15].

In the existing literature there are a great number of generalizations of the Banach contraction principle (see [3, 4] and others). Some generalization of the notion of a metric space have been proposed by some authors, such as, partial metric spaces, probabilistic metric spaces, fuzzy metric spaces, D-metric spaces, cone metric spaces, b-metric spaces and cone b-metric spaces (see, [7, 9, 10, 11, 18, 19, 20, 21, 22, 23, 24, 26, 27, 32, 46]).

Also, as an extension of the fixed point problem there are many results in finding a common fixed point for two self mappings on different types of metric spaces; see, for example, [2], [41], [34], [35], [38], [44] and the references therein. But all of these results were found in real valued metric spaces.

In 2011, Azam *et al.* [5] introduced the notion of complex valued metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. The results proved by Azam *et al.* [5] and Bhatt *et al.* [8] via rational inequality in a complex valued metric space as a contractive condition. Complex valued

[1]Received March 8, 2023, Accepted June 2, 2023.

metric space is very useful in many branches of mathematics, including algebraic geometry, number theory, applied mathematics, applied physics, mechanical engineering, thermodynamics and electrical engineering. After the establishment of complex valued metric spaces, Rouzkardand *et al.* [33] established some common fixed point theorems satisfying certain rational expressions in these spaces which generalize the result of [5]. In 2012, Sintanuvarat and Kumam [42] extend and improve the results of [5] by replacing the constant of contractive conditions to some control functions. Verma and Pathak in [44] introduced the notion of $(E.A)$-property in complex valued metric space and proved some common fixed point results for two pairs of weakly compatible mappings satisfying a "max" type contractive condition. After that many authors have contributed different concepts in this space (see, for example, [29], [36], [37], [42], [39] and many others).

Recently, Mlaiki [28] *(Adv. Fixed Point Theory 4(4) (2014), 509-524)* introduced the concept of complex valued $\mathcal{S}$-metric spaces and investigate the existence and uniqueness of a common fixed point of two self-mappings in such space via various contractive conditions. After Mlaiki's results many authors have established a lot of results in complex valued S-metric space under various contractive conditions (see, for example, [31], [45] and many others).

In this paper, we prove some common fixed point theorems for contractive type conditions involving rational expressions in the framework of complex valued S-metric spaces. Our results extend, generalize and enrich several results from the existing literature.

§2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$;
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$;
(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$;
(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of $(i), (ii)$ and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \lesssim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \precsim z_2,\ z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

In 2014, the following definition was introduced by Mlaiki in [28].

Definition 2.1([28]) *Let X be a nonempty set and $\mathbb{C}$ be the set of all complex numbers. A complex valued $\mathcal{S}$-metric space on X is a function $\mathcal{S}\colon X^3 \to \mathbb{C}$ that satisfies the following conditions, for all $x, y, z, t \in X$:*

($\mathcal{CS}1$) $0 \precsim \mathcal{S}(x,y,z)$;
($\mathcal{CS}2$) $\mathcal{S}(x,y,z) = 0$ *if and only if* $x = y = z$;
($\mathcal{CS}3$) $\mathcal{S}(x,y,z) \precsim \mathcal{S}(x,x,t) + \mathcal{S}(y,y,t) + \mathcal{S}(z,z,t)$.

Then, $\mathcal{S}$ is called a complex valued $\mathcal{S}$-metric on X and the pair $(X,\mathcal{S})$ is called a complex valued $\mathcal{S}$-metric space.

Example 2.2([28]) Let $X = \mathbb{C}$ be the set of complex numbers. Define a mapping $\mathcal{S}\colon \mathbb{C}^3 \to \mathbb{C}$ by $\mathcal{S}(z_1,z_2,z_3) = |\max\{Re(z_1), Re(z_2)\} - Re(z_2)| + i|\max\{Im(z_1), Im(z_2)\} - Im(z_2)|$. Then it is not difficult to verify that $(\mathbb{C},\mathcal{S})$ is a complex valued $\mathcal{S}$-metric space.

Definition 2.3([28]) *If $(X,\mathcal{S})$ is called a complex valued $\mathcal{S}$-metric space, then,*

($\mathbf{\Gamma_1}$) *A sequence $\{u_n\}$ in X converges to u if and only if for every $\varepsilon \in \mathbb{C}$ with $0 \prec \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $\mathcal{S}(u_n,u_n,u) \prec \varepsilon$ and we denote this by $u_n \to u$ or $\lim_{n\to\infty} u_n = u$;*

($\mathbf{\Gamma_2}$) *A sequence $\{u_n\}$ in X is called a Cauchy sequence if for every $\varepsilon \in \mathbb{C}$ with $0 \prec \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $\mathcal{S}(u_n,u_n,u_m) \prec \varepsilon$;*

($\mathbf{\Gamma_3}$) *An $\mathcal{S}$-metric space $(X,\mathcal{S})$ is said to be complete if every Cauchy sequence is convergent.*

Definition 2.4 *Let X be a non-empty set and let $\mathcal{R}, h\colon X \to X$ be two self mappings of X. Then a point $v \in X$ is called a*

($\mathbf{\Lambda_1}$) *fixed point of operator $\mathcal{R}$ if $\mathcal{R}(v) = v$;*
($\mathbf{\Lambda_2}$) *common fixed point of $\mathcal{R}$ and h if $\mathcal{R}(v) = h(v) = v$.*

Definition 2.5([1]) *Let $\mathcal{P}$ and $\mathcal{Q}$ be single valued self-mappings on a set X. If $u = \mathcal{P}z = \mathcal{Q}z$ for some $z \in X$, then z is called a coincidence point point of $\mathcal{P}$ and $\mathcal{Q}$, and u is called a point of coincidence of $\mathcal{P}$ and $\mathcal{Q}$.*

Definition 2.6([16]) *Let $\mathcal{P}$ and $\mathcal{Q}$ be single valued self-mappings on a set X. Mappings $\mathcal{P}$ and $\mathcal{Q}$ are said to be commuting if $\mathcal{P}\mathcal{Q}v = \mathcal{Q}\mathcal{P}v$ for all $v \in X$.*

Definition 2.7([17]) *Let $\mathcal{P}$ and $\mathcal{Q}$ be single valued self-mappings on a set X. Mappings $\mathcal{P}$ and $\mathcal{Q}$ are said to be weakly compatible if they commute at their coincidence points, i.e., if $\mathcal{P}u = \mathcal{Q}u$ for some $u \in X$ implies $\mathcal{P}\mathcal{Q}u = \mathcal{Q}\mathcal{P}u$.*

Definition 2.8([44]) *Let (X,d) be a complex valued metric space and let $\mathcal{R}, \mathcal{Q}\colon X \to X$ be two self mappings of X. The pair $(\mathcal{R},\mathcal{Q})$ is said to satisfy $(E.A)$-property if there exists a sequence $\{r_n\}$ in X such that $\lim_{n\to\infty} \mathcal{R}r_n = \lim_{n\to\infty} \mathcal{Q}r_n = d$ for some $d \in X$.*

Note that weakly compatibility and $(E.A)$-property are independent of each other (see [30] for details).

Example 2.9 Let $X = \mathbb{C}$ and let $\mathcal{R}, \mathcal{Q}\colon X \to X$ be defined by $\mathcal{R}(z) = 4z - 2i$ and $\mathcal{Q}(z) = z + i$ for all $z \in X$. Let $\{z_n\} = \{i + \frac{1}{n}\}_{n\geq 1}$ be the sequence in X. Then

$$\lim_{n\to\infty} \mathcal{R}z_n = \lim_{n\to\infty} \left(4i + \frac{4}{n} - 2i\right) = 2i,$$

and

$$\lim_{n\to\infty} \mathcal{Q}z_n = \lim_{n\to\infty} \Big(i + \frac{1}{n} + i\Big) = 2i.$$

Thus there exists a sequence $\{z_n\}$ in X such that $\lim_{n\to\infty} \mathcal{R}z_n = \lim_{n\to\infty} \mathcal{Q}z_n = 2i \in X$. Hence $\mathcal{R}$ and $\mathcal{Q}$ satisfy $(E.A)$-property.

Liu *et al.* [25] introduced common $(E.A)$-property which is an extension of $(E.A)$-property were define common $(E.A)$-property in the complex valued metric space as follows.

Definition 2.10([25]) *Let (X, d) be a complex valued metric space and let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{T} \colon X \to X$ be four self mappings of X. The pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ satisfy the common $(E.A)$-property if there exist two sequences $\{u_n\}$ and $\{v_n\}$ in X such that*

$$\lim_{n\to\infty} \mathcal{P}u_n = \lim_{n\to\infty} \mathcal{R}u_n = \lim_{n\to\infty} \mathcal{Q}v_n = \lim_{n\to\infty} \mathcal{T}v_n = z \in X.$$

Example 2.11 Let $X = \mathbb{C}$ and let d be a complex valued metric and let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{T} \colon X \to X$ be four self-maps defined by $\mathcal{P}(z) = 3+iz$, $\mathcal{Q}(z) = -i-3z^2$, $\mathcal{R}(z) = -i-3z$ and $\mathcal{T}(z) = 3+(z-2i)$ for all $z \in X$. Let $\{x_n\} = \{-1 + \frac{1}{n}\}_{n\geq 1}$ and $\{y_n\} = \{\frac{1}{n} + i\}_{n\geq 1}$ be two sequences in X. Then

$$\lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \Big(3 - i + \frac{i}{n}\Big) = 3 - i,$$

$$\lim_{n\to\infty} \mathcal{R}x_n = \lim_{n\to\infty} \Big(- i + 3 - \frac{3}{n}\Big) = 3 - i,$$

$$\lim_{n\to\infty} \mathcal{Q}y_n = \lim_{n\to\infty} \Big(- i - 3(\frac{1}{n} + i)^2\Big) = 3 - i,$$

and

$$\lim_{n\to\infty} \mathcal{T}y_n = \lim_{n\to\infty} \Big(3 + (\frac{1}{n} + i - 2i)\Big) = 3 - i.$$

Thus there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{R}x_n = \lim_{n\to\infty} \mathcal{Q}y_n = \lim_{n\to\infty} \mathcal{T}y_n = 3 - i \in X.$$

Hence the pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ satisfy common $(E.A)$-property.

Now, we redefine the common $(E.A)$-property in the setting of complex valued S-metric space as follows.

Definition 2.12 *Let $(X, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{T} \colon X \to X$ be four self mappings of X. The pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ are said to satisfy the common $(E.A)$-property if there exist two sequences $\{p_n\}$ and $\{q_n\}$ in X such that*

$$\lim_{n\to\infty} \mathcal{P}p_n = \lim_{n\to\infty} \mathcal{R}p_n = \lim_{n\to\infty} \mathcal{Q}q_n = \lim_{n\to\infty} \mathcal{T}q_n = t \in X.$$

Example 2.13 Let $X = \mathbb{C}$ and let $\mathcal{S} \colon \mathbb{C}^3 \to \mathbb{C}$ be defined by $\mathcal{S}(z_1, z_2, z_3) = |\max\{Re(z_1), Re(z_2)\} - Re(z_2)| + i|\max\{Im(z_1), Im(z_2)\} - Im(z_2)|$. Then $(\mathbb{C}, \mathcal{S})$ is a complex valued $\mathcal{S}$-metric space.

Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{T}\colon X \to X$ be four self-maps defined by $\mathcal{P}(z) = z + i$, $\mathcal{Q}(z) = z + (1+2i)$, $\mathcal{R}(z) = 3i - z$ and $\mathcal{T}(z) = -z + (2i-1)$ for all $z \in X$. Let $\{p_n\} = \{i + \frac{1}{n}\}_{n\geq 1}$ and $\{q_n\} = \{-1 + \frac{i}{n}\}_{n\geq 1}$ be two sequences in X and that $\mathcal{P}p_n = p_n + i = 2i + \frac{1}{n}$ and $\mathcal{R}p_n = 3i - p_n = 2i - \frac{1}{n}$ for all $n \in \mathbb{N}$. This implies that

$$\mathcal{S}(\mathcal{P}p_n, \mathcal{P}p_n, 0) = \mathcal{S}\Big(2i + \frac{1}{n}, 2i + \frac{1}{n}, 0\Big) \to 0 \text{ as } n \to \infty.$$

This shows that $\mathcal{P}p_n \to 0$ as $n \to \infty$ and by similar way, we have

$$\mathcal{S}(\mathcal{R}p_n, \mathcal{R}p_n, 0) = \mathcal{S}\Big(2i - \frac{1}{n}, 2i - \frac{1}{n}, 0\Big) \to 0 \text{ as } n \to \infty.$$

This shows that $\mathcal{R}p_n \to 0$ as $n \to \infty$. Thus the pair $(\mathcal{P}, \mathcal{R})$ satisfies $(E.A)$-property.

Similarly, note that $\mathcal{Q}q_n = q_n + (1+2i) = 2i + \frac{i}{n}$ and $\mathcal{T}q_n = -q_n + (2i-1) = 2i - \frac{i}{n}$ for all $n \in \mathbb{N}$. This implies that

$$\mathcal{S}(\mathcal{Q}q_n, \mathcal{Q}q_n, 0) = \mathcal{S}\Big(2i + \frac{i}{n}, 2i + \frac{i}{n}, 0\Big) \to 0 \text{ as } n \to \infty.$$

This shows that $\mathcal{Q}q_n \to 0$ as $n \to \infty$ and by similar way, we have

$$\mathcal{S}(\mathcal{T}q_n, \mathcal{T}q_n, 0) = \mathcal{S}\Big(-2i - \frac{i}{n}, -2i - \frac{i}{n}, 0\Big) \to 0 \text{ as } n \to \infty.$$

This shows that $\mathcal{T}q_n \to 0$ as $n \to \infty$. Thus the pair $(\mathcal{Q}, \mathcal{T})$ satisfies $(E.A)$-property. Thus there exist two sequences $\{p_n\}$ and $\{q_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{P}p_n = \lim_{n\to\infty} \mathcal{R}p_n = \lim_{n\to\infty} \mathcal{Q}q_n = \lim_{n\to\infty} \mathcal{T}q_n = 0 \in X.$$

Hence the pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ satisfy common $(E.A)$-property.

Lemma 2.14([28]) *Let $(X, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\{u_n\}$ be a sequence in X. Then $\{u_n\}$ converges to u if and only if* $\lim_{n\to\infty} |\mathcal{S}(u_n, u_n, u)| = 0$ *or* $|\mathcal{S}(u_n, u_n, u)| \to 0$ *as $n \to \infty$.*

Lemma 2.15([28]) *Let $(X, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\{u_n\}$ be a sequence in X. Then $\{u_n\}$ is a Cauchy sequence if and only if* $\lim_{n,m\to\infty} |\mathcal{S}(u_n, u_n, u_{n+m})| = 0$ *or* $|\mathcal{S}(u_n, u_n, u_{n+m})| \to 0$ *as $n, m \to \infty$.*

Lemma 2.16([28]) *Let $(X, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space, then $\mathcal{S}(x, x, y) = \mathcal{S}(y, y, x)$ for all $x, y \in X$.*

§3. **Common Fixed Point Theorems**

In this section, we shall prove some common fixed point theorems under contractive type conditions involving rational expression and satisfies $(E.A)$ property in the framework of complex valued $\mathcal{S}$-metric spaces.

Theorem 3.1 *Let $(X, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{T}\colon X \to X$ be four self-mappings of X satisfying the following conditions:*

(i) For all $u, v \in X$,

$$\begin{aligned}\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) \quad \precsim \quad r \max\Big\{&\mathcal{S}(\mathcal{Q}u, \mathcal{Q}u, \mathcal{T}v), \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u), \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v), \\ &\frac{1}{2}[\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v) + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{Q}u)], \\ &\frac{\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u)[1 + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]}\Big\}, \end{aligned} \tag{3.1}$$

where $r \in [0, 1)$ is a constant;

(ii) The pairs $(\mathcal{A}, \mathcal{Q})$ and $(\mathcal{B}, \mathcal{T})$ are weakly compatible;

(iii) One of the pairs $(\mathcal{A}, \mathcal{Q})$ and $(\mathcal{B}, \mathcal{T})$ satisfy $(E.A)$ property;

(iv) $\mathcal{A}(X) \subseteq \mathcal{T}(X)$ and $\mathcal{B}(X) \subseteq \mathcal{Q}(X)$.

If the range of one of the mappings $\mathcal{Q}(X)$ or $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$ have a unique common fixed point in X.

Proof First, we suppose that the pair $(\mathcal{A}, \mathcal{Q})$ satisfies $(E.A)$ property. Then by Definition 2.8, there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{Q}u_n = t$ for some $t \in X$. Further, since $\mathcal{A}(X) \subseteq \mathcal{T}(X)$, there exists a sequence $\{v_n\}$ in X such that $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{T}v_n$. Hence $\lim_{n\to\infty} \mathcal{T}v_n = t$. We claim that $\lim_{n\to\infty} \mathcal{B}v_n = t$. If not, then putting $u = u_n$, $v = v_n$ in inequality (3.1), using Lemma 2.16 and $(\mathcal{CS}2)$, we have

$$\begin{aligned}\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n) \quad &\precsim \quad r \max\Big\{\mathcal{S}(\mathcal{Q}u_n, \mathcal{Q}u_n, \mathcal{T}v_n), \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n), \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{T}v_n), \\ &\qquad \frac{1}{2}[\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{T}v_n) + \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{Q}u_n)], \\ &\qquad \frac{\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)[1 + \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{T}v_n)]}{[1 + \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)]}\Big\} \\ &= \quad r \max\Big\{\mathcal{S}(\mathcal{Q}u_n, \mathcal{Q}u_n, \mathcal{A}u_n), \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n), \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n), \\ &\qquad \frac{1}{2}[\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{A}u_n) + \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)], \\ &\qquad \frac{\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)[1 + \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)]}{[1 + \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)]}\Big\} \\ &= \quad r \max\Big\{0, \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n), \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n), \\ &\qquad \frac{1}{2}[\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)], \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)\Big\} \\ &\precsim \quad r\, \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n). \end{aligned} \tag{3.2}$$

Thus

$$|\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)| \leq r\, |\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)|,$$

which is a contradiction since $r \in [0,1)$. Letting $n \to \infty$, we have

$$\lim_{n\to\infty} |\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)| \leq r\,.0 = 0,$$

which is a contradiction by condition $(\mathcal{CS}1)$. Thus, we get $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{B}v_n = t$.

Now, first we assume that $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $t = \mathcal{T}p$ for some $p \in X$. Subsequently, we have

$$\lim_{n\to\infty} \mathcal{B}v_n = \lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{Q}u_n = \lim_{n\to\infty} \mathcal{T}v_n = \mathcal{T}p = t.$$

We claim that $\mathcal{B}p = \mathcal{T}p$. For this, putting $u = u_n$ and $v = p$ in inequality (3.1), using Lemma 2.16 and $(\mathcal{CS}2)$, we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}p) \quad \precsim \quad r \max\Big\{ & \mathcal{S}(\mathcal{Q}u_n, \mathcal{Q}u_n, \mathcal{T}p), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{A}u_n), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p), \\ & \frac{1}{2}[\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{T}p) + \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{Q}u_n)], \\ & \frac{\mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{A}u_n)[1 + \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p)]}{[1 + \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}p)]} \Big\}. \end{aligned} \tag{3.3}$$

Letting $n \to \infty$ in (3.3), using Lemma 2.16 and $(\mathcal{CS}2)$, we get

$$\begin{aligned} \mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p) \quad & \precsim \quad r \max\Big\{ \mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{T}p), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p), \\ & \qquad \frac{1}{2}[\mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{T}p) + \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p)], \\ & \qquad \frac{\mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p)[1 + \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p)]}{[1 + \mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p)]} \Big\} \\ & = \quad r \max\Big\{ 0, \mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p), \\ & \qquad \frac{1}{2}[\mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p)], \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p) \Big\} \precsim r\, \mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p). \end{aligned} \tag{3.4}$$

Thus, $|\mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p)| \leq r\,|\mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p)|$, which is a contradiction since $r \in [0,1)$. Hence, we have $\mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p) = 0$, that is, $\mathcal{T}p = \mathcal{B}p = t$. Hence p is a coincidence point of the mappings $\mathcal{B}$ and $\mathcal{T}$, that is, the pair $(\mathcal{B}, \mathcal{T})$. Now, the weak compatibility of the pair $(\mathcal{B}, \mathcal{T})$ implies that $\mathcal{B}\mathcal{T}p = \mathcal{T}\mathcal{B}p$ or $\mathcal{B}t = \mathcal{T}t$.

On the other hand, since $\mathcal{B}(X) \subseteq \mathcal{Q}(X)$, there exists $\nu \in X$ such that $\mathcal{B}p = \mathcal{Q}\nu$. Thus $\mathcal{T}p = \mathcal{B}p = \mathcal{Q}\nu = t$. Let us show that ν is a coincidence point of the pair $(\mathcal{A}, \mathcal{Q})$, that is, $\mathcal{A}\nu = \mathcal{Q}\nu = t$. If not, then putting $u = \nu$ and $v = p$ in inequality (3.1), using Lemma 2.16 and $(\mathcal{CS}2)$, we get

$$\begin{aligned} \mathcal{S}(\mathcal{A}\nu, \mathcal{A}\nu, \mathcal{B}p) \quad \precsim \quad r \max\Big\{ & \mathcal{S}(\mathcal{Q}\nu, \mathcal{Q}\nu, \mathcal{T}p), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{A}\nu), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p), \\ & \frac{1}{2}[\mathcal{S}(\mathcal{A}\nu, \mathcal{A}\nu, \mathcal{T}p) + \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{Q}\nu)], \\ & \frac{\mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{A}\nu)[1 + \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p)]}{[1 + \mathcal{S}(\mathcal{A}\nu, \mathcal{A}\nu, \mathcal{B}p)]} \Big\} \end{aligned}$$

$$
\begin{aligned}
&= \quad r \max\Big\{0, \mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p), 0, \frac{\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p)}{2},\\
&\qquad\qquad \mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p)\Big\}\\
&\precsim\ r\,\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p). \qquad (3.5)
\end{aligned}
$$

Thus, $|\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p)| \le r\,|\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p)|$, which is a contradiction since $r \in [0,1)$. Hence, we have $\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p) = 0$, that is, $\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{Q}\nu) = 0$ and hence $\mathcal{A}\nu = \mathcal{Q}\nu = t$. Thus ν is a coincidence point of the mappings $\mathcal{A}$ and $\mathcal{Q}$, that is, the pair $(\mathcal{A},\mathcal{Q})$. Further, the weak compatibility of the pair $(\mathcal{A},\mathcal{Q})$ implies that $\mathcal{A}\mathcal{Q}\nu = \mathcal{Q}\mathcal{A}\nu$ or $\mathcal{A}t = \mathcal{Q}t$. Hence t is a common coincidence point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$.

Now to show that t is a common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$. For this, we put $u = \nu$ and $v = t$ in (3.1), using Lemma 2.16 and $(\mathcal{CS}2)$, we get

$$
\begin{aligned}
\mathcal{S}(t,t,\mathcal{B}t) &= \mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}t)\\
&\precsim r \max\Big\{\mathcal{S}(\mathcal{Q}\nu,\mathcal{Q}\nu,\mathcal{T}t), \mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{A}\nu), \mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{T}t),\\
&\qquad \frac{1}{2}[\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{T}t) + \mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{Q}\nu)],\\
&\qquad \frac{\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{A}\nu)[1+\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{T}t)]}{[1+\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}t)]}\Big\}\\
&= r \max\Big\{\mathcal{S}(t,t,\mathcal{B}t), \mathcal{S}(\mathcal{B}t,\mathcal{B}t,t), \mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{B}t),\\
&\qquad \frac{1}{2}[\mathcal{S}(t,t,\mathcal{B}t) + \mathcal{S}(\mathcal{B}t,\mathcal{B}t,t)],\\
&\qquad \frac{\mathcal{S}(\mathcal{B}t,\mathcal{B}t,t)[1+\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{B}t)]}{[1+\mathcal{S}(t,t,\mathcal{B}t)]}\Big\}\\
&= r \max\Big\{\mathcal{S}(t,t,\mathcal{B}t), \mathcal{S}(t,t,\mathcal{B}t), 0, \mathcal{S}(t,t,\mathcal{B}t),\\
&\qquad \frac{\mathcal{S}(t,t,\mathcal{B}t)}{[1+\mathcal{S}(t,t,\mathcal{B}t)]}\Big\}\\
&\precsim r \max\Big\{\mathcal{S}(t,t,\mathcal{B}t), \mathcal{S}(t,t,\mathcal{B}t), 0, \mathcal{S}(t,t,\mathcal{B}t), \mathcal{S}(t,t,\mathcal{B}t)\Big\}\\
&\precsim r\,\mathcal{S}(t,t,\mathcal{B}t). \qquad (3.6)
\end{aligned}
$$

Thus, $|\mathcal{S}(t,t,\mathcal{B}t)| \le r\,|\mathcal{S}(t,t,\mathcal{B}t)|$, which is a contradiction since $r \in [0,1)$. Hence, we have $\mathcal{S}(t,t,\mathcal{B}t) = 0$, that is, $\mathcal{B}t = t$. Consequently, $\mathcal{A}t = \mathcal{B}t = \mathcal{Q}t = \mathcal{T}t = t$. This shows that t is a common fixed point of the mappings $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$.

Similar argument arises if we assume that $\mathcal{Q}(X)$ is a complete subspace of $(X,\mathcal{S})$.

Similarly, the property $(E.A)$ of the pair $(\mathcal{B},\mathcal{T})$ will give the similar result.

Now, we show the uniqueness of the common fixed point. For this, let us assume that t' be another common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$ with $t' \ne t$. From inequality (3.1), using Lemma 2.16 and $(\mathcal{CS}2)$ for $u = t'$ and $v = t$, we have

$$
\mathcal{S}(t',t',t) = \mathcal{S}(\mathcal{A}t',\mathcal{A}t',\mathcal{B}t)
$$

$$\begin{aligned}
&\precsim\ r\ \max\Big\{\mathcal{S}(\mathcal{Q}t',\mathcal{Q}t',\mathcal{T}t),\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{A}t'),\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{T}t),\\
&\qquad\qquad \frac{1}{2}[\mathcal{S}(\mathcal{A}t',\mathcal{A}t',\mathcal{T}t)+\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{Q}t')],\\
&\qquad\qquad \frac{\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{A}t')[1+\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{T}t)]}{[1+\mathcal{S}(\mathcal{A}t',\mathcal{A}t',\mathcal{B}t)]}\Big\}\\
&=\ r\ \max\Big\{\mathcal{S}(t',t',t),\mathcal{S}(t,t,t'),\mathcal{S}(t,t,t),\\
&\qquad\qquad \frac{1}{2}[\mathcal{S}(t',t',t)+\mathcal{S}(t,t,t')],\\
&\qquad\qquad \frac{\mathcal{S}(t,t,t')[1+\mathcal{S}(t,t,t)]}{[1+\mathcal{S}(t',t',t)]}\Big\}\\
&=\ r\ \max\Big\{\mathcal{S}(t',t',t),\mathcal{S}(t',t',t),0,\mathcal{S}(t',t',t),\\
&\qquad\qquad \frac{\mathcal{S}(t',t',t)}{[1+\mathcal{S}(t',t',t)]}\Big\}\\
&\precsim\ r\ \max\Big\{\mathcal{S}(t',t',t),\mathcal{S}(t',t',t),0,\mathcal{S}(t',t',t),\\
&\qquad\qquad \mathcal{S}(t',t',t)\Big\}\\
&\precsim\ r\,\mathcal{S}(t',t',t). \qquad (3.7)
\end{aligned}$$

Thus

$$|\mathcal{S}(t',t',t)|\le r\,|\mathcal{S}(t',t',t)|,$$

which is a contradiction since $r\in[0,1)$. Hence, we have

$$\mathcal{S}(t',t',t)=0,$$

that is, $t'=t$. Hence $\mathcal{A}t=\mathcal{B}t=\mathcal{Q}t=\mathcal{T}t=t$ and t is the unique common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$. This completes the proof. □

If we take $\mathcal{A}=\mathcal{B}$ and $\mathcal{Q}=\mathcal{T}$ in Theorem 3.1, then we have the following result.

Corollary 3.2 *Let $(X,\mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\mathcal{A},\mathcal{Q}\colon X\to X$ be two self-mappings of X satisfying the following conditions:*

(i) *For all $u,v\in X$,*

$$\begin{aligned}
\mathcal{S}(\mathcal{A}u,\mathcal{A}u,\mathcal{A}v)\ &\precsim\ r\ \max\Big\{\mathcal{S}(\mathcal{Q}u,\mathcal{Q}u,\mathcal{Q}v),\mathcal{S}(\mathcal{A}v,\mathcal{A}v,\mathcal{A}u),\mathcal{S}(\mathcal{A}v,\mathcal{A}v,\mathcal{Q}v),\\
&\qquad \frac{1}{2}[\mathcal{S}(\mathcal{A}u,\mathcal{A}u,\mathcal{Q}v)+\mathcal{S}(\mathcal{A}v,\mathcal{A}v,\mathcal{Q}u)],\\
&\qquad \frac{\mathcal{S}(\mathcal{A}v,\mathcal{A}v,\mathcal{A}u)[1+\mathcal{S}(\mathcal{A}v,\mathcal{A}v,\mathcal{Q}v)]}{[1+\mathcal{S}(\mathcal{A}u,\mathcal{A}u,\mathcal{A}v)]}\Big\}, \qquad (3.8)
\end{aligned}$$

where $r\in[0,1)$ is a constant;

(ii) *The pairs $(\mathcal{A},\mathcal{Q})$ is weakly compatible;*

(iii) *The pair $(\mathcal{A},\mathcal{Q})$ satisfies $(E.A)$ property;*

(iv) $\mathcal{A}(X)\subseteq\mathcal{Q}(X)$.

If the range of the mapping $\mathcal{Q}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\mathcal{A}$ and $\mathcal{Q}$ have a unique common fixed point in X.

Theorem 3.3 *Let $(X, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{T}$: $X \to X$ be four self-mappings of X satisfying the following conditions:*

(i) For all $u, v \in X$,

$$\begin{aligned}\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) &\precsim n_1\,\mathcal{S}(\mathcal{Q}u, \mathcal{Q}u, \mathcal{T}v) + n_2\,\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u) + n_3\,\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)\\ &\quad + n_4\,\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v)\frac{[1+\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]}\\ &\quad + n_5\,\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u)\frac{[1+\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]}, \end{aligned} \tag{3.9}$$

where $n_1, n_2, n_3, n_4, n_5 > 0$ are nonnegative reals with $n_1 + n_2 + n_3 + n_4 + n_5 < 1$;

(ii) The pairs $(\mathcal{A}, \mathcal{Q})$ and $(\mathcal{B}, \mathcal{T})$ are weakly compatible;

(iii) One of the pairs $(\mathcal{A}, \mathcal{Q})$ and $(\mathcal{B}, \mathcal{T})$ satisfy $(E.A)$ property;

(iv) $\mathcal{A}(X) \subseteq \mathcal{T}(X)$ and $\mathcal{B}(X) \subseteq \mathcal{Q}(X)$.

If the range of one of the mappings $\mathcal{Q}(X)$ or $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$ have a unique common fixed point in X.

Proof First, we suppose that the pair $(\mathcal{A}, \mathcal{Q})$ satisfies $(E.A)$ property. Then by Definition 2.8, there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{Q}u_n = t$ for some $t \in X$. Further, since $\mathcal{A}(X) \subseteq \mathcal{T}(X)$, there exists a sequence $\{v_n\}$ in X such that $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{T}v_n$. Hence $\lim_{n\to\infty} \mathcal{T}v_n = t$. We claim that $\lim_{n\to\infty} \mathcal{B}v_n = t$. If not, then putting $u = u_n$ and $v = v_n$ in inequality (3.9), using Lemma 2.16 and $(\mathcal{CS}2)$, we have

$$\begin{aligned}\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n) &\precsim n_1\,\mathcal{S}(\mathcal{Q}u_n, \mathcal{Q}u_n, \mathcal{T}v_n) + n_2\,\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)\\ &\quad + n_3\,\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{T}v_n)\\ &\quad + n_4\,\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{T}v_n)\frac{[1+\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{T}v_n)]}{[1+\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)]}\\ &\quad + n_5\,\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)\frac{[1+\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{T}v_n)]}{[1+\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)]}\\ &= n_1\,\mathcal{S}(\mathcal{Q}u_n, \mathcal{Q}u_n, \mathcal{Q}u_n) + n_2\,\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)\\ &\quad + n_3\,\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)\\ &\quad + n_4\,\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{A}u_n)\frac{[1+\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)]}{[1+\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)]}\\ &\quad + n_5\,\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)\frac{[1+\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)]}{[1+\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)]}\\ &= n_1\,.0 + n_2\,\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n) + n_3\,\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)\\ &\quad + n_4\,.0 + n_5\,\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)\\ &= (n_2 + n_3 + n_5)\,\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)\\ &\precsim (n_1 + n_2 + n_3 + n_4 + n_5)\,\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)\\ &= m\,\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)\end{aligned}$$

where $m = n_1 + n_2 + n_3 + n_4 + n_5 < 1$. Thus

$$|\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)| \leq m\,|\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)|,$$

which is a contradiction since $m \in [0, 1)$. Letting $n \to \infty$, we have

$$\lim_{n\to\infty} |\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)| \leq m\,.0 = 0,$$

which is a contradiction by condition $(\mathcal{CS}1)$. Thus, we get $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{B}v_n = t$.

Now, first we assume that $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $t = \mathcal{T}p$ for some $p \in X$. Subsequently, we have

$$\lim_{n\to\infty} \mathcal{B}v_n = \lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{Q}u_n = \lim_{n\to\infty} \mathcal{T}v_n = \mathcal{T}p = t.$$

Rest of the proof follows from Theorem 3.1. This completes the proof. □

Theorem 3.4 *Let $(X, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{T}$: $X \to X$ be four self-mappings of X satisfying the following conditions:*

(i) *For all $u, v \in X$,*

$$\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) \precsim \mathcal{R}_1\,\mathcal{D}^{\mathcal{S}}_{\mathcal{C}_1}(u, u, v) + \mathcal{R}_2\,\mathcal{D}^{\mathcal{S}}_{\mathcal{C}_2}(u, u, v), \tag{3.10}$$

where $\mathcal{R}_1, \mathcal{R}_2 > 0$ are nonnegative reals with $\mathcal{R}_1 + \mathcal{R}_2 < 1$ and

$$\mathcal{D}^{\mathcal{S}}_{\mathcal{C}_1}(u, u, v) \quad = \quad \max\Big\{\mathcal{S}(\mathcal{Q}u, \mathcal{Q}u, \mathcal{T}v), \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u), \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)\Big\},$$

$$\begin{aligned}\mathcal{D}^{\mathcal{S}}_{\mathcal{C}_2}(u, u, v) \quad = \quad \max\Big\{&\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v)\frac{[1 + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]},\\ &\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u)\frac{[1 + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]}\Big\},\end{aligned}$$

(ii) *The pairs $(\mathcal{A}, \mathcal{Q})$ and $(\mathcal{B}, \mathcal{T})$ are weakly compatible;*

(iii) *One of the pairs $(\mathcal{A}, \mathcal{Q})$ and $(\mathcal{B}, \mathcal{T})$ satisfy $(E.A)$ property;*

(iv) *$\mathcal{A}(X) \subseteq \mathcal{T}(X)$ and $\mathcal{B}(X) \subseteq \mathcal{Q}(X)$.*

If the range of one of the mappings $\mathcal{Q}(X)$ or $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$ have a unique common fixed point in X.

Proof First, we suppose that the pair $(\mathcal{A}, \mathcal{Q})$ satisfies $(E.A)$ property. Then by Definition 2.8, there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{Q}u_n = t$ for some $t \in X$. Further, since $\mathcal{A}(X) \subseteq \mathcal{T}(X)$, there exists a sequence $\{v_n\}$ in X such that $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{T}v_n$. Hence $\lim_{n\to\infty} \mathcal{T}v_n = t$. We claim that $\lim_{n\to\infty} \mathcal{B}v_n = t$. If not, then putting $u = u_n$ and $v = v_n$ in inequality (3.10), using Lemma 2.16 and $(\mathcal{CS}2)$, we have

$$\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n) \precsim \mathcal{R}_1\,\mathcal{D}^{\mathcal{S}}_{\mathcal{C}_1}(u_n, u_n, v_n) + \mathcal{R}_2\,\mathcal{D}^{\mathcal{S}}_{\mathcal{C}_2}(u_n, u_n, v_n), \tag{3.11}$$

where

$$\begin{aligned}
\mathcal{D}^{\mathcal{S}}_{\mathcal{C}_1}(u_n,u_n,v_n) &= \max\Big\{\mathcal{S}(\mathcal{Q}u_n,\mathcal{Q}u_n,\mathcal{T}v_n),\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n),\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{T}v_n)\Big\}\\
&= \max\Big\{\mathcal{S}(\mathcal{Q}u_n,\mathcal{Q}u_n,\mathcal{Q}u_n),\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n),\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n)\Big\}\\
&= \max\Big\{0,\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n),\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)\Big\}\\
&= \mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n), \qquad (3.12)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}^{\mathcal{S}}_{\mathcal{C}_2}(u_n,u_n,v_n) &= \max\Big\{\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{T}v_n)\frac{[1+\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{T}v_n)]}{[1+\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)]},\\
&\qquad \mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n)\frac{[1+\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{T}v_n)]}{[1+\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)]}\Big\}\\
&= \max\Big\{\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{A}u_n)\frac{[1+\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n)]}{[1+\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)]},\\
&\qquad \mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n)\frac{[1+\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n)]}{[1+\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)]}\Big\}\\
&= \max\Big\{0,\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)\Big\}\\
&= \mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n). \qquad (3.13)
\end{aligned}$$

Using equations (3.12) and (3.13) in equation (3.11), we get

$$\begin{aligned}
\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n) &\precsim \mathcal{R}_1\,\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)+\mathcal{R}_2\,\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)\\
&= \Big(\mathcal{R}_1+\mathcal{R}_2\Big)\,\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)\\
&= \mathcal{W}\,\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n), \qquad (3.14)
\end{aligned}$$

where $\mathcal{W}=\mathcal{R}_1+\mathcal{R}_2<1$.

Thus

$$|\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)|\leq \mathcal{W}\,|\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)|,$$

which is a contradiction since $\mathcal{W}\in[0,1)$. Letting $n\to\infty$, we have

$$\lim_{n\to\infty}|\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)|\leq \mathcal{W}\,.0=0,$$

which is a contradiction by condition $(\mathcal{CS}1)$. Thus, we get $\lim_{n\to\infty}\mathcal{A}u_n=\lim_{n\to\infty}\mathcal{B}v_n=t$.

Now, first we assume that $\mathcal{T}(X)$ is a complete subspace of $(X,\mathcal{S})$, then $t=\mathcal{T}p$ for some $p\in X$. Subsequently, we have

$$\lim_{n\to\infty}\mathcal{B}v_n=\lim_{n\to\infty}\mathcal{A}u_n=\lim_{n\to\infty}\mathcal{Q}u_n=\lim_{n\to\infty}\mathcal{T}v_n=\mathcal{T}p=t.$$

Rest of the proof follows from Theorem 3.1. This completes the proof. □

From Corollary 3.2 we obtain the following special case.

Corollary 3.5 *Let $(X,\mathcal{S})$ be a complete complex valued $\mathcal{S}$-metric space and let $\mathcal{A}\colon X\to X$ be* *a self-mapping of X satisfies the contractive condition:*

$$\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{A}v) \precsim q\,\mathcal{S}(u, u, v),$$

for all $u, v \in X$, where $q \in [0, 1)$ is a constant. Then $\mathcal{A}$ has a unique fixed point in X.

Remark 3.6 Corollary 3.5 extends Theorem 3.1 of Sedghi et al. [40] from complete S-metric space to the setting of complete complex valued S-metric space.

Remark 3.7 Corollary 3.5 also extends the well-known Banach fixed theorem [6] from complete metric space to the setting of complete complex valued S-metric space.

Corollary 3.8([28], Corollary 2.5) *Let $(X, \mathcal{S})$ be a complete complex valued $\mathcal{S}$-metric space and let $\mathcal{A}\colon X \to X$ be a self-mapping of X satisfies the contractive condition:*

$$\mathcal{S}(\mathcal{A}^n u, \mathcal{A}^n u, \mathcal{A}^n v) \precsim q\,\mathcal{S}(u, u, v),$$

for all $u, v \in X$, where n is some positive integer and $q \in [0, 1)$ is a constant. Then $\mathcal{A}$ has a unique fixed point in X.

Proof By Corollary 3.5, there exists $p \in X$ such that $\mathcal{A}^n p = p$. Then

$$\begin{aligned} \mathcal{S}(\mathcal{A}p, \mathcal{A}p, p) &= \mathcal{S}(\mathcal{A}\mathcal{A}^n p, \mathcal{A}\mathcal{A}^n p, \mathcal{A}^n p) \\ &= \mathcal{S}(\mathcal{A}^n \mathcal{A}p, \mathcal{A}^n \mathcal{A}p, \mathcal{A}^n p) \\ &\precsim q\,\mathcal{S}(\mathcal{A}p, \mathcal{A}p, p). \end{aligned}$$

Thus

$$|\mathcal{S}(\mathcal{A}p, \mathcal{A}p, p)| \le q\,|\mathcal{S}(\mathcal{A}p, \mathcal{A}p, p)|,$$

which is a contradiction since $0 \le q < 1$ and so $\mathcal{S}(\mathcal{A}p, \mathcal{A}p, p) = 0$, that is, $\mathcal{A}p = p$. This shows that $\mathcal{A}$ has a unique fixed point in X. This completes the proof. □

Remark 3.9 (i) Completeness of the space X is relaxed in Theorems 3.1, 3.3 and 3.4.

(ii) Continuity of the mappings $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$ is relaxed in Theorems 3.1, 3.3 and 3.4.

Finally, we give the following example which is an application of Corollary 3.5.

Example 3.10 Let $X_1 = \{z \in \mathbb{C} : Re(z) \ge 0, Im(z) = 0\}$ and $X_2 = \{z \in \mathbb{C} : Im(z) \ge 0, Re(z) = 0\}$. Now, let $X = X_1 \cup X_2$ and define a mapping $\mathcal{S}\colon X^3 \to \mathbb{C}$ By:

$$\mathcal{S}(z_1, z_2, z_3) = \begin{cases} \max\{x_1, x_2, x_3\} + i\max\{x_1, x_2, x_3\}, & \text{if } z_1, z_2, z_3 \in X_1, \\ \max\{y_1, y_2, y_3\} + i\max\{y_1, y_2, y_3\}, & \text{if } z_1, z_2, z_3 \in X_2, \\ (\max\{x_1, x_2\} + y_3) + i(\max\{x_1, x_2\} + y_3), & \text{if } z_1, z_2 \in X_1, z_3 \in X_2, \\ (\max\{y_1, y_2\} + x_3) + i(\max\{y_1, y_2\} + x_3), & \text{if } z_1, z_2 \in X_2, z_3 \in X_1, \end{cases}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$. It is very easy to verify that $(X, \mathcal{S})$ is a complete complex valued $\mathcal{S}$-metric space.

Now, we define a self-mapping $\mathcal{A}$ on X (with $z = (x, y)$) as

$$\mathcal{A}(z) = \begin{cases} (\frac{x}{2}, 0), & \text{if } z \in X_1, \\ (0, \frac{y}{2}), & \text{if } z \in X_2. \end{cases}$$

Now, we show that $\mathcal{A}$ satisfies the conditions of Corollary 3.5. Here, we note that

$$0 \precsim \mathcal{S}(z_1, z_2, z_3), \mathcal{S}(\mathcal{A}z_1, \mathcal{A}z_2, \mathcal{A}z_3).$$

. Now, let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Hence, we have the following four cases.

Case 1. If $z_1, z_2 \in X_1$, then we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}z_1, \mathcal{A}z_1, \mathcal{A}z_2) &= \mathcal{S}\Big((\frac{x_1}{2}, 0), (\frac{x_1}{2}, 0), (\frac{x_2}{2}, 0)\Big) \\ &= \max\Big\{\frac{x_1}{2}, \frac{x_2}{2}\Big\} + i \max\Big\{\frac{x_1}{2}, \frac{x_2}{2}\Big\} = \max\Big\{\frac{x_1}{2}, \frac{x_2}{2}\Big\}(1+i) \\ &= \frac{1}{2}\max\{x_1, x_2\}(1+i) \precsim \frac{1}{2}\mathcal{S}(z_1, z_1, z_2) = q\,\mathcal{S}(z_1, z_1, z_2), \end{aligned}$$

Case 2. If $z_1, z_2 \in X_2$, then we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}z_1, \mathcal{A}z_1, \mathcal{A}z_2) &= \mathcal{S}\Big((0, \frac{y_1}{2}), (0, \frac{y_1}{2}), (0, \frac{y_2}{2})\Big) \\ &= \max\Big\{\frac{y_1}{2}, \frac{y_2}{2}\Big\} + i \max\Big\{\frac{y_1}{2}, \frac{y_2}{2}\Big\} = \max\Big\{\frac{y_1}{2}, \frac{y_2}{2}\Big\}(1+i) \\ &= \frac{1}{2}\max\{y_1, y_2\}(1+i) \precsim \frac{1}{2}\mathcal{S}(z_1, z_1, z_2) = q\,\mathcal{S}(z_1, z_1, z_2), \end{aligned}$$

Case 3. If $z_1 \in X_1$, $z_2 \in X_2$, then we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}z_1, \mathcal{A}z_1, \mathcal{A}z_2) &= \mathcal{S}\Big((\frac{x_1}{2}, 0), (\frac{x_1}{2}, 0), (0, \frac{y_2}{2})\Big) = \Big(\frac{x_1}{2} + \frac{y_2}{2}\Big)(1+i) \\ &= \frac{1}{2}(x_1 + y_2)(1+i) \precsim \frac{1}{2}\mathcal{S}(z_1, z_1, z_2) = q\,\mathcal{S}(z_1, z_1, z_2), \end{aligned}$$

Case 4. If $z_2 \in X_1$, $z_1 \in X_2$, then we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}z_1, \mathcal{A}z_1, \mathcal{A}z_2) &= \mathcal{S}\Big((0, \frac{y_1}{2}), (0, \frac{y_1}{2}), (\frac{x_2}{2}, 0)\Big) \\ &= \Big(\frac{x_2}{2} + \frac{y_1}{2}\Big)(1+i) = \frac{1}{2}(x_2 + y_1)(1+i) \\ &\precsim \frac{1}{2}\mathcal{S}(z_1, z_1, z_2) = q\,\mathcal{S}(z_1, z_1, z_2), \end{aligned}$$

where $q = \frac{1}{2}$. If we take $0 \le q < 1$, then all the conditions of Corollary 3.5 are satisfied. Hence by applying Corollary 3.5, $\mathcal{A}$ has a unique fixed point in X. Indeed, in this case $0 \in X$ is the unique fixed point.

§4. Conclusion

In this paper, we prove some common fixed point theorems for contractive type conditions involving rational expressions and using common $(E.A)$ property in the framework of complex-valued S-metric spaces. Also, we give an example in support of the result. The results presented in this paper extend, generalize and enrich several results from the current existing literature.

References

[1] M. Abbas and B. E. Rhoades, Common fixed point results for non-commuting mappings without continuity generalized metric spaces, *Appl. Math. Computation*, 215 (2009), 262-269.

[2] T. Abdeljawad, Meir-Keeler alpha-contractive fixed and common fixed point theorems, *Fixed Point Theory Appl.*, 2013 (2013), Article ID 19.

[3] R. P. Agarwal, D. O'Regan and D. R. Sahu, *Fixed Point Theory for Lipschitzian Type Mappings with Applications*, Series: Topological Fixed Point Theorems and Applications, 6, Springer, New York, 2009.

[4] M. A. Alghamdi, V. Berinde and N. Shahazad, Fixed points non-self almost contractions, *Carpathian J. Math.*, 30(1) (2014), 7-11.

[5] A. Azam, B. Fisher and M. Khan, Common fixed point theorems in complex valued metric spaces, *Numer. Funct. Anal. Optim.*, 3(3) (2011), 243-253.

[6] S. Banach, Surles operation dans les ensembles abstraits et leur application aux equation integrals,*Fund. Math.*, 3 (1922), 133-181.

[7] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, *Funct. Anal. Gos. Ped. Inst. Unianowsk*, 30 (1989), 26-37.

[8] S. Bhatt, S. Chaukiyal and R. C. Dimri, A common fixed point theorem for weakly compatible maps in complex valued metric spaces, *Int. J. Math. Sci. Appl.*, 1 (2011), 1385-1389.

[9] M. Boriceanu, M. Bota and A. Petrusel, A multivalued fractals in b-metric spaces, *Cent. Eur. J. Math.*, 8(2) (2010), 367-377.

[10] M. Bota, A. Moinar and V. Csaba, On Ekeland's variational principle in b-metric spaces, *Fixed Point Theory*, 12 (2011), 21-28.

[11] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, *Atti. Semin. Mat. Fis. Univ. Modena*, 46 (1998), 263-276.

[12] B. K. Dass and S. Gupta, An extension of Banach contraction principle through rational expressions, *Indian J. Pure Appl. Math.*, 6 (1975), 1455-1458.

[13] B. Fisher, Common fixed points and constant mapping satisfying rational inequality, *Math. Sem. Notes* (Univ. Kobe), (1978).

[14] B. Fisher and M. S. Khan, Fixed points, common fixed points and constant mappings, *Studia Sci. Math. Hungar.*, 11 (1978), 467-470.

[15] D. S. Jaggi, Some unique fixed point theorems, *Indian J. Pure Appl. Math.*, 8 (1977), 223-230.

[16] G. Jungck, Compatible mappings and common fixed points, *Int. J. Math. Math. Sci.*, 9 (1986), 771-779.

[17] G. Jungck, Common fixed points for noncontinuous, nonself maps on nonnumetric spaces, *Far East J. Math. Sci.*, 4(2) (1996), 195-215.

[18] O. Hadzic and E. Pap, *Fixed Point Theory in PM-Spaces*, Kluwer Academic Dordrecht, The Netherlands (2011).

[19] H. Huang and S. Xu, Fixed point theorems of contractive mappings in cone b-metric spaces and applications, *Fixed Point Theory Appl.*, 112(2013).

[20] L. -G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, 332(2) (2007), 1468-1476.

[21] N. Hussain and MH. Shah, KKM mappings in cone b-metric spaces, *Comput. Math. Appl.*, 62 (2011), 1677-1684.

[22] D. Ilic and V. Rakocevic, Common fixed points for maps on cone metric space, *J. Math. Anal. Appl.*, 341 (2008), 876-882.

[23] S. Jankovič, Z. Kadelburg and S. Radenovicč, On cone metric spaces: a survey, *Nonlinear Anal.*, 4(7) (2011), 2591-2601.

[24] H. P. A. Kunzi, A note on sequentially compact quasi-pseudo metric spaces, *Monatsheft Math.*, 95 (1983), 219-220.

[25] W. Liu, J. Wu and Z. Li, Common fixed point of single-valued and multi-valued maps, *Int. J. Math. Sci.*, 19 (2005), 3045-3055.

[26] S. G. Matthews, *Partial Metric Topology*, Research report 2012, Dept. Computer Science, University of Warwick, 1992.

[27] S. G. Matthews, Partial metric topology, Proceedings of the 8th Summer Conference on Topology and its Applications, *Annals of the New York Academy of Sciences*, 728 (1994), 183-197.

[28] N. M. Mlaiki, Common fixed points in complex S-metric space, *Adv. Fixed Point Theory*, 4(4) (2014), 509-524.

[29] H. K. Nashine, M. Imdad and M. Hasan, Common fixed point theorems under rational contractions in complvalued metric spaces, *J. Nonlinear Sci. Appl.*, 7 (2014), 42-50.

[30] H. K. Pathak, R. R. Lopez and R. K. Verma, A common fixed point theorem of integral type using implicit relation, *Nonlinear Funct. Anal. Appl.*, 15 (2009), 1-12.

[31] R. A. Rashwan, H. A. Hammad and M. G. Mahmoud, Common fixed point theorems in complex valued S-metric spaces via implicit relations with applications, *Res. Fixed Point Theory Appl.*, 2019, Article ID 2018034, 17 pages.

[32] Sh. Rezapour and R. Hamlbarani, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", *J. Math. Anal. Appl.*, 345(2) (2008), 719-724.

[33] F. Rouzkardand and M. Imdad, Some common fixed point theorems on complex valued metric spaces, *Computers and Mathematics with Applications*, 64 (2012), 1866-1874.

[34] G. S. Saluja, Some fixed point theorems for generalized contractions involving rational expressions in b-metric spaces, *Commun. Optim. Theory*, 2016 (2016), Article ID 17.

[35] G. S. Saluja, Some common fixed point theorems for generalized contraction involving rational expressions in b-metric spaces, *J. Contemp. Appl. Math.*, 6(2) (2016), 67-78.

[36] G. S. Saluja, Fixed point results under generalized contraction involving rational expression in complex valued metric spaces, *International J. Math. Combin.*, 1 (2017), 55-62.

[37] G. S. Saluja, Fixed point theorems under rational contraction in complex valued metric spaces, *Nonlinear Functional Analysis and Applications*, 22(1) (2017), 209-216.

[38] G. S. Saluja, On common fixed point theorems for rational contractions in b-metric spaces, *The Aligarh Bull. Math.*, 37(1-2) (2018), 1-12.

[39] G. S. Saluja, Some common fixed point theorems using rational contraction in complex valued metric spaces, *Palestine J. Math.*, 7(1) (2018), 92-99.

[40] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in S-metric spaces, *Mat. Vesnik*, 64(3) (2012), 258-266.

[41] W. Shatanawi and A. Pitea, Some coupled fixed point theorems in quasi-partial metric spaces, *Fixed Point Theory Appl.*, 2013 (2013), Article ID 153.

[42] W. Sintunavarat and P. Kumam, Generalized common fixed point theorems in complex valued metric spaces and applications, *J. Inequ. Appl.*, 2012:84.

[43] M. Telei and B. Fisher, On a fixed theorem for fuzzy mappings in quasi-metric spaces, *Thai J. Math.*, 2 (2003), 1-8.

[44] R. K. Verma and H. K. Pathak, Common fixed point theorems using property $(E.A)$ in complex valued metric spaces, *Thai J. Math.*, 11(2) (2013), 347-355.

[45] N. Tas and N. Özgür, *Common fixed point results on complex valued S-metric spaces*, (2020), DOI:10.22130/scma.2018.92986.488.

[46] P. Vetro, Common fixed points in cone metric spaces, *Rend. Circ. Mat. Palermo*, (2) 56(3) (2007), 464-468.

International J.Math. Combin. Vol.2(2023), 18-30

On the Core of Second Smarandache Bol Loops

B. Osoba

(Bells University of Technology, Ota, Ogun State, Nigeria)

Y. T. Oyebo

(Department of Mathematics, Lagos State University Ojo, 102101 Nigeria)

E-mail: benardomth@gmail.com, b_osoba@bellsuniversity.edu.ng, yakub.oyebo@lasu.edu.ng

Abstract: Let $(G, \cdot)$ be a loop. A loop $(G_H, \cdot)$ is called a special loop of $(G, \cdot)$ if the pair $(H, \cdot)$ is an arbitrary a non-empty subloop of $(G, \cdot)$. In general, $(G_H, \cdot)$ is called second Smarandache Bol loop (S_{2nd}BL) if it obey the identity $(xs \cdot z)s = x(sz \cdot s)$ for all $s \in H$ and $x, z \in G$. This paper presents some algebraic characterizations of the core of a second Smarandache Bol loop (S_{2nd}BL). Some results in this paper extend or generalize the results of the classical studies of the core of a Bol loop. The conditions for the core of $S_{2nd}BL$ to be left symmetric, left(right) idempotents, left self-distributive, and flexible was shown. A necessary and sufficient condition for a core of (S_{2nd}BL) to be right(left) alternative property was revealed. The characterization of $S-$isotopic and $S-$isomorphic invariance was also presented in this paper.

Key Words: Core, special loop, Smarandache Bol loops.

AMS(2010): Primary 20N05, Secondary 08A05.

§1. Introduction

Let Q be a non -empty set. Define a binary operation " $\cdot$ " on Q. If $x \cdot y \in Q$ for all $x, y \in Q$, then the pair $(Q, \cdot)$ is called a groupoid or magma. If the equations: $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in Q$ for all $a, b \in Q$, then $(Q, \cdot)$ is called a quasigroup. Let $(Q, \cdot)$ be a quasigroup and there exist a unique element $e \in Q$ called the identity element such that for all $x \in Q, x \cdot e = e \cdot x = x$, then $(Q, \cdot)$ is called a loop. At times, we shall write xy instead of $x \cdot y$ and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. Let $(Q, \cdot)$ be a groupoid and a be a fixed element in Q, then the left and right translations L_a and R_a of a are respectively defined by $xL_a = a \cdot x$ and $xR_a = x \cdot a$ for all $x \in Q$. It can now be seen that a groupoid $(Q, \cdot)$ is a quasigroup if its left and right translation mappings are permutations. Since the left and right translation mappings of a quasigroup are bijective, then the inverse mappings L_x^{-1} and R_x^{-1} exist.

Let

$$x \backslash y = yL_x^{-1} = xP_y \qquad \text{and} \qquad x/y = xR_y^{-1} = yP_x^{-1}$$

[1]Received April 2, 2023, Accepted June 4, 2023.

and note that

$$x\backslash y = z \Longleftrightarrow x \cdot z = y \qquad \text{and} \qquad x/y = z \Longleftrightarrow z \cdot y = x.$$

Thus, for any quasigroup $(Q, \cdot)$, we have two new binary operations; right division $(/)$ and left division $(\backslash)$ and middle translation P_a for any fixed $a \in Q$. Consequently, $(Q, \backslash)$ and $(Q, /)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be restated as follows.

Definition 1.1 *A loop $(Q, \cdot, /, \backslash, e)$ is a set G together with three binary operations $(\cdot)$, $(/)$, $(\backslash)$ and one nullary operation e such that*

(i) $x \cdot (x\backslash y) = y$, $(y/x) \cdot x = y$ *for all* $x, y \in Q$*;*
(ii) $x\backslash(x \cdot y) = y$, $(y \cdot x)/x = y$ *for all* $x, y \in Q$*;*
(iii) $x\backslash x = y/y$ *or* $e \cdot x = x$ *for all* $x, y \in Q$*.*

We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\cdot)$ among factors to be multiplied. For instance, $x \cdot y/z$ and $x \cdot y\backslash z$ stand for $x(y/z)$ and $x(y\backslash z)$ respectively.

In a loop $(Q, \cdot)$ with identity element e, the *left inverse element* of $x \in Q$ is the element $xJ_\lambda = x^\lambda \in Q$ such that

$$x^\lambda \cdot x = e$$

while the *right inverse element* of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that

$$x \cdot x^\rho = e.$$

For more on quasigroups and loops, the reader can check Jaiyéọlá [13], Pflugfelder [5] and Shcherbacov [3] for details.

The study of Smarandache concept in groupoid was first introduced by (W. B Vasantha Kandasamy [18], 2002). The paper [20] and her book on Smarandache concept in the study of loops [19], where she initially defined Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operations of the loop have started receiving an attention of researchers.

Smarandache quasigroup was defined by (Muktibodh, [21, 22]), as a non-trivial subset H of a quasigroup $(G, \cdot)$ such that $(H, \cdot)$ is a associative subquasigroup of the quasigroup $(G, \cdot)$.

Immediately after the work of Muktibodh, (Jaiyéọlá [6], 2006) introduced the study of holomorphic structures of a loop under Smarandache quasigroup. It was revealed that a loop is a Smarandache loop if and only if its holomorph is a Smarandache loop and further shown that the statement is also true for some weak Smarandache loops such as inverse property, weak inverse property but false for others(conjugacy closed, Bol, central, extra, Burn, A- homogeneous except if their holomorphs are nuclear or central.

In (Jaiyéọlá [10, 11, 12, 14, 15], 2008), more characterizations of a Smarandache concept in quasigroups and loops are presented. In particular, a Smarandache isotopic quasigroup and holomorphic study of Smarandache automorphism and cross inverse property loops were investigated in the same manner the isotopy theory was carried out for groupoids, quasigroups, and loops. The same author [15], introduced and studied double cryptography using the concept of Smarandache Keedwell Cross inverse quasigroup.

In [16, 17], the author furthered his exploration of Smarandache quasigroups (loops) theory by classifying the algebraic structures into first Smarandache quasigroup (loop) and second Smarandache quasigroup (loop). The author announced that the most comprehensive study in Bol-Moufang type identities called Bol loop falls into the second class of Smarandache loops. Hence, the second Smarandache loop is a particular case of the first Smarandache loops and the second Smarandache Bol loop is a generalization of Bol loops.

In (Jaiyéọlá [8, 9], 2006), the authors studied parastrophic invariants of Smarandache quasigroups, and presented a ground view of the studies of the universality of some Smarandache loops of Bol-Moufang type. His results showed that Smarandache quasigroup (loop) is universal if all its $f, g-$principal isotopes are Smarandache $f, g-$ principal isotopes.

In (Osoba et al. [28, 29], 2018), the authors studied the relationship of multiplication groups and isostrophic quasigroups and some algebraic characterizations of middle Bol loops.

In 2022, Osoba and Jaiyéọlá [24] presented algebraic connections between the middle Bol loop and right Bol loop and their cores. A necessary and sufficient condition for the core of a right Bol loop to be elastic property and right idempotent law was established. It was further revealed that If a middle Bol loop is right (left) symmetric then, the core of its corresponding (RBL) is a medial (semimedial). The results in [16, 17] were extended by the first author of this paper in [27].

In 2023, Jaiyéọlá et al. [23] presented a study on the Bryant-Schneider group of a middle Bol loop. The authors used the concept of the Bryant-Schneider group to link some of the isostrophy-group invariance results of Grecu and Syrbu. In particular, it was established that some subgroups of the Bryant-Schneider group of a middle Bol loop are isomorphic to the automorphism and pseudo-automorphism groups of its corresponding right (left) Bol loop. Some elements of the Bryant-Schneider group of a middle Bol loop were shown to induce automorphisms and middle pseudo-automorphisms. It was discovered that if a middle Bol loop is of exponent two then, its corresponding right (left) Bol loop is a left (right) G-loop while more results on the algebraic properties of a middle Bol loop using its parastrophes were unveiled by Osoba and Oyebo [26] in 2022.

Recently, the characterization of the cry-automorphism group of some quasigroups was studied in [25].

§2. Preliminaries

Definition 2.1 *A groupoid (quasigroup)* $(G, \cdot)$ *is said to have*

(1) *the left inverse property* (LIP) *if there exists a mapping* $J_\lambda : x \mapsto x^\lambda$ *such that* $x^\lambda \cdot xy = y$ *for all* $x, y \in G$*;*

(2) *the right inverse property* (RIP) *if there exists a mapping* $J_\rho : x \mapsto x^\rho$ *such that* $yx \cdot x^\rho = y$ *for all* $x, y \in G$*;*

(3) *the inverse property (IP) if it has both the LIP and RIP;*

(4) *the right alternative property (RAP) if* $y \cdot xx = yx \cdot x$ *for all* $x, y \in G$*;*

(5) *the left alternative property (LAP) if* $y \cdot xx = yx \cdot x$ *for all* $x, y \in G$*;*

(6) *the flexibility or elasticity if* $xy \cdot x = x \cdot yx$ *holds for all* $x, y \in G$*;*

(7) *the cross inverse property* (CIP) *if there exist mapping* $J_\lambda : x \mapsto x^\lambda$ *or* $J_\rho : x \mapsto x^\rho$ *such that* $xy \cdot x^\rho = y$ *or* $x \cdot yx^\rho = y$ *or* $x^\lambda \cdot yx = y$ *or* $x^\lambda y \cdot x = y$ *for all* $x, y \in G$.

Definition 2.2 *A loop* $(G, \cdot)$ *is said to be right power alternative property loop (RPAPL) if its obeys the identity* $xy^n = ((((xy)y)y)y)y...y$ *that is* $R_{y^n} = R_y^n$ *for all* $x, y \in G$.

Definition 2.3 *A special quasigroup(loop)* $(G_H, \cdot)$ *is called:*

(1) *a second Smarandache left inverse property quasigroup(loop)* $S_{2nd}LIPQ(S_{2nd}LIPL)$ *if it obeys the second Smarandache left inverse property* $(S_{2nd}LIP)$ $s^\lambda \cdot sx = x$ *for all* $x \in G$ *and* $s \in H$*;*

(2) *a second Smarandache right inverse property quasigroup(loop)* $S_{2nd}RIPQ(S_{2nd}RIPL)$ *if it obeys the second Smarandache right inverse property* $(S_{2nd}LIP)$ $xs \cdot s^\rho = x$ *for all* $x \in G$ *and* $s \in H$*;*

(3) *a second Smarandache inverse property quasigroup(loop)* $S_{2nd}IP$ *if it has both the* $S_{2nd}RIP$ *and* $S_{2nd}LIP$*;*

(4) *a second Smarandache right alternative property quasigroup(loop)* $S_{2nd}RAPQ(S_{2nd}RAPL)$ *if* $x \cdot ss = xs \cdot s$ *for all* $x \in G$ *and* $s \in H$*;*

(5) *a second Smarandache left alternative property quasigroup(loop)* $S_{2nd}LAPQ(S_{2nd}LAPL$ *if* $ss \cdot x = s \cdot sx$ *for all* $x \in G$ *and* $s \in H$ *for all* $x, y \in G$*;*

(6) *a second Smarandache flexible or elastic quasigroup(loop) if* $sx \cdot s = s \cdot xs$ *holds for all* $x \in G$ *and* $s \in H$*;*

(7) *a second Smarandache right power alternative property loop* $S_{2nd}RPAPL$ *if its obeys the identity* $sx^n = ((((xs)s)s)s)s...s$ *that is* $R_{s^n} = R_s^n$ *for all* $x \in G$ *and* $s \in H$.

Definition 2.4 *A Smarandache groupoid (quasigroup)* $(Q, \cdot)$ *is called:*

(1) *the second Smarandache right symmetric (*$S_{2nd}RS$*) if* $xs \cdot s = x$ *for all* $x \in Q$ *and* $s \in H$*;*

(2) *the second Smarandache left symmetric (*$S_{2nd}LS$*) if* $s \cdot sx = x$ *for all* $x \in Q$ *and* $s \in H$*;*

(3) *the second Smarandache middle symmetric (*$S_{2nd}MS$*) if* $s \cdot xs = x$ *or* $xs \cdot x = x$ *for all* $x \in Q$ *and* $s \in H$*;*

(4) *the third Smarandache middle symmetric (*$S_{3nd}MS$*) if* $xs \cdot x = s$ *or* $x \cdot sx = s$ *for all* $x \in Q$ *and* $s \in H$*;*

(5) *the second Smarandache idempotent (*$S_{2nd}I$*) if* $s \cdot s = s$ *for all* $x \in Q$ *and* $s \in H$*;*

(6) *the second Smarandache left idempotent (*S_2ndLI*) if* $ss \cdot x = sx$ *for all* $x \in Q$ *and* $s \in H$*;*

(7) *the second Smarandache right idempotent (*$S_{2nd}RI$*) if* $x \cdot ss = sx$ *for all* $x \in Q$ *and* $s \in H$*;*

(8) *the second Smarandache commutative (*$S_{2nd}CP$*) if* $x \cdot s = s \cdot x$ *for all* $x \in Q$ *and* $s \in H$*;*

(9) *the second Smarandache anti-automorphic inverse property (*$S_{2nd}AAIP$*) if* $(x \cdot s)^\rho = (s \cdot x)^\rho$ *or* $(x \cdot s)^\lambda = (s \cdot x)^\lambda$ *for all* $x \in Q$ *and* $s \in H$*;*

(10) *the second Smarandache totally quasigroup ($S_{2nd}TQ$) if and only if (1) or (2) and (8) hold.*

Definition 2.5 *Let $(Q,\cdot)$ be a $S_{2nd}TQ$. If $(Q,\cdot)$ is a special loop, then it is called second Smarandache Steiner loop ($S_{2nd}SL$).*

Theorem 2.6 (Jaiyeola [16]) *Let the special loop $(G_H,\cdot)$ be a $S_{2nd}BL$. Then, $S_{2nd}BL$ is satisfies $S_{2nd}RIPL$ and $S_{2nd}RAPL$.*

Theorem 2.7 (Jaiyeola [16]) *If the special loop $(G_H,\cdot)$ is a $S_{2nd}BL$. Then,*

$$xs^n = xs^{n-1}\cdot s = xs\cdot s^{n-1}$$

for all $n\in\mathbb{Z}, s\in H$ and $x\in G$.

Theorem 2.8 (Jaiyeola [16]) *If the special loop $(G_H,\cdot)$ is a $S_{2nd}BL$. Then, $xs^m\cdot s^n = xs^{m+n}$ for all $m,n\in\mathbb{Z}, s\in H$ and $x\in G$.*

Theorem 2.9 (Jaiyeola [16]) *If the special loop $(G_H,\cdot)$ is a $S_{2nd}BL$. Then, G_H is a $S_{2nd}SAIPL$ if and only if G_H is a $S_{3rd}RIPL$.*

Corollary 2.10 (Jaiyeola [16]) *Every $S_{2nd}BL$ is a Smarandache right power associative property loop.*

Definition 2.11 (Jaiyeola [17]) *Let $(G_H,\cdot)$ and $(Q_N,\circ)$ be spacial groupiods and let G_H, Q_N be Smarandache isotopes (S-isotopes). Then, $(Q_N,\circ)$ is a Smarandache isotopic of $(G_H,\cdot)$ if and only if there is a bijective $(A,B,C): H\mapsto N$ such that the triple $(A,B,C):(G_H,\cdot)\mapsto(Q_N,\cdot)$ is isotopism. Suppose that the triple $A=B=C$, then $(G_H,\cdot)$ and $(Q_N,\circ)$ are said to be Smarandache isomorphic (S-isomorphic).*

Definition 2.12 *Let the spacial loop $(G_H,\cdot)$ be a $S_{2nd}BL$. The groupoid $(G_H,+)$ called the core of $(G_H,\cdot)$ is define as $x+y=xy^\lambda\cdot x$ for all $x\in H$ and $y\in G$.*

Definition 2.13 *A special groupoid $(Q,+)$ is called:*

(1) *Smarandache left self distributive (SLSD) if $s+(y+z)=(s+y)+(s+z)$ for all $y,z\in Q$ and $s\in H$;*

(2) *Smarandache left distributive(SLD) if $s(y+z)=(sy)+(sz)$ for all $y,z\in Q$ and $s\in H$;*

(3) *Smarandache right distributive(SRD) if $(y+z)s=(ys)+(zs)$ for all $y,z\in Q$ and $s\in H$.*

Definition 2.14 (Jaiyeola [17], 2011) *Let $(G_H,\cdot)$ is called a special loop with special subloop $(H,\cdot)$. If $(H,\cdot)$ is of exponent 2, then $(G_H,\cdot)$ is called a special loop of Smarandache exponent two.*

Definition 2.15 *Let $(G_H,\cdot)$ be a special loop. H is called an ideal of $(G,\cdot)$ if $sx\in H$ for all $s\in H$, and $x\in G$*

Furtherance to the past research, this paper is posted to extend the results in [16, 24]. Some new definitions were established and were used to characterize the core of the second Smarandache Bol loop.

§3. Main Results

Lemma 3.1 *Let $(G_H, \cdot)$ be a special quasigroup.*

(1) *if $(G_H, \cdot)$ is a $S_{3nd}RIP$ and H is a right ideal of $(G, \cdot)$, then $x^{\rho^2} = x$ and $x^\rho = x^\lambda$ for all $x \in G$;*

(2) *if $(G_H, \cdot)$ is a $S_{3nd}LIP$ and H is a left ideal of $(G, \cdot)$, then $x^{\lambda^2} = x$ and $x^\rho = x^\lambda$ for all $x \in G$;*

(3) *if $(G_H, \cdot)$ is a $S_{2nd}LIP$, then $sx = b \Rightarrow x = s^\lambda b$ for all $s \in H$ and $x \in G$;*

(4) *if $(G_H, \cdot)$ is a $S_{3nd}RIP$, then $xs = b \Rightarrow x = bs^\rho$ for all $s \in H$ and $x \in G$;*

(5) *if $(G_H, \cdot)$ is a $S_{2nd}RIP$, then $ys = b \Rightarrow y = bs^\rho$ for all $s \in H$ and $x \in G$;*

(6) *if $(G_H, \cdot)$ is a $S_{3nd}LIP$, then $ys = b \Rightarrow s = y^\lambda b$ for all $s \in H$ and $x \in G$;*

(7) *if $(G_H, \cdot)$ is a $S_{2nd}RIP$ and $S_{3nd}LIP$, then $s^\lambda = (as)^\lambda a$ $-S_{3rd}LWIP$ for all $s \in H$ and $a \in G$;*

(8) *if $(G_H, \cdot)$ is a $S_{2nd}RIP$, $S_{3rd}LIP$, $S_{3nd}RIP$ and H is $\lambda-$ideal, then $s^{-1}a^{-1} = (as)^{-1}$ for all $s \in H$ and $a \in G$;*

(9) *if $(G_H, \cdot)$ is a $S_{2nd}LIP$ and $S_{3nd}RIP$, then $s^\rho = b(sb)^\rho-$ $S_{3rd}RWIP$ for all $s \in H$ and $b \in G$;*

(10) *if $(G_H, \cdot)$ is a $S_{2nd}LIP$, $S_{3nd}RIP$, $S_{3rd}LIP$ and H is $\rho-$ideal, then $b^{-1}s^{-1} = (sb)^{-1}$ for all $s \in H$ and $b \in G$;*

(11) *$(G_H, \cdot)$ has $S_{2nd}RIP \Leftrightarrow R_{s^{-1}} = R_s^{-1}$;*

(12) *$(G_H, \cdot)$ has $S_{2nd}LIP \Leftrightarrow L_{s^{-1}} = L_s^{-1}$;*

(13) *if $(G_H, \cdot)$ is a $S_{2nd}RIP$, $S_{3nd}IP$ and $\lambda-$ideal, $J_\lambda R_s J_\rho = L_{s^{-1}}$ for all $s \in H$;*

(14) *if $(G_H, \cdot)$ is a $S_{2nd}LIP$, $S_{3nd}IP$ and $\rho-$ideal, $J_\lambda L_s J_\rho = R_{s^{-1}}$ for all $s \in H$.*

Proof (1) Consider the expression $(sx \cdot x^\rho)(x^\rho)^\rho$, then

$$(sx \cdot x^\rho)(x^\rho)^\rho \underbrace{=}_{3_{nd}RIP} s(x^\rho)^\rho = sx \Rightarrow x^{\rho^2} = x \Rightarrow J_\rho^2 = I \Rightarrow J_\rho^{-1} = J_\rho \Rightarrow J_\lambda = J_\rho.$$

(2) Consider the expression $(x^\lambda \cdot xs)(x^\lambda)^\lambda$, then

$$(x^\lambda \cdot xs)(x^\lambda)^\lambda \underbrace{=}_{3_{nd}LIP} (x^\lambda)^\lambda s = xs \Rightarrow x^{\lambda^2} = x \Rightarrow J_\lambda^2 = I \Rightarrow J_\lambda{-}1 = J_\rho \Rightarrow J_\lambda = J_\rho.$$

(3) Let $sx = b$. Multiplying both sides by s^λ on the left, we have $x = \underbrace{=}_{2_{nd}LIP} s^\lambda b$.

(4) Let $xs = b$. Multiplying both sides by s^ρ on the right, we have $x = \underbrace{=}_{2_{nd}RIP} bs^\rho$.

(5) Let $ys = b$. Multiplying both sides by s^ρ on the right, we have $y = \underbrace{=}_{2_{nd}RIP} bs^\rho$.

(6) Let $ys = b$. Multiplying both sides by y^{λ} on the left, we have $s = \underbrace{=}_{3_{nd}LIP} y^{\lambda}b$.

(7) Let $as = c$, then $a \underbrace{=}_{2_{nd}RIP} cs^{\rho} \underbrace{\Rightarrow}_{S_{3nd}LIP} s^{\rho} = (as)^{\lambda}a \Rightarrow s^{\lambda} = (as)^{\lambda}a$.

(8) So, $s^{\rho} = (as)^{\lambda}a \Rightarrow s^{\lambda} = (as)^{\lambda}a \underbrace{\overset{\lambda-ideal}{\Rightarrow}}_{S_{3rd}RIP} s^{\lambda}a^{\rho} = (as)^{\lambda} \Rightarrow s^{-1}a^{-1} = (as)^{-1}$.

(9) Let $sb = c \underbrace{\Rightarrow}_{S_{2nd}LIP} b = s^{\lambda}c \underbrace{\Rightarrow}_{S_{3nd}RIP} bc^{\rho}s^{\lambda} \Rightarrow b(sb)^{\rho} = s^{\rho}$.

(10) So, $b(sb)^{\rho} = s^{\lambda} \Rightarrow b(sb)^{\rho} = s^{\rho} \underbrace{\overset{\rho-ideal}{\Rightarrow}}_{S_{3rd}LIP} b^{\lambda}s^{\rho} = (bs)^{\rho} \Rightarrow b^{-1}s^{-1} = (sb)^{-1}$.

(11) $ys \cdot s^{-1} = y \Leftrightarrow yR_sR_{s^{-1}} = y \Leftrightarrow R_sR_{s^{-1}} = I \Leftrightarrow R_s^{-1} = R_{s^{-1}}$.

(12) $s^{\lambda} \cdot sx = x \Leftrightarrow xL_sL_{s^{-1}} = x \Leftrightarrow L_sL_{s^{-1}} = I \Leftrightarrow L_s^{-1} = L_{s^{-1}}$.

(13) $xJ_{\lambda}R_sJ_{\rho} = (x^{\lambda}s)^{\rho} \underbrace{\overset{S_{3rd}IP}{\Rightarrow}}_{S_{2rd}RIP} s^{-1}(x^{-1})^{-1} = s^{-1}x = xL_{s^{-1}}$.

(14) $xJ_{\lambda}L_sJ_{\rho} = (sx^{\lambda})^{\rho} \underbrace{\overset{S_{2nd}LIP}{\Rightarrow}}_{S_{3rd}IP} (x^{-1})^{-1}s^{-1} = xs^{-1} = xR_{s^{-1}}$.

This completes the proof. □

Theorem 3.2 *Let the spacial loop $(G_H, \cdot)$ be a $S_{2nd}BL$. Then,*

(1) $(G_H, +)$ is (S_{2nd} LS) if $(G_H, \cdot)$ is a (S_{2nd} RIPL) and H is $\rho-$ideal of $(G, \cdot)$;

(2) $(G_H, +)$ is (S_{2nd} LI);

(3) $(G_H, +)$ is (S_{2nd} RI);

(4) if $(G_H, \cdot)$ is (S_{2nd} RIP), S_{2nd}elastic, S_{3rd}RIP and H is $\rho-$ideal, then $(G_H, \cdot)$ satisfies commutative if and only if $(G_H, +)$ is S_{2nd} middle symmetric;

(5) if $(G_H, \cdot)$ is S_{3rd}RIP and H is $\rho-$ideal, then $(G_H, +)$ is S_{2rd} LD if and only if $(sy^{\rho} \cdot z)y^{\rho} = s(yz^{\rho} \cdot y)^{\rho}$ for all $s \in H$ and $y, z \in G$;

(6) if $(G_H, \cdot)$ is S_{3rd}SAIP then $(G_H, +)$ is S_{2nd} flexible if and only if $(sy^{\rho} \cdot s)y^{\rho} = s(y^{\rho}s \cdot y^{\rho})$ for all $y \in G$, and $s \in H$.

Proof (1) By symmetric property, we have $s + (s + y) = s(s + y)^{\lambda} \cdot s = [s(sy^{\lambda} \cdot s)^{\lambda}]s = [s(s^{\lambda}y^{\lambda^2} \cdot s^{\lambda})]s$. Apply Theorem 2.9, we have $((ss^{\lambda} \cdot y^{\lambda^2})s^{\lambda})s = y^{\lambda^2}s^{\lambda} \cdot s = y$.

(2) By left idempotent: $(s+s)+y = (s+s)y^{\lambda} \cdot (s+s) = [(ss^{\lambda} \cdot s)y^{\lambda}]ss^{\lambda} \cdot s = sy^{\lambda} \cdot s = s+y$.

(3) By right idempotent: $y + (s + s) = y(s + s)^{\lambda} \cdot y = y(ss^{\lambda} \cdot s)^{\lambda} \cdot y = ys^{\lambda} \cdot y = y + s$.

(4) By middle symmetric:

$$\begin{aligned} s + (x + s) = x \quad &\Leftrightarrow \quad [s(x + s)^{\lambda}]s = x \\ &\Leftrightarrow \quad [s(xs^{-1} \cdot x)^{\lambda}]s = x \underbrace{\Leftrightarrow}_{S_{2nd}IP} (xs^{\lambda} \cdot x)^{\lambda} = s^{\lambda} \cdot xs^{\lambda} \\ &\underbrace{\Leftrightarrow} \quad {\scriptstyle S_{3rd} \text{ (RIP) and H is } \rho-\text{ideal}} \ (xs^{\lambda} \cdot x)^{\lambda} = (s \cdot x^{\lambda}s)^{\lambda} \\ &\underbrace{\Leftrightarrow} \quad {\scriptstyle S_{2nd}\text{(elasticity)}} \ (xs^{\lambda} \cdot x)^{\lambda} = (sx^{\lambda} \cdot s)^{\lambda} \\ &\Leftrightarrow \quad (x + s)^{\lambda} = (s + x)^{\lambda} \Leftrightarrow x + s = s + x \end{aligned}$$

for all $x \in G$ and $s \in H$.

(5) By left self distributive: $s+(y+z)=(s+y)+(s+z)$. For all $s\in H$ and $y,z\in G$, we have

$$
\begin{array}{lcl}
LHS=(s+y)+(s+z) & = & [(sy^\lambda\cdot s)(sz^\lambda\cdot s)^\lambda](sy^\lambda\cdot s)\\
 & \underbrace{=}_{S_{3rd}\text{ RIP and H is }\rho-\text{ideal}} & [(sy^\lambda\cdot s)(s^{-1}z\cdot s^{-1})](sy^\lambda\cdot s)\\
 & = & \left[\left(((sy^\lambda\cdot s)s^\lambda)z\right)s^\lambda\right](sy^\lambda\cdot s)\\
 & = & \left[((sy^\lambda\cdot z))s^\lambda\right](sy^\lambda\cdot s)\\
 & = & \left[\left(((sy^\lambda\cdot z)s^{-1})s\right)y^\lambda\right]s\\
 & = & (sy^\lambda\cdot z)y^\lambda\cdot s
\end{array}
$$

and $RHS=s+(y+z)=s(y+z)^\rho\cdot s=s(yz^\lambda\cdot y)^\rho\cdot s$. So, $(G_H,+)$ has S_{2rd}LSD $\Leftrightarrow (sy^\rho\cdot z)y^\rho=s(yz^\rho\cdot y)^\rho$ for all $s\in H$ and $y,z\in G$.

(6) $(G_H,+)$ is S_{2nd} flexible $\Leftrightarrow (s+y)+s=s+(y+s)$ for all $y\in G$, and $s\in H$.

$$
\begin{array}{rcl}
RHS & = & LSH\\
 & \Leftrightarrow & (s+y)s^\lambda\cdot(s+y)=s(y+s)^\lambda\cdot s\\
 & \Leftrightarrow & [(sy^\lambda\cdot s)s^\lambda](sy^\lambda\cdot s)=(s(ys^\lambda\cdot y)^\lambda)s\\
 & \Leftrightarrow & [(sy^\lambda\cdot s)s^\lambda](sy^\lambda\cdot s)\underbrace{=}_{S_{3rd}SAIP}(s(y^\lambda s\cdot y^\lambda))s\\
 & \Leftrightarrow & sy^\lambda(sy^\lambda\cdot s)=s(y^\lambda s\cdot y^\lambda)\cdot s\\
 & \Leftrightarrow & (sy^\lambda\cdot s)y^\lambda\cdot s=s(y^\lambda s\cdot y^\lambda)\cdot s\\
 & \Leftrightarrow & (sy^\lambda\cdot s)y^\lambda=s(y^\lambda s\cdot y^\lambda).
\end{array}
$$

This completes the proof. □

Theorem 3.3 *Let the spacial loop $(G_H,\cdot)$ be a $S_{2nd}BL$. Then,*

(1) *$(G_H,+)$ is $S_{2nd}RAP$ if and only if $y+s=s$ for all $s\in H$;*

(2) *$(G_H,+)$ is $S_{2nd}LAP$ if and only if $s+y=y$ for all $s\in H$;*

(3) *if $(s+y)x=sx+yx$ for all $s\in H$ and $x,y\in G$, then $(G_H,\cdot)$ is satisfies $S_{2nd}LAP$ if and only if it satisfies $S_{3nd}RIP$;*

(4) *if $x(s+y)=xs+xy$, then $(G_H,\cdot)$ is satisfies $SAAIP$ for all $s\in H$ and $x,y\in G$.*

Proof (1) Notice that

$$\begin{aligned}(G_H,+) \text{ is } S_{2nd}RAP &\Leftrightarrow (y+s)+s = y+(s+s)\\ &\Leftrightarrow (y+s)s^\lambda(y+s) = y(s+s)^\lambda \cdot y\\ &\Leftrightarrow [(ys^\lambda \cdot y)s^\lambda](ys^{-1}\cdot y) = [y(ss^\lambda \cdot s)^\lambda]y\\ &\Leftrightarrow [(ys^\lambda \cdot y)s^\lambda](ys^\lambda \cdot y) = ys^\lambda \cdot y\\ &\Leftrightarrow (ys^\lambda \cdot y)s^\lambda = e\\ &\Leftrightarrow ys^\lambda \cdot y = s^{(\lambda)^\rho}\\ &\Leftrightarrow y+s = s.\end{aligned}$$

(2) Notice that

$$\begin{aligned}(G_H,+) \text{ is } S_{2nd}LAP &\Leftrightarrow (s+s)+y = s+(s+y)\\ &\Leftrightarrow [(s+s)y^\lambda](s+s) = [s(s+y)^\lambda]s\\ &\Leftrightarrow [(ss^{-1}\cdot s)y^\lambda](ss^{-1}\cdot s) \underbrace{=}_{S_{3rd}\text{RIP and H is } \rho-\text{ideal}} [s(sy^\lambda \cdot s)^\lambda]s\\ &\Leftrightarrow sy^\lambda \cdot s = [s(s^\lambda y \cdot s^\lambda)]s\\ &\Leftrightarrow sy^\lambda \cdot s = ((ss^\lambda \cdot y)s^\lambda)s\\ &\Leftrightarrow sy^\lambda \cdot s = ys^\lambda \cdot s\\ &\Leftrightarrow sy^\lambda \cdot s = y\\ &\Leftrightarrow s+y = y.\end{aligned}$$

(3) if $(s+y)x = sx+yx$ then,

$$(sy^\lambda \cdot s)x = [(sx)(yx)^\lambda](sx)$$

Put $y = e$, the identity element in G, we have $ss\cdot x = (sx\cdot x^\lambda)(sx) \underbrace{\Rightarrow}_{S_{3nd}\text{RIP}} ss\cdot x = s\cdot sx$ for all $x \in G$ and $s \in H$.

(4) if $x(s+y) = xs+yx$ then,

$$\begin{aligned}x(sy^\lambda \cdot s) &= [(xs)(xy)\lambda](xs)\\ &\Rightarrow (xs\cdot y^\lambda)s = [(xs)(xy)^\lambda](xs).\end{aligned}$$

Let $x = s^\lambda$ for all $s \in H$, get $y^\rho s = (s^\lambda y)^\rho$. □

Corollary 3.4 *Let $(G_H,+)$ be a $S_{2nd}RAP(LAP)$ of $S_{2nd}BL$. Then, $(G_H,\cdot)$ is $S_{3nd}MS(S_{2nd}MS)$ respectively if and only if it is Smarandache exponent two.*

Proof The proof follows from Theorem 3.3. □

Corollary 3.5 *Let $(G_H,+)$ be an alternative property of $S_{2nd}BL$. Then, $(G_H,\cdot)$ is S_2ndC if* *and only if it $S_{2nd}RIP$ and Smarandache exponent two.*

Proof By Theorem 3.3, we have $sy^\rho \cdot s = y \Leftrightarrow s \cdot y^\rho = y \cdot s^\rho \Leftrightarrow s \cdot y = y \cdot s$. $\square$

Corollary 3.6 *Let $(G_H, +)$ be an alternative property of $S_{2nd}BL$. Then, $(G_H, \cdot)$ is a second Smarandache Steiner loop if and only if it $S_{2nd}RIP$ and Smarandache exponent two.*

Proof Following from Theorem 3.3, this is true base on Corollaries 3.4 and 3.5. $\square$

Theorem 3.7 *Let the spacial loop $(G_H, \cdot)$ be a $S_{2nd}BL$ with Smarandache AIPL and $(G_H, +)$ be its core.*

(1) *If $(s + y)x = sx + yx$, then $(G_H, \cdot)$ is Smarandache loop. For all $x, z \in G$ and $s \in H$;*

(2) *If $x(s + y) = xs + yx$, then $(G_H, \cdot)$ is S_{2nd} flexible. For all $x, z \in G$ and $s \in H$.*

Proof (1) By using Theorems 3.3 and 2.6, we have $(sy^\rho \cdot s)x = [sx(yx)^\rho](sx) \Leftrightarrow (sy^\rho \cdot s)x = [sx(x^{-1}y^\rho)](sx) \Leftrightarrow (sy^\rho \cdot s)x = (sy^\rho)(sx)$. Let $y^\rho = t$, for all $s \in H$ and $x, t \in G$, we have $tL_sR_s \cdot x = tL_s \cdot xL_s$. Let $t = tL_s^{-1}$, then $tR_s \cdot x = t \cdot xL_s \Leftrightarrow ts \cdot x = t \cdot sx$. For all $s \in H$.

(2) By using Theorems 3.3 and 2.6, we have $x(sy^\rho \cdot s) = [xs(xy)^\rho](xs) \Leftrightarrow x(sy^\rho \cdot s) = [xs(y^\rho x^{-1}](xs) \Leftrightarrow x(sy^\rho \cdot s) = (xs)[(y^\rho x^{-1} \cdot (xs)] \Leftrightarrow x(sy^\rho \cdot s) = (xs)(y^\rho s)$. Let $t = y^\rho$, then $x(st \cdot s) = (xs)(ts) \Leftrightarrow x \cdot tL_sR_s = xL_s \cdot tR_s$. Put $x = xL_s^{-1}$, get $xL_s^{-1} \cdot tL_sR_s = x \cdot tR_s \Leftrightarrow sy \cdot s = s \cdot ys$. By letting $x = s$ for all $s \in H$ and $y \in G$. $\square$

Theorem 3.8 *Let the spacial loop $(G_H, \cdot)$ be a $S_{2nd}BL$. Let $(G_H, \circ)$ be a $S-$principal isotope of $(G_H, \cdot)$, where $x \circ y = xR_g \cdot yL_g^{-1}$ for all $x, y \in G$ and some $g \in H$. Let $(G_H, +)$ and $(G_H, \oplus)$ be the cores of $(G_H, \cdot)$ and $(G_H, \circ)$ respectively. Then $s\phi \oplus y\phi = (s + y)\phi$ if and only if*

$$((gs \cdot t^{-1})s)\phi^{-1} = [(gs)\phi^{-1} \cdot (gt)\phi^{-1}J] \cdot (gs)\phi^{-1}$$

where ϕ is $S-$permutation in G_H.

Proof Let $(G_H, \cdot)$ be a $S_{2nd}BL$. By Theorem 2.6, $(G_H, \cdot)$ is a $S_{2nd}RIPL$. Let $(G_H, \circ)$ be a S-principal isotope of $(G_H, \cdot)$ defined as $x \circ y = xR_g \cdot yL_g^{-1}$ for all $g \in H$ is the identity element in G_H. Then $y \circ y^\rho = yR_g \cdot yJ_\rho = e \Rightarrow yR_fJ = yJ_\rho L_g^{-1} \Rightarrow R_gJL_g = J_\rho$, where $J : y \mapsto y^{-1}$ and $y^\rho = yJ_\rho$ the right inverse element in G_H.

$$\begin{aligned} s \oplus y &= (s \circ yJ_\rho) \circ s \\ &= (sR_g \cdot yJ_\rho L_g^{-1})R_g \cdot sL_g^{-1} \\ &= (sR_g \cdot yR_gJL_fL_g^{-1})R_g \cdot sL_g^{-1} \\ &= (sR_g \cdot yR_fJ)R_g \cdot sL_g^{-1} \end{aligned}$$

So, $s\phi \oplus y\phi = (s + y)\phi \Leftrightarrow (sR_g\phi \cdot y\phi R_fJ)R_g \cdot s\phi L_g^{-1} = (sy^{-1} \cdot s)\phi$ for all $s \in H$ and $y \in G$.

Doing the following steps: Replace s by $sL_g\phi^{-1}$ and y by

$$\begin{aligned}
yR_g^{-1}\phi^{-1} \quad &\Leftrightarrow \quad (sLgR_g \cdot yJ)R_g \cdot s = [(sL_g\phi^{-1} \cdot yR_g^{-1}\phi^{-1}J)sL_g\phi^{-1}]\phi \\
&\Leftrightarrow \quad ((sLgR_g \cdot yJ)R_g \cdot s)\phi^{-1} = [(sL_g\phi^{-1} \cdot yR_g^{-1}\phi^{-1}J)sL_g\phi^{-1}] \\
&\Leftrightarrow \quad [((gs)(gy^{-1} \cdot g)s]\phi^{-1} = [(gs)\phi^{-1} \cdot (gg^{-1} \cdot yg^{-1})\phi^{-1}J] \cdot (gs)\phi^{-1} \\
&\Leftrightarrow \quad [((gs)(gy^{-1} \cdot g)s]\phi^{-1} = [(gs)\phi^{-1} \cdot g(g^{-1}y \cdot g^{-1})\phi^{-1}J] \cdot (gs)\phi^{-1}.
\end{aligned}$$

Now, set $t = g^{-1}y \cdot g^{-1}$, we have

$$\Leftrightarrow [(gs \cdot t^{-1})s]\phi^{-1} = [(gs)\phi^{-1} \cdot (gt)\phi^{-1}J] \cdot (gs)\phi^{-1}$$

for any $t \in G_H$. □

Theorem 3.9 *The core $(G_H, +)$ is $S-$isotopic invariant for $S_{2nd}BL$ $(G_H, \cdot)$. That is $S-$isotopic $(G_H, \cdot)$ have $S-$isomorphic $(G_H, +)$.*

Proof Use Theorem 3.8, we consider the those S-isotopes $(G_H, \circ)$, where $x \circ y = xR_g \cdot yL_g^{-1}$ for all x, y, G_H and some $g \in H$. Let $(G_H, +)$ and $(G_H, \oplus)$ be the $S_{2nd}cores$ of $(G_H, \cdot)$ and $(G_H, \circ)$ respectively. Since $(G_H, \cdot)$ is a $S_{2nd}BL$, we have $(gs \cdot t^{-1})s = g(st^{-1} \cdot s)$ for all $t \in G_H$, and $s \in H$. Using Theorem 3.8 by replacing ϕ^{-1} by L_g, we have

$$\begin{aligned}
[gs \cdot t^{-1})s]L_g^{-1} \quad &= \quad st^{-1} \cdot s \\
&\Rightarrow \quad [(gs \cdot t^{-1})s]L_g^{-1} = [(gs)L_g^{-1} \cdot (gt)L_g^{-1}J] \cdot (gs)L_g^{-1}
\end{aligned}$$

for some $g \in H$. So, by Theorem 3.8, $(G_H, +)$ and $(G_H, \oplus)$ are S-isomorphic. □

Corollary 3.10 *A $S_{2nd}BL$ is SAAIPL if and only if for each S-principal isotope of $(G_H, \circ)$, where $x \circ y = xR_g \cdot yL_g^{-1}$ for all $x, y \in G$ and some $g \in H$, $s \oplus y = (s + y)$ for all $s \in H$, and $y \in G$ where $(G_H, +)$ and $(G_H, \oplus)$ are the cores of $(G_H, \cdot)$ and $(G_H, \circ)$ respectively.*

Proof Put $\phi = e$ in Theorem 3.9, we have $((gs \cdot t^{-1})s) = [(gs) \cdot (gt)^{-1}] \cdot (gs)$ for all $s \in H$ and $t \in G_H$. Put $s = g^{-1}$, then $t^{-1}g^{-1} = (gt)^{-1}$ for all $t \in G_H$ and $g \in H$. □

References

[1] D. A. Robinson, *Bol Loops*, Ph. D thesis, University of Wisconsin, Madison, Wisconsin, 1964.

[2] D. A. Robinson, Bol loops, *Trans. Amer. Math. Soc.*, 123(1966), No. 2, 341-354

[3] Shcherbacov V. A. (2017), Elements of quasigroup theory and applications, *Monographs and Research Notes in Mathematics*, CRC Press, Boca Raton, FL.

[4] Belousov, V. D. (1967), *Foundations of the Theory of Quasigroups and Loops* (Russian), Izdat. "Nauka", Moscow 223pp.

[5] Pflugfelder, Hala O. (1990), *Quasigroups and Loops: Introduction*, Sigma Series in Pure Mathematics 7, Heldermann Verlag, Berlin. viii+147.

[6] T. G. Jaiyeola (2006), An holomorphic study of the Smarandache concept in loops, *Scientia Magna Journal*, Vol.2, No.1, pp. 1-8. MR2255667, Zbl 1157.20337. China/USA. http://doi.org/10.5281/zenodo.32297.

[7] T. G. Jaiyeola (2006), Parastrophic invariance of Smarandache quasigroups, *Scientia Magna Journal*, Vol. 2, No.3, 48-53. MR2321213, Zbl 1157.20338. China/USA. http://doi.org/10.5281/zenodo.32325.

[8] T. G. Jaiyeola (2006), Palindromic permutations and generalized Smarandache palindromic permutations, *Scientia Magna Journal*, Vol. 2, No. 3., pp. 65-77. MR2321215. China/USA. http://doi.org/10.5281/zenodo.32322.

[9] T. G. Jaiyeola (2006), On the universality of some Smarandache loops of Bol-Moufang type, *Scientia Magna Journal*, Vol. 2, No. 4, 45-58. MR2298502. China/USA. http://doi.org/10.5281/zenodo.9639 http://doi.org/10.5281/zenodo.9033.

[10] T. G. Jaiyeola (2008), A pair of Smarandachely isotopic quasigroups and loops of the same variety, *International J.Math.Combin.*, Vol. 1, 36-44. MR2404000, Zbl 1144.08300. China/USA. http://doi.org/10.5281/zenodo.32301.

[11] T.G. Jaiyeola (2008), An holomorphic study of Smarandache automorphic and cross inverse property loops, Proceedings of the 4th International Conference on Number Theory and Smarandache Problems, *Scientia Magna Journal*, Vol. 4, No. 1, 102-108. MR2440545 (2009i:20132). China/USA. https://doi.org/10.5281/zenodo.9150.

[12] T.G. Jaiyeola (2008), On Smarandache Bryant Schneider group of a Smarandache loop, *International J.Math.Combin.*, Vol. 2, 51-63. MR2435816 (2009i:20131), Zbl 1152.20057. China/USA. http://doi.org/10.5281/zenodo.820935 .

[13] T. G. Jaiyéọlá (2009), *A Study of New Concepts in Smarandache Quasigroups and Loop*, ProQuest Information and Learning(ILQ), Ann Arbor, 127pp.

[14] T. G. Jaiyeola (2008), Smarandache Isotopy Theory of Smarandache: Quasigroups and Loops, Proceedings of the 4th International Conference on Number Theory and Smarandache Problems, *Scientia Magna Journal*, Vol. 4, No. 1, 168-177. MR2440557. China/USA. http://doi.org/10.5281/zenodo.32337.

[15] T.G. Jaiyeola (2008), A double cryptography using the Smarandache keedwell cross inverse quasigroup, *International J. Math. Combin.*, Vol. 3, 28-33. MR2464630. China/USA. http://doi.org/10.5281/zenodo.823493.

[16] T.G. Jaiyeola (2009), Basic properties of second Smarandache Bol loops, *International J.Math.Combin.*, Vol. 2, 11-20. MR2555374, Zbl 1197.20057. China/USA. http://doi.org/10.5281/zenodo.32303.

[17] T. G. Jaiyeola (2011), Smarandache isotopy Of second Smarandache Bol loops, *Scientia Magna Journal*, Vol. 7, No. 1., 82-93. Zbl 1216.20060. China/USA. http://doi.org/10.5281/zenodo.234114.

[18] W. B. Vasantha Kandasamy (2002), *Smarandache Loops*, Department of Mathematics, Indian Institute of Technology, Madras, India, 128pp.

[19] W. B. Vasantha Kandasamy (2002), Smarandache loops, *Smarandache Notions Journal*, 13, 252C258.

[20] W. B. Vasantha Kandasamy (2005), Smarandache groupoids, *Scientia Magana Journal*,

No.1-2, 27C35.

[21] A. S. Muktibodh (2005), Smarandache quasigroups rings, *Scientia Magna Journal*, No.1-2, 139C144.

[22] A. S. Muktibodh (2006), Smarandache quasigroups, *Scientia Magna Journal*, No.1-2, 13C 19.

[23] T. G Jaiyéọlá, B. Osoba, and A. Oyem. (2022), Isostrophy Bryant-Schneider group-invariant of Bol loops, *Buletinul Academiei De Stiinte a Republicii Moldova*, 9(99), 3–18.

[24] B. Osoba, and T. G Jaiyéọlá. (2022), Algebraic connections between right and middle Bol loops and their cores, *Quasigroups and Related Systems*, 30(1), 149-160.

[25] T. Y Oyebo, B. Osoba and T. G Jaiyéọlá, Crypto-automorphism Group of some quasigroups, *Discussiones Mathematicae-General Algebra and Applications* (Accepted for publication).

[26] B. Osoba and Y. T. Oyebo (2022), More results on the algebraic properties of middle Bol loops, *Journal of the Nigerian Mathematical Society*, 41(2), 129-142.

[27] B. Osoba (2022), Smarandache nuclei of second Smarandache Bol loops, *Scientia Magna Journal*, 17(1), 11-21.

[28] B. Osoba and Y. T . Oyebo(2018), On relationship of multiplication groups and isostrophic quasogroups, *International Journal of Mathematics Trends and Technology* (IJMTT) C Volume 58 Issue 2 C June 2018.

[29] B.Osoba and Y. T. Oyebo (2018), On multiplication groups of middle Bol loop related to left Bol loop, *Int. J. Math. And Appl.*, 6(4), 149C155

International J.Math. Combin. Vol.2(2023), 31-37

Trees with Large Roman Domination Number

C.Sivagnanam

Department of General Requirements, University of Technology and Applied Sciences-Sur

Sultanate of Oman

E-mail: choshi71@gmail.com

Abstract: A Roman dominating function on a graph G is a function $f : V(G) \longrightarrow \{0,1,2\}$ satisfying the condition that every vertex $v \in V(G)$ for which $f(v) = 0$, is adjacent to at least one vertex u with $f(u) = 2$. The weight of a Roman dominating function f is the value $w(f) = \sum\limits_{v \in V} f(v)$. The minimum weight of a Roman dominating function is called the Roman domination number of G and is denoted by $\gamma_R(G)$. In this paper, we characterize trees with $\gamma_R \geq n - \Delta$.

Key Words: Tree, domination number, Roman dominating function, Smarandache-Roman k-dominating function, Roman domination number.

AMS(2010): 05C69.

§1. Introduction

The graph $G = (V, E)$ we mean a finite, undirected, connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. The degree of a vertex u in G is the number of edges incident with u and is denoted by $d_G(u)$, simply $d(u)$. The minimum and maximum degree of a graph G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1] and Haynes et.al [4,5].

Let $v \in V$. The open neighborhood and closed neighborhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$. If $S \subseteq V$ then $N(S) = \bigcup\limits_{v \in S} N(v)$ for all $v \in S$ and $N[S] = N(S) \cup S$. If $S \subseteq V$ and $u \in S$ then the private neighbor set of u with respect to S is defined by $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$. For any set $S \subseteq V$, the subgraph induced by S is the maximal subgraph of G with vertex set S and is denoted by $\langle S \rangle$.The vertex has degree one is called a pendant vertex. The set of all pendant vertices of a graph G is denoted as $l(G)$. A support is a vertex which is adjacent to a pendant vertex. A weak support is a vertex which is adjacent to exactly one pendant vertex. A strong support is a vertex which is adjacent to at least two pendant vertices. An unicyclic graph is a graph with exactly one cycle. A graph without cycle is called acyclic graph and a connected acyclic graph is called a tree.

A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. E.J.Cockayne et.al [2] studied the concept of Roman domination

[1]Received March 7, 2023, Accepted June 5, 2023.

first. A Roman dominating function on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $v \in V$ for which $f(v) = 0$ is adjacent to at least one vertex $u \in V$ with $f(u) = 2$. Generally, if every vertex $v \in V$ for which $f(v) = 0$ is adjacent to at least k vertices $u \in V(G)$ with $f(u) = 2$ for a function $f : V(G) \longrightarrow \{0, 1, 2\}$, such a function f is said to be a Smarandache-Roman k-dominating function, where $k \geq 1$ is an integer. Clearly, if $k = 1$, such a Smarandache-Roman k-dominating function is nothing else but the Roman dominating function. The weight of a Roman dominating function is the value $w(f) = \sum\limits_{v \in V} f(v)$. The minimum weight of a Roman dominating function is called the roman dominating number of G and is denoted by $\gamma_R(G)$.

For a graph G, let $f : V \longrightarrow \{0, 1, 2\}$ and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V : f(v) = i\}$. Note that there exists an one to one correspondence between the function $f : V \longrightarrow \{0, 1, 2\}$ and the ordered partition (V_0, V_1, V_2) of V. Thus we will write $f = (V_0, V_1, V_2)$. We say that a function $f = (V_0, V_1, V_2)$ is a γ_R-function if it is a Roman dominating function and $w(f) = \gamma_R(G)$. Also $w(f) = |V_1| + 2|V_2|$.

Erin W. Chambers et.al [3] proved that $\gamma_R(G) \leq n - \Delta + 1$. In this paper we characterize the trees with $\gamma_R \geq n - \Delta$.

§2. Family of Trees $\mathscr{G}$

Notation 2.1 The family of trees $\mathscr{G}_{33}$ is obtained from $K_{1,\Delta}$ by attaching a path on three vertices twice to a pendant vertex.

Notation 2.2 The family of trees $\mathscr{G}_{23}$ is obtained from $K_{1,\Delta}$ by attaching a path on three vertices and a path on two vertices to a pendant vertex.

Notation 2.3 The family of trees $\mathscr{G}_1$ is obtained from a tree in $\mathscr{G}_{33} \cup \mathscr{G}_{23}$ by attaching a path on three vertices twice or a path on two vertices twice or a path on three vertices and a path on two vertices to at most $\Delta - 3$ pendant vertices whose support has degree Δ.

Notation 2.4 The family of trees $\mathscr{G}_{(1)}$ is obtained from a tree in $\mathscr{G}_1$ by attaching a path $P_k, k = 1$ or 2 to the pendant vertices whose support has degree Δ.

Notation 2.5 The family of trees $\mathscr{G}_{(2)}$ is obtained from $K_{1,\Delta}$ by subdividing $\Delta - 1$ edges twice.

Notation 2.6 The family of trees $\mathscr{G}_{(3)}$ is obtained from $K_{1,\Delta}$ by subdividing twice $i, 1 \leq i \leq \Delta - 2$ edges and subdividing once $k, 0 \leq k \leq \Delta - i$ edges.

Notation 2.7 The family of trees $\mathscr{G}_{(4)}$ is obtained from $K_{1,\Delta}$ by attaching twice a path on two vertices to $i, 0 \leq i \leq \Delta - 2$ pendant vertices and attaching a path on two vertices to $k, 0 \leq k \leq \Delta - i$ pendant vertices.

Notation 2.8 The family of trees $\mathscr{G} = \{K_{1,\Delta}\} \cup \mathscr{G}_{(1)} \cup \mathscr{G}_{(2)} \cup \mathscr{G}_{(3)} \cup \mathscr{G}_{(4)}$

§3. Trees with $\gamma_R = n - \Delta + 1$

Theorem 3.1 *For a tree T, $\gamma_R(T) = n - \Delta + 1$ if and only if $T \in \mathscr{G}$.*

Proof Let T be a tree with $\gamma_R(T) = n - \Delta + 1$. Let $v \in V(T)$ such that $d(v) = \Delta$. If $\Delta = n-1$, then T is a star. Suppose $\Delta < n-1$. Let $N(v) = \{v_1, v_2, \cdots, v_\Delta\}$ and let $T_1 = \langle V - N[v]\rangle$.

Case 1. $E(T_1) = \phi$.

Then every vertex of T_1 is adjacent to a vertex in $N(v)$. Suppose $d(v_i) \geq 4$ for some $i, 1 \leq i \leq \Delta$. Let $w_1, w_2, w_3 \in N(v_i) \cap V(T_1)$. Then $f = ([N(v) - \{v_i\}] \cup \{w_1, w_2, w_3\}, V - [N[v] \cup \{w_1, w_2, w_3\}], \{v, v_i\})$ is a Roman dominating function with $w(f) = n - (\Delta+4) + 4 = n - \Delta$, which is a contradiction. Hence $d(v_i) \leq 3$ for all $i, 1 \leq i \leq \Delta$. Suppose $d(v_i) = 3$ for all $i, 1 \leq i \leq \Delta$. Then $f = (V - N(v), \phi, N(v))$ is a Roman dominating function with $w(f) = 2\Delta = n - \Delta - 1$, which is a contradiction. Hence $d(v_i) \leq 2$ for some $i, 1 \leq i \leq \Delta$.

Suppose $d(v_i) = 3, 1 \leq i \leq \Delta - 1$ and $d(v_\Delta) \leq 2$. Then

$$f = \begin{cases} (V - N(v), \phi, N(v)) & if\ d(v_\Delta) = 2 \\ (V - N(v), \{v_\Delta\}, N(v) - \{v_\Delta\}) & if\ d(v_\Delta) = 1 \end{cases}$$

is a Roman dominating function with $w(f) < n - \Delta + 1$ which is a contradiction. Hence at most $\Delta - 2$ vertices of $N(v)$ have degree 3.Thus T is isomorphic to a tree obtained from $K_{1,\Delta}$ by attaching twice a path on two vertices to $i, 0 \leq i \leq \Delta - 2$ pendant vertices and attaching a path on two vertices to $k, 0 \leq k \leq \Delta - i$ pendant vertices. Hence, $T \in \mathscr{G}_{(4)}$

Case 2. $E(T_1) \neq \phi$.

Let G_1 be any non trivial component of T_1 and we may assume without loss generality that $v_1 \in N(V(G_1))$. Suppose G_1 contains more than one pendant vertex of T. Let $w_1, w_2 \in V(G_1)$ such that $d(w_i) = 1$. Let $P = (w_1, u_1, u_2, \cdots, u_i, w_2), i \geq 1$ is a $w_1 - w_2$ path in G_1. Let $V_0 = N(v) \cup \{w_1, u_2\}, V_1 = V - [N(v) \cup \{v, w_1, u_1, u_2\}, V_2 = \{v, u_1\}$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function of T with $w(f) = n - (\Delta + 4) + 4 = n - \Delta$ which is a contradiction. Thus G_1 has exactly one pendant vertex of T and hence G_1 is a path. Let $G_1 = (x_1, x_2, \cdots, x_r)$ such that $v_1 \in N(x_1)$. If $r > 2$, then

$$f = (N(v) \bigcup \{x_1, x_3\}, V - [N(v) \cup \{x_1, x_2, x_3, v\}], \{v, x_2\})$$

is a Roman dominating function of T with $w(f) = n - (\Delta+4) + 4 = n - \Delta$ which is a contradiction. Hence $r \leq 2$. Then $G_1 = P_2$. Suppose $d(v_i) \geq 4$. Let $x_1, x_2, x_3 \in N(v_1), x_i \neq v, 1 \leq i \leq 3$. Then

$$f = ([N(v) - \{v_1\}] \cup \{x_1, x_2, x_3\}, V - [N[v] \bigcup \{x_1, x_2, x_3\}], \{v, v_1\})$$

is a Roman dominating function with $w(f) = n - \Delta$, which is a contradiction. Hence $d(v_i) \leq 3$ for all i.

Suppose $d(v_i) = 3$ for all $i, 1 \leq i \leq \Delta - 1$. Then $d(v_\Delta) \leq 2$ and then

$$f = (\bigcup_{i=1}^{\Delta-1} N(v_i), V - [\bigcup_{i=1}^{\Delta-1} N[v_i]], \bigcup_{i=1}^{\Delta-1} \{v_i\})$$

is a Roman dominating function with $w(f) = n - \Delta$, which is a contradiction. Hence at most $\Delta - 2$ vertices of $N(v)$ have degree three. If at least one vertex in $N(v)$ has degree three then $T \in \mathscr{G}_{(1)}$.

Suppose $d(v_i) \leq 2$ for all $i, 1 \leq i \leq \Delta$. Then T_1 contains maximum of Δ nontrivial components. Suppose T_1 contains Δ non trivial components $G_1, G_2, \cdots, G_\Delta$. Let $V(G_i) = \{x_{i1}, x_{i2}\}$ such that $v_i \in N(x_{i1})$. Then

$$f = (N(v) \cup \{x_{i2} : 1 \leq i \leq \Delta\}, \{v\}, \{x_{i1} : 1 \leq i \leq \Delta\})$$

is a Roman dominating function of T with $w(f) = 1 + 2\Delta = n - \Delta$ which is a contradiction. Hence T_1 contains at most $\Delta - 1$ non trivial components.

Suppose T_1 contains exactly $\Delta - 1$ non trivial components. Let $G_1, G_2, \cdots, G_{\Delta-1}$ be the non trivial components of T_1. If G_Δ is trivial component of T_1, then

$$f = ((N[v] - \{v_\Delta\}) \cup \{x_{i2} : 1 \leq i \leq \Delta - 1\}, \phi, \{x_{i1} : 1 \leq i \leq \Delta - 1\})$$

is a Roman dominating function of T with $w(f) = 2(\Delta - 1) = 2\Delta - 2 = n - \Delta - 2$ which is a contradiction. Hence T is isomorphic to a tree obtained from $K_{1,\Delta}$ by subdividing $\Delta - 1$ edges twice. Thus $T \in \mathscr{G}_{(2)}$

If T_1 contains $i, 1 \leq i \leq \Delta - 2$ non trivial components, then T is isomorphic to a tree obtained from $K_{1,\Delta}$ by subdividing twice $i, 1 \leq i \leq \Delta - 2$ edges and subdividing once $k, 0 \leq k \leq \Delta - i$ edges. Hence $T \in \mathscr{G}_{(3)}$. The converse is obvious. □

§4. Family of Trees $\mathscr{F}$

Notation 4.1 The family of trees $\mathscr{T}_1$ is obtained from $K_{1,\Delta}$ by attaching thrice a path on two vertices to a pendant vertex, attaching twice a path on two vertices to $i, 0 \leq i \leq \Delta - 3$ pendant vertices and attaching a path on two vertices to $k, 0 \leq k \leq \Delta - 1 - i$ pendant vertices.

Notation 4.2 The family of trees $\mathscr{T}_2$ is obtained from $K_{1,\Delta}$ by attaching twice a path on two vertices to $\Delta - 1$ pendant vertices and attaching a path $P_k, k = 1$ or 2 to a pendant vertex.

Notation 4.3 Let v be a vertex of degree Δ in a star graph $K_{1,\Delta}$ and let $N(v) = \{v_1, v_2, \cdots, v_\Delta\}$ The family of trees $\mathscr{T}^{(a)}$ is obtained from $K_{1,\Delta}$ by subdividing $a, 2 \leq a \leq 5$ times the edge vv_1.

Notation 4.4 The family of trees $\mathscr{T}_{1i}^{(a)}$ is obtained from a tree in $\mathscr{T}^{(a)}, 2 \leq a \leq 4$ by attaching a path $P_i, 2 \leq i \leq 4$ to the vertex of distance two from center vertex v.The family of trees $\mathscr{T}_{2i}^{(a)}$ is obtained from a tree in $\mathscr{T}^{(a)}, 4 \leq a \leq 5$ by attaching a path $P_i, 1 \leq i \leq 3$ to the vertex v_1.

Notation 4.5 The family of trees $\mathscr{T}_{1(i,j,k)}^{(a)}$ is obtained from a tree in $\mathscr{T}_{1i}^{(a)}$ by attaching a path P_3 to at most two times to some or all the vertices of $v_1, v_2, \cdots, v_j, j \leq \Delta - 3$, attaching a path P_2 at most two times to some or all the vertices of $v_{j+1}, v_{j+2}, \cdots, v_k$, $k \leq \Delta - 3$ and attaching a path P_2 at most one time to the vertices $v_{k+1}, v_{k+2}, \cdots, v_\Delta$.

Notation 4.6 The family of trees $\mathscr{T}_{2(i,j,k)}^{(a)}$ is obtained from a tree in $\mathscr{T}_{2i}^{(a)}$ by attaching a path

P_3 to at most two times to some or all the vertices of $v_2, v_3, \cdots, v_j, j \leq \Delta - 3$, attaching a path P_2 at most two times to some or all the vertices of $v_{j+1}, v_{j+2}, \cdots, v_k$, $k \leq \Delta - 3$ and attaching a path P_2 at most one time to the vertices $v_{k+1}, v_{k+2}, \cdots, v_\Delta$.

Notation 4.7 The family of trees $\mathscr{T}_i^{(3)}$ is obtained from a tree in $\mathscr{T}^{(3)}$ by attaching a path $P_i, 1 \leq i \leq 3$ to the vertex v_1.The family of trees $\mathscr{T}_{(i,j,k)}^{(3)}$ is obtained from a tree in $\mathscr{T}_i^{(3)}$ by attaching a path P_3 to at most two times to some or all the vertices of $v_2, v_3, \cdots, v_j, j \leq \Delta - 3$, attaching a path P_2 at most two times to some or all the vertices of $v_{j+1}, v_{j+2}, \cdots, v_k, k \leq \Delta - 3$ and attaching a path P_2 at most one time to the vertices $v_{k+1}, v_{k+2}, \cdots, v_\Delta$.

Notation 4.8 The family of trees $\mathscr{T}_{bc}^{(3)}$ is obtained from a tree in $\mathscr{T}^{(3)}$ by attaching the paths P_b and P_c, $2 \leq b \leq 3$, $2 \leq c \leq 3$ to the vertex v_1.The family of trees $\mathscr{T}_{(bc,j,k)}^{(3)}$ is obtained from a tree in $\mathscr{T}_{bc}^{(3)}$ by attaching a path P_3 to at most two times to some or all the vertices of $v_2, v_3, \cdots, v_j, j \leq \Delta - 3$, attaching a path P_2 at most two times to some or all the vertices of $v_{j+1}, v_{j+2}, \cdots, v_k, k \leq \Delta - 3$ and attaching a path P_2 at most one time to the vertices $v_{k+1}, v_{k+2}, \cdots, v_\Delta$.

Notation 4.9 The family of trees $\mathscr{T}_{23}^{(2)}$ is obtained from the tree $\mathscr{T}^{(2)}$ by attaching the paths P_2 and P_3, to the vertex v_1.The family of trees $\mathscr{T}_{(23,j,k)}^{(2)}$ is obtained from a tree in $\mathscr{T}_{23}^{(2)}$ by attaching a path P_3 to at most two times to some or all the vertices of $v_2, v_3, \cdots, v_j, j \leq \Delta - 3$, attaching a path P_2 at most two times to some or all the vertices of $v_{j+1}, v_{j+2}, \cdots, v_k$, $k \leq \Delta - 3$ and attaching a path P_2 at most one time to the vertices $v_{k+1}, v_{k+2}, \cdots, v_\Delta$.

Notation 4.10 The family of trees

$$\mathscr{F} = \mathscr{T}_1 \cup \mathscr{T}_2 \cup \mathscr{T}_{1(i,j,k)}^{(a)} \cup \mathscr{T}_{2(i,j,k)}^{(a)} \cup \mathscr{T}_{(i,j,k)}^{(3)} \cup \mathscr{T}_{(bc,j,k)}^{(3)} \cup \mathscr{T}_{(23,j,k)}^{(2)}.$$

§5. Trees with $\gamma_R = n - \Delta$

Theorem 5.1 *For a tree T, $\gamma_R(T) = n - \Delta$ if and only if $T \in \mathscr{F}$.*

Proof Let T be a tree with $\gamma_R(G) = n - \Delta$. Let $v \in V(T)$ such that $d(v) = \Delta$. It is clear that $\Delta < n - 1$. Let $N(v) = \{v_1, v_2, \cdots, v_\Delta\}$ and let $T_1 = \langle V - N[v] \rangle$.

Case 1. $E(T_1) = \phi$.

Then every vertex of T_1 is adjacent to a vertex in $N(v)$. Suppose $d(v_i) \geq 5$ for some $i, 1 \leq i \leq \Delta$. Let $V_0 = (N(v) \cup N(v_i)) - \{v, v_i\}, V_1 = V - [N(v) \cup N(v_i)], V_2 = \{v, v_i\}$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function with $w(f) \leq n - (\Delta + 5) + 4 = n - \Delta - 1$ which is a contradiction. Hence $d(v_i) \leq 4$ for all $i, 1 \leq i \leq \Delta$. Suppose $d(v_1) = d(v_2) = 4$. Let $N(v_1) = \{v, u_1, u_2, u_3\}$ and $N(v_2) = \{v, w_1, w_2, w_3\}$. Now we assume $V_0 = (N(v) \cup N(v_1) \cup N(v_2)) - \{v, v_1, v_2\}, V_1 = V - [N(v) \cup N(v_1) \cup N(v_2)], V_2 = \{v, v_1, v_2\}$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function with $w(f) = n - (\Delta + 4 + 3) + 6 = n - \Delta - 1$ which is a contradiction. Hence at most one vertex in $N(v)$ has degree 4.

Let $d(v_1) = 4$ and $d(v_i) \leq 3, 2 \leq i \leq \Delta$. Suppose $d(v_i) = 3$ for all $i, 2 \leq i \leq \Delta$. Then

$$f = (V - N(v), \phi, N(v))$$

is a Roman dominating function with $w(f) = 2\Delta = n - \Delta - 2$ which is a contradiction. Hence $d(v_i) = 3$ for all $i, 2 \leq i \leq \Delta - 1$ and $d(v_\Delta) \leq 2$. Then

$$f = \begin{cases} (V - N(v), \phi, N(v)) & if\ d(v_\Delta) = 2 \\ (V - N(v), \{v_\Delta\}, N(v) - \{v_\Delta\}) & if\ d(v_\Delta) = 1 \end{cases}$$

is a Roman dominating function with $w(f) = n - \Delta - 1$ which is a contradiction. Hence at most $\Delta - 3$ vertices of $N(v)$ have degree 3. Thus T is isomorphic to a tree obtained from $K_{1,\Delta}$ by attaching thrice a path on two vertices to a pendant vertex, attaching twice a path on two vertices to $i, 0 \leq i \leq \Delta - 3$ pendant vertices and attaching a path on two vertices to $k, 0 \leq k \leq \Delta - 1 - i$ pendant vertices. Thus $T \in \mathscr{T}_1$.

Suppose $d(v_i) \leq 3$ for all $i, 1 \leq i \leq \Delta$. If $d(v_i) = 3$ for all $i, 1 \leq i \leq \Delta$. Then $f = (V - N(v), \phi, N(v))$ is a Roman dominating function with $w(f) = 2\Delta = n - \Delta - 1$, which is a contradiction. Hence at least one vertex in $N(v)$ has degree less than 3. If more than two vertices of $N(v)$ have degree less than 3 then by proof as in case 1 we get a contradiction. Hence $d(v_i) = 3$ for all $i, 1 \leq i \leq \Delta - 1$. Thus T is isomorphic to a tree obtained from $K_{1,\Delta}$ by attaching twice a path on two vertices to $\Delta - 1$ pendant vertices and attaching a path $P_k, k = 1$ or 2 to a pendant vertex. Thus $T \in \mathscr{T}_2$.

Case 2. $E(T_1) \neq \phi$.

Let G_1 be any nontrivial component of T_1 and we may assume without loss of generality $v_1 \in N(V(G_1))$. Suppose G_1 contains more than two pendant vertices of T. Let $w_1, w_2, w_3 \in V(G_1)$ such that $d(w_i) = 1, 1 \leq i \leq 3$. Then there is a vertex $u \in G_1$ such that $d_{G_1}(u) \geq 3$. Let $x_1, x_2, x_3 \in N(u) \cap V(G_1)$. Then

$$f = (N(v) \cup \{x_1, x_2, x_3\}, V - (N[v] \cup \{x_1, x_2, x_3\}, \{u, v\})$$

is a Roman dominating function of T with $w(f) = n - (\Delta + 1 + 4) + 4 = n - \Delta - 1$, which is a contradiction. Hence G_1 is a path.

Subcase 2.1 $|V(G_1) \cap l(T)| = 2$.

Let $w_1, w_2 \in V(G_1)$ such that $d_T(w_i) = 1$. Let $G = (w_1, u_1, u_2, \cdots, u_k, w_2)$. Suppose $d(v_1, G_1) \geq 2$. Let $(v_1, x_1, x_2, \cdots, x_i, u_j), j \leq k$, be the shortest $v_1 - G_1$ path. Then

$$f = (N(v) \cup \{x_i, u_{j-1}, u_{j+1}\}, V - (N[v] \cup \{x_i, u_{j-1}, u_{j+1}\}, \{u_j, v\})$$

is a Roman dominating function of T with $w(f) = n - \Delta - 1$, which is a contradiction. Hence $d(v_1, G_1) = 1$. Thus $v_1u_j \in E$. Suppose $d(v_1, w_i) \geq 5, i = 1$ or 2. Let $V_0 = N(v) \cup \{u_{j-1}, u_{j+1}, u_{j+2}, u_{j+4}\}, V_2 = \{v, u_j, u_{j+3}\}, V_1 = V - (V_0 \cup V_2)$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function with $w(f) = n - (\Delta + 4 + 3) + 6 = n - \Delta - 1$, which is a contradiction.

Hence $G_1 = (w_1, u_1, u_2, \cdots, u_i, w_2), i \le 5$. If $i = 5$ then $v_1u_3 \in E$. If $i = 4$ then $v_1u_2 \in E$. If $i = 3$ then either $v_1u_1 \in E$ or $v_1u_2 \in E$. If $i = 2$ then $v_1u_1 \in E$.

Let $G_2(\neq G_1)$ be a nontrivial component of T_1. If G_2 contains more than one pendant vertex of T then there is a vertex $y_1 \in G_2$ such that $d_{G_2}(y_1) \ge 2$. Let $y_2, y_3 \in N(y_1) \cap V(G_2)$. We assume $V_0 = N(v) \cup \{u_{j-1}, u_{j+1}, y_2, y_3\}, V_2 = \{v, u_j, y_1\}$ and $V_1 = V - (V_0 \cup V_2)$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function of T with $w(f) = n-\Delta-1$, which is a contradiction. Hence every nontrivial component of T_1 except G_1 is a path. Let $G_2 = (x_1, x_2, \cdots, x_r)$ such that $v_i \in N(x_1)$ for some i. Suppose $r \ge 3$. Let $V_0 = N(v) \cup \{u_{j-1}, u_{j+1}, x_1, x_3\}, V_2 = \{v, u_j, x_2\}$ and $V_1 = V - (V_0 \cup V_2)$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function of T with $w(f) = n - \Delta - 1$, which is a contradiction. Hence $r = 2$. If all the components of T_1 are nontrivial then by similar arguments as above we get $\gamma_R \le n - \Delta - 1$, which is a contradiction and hence $T \in \mathscr{T}^{(a)}_{1(i,j,k)}$.

Subcase 2.2 $|V(G_1) \cap l(T)| = 1$.

Let $G_1 = (u_1, u_2, \cdots, u_r, w_1)$ with $d(w_1) = 1$ and let $v_1u_1 \in E$. If $r \ge 5$ then $f = (N(v) \cup \{u_1, u_3, u_4, u_6\}, V - (N[v] \cup \{u_1, u_2, u_3, u_4, u_5, u_6\}, \{v, u_2, u_5\})$ is a Roman dominating function with $w(f) = n-(\Delta+1+6)-6 = n-\Delta-1$, which is a contradiction. Hence $r \le 4$. Let $3 \le r \le 4$. Suppose $d(v_1) \ge 4$. Let $u_1, x_1, x_2 \in N(v_1)$ and let $V_0 = [N(v) \cup \{x_1, x_2, u_1, u_2, u_4\}] - \{v_1\}, V_1 = V - [N[v] \cup \{x_1, x_2, u_1, u_2, u_3, , u_4\}, V_2 = \{v, v_1, u_3\}$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function with $w(f) = n - [\Delta + 1 + 6] + 6 = n - \Delta - 1$, which is a contradiction. Hence $d(v_1) = 2$ or 3. If $d(v_1) = 3$ then there exists a path $P_j(\neq G_1), j \ge 1$ attached to v_1. Suppose $P_j = (v_1, x_1, x_2, \cdots, x_j), j \ge 3$. Now, let $V_0 = N(v) \cup \{u_1, u_3, x_1, x_3\}, V_1 = V - [N[v] \cup \{u_1, u_2, u_3, x_1, x_2, x_3\}], V_2 = \{v, u_2, x_2\}$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function with $w(f) = n - \Delta - 1$, which is a contradiction. Hence $j \le 2$. Hence by similar arguments as in case 1 we have $T \in \mathscr{T}^{(a)}_{2(i,j,k)}$. If $r = 2$ then by similar arguments as above we have $T \in \mathscr{T}^{(3)}_{(i,j,k)} \cup \mathscr{T}^{(3)}_{(bc,j,k)}$. If $r = 1$ then $T \in \mathscr{T}^{(2)}_{(23,j,k)}$. The converse is obvious. □

References

[1] G.Chartrand and L.Lesniak, *Graphs and Digraphs*, Chapman and Hall, CRC, 2005.

[2] E.J.Cockayne, P.A.Dreyer Jr, S.M.Hedetniemi and S.T.Hedetniemi, Roman domination in graphs, *Discrete Mathematics*, 278(2004), 11-22.

[3] Erin W. Chambers, Bill Kinnersley, Noah prince and Douglas B. West, extremal problems for Roman domination, *SIAM Journal on Discrete Mathematics*, 23(3)(2009), 1575-1586.

[4] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Fundamentals of Domination in Graphs,* Marcel Dekker, Inc.,New York, 1998.

[5] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Domination in Graphs- Advanced Topics,* Marcel Dekker, Inc., New York, 1998.

International J.Math. Combin. Vol.2(2023), 38-50

On the M-Polynomials and Degree-Based Topological Indices of an Important Class of Graphs

Shilpa H. C[1], Gayathri K[1], Nagesh H. M[2] and Narahari N[3]

1. Department of Mathematics, School of Applied Sciences, REVA University, Kattigenahalli, Bangalore, India
2. Department of Science & Humanities, PES University - Electronic City Campus, Hosur Road, Bangalore, India
3. Department of Mathematics, University College of Science, Tumkuru University, Tumakuru, India

E-mail: shilpahc539@gmail.com, gayathri.k@reva.edu.in, nageshhm@pes.edu, narahari@tumkuruniversity.in

Abstract: For a graph G, the M-polynomial is defined as

$$M(G;x,y)=\sum_{\delta\le i\le j\le\Delta} m_{ij}(G)x^iy^j,$$

where $m_{ij},(i,j\ge 1)$, is the number of edges uv of G such that $deg_G(u)=i$ and $deg_G(v)=j$; and δ and Δ are the minimum and maximum degree of G, respectively. The topological indices play an important role in determining physio-chemical properties of chemical graphs, among them the degree-based topological indices can be driven from an algebraic formula corresponding to the chemical graphs called M-polynomial. In this paper, we compute the closed forms of M-polynomial for cycle-star graph and the line graph of cycle-star graph. Further, we give the graphical representation of M-polynomial and derive some degree-based topological indices from M-polynomial.

Key Words: M-polynomial, degree-based topological indices, cycle-star graph, line graph.

AMS(2010): 05C07, 05C31.

§1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [5]. Throughout this paper, by a graph $G=(V,E)$, we mean a simple, undirected, finite graph of order n and size m. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. A *chemical graph (or, molecular graph)* is a labeled graph whose vertices and edges correspond to the atoms and chemical bonds of the compound, respectively. The numerical parameters of a graph which describe its topology are said to be the *topological indices* or *graph invariants*. The topological indices of a chemical or molecular graph helps us to investigate the physio-chemical properties and boiling activities.

The study of topological indices was first initiated by H. Wiener [13] in the year 1947. He introduced Weiner index in order to understand the correlation of the measured properties of

[1]Received March 1, 2023, Accepted June 6, 2023.

molecules in a compound with their structural properties. In the year 1972, the Weiner index was interpreted by Hosoya [6] using distances between vertices in a graph. Over the last decade, various topological indices were introduced and studied by different authors [1,3, 4].

There are many algebraic polynomials such as Hosoya polynomial (or, Weiner polynomial), which plays an important role in determining distance-based topological indices. Among many other algebraic polynomials, M-polynomial [2] introduced in 2015 plays an important role in determining the closed form of many degree-based topological indices. Related papers on finding topological indices using M-polynomials can be found in [7,8,9,10,11].

Sedlar [12] introduced the concept of cycle-star graph while studying additively weighted Harary index for extremal unicyclic graphs.

Definition 1.1 *A cycle-star graph, written $CS_{k,n-k}$, is a graph with n vertices consisting of the cycle graph of length k and $n-k$ leafs appended to the same vertex of the cycle.*

Clearly, cycle-star graphs are the unicyclic graphs (i.e., connected graphs containing exactly one cycle). Recently, the topological indices of unicyclic graphs attracted much attention. Studies along this line include general multiplicative Zagreb indices of unicyclic graphs, Zagreb eccentricity indices of unicyclic graphs, Maximal hyper-Zagreb index of unicyclic graphs with a given order, and matching number. However, the studies on the topological indices of the intersection graph on the vertex set of cycle-star graph was not attempted. In this paper we have made an attempt to fill this gap and study the topological indices of the cycle-star graph and line graph of the cycle-star graph through a polynomial approach.

The cycle-star graphs $CS_{3,4}$ and $CS_{4,3}$ are shown in Figure 1.

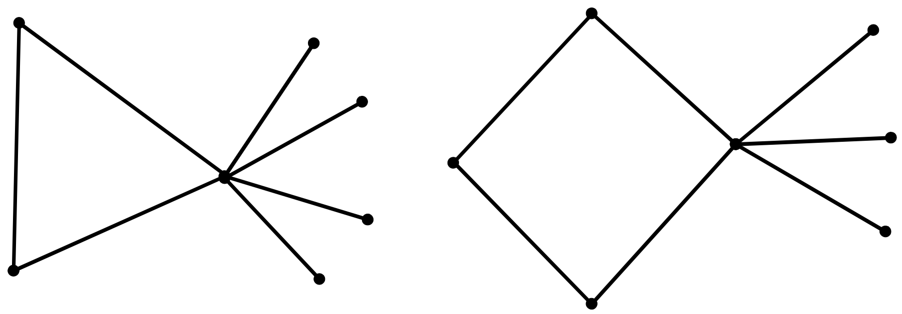


Figure 1

§2. Methodology

We first divide the edge set of cycle-star graph and the line graph of cycle-star graph into different classes based on the degree of end vertices. With the help of this edge division, we compute the M-polynomial of cycle-star graph and the line graph of cycle-star graph. Further, by using M-polynomial, we compute the degree-based topological indices as listed in Table 1. The 3-D graph of M-polynomials are sketched by using MATLAB.

§3. Preliminaries

Definition 3.1 *For a graph G, the* M-polynomial *is defined as*

$$M(G;x,y)=\sum_{\delta\le i\le j\le\Delta} m_{ij}(G)x^iy^j,$$

where $m_{ij},(i,j\ge 1)$, is the number of edges uv of G such that $deg_G(u)=i$ and $deg_G(v)=j$; and δ and Δ are the minimum and maximum degree of G, respectively.

Table 1. Operations to derive degree-based topological indices from M-polynomial

Notation	Topological Index	Derivation from $M(G;x,y)$
$M_1(G)$	First Zagreb index	$(D_x+D_y)(M(G;x,y))\|_{x=y=1}$
$M_2(G)$	Second Zagreb index	$(D_xD_y)(M(G;x,y))\|_{x=y=1}$
${}^mM_2(G)$	Second modified Zagreb index	$(S_xS_y)(M(G;x,y))\|_{x=y=1}$
$R_\alpha(G)$	Randić index	$(D_x^\alpha D_y^\alpha)(M(G;x,y))\|_{x=y=1}$
$RR_\alpha(G)$	Inverse Randić index	$(S_x^\alpha S_y^\alpha)(M(G;x,y))\|_{x=y=1}$
$SSD(G)$	Symmetric division index	$(D_xS_y+D_yS_x)(M(G;x,y))\|_{x=y=1}$
$H(G)$	Harmonic index	$2S_xJ(M(G;x,y))\|_{x=1}$
$I(G)$	Inverse sum index	$S_xJD_xD_y(M(G;x,y))\|_{x=1}$
$A(G)$	Augmented Zagreb index	$S_x^3Q_{-2}JD_x^3D_y^3(M(G;x,y))\|_{x=1}$

Here,

$$\begin{aligned}
M(G;x,y) &= f(x,y),\ D_x(f(x,y))=x\frac{\partial f(x,y)}{\partial x},\ D_y(f(x,y))=y\frac{\partial f(x,y)}{\partial y},\\
S_x(f(x,y)) &= \int_o^x \frac{f(t,y)}{t}dt,\ S_y(f(x,y))=\int_o^y \frac{f(x,t)}{t}dt,\\
J(f(x,y)) &= f(x,x) \text{ and } Q_\alpha f(x,y)=x^\alpha f(x,y).
\end{aligned}$$

are the operators.

As discussed in [2], each of these topological indices can be found using M-polynomials as given in Table 1.

§4. M-Polynomial of Cycle-Star Graph $CS_{k,n-k}$

In this section, we find the M-polynomial of cycle-star graph $CS_{k,n-k}$.

Theorem 4.1 *Let $G=CS_{k,n-k}$ be the cycle-star graph. Then the M-polynomial of G is*

$$M(G;x,y)=(k-2)x^2y^2+2x^2y^{n-k+2}+(n-k)xy^{n-k+2}.$$

Proof Let $G = CS_{k,n-k}$ be the cycle-star graph. It is easy to see from Figure 1 that $|V(G)| = n$ and $|E(G)| = k + n - k = n$. Since each of the vertices of G is of degree either 1 or 2 or $n - k + 2$, the vertex set of G has three partitions with respect to degree:

$V_1(G) = \{u \in V(G) : deg_G(u) = 1\}$;
$V_2(G) = \{u \in V(G) : deg_G(u) = 2\}$;
$V_3(G) = \{u \in V(G) : deg_G(u) = n - k + 2\}$.

Clearly, $|V_1(G)| = n - k; |V_2(G)| = k - 1; |V_3(G)| = 1$.

Further, the edge set of G has three partitions based on the degree of end vertices.

$E_1(G) = \{e = uv \in E(G) : deg_G(u) = 2, deg_G(v) = 2\}$;
$E_2(G) = \{e = uv \in E(G) : deg_G(u) = 2, deg_G(v) = n - k + 2\}$;
$E_3(G) = \{e = uv \in E(G) : deg_G(u) = 1, deg_G(v) = n - k + 2\}$.

Clearly, $|E_1(G)| = k - 2; |E_2(G)| = 2; |E_3(G)| = n - k$. Now, from the definition of M-polynomial,

$$\begin{aligned} M(G; x, y) &= \sum_{\delta \le i \le j \le \Delta} m_{ij}(G) x^i y^j \\ &= m_{22}(G) x^2 y^2 + m_{2(n-k+2)}(G) x^2 y^{n-k+2} + m_{1(n-k+2)}(G) x y^{n-k+2} \\ &= (k-2) x^2 y^2 + 2x^2 y^{n-k+2} + (n-k) x y^{n-k+2}. \end{aligned}$$

This completes the proof. □

We now compute some degree-based topological indices of the cycle-star graph using this M-polynomial.

Theorem 4.2 *Let $G = CS_{k,n-k}$ be the cycle-star graph. Then,*

$$\begin{aligned} M_1(G) &= n^2 + (5 - 2k)n + k^2 - k, \\ M_2(G) &= n^2 + (6 - 2k)n + k^2 - 2k, \\ {}^mM_2(G) &= \frac{(k+2)n - k^2}{4n - 4k + 8}, \\ R_\alpha(G) &= 4^\alpha(n - k + 2) + (n - k)(n - k + 2) + 4^\alpha(k - 2), \\ RR_\alpha(G) &= \frac{1}{n - k + 2} + 4^\alpha(n - 4) + (k - 2)n - k^2 + 4k - 4, \\ SSD(G) &= \frac{n^3 + (5 - 3k)n^2 + (3k^2 - 8k + 5)n - k^3 + 3k^2 - k}{n - k + 2}, \\ H(G) &= \frac{2\left((k+2)n^2 + (-2k^2 + 11k - 14)n + (16 - 8k)n + 8n + k^3 - 5k^2 + 2k\right)}{4n^2 + (28 - 8k)n + 4k^2 - 28k + 48}, \\ I(G) &= \frac{n^3 + (8 - 2k)n^2 + (k^2 - 9k + 14)n + k^2 - 2k}{n^2 + (7 - 2k)n + k^2 - 7k + 12}, \\ A(G) &= \frac{n^4 + (4k + 6)n^3 + (-18k^2 + 6k + 12)n^2 + (20k^3 - 30k^2 + 8)n - 7k^4 + 18k^3 - 12k^2}{n^3 + (3 - 3k)n^2 + (3k^2 - 6k + 3)n - k^3 + 3k^2 - 3k + 1}. \end{aligned}$$

Proof From Theorem 4.1, we have

$$M(G;x,y)=f(x,y)=(k-2)x^2y^2+2x^2y^{n-k+2}+(n-k)xy^{n-k+2}.$$

Then, we have the following:

$$D_x(f(x,y))=4x^2y^{n-k+2}+(n-k)xy^{n-k+2}+2(k-2)x^2y^2,$$
$$D_y((f(x,y))=2(n-k+2)x^2y^{n-k+2}+(n-k)(n-k+2)xy^{n-k+2}+2(k-2)x^2y^2,$$
$$(D_yD_x)(f(x,y))=4(n-k+2)x^2y^{n-k+2}+(n-k)(n-k+2)xy^{n-k+2}+4(k-2)x^2y^2,$$
$$S_x(f(x,y))=x^2y^{n-k+2}+(n-k)xy^{n-k+2}+\tfrac{1}{2}(k-2)x^2y^2,$$
$$S_y(f(x,y))=\tfrac{1}{2n-2k+4}\left(4x^2y^{n-k+2}+(2n-2k)xy^{n-k+2}+((k-2)n-k^2+4k-4)x^2y^2\right),$$
$$S_xS_y(f(x,y))=\tfrac{1}{4n-4k+8}\left(4x^2y^{n-k+2}+(4n-4k)xy^{n-k+2}+((k-2)n-k^2+4k-4)x^2y^2\right),$$
$$S_yD_x(f(x,y))=\tfrac{1}{n-k+2}\left((4x+n-k)xy^{n-k+2}+((k-2)n-k^2+4k-4)x^2y^2\right),$$
$$S_xD_y(f(x,y))=(n-k+2)x^2y^{n-k+2}+\left(n^2+(2-2k)n+k^2-2k\right)xy^{n-k+2}+(k-2)x^2y^2,$$

$$\begin{aligned}2S_xJ(f(x,y)) &= \frac{2((k-2)n^2+(-2k^2+11k-14)n+k^3-9k^2+26k-24)x^4}{4n^2+(28-8k)n+4k^2-28k+48}\\ &\quad+\frac{(8n-8k+24)x^{n+4-k}+(4n^2+(16-8k)n+4k^2-16k)x^{n+3-k}}{4n^2+(28-8k)n+4k^2-28k+48},\end{aligned}$$

$$\begin{aligned}S_xJD_xD_y(f(x,y)) &= \frac{\left((k-2)n^2+(-2k^2+11k-14)n+k^3-9k^2+26k-24\right)x^4}{n^2+(7-2k)n+k^2-7k+12}\\ &\quad+\frac{\left(4n^2+(20-8k)n+4k^2-20k+24\right)x^{n+4-k}}{n^2+(7-2k)n+k^2-7k+12}\\ &\quad+\frac{\left(n^3+(6-3k)n^2+(3k^2-12k+8)n-k^3+6k^2-8k\right)x^{n+3-k}}{n^2+(7-2k)n+k^2-7k+12},\end{aligned}$$

$$\begin{aligned}S_x^3Q_{-2}JD_x^3D_y^3(f(x,y)) &= \frac{\left((8k-16)n^3+(-24k^2+72k-48)n^2+(24k^3-96k^2+120k-48)n\right)x^2}{n^3+(3-3k)n^2+(3k^2-6k+3)n-k^3+3k^2-3k+1}\\ &\quad+\frac{\left(-8k^4+40k^3-72k^2+56k-16\right)x^2}{n^3+(3-3k)n^2+(3k^2-6k+3)n-k^3+3k^2-3k+1}\\ &\quad+\frac{\left(16n^3+(48-48k)n^2+(48k^2-96k+48)n\right)x^{n-k+2}}{n^3+(3-3k)n^2+(3k^2-6k+3)n-k^3+3k^2-3k+1}\\ &\quad+\frac{\left(-16k^3+48k^2-48k+16\right)x^{n-k+2}}{n^3+(3-3k)n^2+(3k^2-6k+3)n-k^3+3k^2-3k+1}\\ &\quad+\frac{\left(n^4+(6-4k)n^3+(6k^2-18k+12)n^2+(-4k^3+18k^2-24k+8)n\right)x^{n-k+1}}{n^3+(3-3k)n^2+(3k^2-6k+3)n-k^3+3k^2-3k+1}\\ &\quad+\frac{\left(k^4-6k^3+12k^2-8k\right)x^{n-k+1}}{n^3+(3-3k)n^2+(3k^2-6k+3)n-k^3+3k^2-3k+1}.\end{aligned}$$

Now, we have the following from Table 1:

1. First Zagreb index

$$M_1(G) = (D_x + D_y)(f(x,y))|_{x=y=1} = n^2 + (5-2k)n + k^2 - k.$$

2. Second Zagreb index

$$M_2(G) = (D_x D_y)(f(x,y))|_{x=y=1} = n^2 + (6-2k)n + k^2 - 2k.$$

3. Second modified Zagreb index

$${}^mM_2(G) = (S_x S_y)(f(x,y))|_{x=y=1} = \frac{(k+2)n - k^2}{4n - 4k + 8}.$$

4. Randić index

$$R_\alpha(G) = (D_x^\alpha D_y^\alpha)(f(x,y))|_{x=y=1} = 4^\alpha(n-k+2) + (n-k)(n-k+2) + 4^\alpha(k-2).$$

5. Inverse Randić index

$$\begin{aligned} RR_\alpha(G) &= (S_x^\alpha S_y^\alpha)(f(x,y))|_{x=y=1} \\ &= \frac{1}{n-k+2} + 4^\alpha(n-4) + (k-2)n - k^2 + 4k - 4. \end{aligned}$$

6. Symmetric division index

$$\begin{aligned} SSD(G) &= (D_x S_y + D_y S_x)(f(x,y))|_{x=y=1} \\ &= \frac{n^3 + (5-3k)n^2 + (3k^2 - 8k + 5)n - k^3 + 3k^2 - k}{n-k+2}. \end{aligned}$$

7. Harmonic index

$$\begin{aligned} H(G) &= 2S_x J(f(x,y))|_{x=1} \\ &= \frac{2\left((k+2)n^2 + (-2k^2 + 11k - 14)n + (16-8k)n + 8n + k^3 - 5k^2 + 2k\right)}{4n^2 + (28-8k)n + 4k^2 - 28k + 48}. \end{aligned}$$

8. Inverse sum index

$$I(G) = S_x J D_x D_y(f(x,y))|_{x=1} = \frac{n^3 + (8-2k)n^2 + (k^2 - 9k + 14)n + k^2 - 2k}{n^2 + (7-2k)n + k^2 - 7k + 12}.$$

9. Augmented Zagreb index

$$\begin{aligned} A(G) &= S_x^3 Q_{-2} J D_x^3 D_y^3(f(x,y))|_{x=1} \\ &= \frac{n^4 + (4k+6)n^3 + (-18k^2 + 6k + 12)n^2 + (20k^3 - 30k^2 + 8)n - 7k^4 + 18k^3 - 12k^2}{n^3 + (3-3k)n^2 + (3k^2 - 6k + 3)n - k^3 + 3k^2 - 3k + 1}. \end{aligned}$$

This completes the proof. □

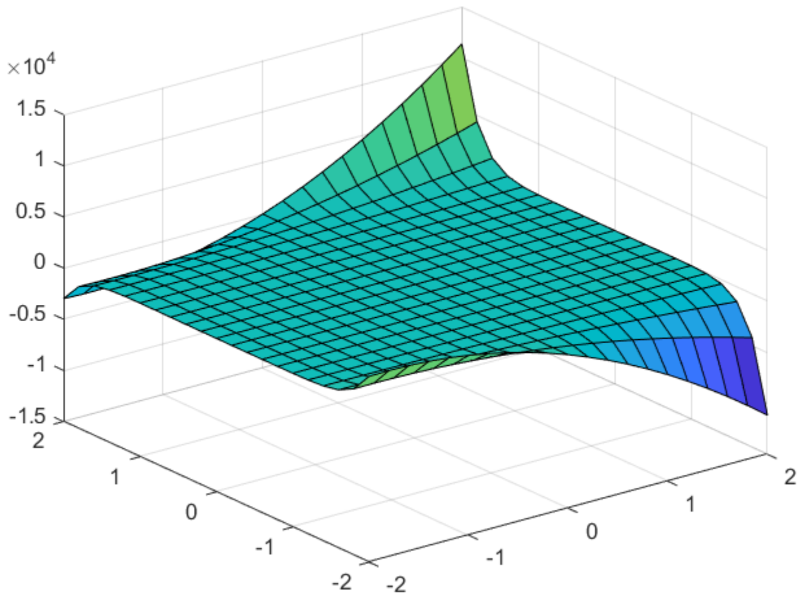


Figure 2. Plot of M-polynomial of the cycle-star graph $CS_{7,7}$

§5. M-Polynomial of Line Graph of Cycle-Star Graph $CS_{k,n-k}$

There are many graph operators with which one can construct a new graph from a given graph, such as the line graphs, total graphs, middle graphs, and their generalizations.

Definition 5.1 *A line graph of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common.*

In the next Theorem, we find the M-polynomial of the line graph of cycle-star graph.

Theorem 5.1 *Let $G = CS_{k,n-k}$ be the cycle-star graph. Then the M-polynomial of $L(G)$ is*

$$\begin{aligned} M(L(G); x, y) &= (k-3)x^2y^2 + 2x^2y^{n-k+2} + x^{n-k+2}y^{n-k+2} + 2(n-k)x^{n-k+1}y^{n-k+2} \\ &\quad + \binom{n-k}{2} x^{n-k+1}y^{n-k+1}. \end{aligned}$$

Proof Let $G = CS_{k,n-k}$ be the cycle-star graph. Then, $|V(L(G))| = n$ and $|E(L(G))| = \frac{1}{2}\left(n^2 + k^2 - 2nk + 3n - k\right)$. Since each of the vertices of $L(G)$ is of degree either 2 or $n-k+1$

or $n-k+2$, the vertex set of $L(G)$ has three partitions with respect to degree:

$$\begin{aligned}
V_1(L(G)) &= \{u \in V(L(G)) : deg_{L(G)}(u) = 2\};\\
V_2(L(G)) &= \{u \in V(L(G)) : deg_{L(G)}(u) = n-k+1\};\\
V_3(L(G)) &= \{u \in V(L(G)) : deg_{L(G)}(u) = n-k+2\}.
\end{aligned}$$

Clearly, $|V_1(L(G))| = k-2; |V_2(L(G))| = n-k; |V_3(L(G))| = 2$.

Furthermore, the edge set of $L(G)$ has five partitions based on the degree of the end vertices.

$$\begin{aligned}
E_1(L(G)) &= \{e = uv \in E(L(G)) : deg_{L(G)}(u) = 2, deg_{L(G)}(v) = 2\};\\
E_2(L(G)) &= \{e = uv \in E(L(G)) : deg_{L(G)}(u) = 2, deg_{L(G)}(v) = n-k+2\};\\
E_3(L(G)) &= \{e = uv \in E(L(G)) : deg_{L(G)}(u) = n-k+2, deg_{L(G)}(v) = n-k+2\};\\
E_4(L(G)) &= \{e = uv \in E(L(G)) : deg_{L(G)}(u) = n-k+1, deg_{L(G)}(v) = n-k+2\};\\
E_5(L(G)) &= \{e = uv \in E(L(G)) : deg_{L(G)}(u) = n-k+1, deg_{L(G)}(v) = n-k+1\}.
\end{aligned}$$

Clearly,

$$|E_1(L(G))| = k-3,\ |E_2(L(G))| = 2,\ |E_3(L(G))| = 1,$$

$$|E_4(L(G))| = 2(n-k) \text{ and } |E_5(L(G))| = \binom{n-k}{2}.$$

Now, from the definition of M-polynomial,

$$\begin{aligned}
M(L(G); x, y) &= \sum_{\delta \le i \le j \le \Delta} m_{ij}(G)x^i y^j = m_{22}(L(G))x^2y^2\\
&\quad + m_{2(n-k+2)}(L(G))x^2y^{n-k+2} + m_{(n-k+2)(n-k+2)}(L(G))x^{n-k+2}y^{n-k+2}\\
&\quad + m_{(n-k+1)(n-k+2)}(G)x^{n-k+1}y^{n-k+2}\\
&\quad + m_{(n-k+1)(n-k+1)}(L(G))x^{n-k+1}y^{n-k+1}\\
&= (k-3)x^2y^2 + 2x^2y^{n-k+2} + x^{n-k+2}y^{n-k+2}\\
&\quad + 2(n-k)x^{n-k+1}y^{n-k+2} + \binom{n-k}{2}x^{n-k+1}y^{n-k+1}
\end{aligned}$$

This completes the proof. □

We now compute some degree-based topological indices of the line graph of cycle-star graph using this M-polynomial.

Theorem 5.2 *Let $G = CS_{k,n-k}$ be the cycle-star graph. Then,*

$$\begin{aligned}
M_1(L(G)) &= 4n^2 + (-8k + 2a + 10)n + 4k^2 + (-2a - 6)k + 2a,\\
M_2(L(G)) &= 2n^3 + (-6k + a + 7)n^2 + (6k^2 + (-2a - 14)k + 2a + 12)n - 2k^3\\
&\quad +(a + 7)k^2 + (-2a - 8)k + a,\\
{}^mM_2(G) &= \frac{A_1}{B_1} + \frac{A_2}{B_2},\\
R_\alpha(G) &= 4^\alpha(n - k + 2) + (n - k)(n - k + 2) + 4^\alpha(k - 2),\\
RR_\alpha(G) &= \frac{1}{n - k + 2} + 4^\alpha(n - 4) + (k - 2)n - k^2 + 4k - 4,\\
SSD(G) &= \frac{C_1}{D_1},\\
H(G) &= \frac{E_1}{F_1} + \frac{E_2}{F_2},\\
I(G) &= \frac{G_1}{H_1} + \frac{G_2}{H_2},
\end{aligned}$$

where, $A_1 = (k-3)n^4 + (-4k^2 + 18k - 6)n^3 + (6k^3 - 36k^2 + 31k + 4a + 5)n^2$, $B_1 = 4n^4 + (24 - 16k)n^3 + (24k^2 - 72k + 52)n^2 + (-16k^3 + 72k^2 - 104k + 48)n + 4k^4 - 24k^3 + 52k^2 - 48k + 16$, $A_2 = (-4k^4 + 30k^3 - 44k^2 + (2-8a)k + 16a + 8)n + k^5 - 9k^4 + 19k^3 + (4a-7)k^2 + (-16a-4)k + 16a$, $B_2 = 4n^4 + (24-16k)n^3 + (24k^2 - 72k + 52)n^2 + (-16k^3 + 72k^2 - 104k + 48)n + 4k^4 - 24k^3 + 52k^2 - 48k + 16$, $C_1 = 5n^3 + (-13k + 2a + 13)n^2 + (11k^2 + (-4a-20)k + 6a + 10)n - 3k^3 + (2a+7)k^2 + (-6a-6)k + 4a$, $D_1 = n^2 + (3 - 2k)n + k^2 - 3k + 2$, $E_1 = (2k + 2)n^4 + (-8k^2 + 9k + 4a + 25)n^3 + (12k^3 - 39k^2 + (-12a - 26)k + 30a + 63)n^2$, $F_1 = 4n^4 + (34 - 16k)n^3 + (24k^2 - 102k + 98)n^2 + (-16k^3 + 102k^2 - 196k + 116)n + 4k^4 - 34k^3 + 98k^2 - 116k + 48$, $E_2 = (-8k^4 + 43k^3 + (12a - 23)k^2 + (-60a - 68)k + 68a + 40)n + 2k^5 - 15k^4 + (24 - 4a)k^3 + (30a + 5)k^2 + (-68a - 16)k + 48a$, $F_2 = 4n^4 + (34 - 16k)n^3 + (24k^2 - 102k + 98)n^2 + (-16k^3 + 102k^2 - 196k + 116)n + 4k^4 - 34k^3 + 98k^2 - 116k + 48$, $G_1 = 4n^4 + (-16k + 2a + 30)n^3 + (24k^2 + (-6a - 86)k + 13a + 75)n^2$, $H_1 = 4n^2 + (22 - 8k)n + 4k^2 - 22k + 24$, $G_2 = (-16k^3 + (6a+82)k^2 + (-26a-128)k + 23a + 56)n + 4k^4 + (-2a-26)k^3 + (13a+53)k^2 + (-23a-32)k + 12a$, $H_2 = 4n^2 + (22-8k)n + 4k^2 - 22k + 24$.

Proof From Theorem 5.1, we have

$$\begin{aligned}
M(L(G); x, y) &= (k - 3)x^2y^2 + 2x^2y^{n-k+2} + x^{n-k+2}y^{n-k+2}\\
&\quad +2(n - k)x^{n-k+1}y^{n-k+2} + ax^{n-k+1}y^{n-k+1},
\end{aligned}$$

where $a = \binom{n-k}{2}$. Then, we have the following:

$$\begin{aligned}
D_x(f(x, y)) &= (n - k + 2)x^{n-k+2}y^{n-k+2} + 2(n - k)(n - k + 1)x^{n-k+1}y^{n-k+2}\\
&\quad +4x^2y^{n-k+2} + a(n - k + 1)x^{n-k+1}y^{n-k+1} + 2(k - 3)x^2y^2;\\
D_y((f(x, y)) &= (n - k + 2)x^{n-k+2}y^{n-k+2} + 2(n - k)(n - k + 2)x^{n-k+1}y^{n-k+2}\\
&\quad +2(n - k + 2)x^2y^{n-k+2} + a(n - k + 1)x^{n-k+1}y^{n-k+1} + 2(k - 3)x^2y^2;
\end{aligned}$$

$$
\begin{aligned}
(D_yD_x)(f(x,y)) &= (n-k+2)^2x^{n-k+2}y^{n-k+2} \\
&\quad +2(n-k)(n-k+1)(n-k+2)x^{n-k+1}y^{n-k+2} \\
&\quad +4(n-k+2)x^2y^{n-k+2}+a(n-k+1)^2x^{n-k+1}y^{n-k+1}+4(k-3)x^2y^2; \\
S_x(f(x,y)) &= \frac{(k-3)x^2y^2}{2}+x^2y^{n-k+2}+\frac{x^{n-k+2}y^{n-k+2}}{n-k+2} \\
&\quad +\frac{2(n-k)x^{n-k+1}y^{n-k+2}}{n-k+1}+a\frac{x^{n-k+1}y^{n-k+1}}{n-k+1};
\end{aligned}
$$

$$
\begin{aligned}
S_y(f(x,y)) &= \frac{(k-3)x^2y^2}{2}+\frac{2x^2y^{n-k+2}}{n-k+2}+\frac{x^{n-k+2}y^{n-k+2}}{n-k+2} \\
&\quad +\frac{2(n-k)x^{n-k+1}y^{n-k+2}}{n-k+2}+a\frac{x^{n-k+1}y^{n-k+1}}{n-k+1}; \\
S_xS_y(f(x,y)) &= \frac{(k-3)x^2y^2}{4}+\frac{x^2y^{n-k+2}}{n-k+2}+\frac{x^{n-k+2}y^{n-k+2}}{(n-k+2)^2} \\
&\quad +\frac{2(n-k)x^{n-k+1}y^{n-k+2}}{(n-k+1)(n-k+2)}+a\frac{x^{n-k+1}y^{n-k+1}}{(n-k+1)^2}; \\
S_yD_x(f(x,y)) &= \frac{I}{(n-k+2)y^k}; \\
S_xD_y(f(x,y)) &= \frac{((k-3)n-k^2+4k-3)x^{k+2}y^2}{(n-k+1)x^2}+\frac{J}{(n-k+1)x^2};
\end{aligned}
$$

where, $I = x^{1-k}((((k-3)n-k^2+5k-6)x^{k+1}y^{k+2}+y^n((4x^{k+1}+x^n((n-k+2)x+2n^2+(2-4k)n+2k^2-2k))y^2+(an-ak+2a)x^ny))$, $J = y^n((((n^2+(3-2k)n+k^2-3k+2)x^{k+2}+x^n((n-k+1)x^2+(2n^2+(4-4k)n+2k^2-4k)x))y+(an-ak+a)x^{n+1})$,

$$
2S_xJ(f(x,y)) = \frac{K_1}{L_1}+\frac{K_2}{L_2}+\frac{K_3}{L_3}+\frac{K_4}{L_4}+\frac{K_5}{L_5}+\frac{K_6}{L_6}+\frac{K_7}{L_7},
$$

where, $K_1 = 2((16n^3+(72-48k)n^2+(48k^2-144k+104)n-16k^3+72k^2-104k+48)x^{n+k+4}$, $L_1 = (8n^4+\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $K_2 = ((2k-6)n^4+(-8k^2+41k-51)n^3+(12k^3-87k^2+202k-147)n^2)x^{2k+4}$, $L_2 = (8n^4+\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $K_3 = ((-8k^4+75k^3-251k^2+352k-174)n+2k^5-23k^4+100k^3-205k^2+198k-72)x^{2k+4})$, $L_3 = (8n^4+\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $K_4 = x^{2n}((4n^3+(26-12k)n^2+(12k^2-52k+46)n-4k^3+26k^2-46k+24)x^4$, $L_4 = (8n^4+\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $K_5 = (8n^4+(56-32k)n^3+(48k^2-168k+112)n^2+(-32k^3+168k^2-224k+64)n)x^3$, $L_5 = (8n^4+\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $K_6 = (8k^4-56k^3+112k^2-64k)x^3$, $L_6 = (\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $K_7 = (4an^3+(30a-12ak)n^2+(12ak^2-60ak+68a)n-4ak^3+30ak^2-68ak+48a)x^2)$, $L_7 = (8n^4+\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $\alpha = 68-32k, \beta =$

$48k^2-204k+196, \gamma=-32k^3+204k^2-392k+232$.

$$S_xJD_xD_y(f(x,y)) = \frac{(16n^2+(56-32k)n+16k^2-56k+48)x^{n+k+4}}{(4n^2+(22-8k)n+4k^2-22k+24)x^{2k}} + \frac{M_1}{N_1}+\frac{M_2}{N_2}+\frac{M_3}{N_3}+\frac{M_4}{N_4}+\frac{M_5}{N_5},$$

where, $M_1=((4k-12)n^2+(-8k^2+46k-66)n+4k^3-34k^2+90k-72)x^{2k+4}$, $N_1=(4n^2+(22-8k)n+4k^2-22k+24)x^{2k}$, $M_2=x^{2n}((2n^3+(15-6k)n^2+(6k^2-30k+34)n-2k^3+15k^2-34k+24)x^4$, $N_2=(4n^2+(22-8k)n+4k^2-22k+24)x^{2k}$, $M_3=4n^4+(28-16k)n^3+(24k^2-84k+56)n^2+(-16k^3+84k^2-112k+32)n$, $N_3=(4n^2+(22-8k)n+4k^2-22k+24)x^{2k}$, $M_4=4k^4-28k^3+56k^2-32k$, $N_4=(4n^2+(22-8k)n+4k^2-22k+24)x^{2k}$, $M_5=(2an^3+(13a-6ak)n^2+(6ak^2-26ak+23a)n-2ak^3+13ak^2-23ak+12a)x^2)$, $N_5=(4n^2+(22-8k)n+4k^2-22k+24)x^{2k}$.

Now, we have the following from Table 1:

1. First Zagreb index

$$M_1(L(G))=(D_x+D_y)(f(x,y))|_{x=y=1}=4n^2+(-8k+2a+10)n+4k^2+(-2a-6)k+2a.$$

2. Second Zagreb index

$$M_2(L(G)) = (D_xD_y)(f(x,y))|_{x=y=1}=2n^3+(-6k+a+7)n^2 +(6k^2+(-2a-14)k+2a+12)n-2k^3+(a+7)k^2+(-2a-8)k+a.$$

3. Second modified Zagreb index

$${}^mM_2(L(G))=(S_xS_y)(f(x,y))|_{x=y=1} = \frac{O_1}{P_1}+\frac{O_2}{P_2},$$

where, $O_1=(k-3)n^4+(-4k^2+18k-6)n^3+(6k^3-36k^2+31k+4a+5)n^2$, $P_1=4n^4+(24-16k)n^3+(24k^2-72k+52)n^2+(-16k^3+72k^2-104k+48)n+4k^4-24k^3+52k^2-48k+16$, $O_2=(-4k^4+30k^3-44k^2+(2-8a)k+16a+8)n+k^5-9k^4+19k^3+(4a-7)k^2+(-16a-4)k+16a$, $P_2=4n^4+(24-16k)n^3+(24k^2-72k+52)n^2+(-16k^3+72k^2-104k+48)n+4k^4-24k^3+52k^2-48k+16$.

4. Randić index

$$R_\alpha(L(G)) = (D_x^\alpha D_y^\alpha)(f(x,y))|_{x=y=1}=(n-k+2)^2+2^\alpha(n-k)(n-k+1)(n-k+2) +4^\alpha(n-k+2)+a(n-k+1)^2+4^\alpha(k-3).$$

5. Inverse Randić index

$$RR_\alpha(L(G)) = (S_x^\alpha S_y^\alpha)(f(x,y))|_{x=y=1}=\frac{k-3}{4^\alpha}+\frac{1}{n-k+2}+\frac{1}{(n-k+2)^2} +\frac{2^\alpha(n-k)}{(n-k+1)(n-k+2)}+a\frac{1}{(n-k+1)^2}.$$

6. Symmetric division index

$$\begin{aligned} SSD(L(G)) &= (D_xS_y + D_yS_x)(f(x,y))|_{x=y=1} \\ &= \frac{Q}{n^2+(3-2k)n+k^2-3k+2}, \end{aligned}$$

where $Q = 5n^3 + (-13k+2a+13)n^2 + (11k^2 + (-4a-20)k + 6a + 10)n - 3k^3 + (2a+7)k^2 + (-6a-6)k + 4a$.

7. Harmonic index

$$H(L(G)) = 2S_xJ(f(x,y))|_{x=1} = \frac{R_1}{S_1} + \frac{R_2}{S_2},$$

where, $R_1 = (2k+2)n^4 + (-8k^2+9k+4a+25)n^3 + (12k^3 - 39k^2 + (-12a-26)k + 30a + 63)n^2$, $S_1 = 4n^4 + (34-16k)n^3 + (24k^2 - 102k + 98)n^2 + (-16k^3 + 102k^2 - 196k + 116)n + 4k^4 - 34k^3 + 98k^2 - 116k + 48$, $R_2 = (-8k^4 + 43k^3 + (12a-23)k^2 + (-60a-68)k + 68a + 40)n + 2k^5 - 15k^4 + (24-4a)k^3 + (30a+5)k^2 + (-68a-16)k + 48a$, $S_2 = 4n^4 + (34-16k)n^3 + (24k^2 - 102k + 98)n^2 + (-16k^3 + 102k^2 - 196k + 116)n + 4k^4 - 34k^3 + 98k^2 - 116k + 48$.

8. Inverse sum index

$$I(L(G)) = S_xJD_xD_y(f(x,y))|_{x=1} = \frac{U_1}{V_1} + \frac{U_2}{V_2},$$

where, $U_1 = 4n^4 + (-16k+2a+30)n^3 + (24k^2 + (-6a-86)k + 13a + 75)n^2$, $V_1 = 4n^2 + (22-8k)n + 4k^2 - 22k + 24$, $U_2 = (-16k^3 + (6a+82)k^2 + (-26a-128)k + 23a + 56)n + 4k^4 + (-2a-26)k^3 + (13a+53)k^2 + (-23a-32)k + 12a$, $V_2 = 4n^2 + (22-8k)n + 4k^2 - 22k + 24$.

This completes the proof. □

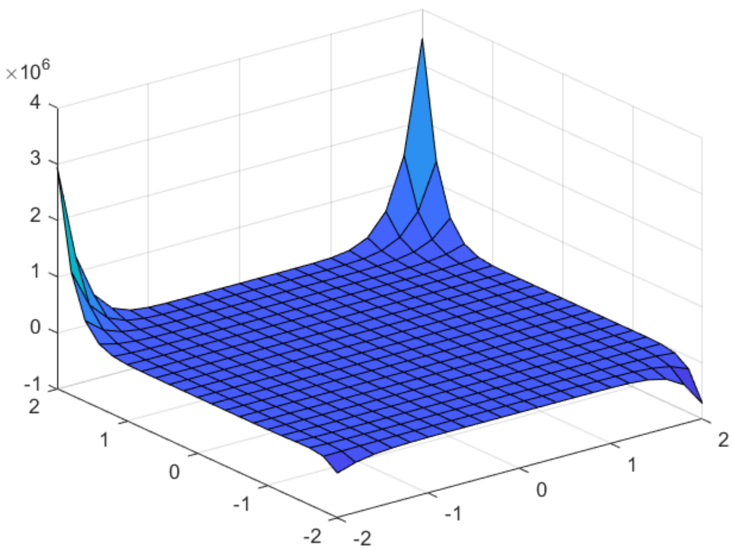


Figure 3. Plot of M-polynomial of the line graph of cycle-star graph $CS_{7,7}$

§6. Conclusion

Topological indices play an important role in understanding many physical and chemical properties of a chemical compound. Some of the degree-based topological indices can be found by means of the M-polynomial of the corresponding chemical graph. In this paper, we have determined some of these topological indices using the closed form of the M-polynomial of cycle-star graph and the line graph of cycle-star graph. The study on M-polynomials with respect to different types of graph operators seem to be much promising.

References

[1] Bruckler, F, M., Došlić, T., Graovac. A., & Gutman, I. (2011), On a class of distance - based molecular structure descriptors, *Chemical Physics Letters*, 503, (4-6), 336-338.

[2] Deutsch, E., & Klavźar, S. (2015), M-polynomial and degree-based topological indices, *Iranian Journal of Mathematical Chemistry*, 6(2), 93-102.

[3] Das, K. C., & Trinajstić, N. (2010), Comparison between first geometric - arithmetic index and atom-bond connectivity index, *Chem. Phys. Lett*, 497, 149-151.

[4] Fath-Tabar, G., Furtula, B., & Gutman, I. (2010), A new geometric-arithmetic index, *J. Math. Chem*, 47, 477-486.

[5] Harary, F. (1969), *Graph Theory*, Addison Wesley, Reading.

[6] Hosoya, H. (1971), Topological index, a newly proposed quantity characterizing the topological nature of structure isomers of saturated hydrocarbons, *Bull. Chem. Soc. Jpn*, 44, 2332-2339.

[7] Khalaf, A, J, M., Hussain, S., Afzal, D., Afzal, F., & Maqbool, A. (2020), M-polynomial and topological indices of book graph, *Journal of Discrete Mathematical Sciences and Cryptography*, 23(6), 1217-1237.

[8] Kwun, Y. C., Munir, M., Nazeer, W., Rafique, S., & Min Kang, S. (2017), M-Polynomials and topological indices of V-Phenylenic nanotubes and nanotori, *Scientific reports*, 7(1), 1-9.

[9] Munir, M., Nazeer, W., Rafique, S., & Kang, S. M. (2016), M-polynomial and degree-based topological indices of polyhex nanotubes, *Symmetry*, 8(12), 149.

[10] Munir, M., Nazeer, W., Nizami, A. R., Rafique, S., & Kang, S. M. (2016), M-polynomials and topological indices of titania nanotubes. *Symmetry*, 8(11), 117.

[11] Swamy, N. N., Gangappa, C. K., Poojary, P., Sooryanarayana, B., & Nagesh, H. M. (2022), Topological indices of the subdivision graphs of the nanostructure $TUC_4C_8(R)$ using M-polynomials, *Journal of Discrete Mathematical Sciences and Cryptography*, 25(1), 265-282.

[12] Sedlar, J. (2013), Extremal unicyclic graphs with respect to additively weighted Harary index, *Miskolic mathematical Notes*, 16(2), 1-16.

[13] Wiener, H. (1947), Structural determination of the paraffin boiling points, *J. Am. Chem. Soc*, 69, 17-20.

International J.Math. Combin. Vol.2(2023), 51-58

Further Results on 4-Total Mean Cordial Graphs

R. Ponraj

Department of Mathematics, Sri Paramakalyani College, Alwarkurichi – 627 412, India

S.Subbulakshmi

Research Scholar, Reg.No: 19124012092011

Department of Mathematics, Manonmaniam Sundaranar University, Abhishekapati, Tirunelveli–627 012, India

M. Sivakumar

Department of Mathematics, Government Arts and Science College,Tittagudi-606106, Tamilnadu, India

E-mail: ponrajmaths@gmail.com, ssubbulakshmis@gmail.com, sivamaths1975@gmail.com

Abstract: Let G be a graph. Let $f : V(G) \to \{0, 1, 2, \ldots, k-1\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label

$$f(uv) = \left\lceil \frac{f(u) + f(v)}{2} \right\rceil$$

and f is called a k-total mean cordial labeling of G if $|t_{mf}(i) - t_{mf}(j)| \leq 1$, for all $i, j \in \{0, 1, 2, \ldots, k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0, 1, 2, \ldots, k-1\}$. A graph with admit a k-total mean cordial labeling is called k-total mean cordial graph. In this paper we investigate the 4-Total mean cordial labeling behavior of some graphs like $C_4 \times P_n$, middle graph of P_n, total graph of P_n, middle graph of C_n, total graph of C_n and kayak paddale graph.

Key Words: Total mean cordial labelling, Smarandachely total mean cordial labeling, middle graph, total graph.

AMS(2010): 05C78.

§1. Introduction

In this paper we consider simple, finite and undirected graphs only. Cordial labeling was introduced by Cahit [1]. The notion of k-total mean cordial labeling has been introduced in [5]. The 4-total mean cordial labeling behaviour of several graphs like cycle, complete graph, star, bistar, comb and crown have been studied in [5, 6, 7, 8, 9, 10, 11, 12, 13]. Edge-Odd gracefulness of middle graphs and total graphs of certain graphs was studied in [4]. In this paper we investigate the 4- total mean cordial labeling of middle graph of the path P_n, total graph of the path P_n, middle graph of the cycle C_n, total graph of the cycle C_n, $C_4 \times P_n$ and kayak paddale graph. Let x be any real number. Then $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Terms are not defined here follow from Harary [3] and Gallian [2].

[1]Received February 5, 2023, Accepted June 7, 2023.

.

§2. k-Total Mean Cordial Graph

Definition 2.1 *Let G be a graph. Let $f : V(G) \to \{0, 1, 2, \ldots, k-1\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a k-total mean cordial labeling of G if $|t_{mf}(i) - t_{mf}(j)| \leq 1$, for all $i, j \in \{0, 1, 2, \ldots, k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0, 1, 2, \ldots, k-1\}$. A graph with admit a k-total mean cordial labeling is called k-total mean cordial graph.*

Such a labeling f is called a Smarandachely k-total mean cordial labeling of G if there are integers $i, j \in \{0, 1, 2, \cdots, k-1\}$ hold with $|t_{mf}(i) - t_{mf}(j)| \geq 2$ and G is called a Smarandachely k-total mean cordial graph.

§3. Preliminaries

Definition 3.1([3]) *A middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.*

Definition 3.2([3]) *A total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent whenever they are either adjacent or incident in G.*

Definition 3.3([3]) *A Kayak Paddale $KP(m, n, l)$ is the graph obtained by joining the cycles C_m and C_n with the path P_{l+1} of length l. Let C_m be the cycle $u_1\ u_2\ \ldots\ u_n\ u_1$ and C_n be the cycle $v_1\ v_2\ \ldots\ v_n\ v_1$. Let P_{l+1} be the path $w_1\ w_2\ \ldots\ w_n$. Identify u_1 with w_1 and w_n with v_1.*

§4. Main Results

Theorem 4.1 *A graph $C_4 \times P_n$ is a 4-total mean cordial for all $n \geq 2$.*

Proof Let $V(C_4 \times P_n) = \{a_i, b_i, c_i, d_i : 1 \leq i \leq n\}$ and $E(C_4 \times P_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, d_i d_{i+1} : 1 \leq i \leq n-1\} \bigcup \{a_i b_i, b_i c_i, c_i d_i, d_i a_i : 1 \leq i \leq n\}$. Obviously,

$$|V(C_4 \times P_n)| + |E(C_4 \times P_n)| = 12n - 4.$$

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4r$, $r \in \mathbb{N}$. Assign the label 0 to the $4r-1$ vertices a_1, a_2, $\cdots$, a_{4r-1}. Now we assign the label 2 to the vertex a_{4r}. Next we assign the label 3 to the $4r$ vertices b_1, b_2, $\cdots$, b_{4r}. We now assign the label 0 to the $2r$ vertices c_1, c_2, $\ldots$, c_{2r}. Now we assign the label 1 to the $r-1$ vertices c_{2r+1}, c_{2r+2}, $\cdots$, c_{3r-1}. Next we assign the label 3 to the r vertices c_{3r}, c_{3r+1}, $\cdots$, c_{4r-1}. Now we assign the label 0 to the vertex c_{4r}. We now assign the label 1 to the $2r$ vertices d_1, d_2, $\cdots$, d_{2r}. Now we assign the label 2 to the $2r-1$ vertices d_{2r+1}, d_{2r+2}, $\cdots$, d_{4r-1}. Finally we assign the label 0 to the vertex d_{4r}.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r+1$, $r \in \mathbb{N}$. Assign the label 0 to the $4r+1$ vertices $a_1, a_2, \cdots, a_{4r+1}$. Next we assign the label 3 to the $4r+1$ vertices $b_1, b_2, \cdots, b_{4r+1}$. Now we assign the label 0 to the $2r+1$ vertices $c_1, c_2, \cdots, c_{2r+1}$. We now assign the label 1 to the r vertices $c_{2r+2}, c_{2r+3}, \cdots, c_{3r+1}$. Next we assign the label 3 to the r vertices $c_{3r+2}, c_{3r+3}, \cdots, c_{4r+1}$. We now assign the label 1 to the $2r+1$ vertices $d_1, d_2, \cdots, d_{2r+1}$. Now we assign the label 2 to the $2r-1$ vertices $d_{2r+2}, d_{2r+3}, \cdots, d_{4r}$. Finally we assign the label 3 to the vertex d_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r+2$, $r \geq 0$. Label the vertices a_i, b_i, c_i, d_i $(1 \leq i \leq 4r+1)$ as in Case 1. Next we assign the labels 2, 3, 0, 0 to the vertices $a_{4r+2}, b_{4r+2}, c_{4r+2}, d_{4r+2}$.

Case 4. $n \equiv 3 \pmod 2$.

Let $n = 4r+3$, $r \geq 0$. Assign the label 0 to the $4r+3$ vertices $a_1, a_2, \cdots, a_{4r+3}$. Now we assign the label 3 to the $4r+3$ vertices $b_1, b_2, \cdots, b_{4r+3}$. Next we assign the label 0 to the $2r+2$ vertices $c_1, c_2, \cdots, c_{2r+2}$. Now we assign the label 1 to the r vertices $c_{2r+3}, c_{2r+4}, \cdots, c_{3r+2}$. Next we assign the label 3 to the $r+1$ vertices $c_{3r+3}, c_{3r+4}, \ldots, c_{4r+3}$. We now assign the label 1 to the $2r+2$ vertices $d_1, d_2, \cdots, d_{2r+2}$. Now we assign the label 2 to the $2r+1$ vertices $d_{2r+3}, d_{2r+4}, \cdots, d_{4r+3}$.

Thus, this vertex labeling f is a 4-total mean cordial labeling follows from the Table 1.

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$12r-1$	$12r-1$	$12r-1$	$12r-1$
$n = 4r+1$	$12r+2$	$12r+2$	$12r+2$	$12r+2$
$n = 4r+2$	$12r+5$	$12r+5$	$12r+5$	$12r+5$
$n = 4r+3$	$12r+8$	$12r+8$	$12r+8$	$12r+8$

Table 1

This completes the proof. □

Theorem 4.2 *A middle graph of the path P_n, $M(P_n)$ is a 4-total mean cordial for all values of $n \geq 2$.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of path P_n and let $v_1, v_2, \cdots, v_{n-1}$ be the added vertices corresponding to the edges $e_1, e_2, \cdots, e_n$ of P_n to obtain $M(P_n)$. Let $V(M(P_n)) = \{u_i : 1 \leq i \leq n\} \bigcup \{v_i : 1 \leq i \leq n-1\}$, $E(M(P_n)) = \{u_iv_i, v_iu_{i+1} : 1 \leq i \leq n-1\} \bigcup \{v_iv_{i+1} : 1 \leq i \leq n-2\}$. Clearly, $|V(M(P_n))| + |E(M(P_n))| = 5n-5$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4r$, $r \geq 1$. Assign the label 0 to the r vertices $u_1, u_2, \cdots, u_r$. Now we assign the label 1 to the r vertices $u_{r+1}, u_{r+2}, \cdots, u_{2r}$. Next we assign the label 2 to the $r-1$ vertices $u_{2r+1}, u_{2r+2}, \cdots, u_{3r-1}$. We now assign the label 3 to the r vertices $u_{3r}, u_{3r+1}, \cdots, u_{4r-1}$. Next we assign the label 0 to the vertex u_{4r}.

Now we assign the label 0 to the r vertices v_1, v_2, $\cdots$, v_r. Next we assign the label 1 to the $r-1$ vertices v_{r+1}, v_{r+2}, $\cdots$, v_{2r-1}. We now assign the label 2 to the r vertices v_{2r}, v_{2r+1}, $\cdots$, v_{3r-1}. Next we assign the label 3 to the $r-1$ vertices v_{3r}, v_{3r+1}, $\cdots$, v_{4r-2}. Finally, we assign the label 2 to the vertex v_{4r-1}.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r+1$, $r \geq 1$. Assign the label 0 to the $r+1$ vertices u_1, u_2, $\cdots$, u_{r+1}. Next we assign the label 1 to the r vertices u_{r+2}, u_{r+3}, $\cdots$, u_{2r+1}. Now we assign the label 2 to the r vertices u_{2r+2}, u_{2r+3}, $\cdots$, u_{3r+1}. We now assign the label 3 to the r vertices u_{3r+2}, u_{3r+3}, $\cdots$, u_{4r+1}.

Next we assign the label 0 to the r vertices v_1, v_2, $\cdots$, v_r. Now we assign the label 1 to the r vertices v_{r+1}, v_{r+2}, $\cdots$, v_{2r}. We now assign the label 2 to the r vertices v_{2r+1}, v_{2r+2}, $\cdots$, v_{3r}. Next we assign the label 3 to the r vertices v_{3r+1}, v_{3r+2}, $\cdots$, v_{4r}.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r+2$, $r \geq 1$. Label the vertices u_i $(1 \leq i \leq 4r+1)$, v_i $(1 \leq i \leq 4r)$ as in Case 2. Now we assign the labels 0, 2 to the vertex u_{4r+2},v_{4r+1}.

Case 4. $n \equiv 3 \pmod 2$.

Let $n = 4r+3$, $r \geq 1$. Assign the label 0 to the $r+1$ vertices u_1, u_2, $\cdots$, u_{r+1}. Now we assign the label 1 to the $r+1$ vertices u_{r+2}, u_{r+3}, $\cdots$, u_{2r+2}. We now assign the label 2 to the r vertices u_{2r+3}, u_{2r+4}, $\cdots$, u_{3r+2}. Next we assign the label 3 to the $r+1$ vertices u_{3r+3}, u_{3r+4}, $\cdots$, u_{4r+3}.

Now we assign the label 0 to the $r+1$ vertices v_1, v_2, $\cdots$, v_{r+1}. Next we assign the label 1 to the r vertices v_{r+2}, v_{r+3}, $\cdots$, v_{2r+1}. We now assign the label 2 to the $r+1$ vertices v_{2r+2}, v_{2r+3}, $\cdots$, v_{3r+2}. Finally, we assign the label 3 to the r vertices v_{3r+3}, v_{3r+4}, $\ldots$, v_{4r+2}.

Thus, this vertex labeling f is a 4-total mean cordial labeling follows from the Table 2.

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$5r-1$	$5r-2$	$5r-1$	$5r-1$
$n = 4r+1$	$5r$	$5r$	$5r$	$5r$
$n = 4r+2$	$5r+1$	$5r+1$	$5r+1$	$5r+2$
$n = 4r+3$	$5r+3$	$5r+2$	$5r+3$	$5r+2$

Table 2

Case 5. $n = 2$ or 3.

A 4-total mean cordial labeling of $M(P_n)$ is given in Tabel 3.

Value of n	u_1	u_2	u_3	v_1	v_2
2	0	3		2	
3	0	1	3	0	3

Table 3

This completes the proof. $\square$

Theorem 4.3 *A total graph of the path P_n, $T(P_n)$ is a 4-total mean cordial for all values of $n \geq 2$.*

Proof Clearly, the vertex labeling of Theorem 4.2 is also a 4-total mean cordial labeling of $T(P_n)$. □

Theorem 4.4 *A middle graph of the cycle C_n, $M(C_n)$ is a 4-total mean cordial for all values of $n \geq 3$.*

Proof Let u_1, u_2, $\cdots$, u_n be the vertices of cycle C_n and let v_1, v_2, $\cdots$, v_n be the added vertices corresponding to the edges e_1, e_2, $\cdots$, e_n of C_n to obtain $M(C_n)$. Let $V(M(C_n)) = \{u_i, v_i : 1 \leq i \leq n\}$ and let $E(M(C_n)) = \{u_iv_i : 1 \leq i \leq n\} \bigcup \{v_iv_{i+1}, v_iu_{i+1} : 1 \leq i \leq n-1\} \bigcup \{v_1v_n, v_nu_1\}$. Notice that $|V(M(C_n))| + |E(M(C_n))| = 5n$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4r$, $r \geq 1$. Assign the label 0 to the r vertices v_1, v_2, $\cdots$, v_r. Next we assign the label 1 to the r vertices v_{r+1}, v_{r+2}, ..., v_{2r}. We now assign the label 2 to the r vertices v_{2r+1}, v_{2r+2}, $\cdots$, v_{3r}. Now we assign the label 3 to the r vertices v_{3r+1}, v_{3r+2}, $\cdots$, v_{4r}.

Next we assign the label 0 to the r vertices u_1, u_2, $\cdots$, u_r. Now we assign the label 1 to the r vertices u_{r+1}, u_{r+2}, $\cdots$, u_{2r}. We now assign the label 2 to the $r-1$ vertices u_{2r+1}, u_{2r+2}, $\cdots$, u_{3r-1}. Next we assign the label 3 to the r vertices u_{3r}, u_{3r+1}, $\cdots$, u_{4r-1}. Finally we assign the label 0 to the vertex u_{4r}.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r+1$, $r \geq 1$. Assign the label 0 to the $r+1$ vertices v_1, v_2, $\cdots$, v_{r+1}. Now we assign the label 1 to the r vertices v_{r+2}, v_{r+3}, ..., v_{2r+1}. Next we assign the label 2 to the r vertices v_{2r+2}, v_{2r+3}, $\cdots$, v_{3r+1}. We now assign the label 3 to the r vertices v_{3r+2}, v_{3r+3}, $\cdots$, v_{4r+1}.

Now we assign the label 0 to the r vertices u_1, u_2, $\cdots$, u_r. Next we assign the label 1 to the r vertices u_{r+1}, u_{r+2}, $\cdots$, u_{2r}. We now assign the label 2 to the r vertices u_{2r+1}, u_{2r+2}, $\cdots$, u_{3r}. Now we assign the label 3 to the r vertices u_{3r+1}, u_{3r+2}, $\cdots$, u_{4r}. Finally we assign the label 2 to the vertex u_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r+2$, $r \geq 1$. Assign the label 0 to the $r+1$ vertices v_1, v_2, $\cdots$, v_{r+1}. Next we assign the label 1 to the $r+1$ vertices v_{r+2}, v_{r+3}, $\cdots$, v_{2r+2}. We now assign the label 2 to the r vertices v_{2r+3}, v_{2r+4}, $\cdots$, v_{3r+2}. Now we assign the label 3 to the r vertices v_{3r+3}, v_{3r+4}, $\cdots$, v_{4r+2}.

Next we assign the label 0 to the $r+1$ vertices u_1, u_2, $\cdots$, u_{r+1}. Now we assign the label 1 to the r vertices u_{r+2}, u_{r+3}, $\cdots$, u_{2r+1}. We now assign the label 2 to the r vertices u_{2r+2}, u_{2r+3}, $\cdots$, u_{3r+2}. Finally we assign the label 3 to the $r+1$ vertices u_{3r+3}, u_{3r+4}, $\cdots$, u_{4r+2}.

Case 4. $n \equiv 3 \pmod 2$.

Let $n = 4r+3$, $r \geq 1$. Assign the label 0 to the $r+1$ vertices v_1, v_2, $\cdots$, v_{r+1}. Now we assign the label 1 to the $r+1$ vertices v_{r+2}, v_{r+3}, $\cdots$, v_{2r+2}. We now assign the label 2 to the r vertices v_{2r+3}, v_{2r+4}, $\cdots$, v_{3r+2}. Now we assign the label 3 to the $r+1$ vertices v_{3r+3}, v_{3r+4}, $\cdots$, v_{4r+3}.

Next we assign the label 0 to the $r+1$ vertices u_1, u_2, $\cdots$, u_{r+1}. Now we assign the label 1 to the r vertices u_{r+2}, u_{r+3}, $\cdots$, u_{2r+1}. We now assign the label 2 to the $r+1$ vertices u_{2r+2}, u_{2r+3}, $\cdots$, u_{3r+2}. Next we assign the label 3 to the r vertices u_{3r+3}, u_{3r+4}, $\cdots$, u_{4r+2}. Finally we assign the label 2 to the vertex u_{4r+3}.

Thus this vertex labeling f is a 4-total mean cordial labeling follows from the Table 4.

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n=4r$	$5r$	$5r$	$5r$	$5r$
$n=4r+1$	$5r+1$	$5r+1$	$5r+2$	$5r+1$
$n=4r+2$	$5r+3$	$5r+3$	$5r+2$	$5r+2$
$n=4r+3$	$5r+3$	$5r+4$	$5r+4$	$5r+4$

Table 4

This completes the proof. □

Theorem 4.5 *A total graph of the cycle C_n, $T(C_n)$ is a 4-total mean cordial if $n \equiv 0, 2 \pmod 4$.*

Proof Obviously, the vertex labeling of Theorem **??** is also a 4-total mean cordial labeling of $T(C_n)$. □

Theorem 4.6 *A Kayak Paddale $KP(n,n,n)$ is a 4-total mean cordial for all values of $n \geq 3$.*

Proof Let $V(KP(n,n,n)) = \{u_i, v_i : 1 \leq i \leq n\} \bigcup \{u_1 = w_{1,v_1=w_{n+1}}\} \bigcup \{w_i : 2 \leq i \leq n\}$ and let $E(KP(n,n,n)) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n\} \bigcup \{u_1u_n, v_1v_n\} \bigcup \{w_{1-1}w_i : 2 \leq i \leq n\}$. Notice that $|V(KP(n,n,n))| + |E(KP(n,n,n))| = 6n-1$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4r$, $r \geq 1$. Assign the label 0 to the $r+1$ vertices u_1, u_2, $\cdots$, u_{r+1}. Next we assign the label 1 to the $3r-1$ vertices u_{r+2}, u_{r+3}, $\cdots$, u_{4r}.
Now we assign the label 3 to the r vertices v_1, v_2, $\cdots$, v_r. Now we assign the label 2 to the $3r-1$ vertices v_{r+1}, v_{r+2}, $\cdots$, v_{4r-1}. Then we assign the label 0 to the vertex v_{4r}.

Next we assign the label 0 to the $2r-1$ vertices w_2, w_2, $\cdots$, w_{2r}. Finally we assign the label 3 to the $2r$ vertices w_{2r+1}, w_{2r+2}, $\cdots$, w_{4r}.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r+1$, $r \geq 1$. Now we assign the label 0 to the $r+1$ vertices u_1, u_2, $\cdots$, u_{r+1}. Next we assign the label 1 to the $3r$ vertices u_{r+2}, u_{r+3}, $\cdots$, u_{4r+1}.

Next we assign the label 2 to the $r+1$ vertices v_1, v_2, $\cdots$, v_{r+1}. We now assign the label 3 to the $3r$ vertices v_{r+2}, v_{r+3}, $\cdots$, v_{4r+1}.

Now we assign the label 0 to the $2r$ vertices $w_2, w_3, \cdots, w_{2r+1}$. Next we assign the label 2 to the $2r$ vertices $w_{2r+2}, w_{2r+3}, \cdots, w_{4r+1}$.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r + 2$, $r \geq 1$. We now assign the label 0 to the $r + 2$ vertices $u_1, u_2, \cdots, u_{r+2}$. Next we assign the label 1 to the $3r$ vertices $u_{r+3}, u_{r+4}, \cdots, u_{4r+2}$.

Now we assign the label 2 to the $r + 1$ vertices $v_1, v_2, \cdots, v_{r+1}$. Next we assign the label 3 to the $3r + 1$ vertices $v_{r+2}, v_{r+3}, \cdots, v_{4r+2}$.

We now assign the label 0 to the $2r$ vertices $w_2, w_3, \cdots, w_{2r+1}$. Finally we assign the label 2 to the $2r + 1$ vertices $w_{2r+2}, w_{2r+3}, \cdots, w_{4r+2}$.

Case 4. $n \equiv 3 \pmod 2$.

Let $n = 4r + 3$, $r \geq 1$. Assign the label 0 to the $r + 2$ vertices $u_1, u_2, \cdots, u_{r+2}$. Now we assign the label 1 to the $3r + 1$ vertices $u_{r+3}, u_{r+4}, \cdots, u_{4r+3}$.
We now assign the label 3 to the $r + 1$ vertices $v_1, v_2, \cdots, v_{r+1}$. Next we assign the label 2 to the $3r + 1$ vertices $v_{r+2}, v_{r+3}, \cdots, v_{4r+2}$. Now we assign the label 1 to the vertex v_{4r+3}.

Next we assign the label 0 to the $2r + 1$ vertices $w_2, w_3, \cdots, w_{2r+2}$. Finally we assign the label 3 to the $2r + 1$ vertices $w_{2r+3}, w_{2r+4}, \cdots, w_{4r+3}$.

Thus, this vertex labeling f is a 4-total mean cordial labeling follows from the Table 5.

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$6r$	$6r$	$6r - 1$	$6r$
$n = 4r + 1$	$6r + 1$	$6r + 2$	$6r + 1$	$6r + 1$
$n = 4r + 2$	$6r + 3$	$6r + 2$	$6r + 3$	$6r + 3$
$n = 4r + 3$	$6r + 5$	$6r + 4$	$6r + 4$	$6r + 4$

Table 5

This completes the proof. □

References

[1] I.Cahit, Cordial graphs: A weaker version of graceful and harmonious graphs, *Ars combin.*, 23 (1987) 201-207.

[2] J.A.Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 19 (2016) #Ds6.

[3] Harary, *Graph Theory*, Addision Wesley, New Delhi (1969).

[4] S.Padmapriya, B.Sooryanarayana, Edge-odd gracefulness of middle graphs and total graphs of paths and cycles, *International Journal of Engineering Research and Technology*, Vol. 2 (2013) 1433-1436.

[5] R.Ponraj, S.Subbulakshmi, S.Somasundaram, k-Total mean cordial graphs, *J.Math.Comput.Sci.*, 10(2020), No.5, 1697-1711.

[6] R.Ponraj, S.Subbulakshmi, S.Somasundaram, 4-Total mean cordial graphs derived from paths, *J.Appl and Pure Math.*, Vol 2(2020), 319-329.

[7] R.Ponraj, S.Subbulakshmi, S.Somasundaram, 4-Total mean cordial labeling in subdivision graphs, *Journal of Algorithms and Computation*, 52(2020), 1-11.

[8] R.Ponraj, S.Subbulakshmi, S.Somasundaram, Some 4-total mean cordial graphs derived from wheel, *J. Math. Comput. Sci.*, 11(2021), 467-476.

[9] R.Ponraj, S.Subbulakshmi, S.Somasundaram, 4-Total mean cordial graphs with star and bistar, *Turkish Journal of Computer and Mathematics Education*, 12(2021), 951-956.

[10] R.Ponraj, S.Subbulakshmi, S.Somasundaram, On 4-total mean cordial graphs, *J. Appl. Math and Informatics,* Vol 39(2021), 497-506.

[11] R.Ponraj, S.Subbulakshmi, S.Somasundaram, 4-Total mean cordial labeling of special graphs, *Journal of Algorithms and Computation,* 53(2021), 13-22.

[12] R.Ponraj, S.Subbulakshmi, S.Somasundaram, 4-Total mean cordial labeling of union of some graphs with the complete bipartite graph $K_{2,n}$, *Journal of Algorithms and Computation,* 54(2022), 35-46.

[13] R.Ponraj, S.Subbulakshmi, S.Somasundaram, 4-Total mean cordial labeling of some graphs derived from H-graph and star, *International J. Math.Combin,* Vol 3(2022), 99-106.

International J.Math. Combin. Vol.2(2023), 59-64

On Full Block Signed Graphs

Swamy[1], P. Somashekar[2] and Khaled A. A. Alloush[3]

1. Department of Mathematics, Maharani's Science College for Women, Mysore-570 005, India

2. Department of Mathematics, Government First Grade College, Ooty Road, Nanjangud-571 301, India

3. Department of Mathematics, Dar Al-Uloom University, Riyadh-13314, Saudi Arabia

E-mail: sgautham48@gmail.com, somashekar2224@gmail.com, khaledindia@gmail.com

Abstract: In this paper we introduced the new notion called full block signed graph of a signed graph and its properties are studied. Also, we obtained the structural characterization of this new notion and presented some switching equivalent characterizations.

Key Words: Smarandachely signed graph, signed graphs, balance, switching, full signed graph, full line Signed graph, full block signed graph.

AMS(2010): 05C22.

§1. Introduction

For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [1]. The non-standard will be given in this paper as and when required.

To model individuals' preferences towards each other in a group, Harary [2] introduced the concept of signed graphs in 1953. A signed graph $S = (G, \sigma)$ is a graph $G = (V, E)$ whose edges are labeled with positive and negative signs (i.e., $\sigma : E(G) \to \{+, -\}$). The vertices of a graph represent people and an edge connecting two nodes signifies a relationship between individuals. The signed graph captures the attitudes between people, where a positive (negative edge) represents liking (disliking). An unsigned graph is a signed graph with the signs removed. Similar to an unsigned graph, there are many active areas of research for signed graphs.

The sign of a cycle (this is the edge set of a simple cycle) is defined to be the product of the signs of its edges; in other words, a cycle is positive if it contains an even number of negative edges and negative if it contains an odd number of negative edges. A signed graph S is said to be balanced if every cycle in it is positive. A signed graph S is called totally unbalanced if every cycle in S is negative. Otherwise, such a signed graph G is Smarandachely, i.e., both of the positive and negative cycles appeared in it. A chord is an edge joining two non adjacent vertices in a cycle.

A *marking* of S is a function $\zeta : V(G) \to \{+, -\}$. Given a signed graph S one can easily

[1]Received March 15, 2023, Accepted June 8, 2023.

[2]Corresponding author: somashekar2224@gmail.com

define a marking ζ of S as follows: For any vertex $v \in V(S)$,

$$\zeta(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking ζ of S is called *canonical marking* of S. For more new notions on signed graphs refer the papers (see [6, 8, 9, 13C17, 17C26]).

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1 *A signed graph S is balanced if and only if either of the following equivalent conditions is satisfied:*

(i) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2* (Harary [2]).

(ii) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$.* (Sampathkumar [10]).

A switching S with respect to a marking ζ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs.

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be *weakly isomorphic* (see [28]) or *cycle isomorphic* (see [29]) if there exists an isomorphism $\phi : G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (see [29]):

Theorem 1.2(T. Zaslavsky, [29]) *Given a graph G, any two signed graphs in $\psi(G)$, where $\psi(G)$ denotes the set of all the signed graphs possible for a graph G, are switching equivalent if and only if they are cycle isomorphic.*

§2. Full Block Signed Graph of a Signed Graph

The full block graph $\mathcal{FB}(G)$ of a graph G is the graph whose vertex set is the union of the set of vertices, edges and blocks of G in which two vertices are adjacent if the corresponding vertices and blocks of G are adjacent or the corresponding members of G are incident (See [5]).

Motivated by the existing definition of complement of a signed graph, we now extend the notion of full block graphs to signed graphs as follows: The *Full block signed graph* $\mathcal{FB}(S) = (\mathcal{FB}(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{FB}(G)$ and sign of any edge uv is $\mathcal{FB}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called a full block signed graph, if $S \cong \mathcal{FB}(S')$ for some signed graph S'. The following result restricts the class of full line signed graphs.

Theorem 2.1 *For any signed graph $S = (G, \sigma)$, its full block signed graph $\mathcal{FB}(S)$ is balanced.*

Proof Since sign of any edge $e = uv$ in $\mathcal{FB}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S, by Theorem 1.1, $\mathcal{FB}(S)$ is balanced. $\square$

For any positive integer k, the k^{th} iterated full block signed graph, $\mathcal{FB}^k(S)$ of S is defined as follows:

$$\mathcal{FB}^0(S) = S, \quad \mathcal{FB}^k(S) = \mathcal{FB}(\mathcal{FB}^{k-1}(S)).$$

Corollary 2.2 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $\mathcal{FB}^k(S)$ is balanced.*

Corollary 2.3 *For any two signed graphs S_1 and S_2 with the same underlying graph, $\mathcal{FB}(S_1) \sim \mathcal{FB}(S_2)$.*

The following result characterize signed graphs which are full line signed graphs.

Theorem 2.4 *A signed graph $S = (G, \sigma)$ is a full block signed graph if, and only if, S is balanced signed graph and its underlying graph G is a full block graph.*

Proof Suppose that S is balanced and G is a full block graph. Then there exists a graph G' such that $\mathcal{FB}(G') \cong G$. Since S is balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge uv in S satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $\mathcal{FB}(S') \cong S$. Hence S is a full block signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a full block signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $\mathcal{FB}(S') \cong S$. Hence, G is the full block graph of G' and by Theorem 2.1, S is balanced. $\square$

The notion of *negation* $\eta(S)$ of a given signed graph S defined to be $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S in [3]. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the $\mathcal{FB}(S)$ is balanced (Theorem 1.1). We now examine, the conditions under which negation $\eta(S)$ of $\mathcal{FB}(S)$ is balanced.

Proposition 2.5 *Let $S = (G, \sigma)$ be a signed graph. If $\mathcal{FB}(G)$ is bipartite then $\eta(\mathcal{FB}(S))$ is balanced.*

Proof Since, by Theorem 1.1, $\mathcal{FB}(S)$ is balanced, it follows that each cycle C in $\mathcal{FLS}(S)$ contains even number of negative edges. Also, since $\mathcal{FB}(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{FB}(S)$ is also even. Hence $\eta(\mathcal{FB}(S))$ is balanced. $\square$

§3. Switching Equivalence of Full Block Signed Graphs and Full Signed Graphs

In [27], we defined the full signed graph of a signed graph as follows: The *full signed graph* $\mathcal{FS}(S) = (\mathcal{FG}(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{FG}(G)$ and sign of any edge uv is $\mathcal{FS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called a full signed graph, if $S \cong \mathcal{FS}(S')$ for some signed graph S'. The following result restricts the class of full signed graphs.

Theorem 3.1(Swamy et al., [27]) *For any signed graph $S = (G, \sigma)$, its full signed graph $\mathcal{FS}(S)$ is balanced.*

In [5], the authors remarked that $\mathcal{FB}(G)$ and $\mathcal{FG}(G)$ are isomorphic if and only if G is a P_2. We now give a characterization of signed graphs whose full block signed graphs are switching equivalent to their full signed graphs.

Theorem 3.2 *For any nontrivial connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(S) \sim \mathcal{FS}(S)$ if and only if G is a P_2.*

Proof Suppose $\mathcal{FB}(S) \sim \mathcal{FS}(S)$. This implies, $\mathcal{FB}(G) \cong \mathcal{FG}(G)$ and hence G is a P_2.

Conversely, suppose that G is a P_2. Then $\mathcal{FB}(G) \cong \mathcal{FG}(G)$. Now, if S any signed graph with G is a P_2, By Theorem 2.1 and 3.1, $\mathcal{FB}(S)$ and $\mathcal{FS}(S)$ are balanced and hence, the result follows from Theorem 1.2. This completes the proof. □

§4. Switching Equivalence of Full Block Signed Graphs and Full Line Signed Graphs

In [27], we defined the full line signed graph of a signed graph as follows: The *full line signed graph* $\mathcal{FLS}(S) = (\mathcal{FLG}(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{FLG}(G)$ and sign of any edge uv is $\mathcal{FLS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called a full line signed graph, if $S \cong \mathcal{FLS}(S')$ for some signed graph S'. The following result restricts the class of full line signed graphs.

Theorem 4.1(Swamy et al., [27]) *For any signed graph $S = (G, \sigma)$, its full line signed graph $\mathcal{FLS}(S)$ is balanced.*

In [5], the authors remarked that $\mathcal{FB}(G)$ and $\mathcal{FLG}(G)$ are isomorphic if and only if G is a tree. We now give a characterization of signed graphs whose full block signed graphs are switching equivalent to their full line signed graphs.

Theorem 4.2 *For any nontrivial connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(S) \sim \mathcal{FLS}(S)$ if and only if G is a P_2.*

Proof Suppose $\mathcal{FB}(S) \sim \mathcal{FLS}(S)$. This implies, $\mathcal{FB}(G) \cong \mathcal{FLG}(G)$ and hence G is a tree.

Conversely, suppose that G is a tree. Then $\mathcal{FB}(G) \cong \mathcal{FLG}(G)$. Now, if S any signed graph with G is a tree, By Theorem 2.1 and 4.1, $\mathcal{FB}(S)$ and $\mathcal{FLS}(S)$ are balanced and hence, the result follows from Theorem 1.2. This completes the proof. □

In view of the negation operator introduced by Harary [3], we have the following cycle isomorphic characterizations.

Corollary 4.3 *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $\eta(\mathcal{FB}(S_1)) \sim \eta(\mathcal{FB}(S_2))$, if G_1 and G_2 are isomorphic.*

Corollary 4.4 *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $\mathcal{FB}(\eta(S_1))$ and $\mathcal{FB}(\eta(S_2))$ are cycle isomorphic, if G_1 and G_2 are isomorphic.*

Corollary 4.5 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(\eta(S)) \sim \mathcal{FS}(S)$ if and only if G is a P_2.*

Corollary 4.6 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(S) \sim \mathcal{FS}(\eta(S))$ if and only if G is a P_2.*

Corollary 4.7 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(\eta(S)) \sim \mathcal{FS}(\eta(S))$ if and only if G is a P_2.*

Corollary 4.8 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(\eta(S)) \sim \mathcal{FLS}(S)$ if and only if G is a tree.*

Corollary 4.9 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(S) \sim \mathcal{FLS}(\eta(S))$ if and only if G is a tree.*

Corollary 4.10 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(\eta(S)) \sim \mathcal{FLS}(\eta(S))$ if and only if G is a tree.*

Acknowledgements

The authors gratefully thank to the referee for the constructive comments and recommendations which definitely help to improve the readability and quality of the paper.

References

[1] F. Harary, *Graph Theory*, Addison Wesley, Reading, Mass, (1972).

[2] F. Harary, On the notion of balance of a sigraph, *Michigan Math. J.*, 2(1953), 143-146.

[3] F. Harary, Structural duality, *Behav. Sci.*, 2(4) (1957), 255-265.

[4] V. R. Kulli, On Full Graphs, *Journal of Computer and Mathematical Sciences*, 6(5) (2015), 261–267.

[5] V. R. Kulli, The full line graph and the full block graph of a graph, *International Journal of Mathematical Archive*, 6(8) (2015), 91-95.

[6] V. Lokesha, P. Siva Kota Reddy and S. Vijay, The triangular line n-sigraph of a symmetric n-sigraph, *Advn. Stud. Contemp. Math.*, 19(1) (2009), 123-129.

[7] R. Rajendra and P. Siva Kota Reddy, Tosha-Degree Equivalence Signed Graphs, *Vladikavkaz Mathematical Journal*, 22(2) (2020), 48-52.

[8] R. Rangarajan, P. Siva Kota Reddy and M. S. Subramanya, Switching equivalence in symmetric n-sigraphs, *Advn. Stud. Contemp. Math.*, 18(1) (2009), 79-85.

[9] R. Rangarajan, P. Siva Kota Reddy and N. D. Soner, Switching equivalence in symmetric n-sigraphs-II, *J. Orissa Math. Sco.*, 28 (1 & 2) (2009), 1-12.

[10] E. Sampathkumar, Point signed and line signed graphs, *Nat. Acad. Sci. Letters*, 7(3) (1984), 91-93.

[11] E. Sampathkumar, P. Siva Kota Reddy and M. S. Subramanya, The Line n-sigraph of a symmetric n-sigraph, *Southeast Asian Bull. Math.*, 34(5) (2010), 953-958.

[12] E. Sampathkumar, M. S. Subramanya and P. Siva Kota Reddy, Characterization of line sidigraphs, *Southeast Asian Bull. Math.*, 35(2) (2011), 297-304.

[13] P. Siva Kota Reddy, S. Vijay and V. Lokesha, n^{th} Power signed graphs, Proceedings of the Jangjeon Math. Soc., 12(3) (2009), 307-313.

[14] P. Siva Kota Reddy and M. S. Subramanya, Signed graph equation $L^k(S) \sim \overline{S}$, *International J. Math. Combin.*, 4 (2009), 84-88.

[15] P. Siva Kota Reddy, S. Vijay and H. C. Savithri, A Note on Path Sidigraphs, *International J. Math. Combin.*, 1 (2010), 42-46.

[16] P. Siva Kota Reddy and S. Vijay, Total minimal dominating signed graph, *International J.Math. Combin.*, 3 (2010), 11-16.

[17] P. Siva Kota Reddy, K. Shivashankara and K. V. Madhusudhan, Negation switching equivalence in signed graphs, *International J.Math. Combin.*, 3 (2010), 85-90.

[18] P. Siva Kota Reddy, E. Sampathkumar and M. S. Subramanya, Common-edge signed graph of a signed graph, *J. Indones. Math. Soc.*, 16(2) (2010), 105-112.

[19] P. Siva Kota Reddy, t-Path Sigraphs, Tamsui Oxford J. of Math. Sciences, 26(4) (2010), 433-441.

[20] P. Siva Kota Reddy, B. Prashanth, and T. R. Vasanth Kumar, Antipodal Signed Directed Graphs, Advn. Stud. Contemp. Math., 21(4) (2011), 355-360.

[21] P. Siva Kota Reddy and B. Prashanth, $\mathcal{S}$-Antipodal Signed Graphs, Tamsui Oxf. J. Inf. Math. Sci., 28(2) (2012), 165-174.

[22] P. Siva Kota Reddy and S. Vijay, The super line signed graph $\mathcal{L}_r(S)$ of a signed graph, *Southeast Asian Bulletin of Mathematics*, 36(6) (2012), 875-882.

[23] P. Siva Kota Reddy and U. K. Misra, The Equitable Associate Signed Graphs, *Bull. Int. Math. Virtual Inst.*, 3(1) (2013), 15-20.

[24] P. Siva Kota Reddy and U. K. Misra, Graphoidal Signed Graphs, Advn. Stud. Contemp. Math., 23(3) (2013), 451-460.

[25] P. Siva Kota Reddy and U. K. Misra, Restricted super line signed graph $\mathcal{RL}_r(S)$, *Notes on Number Theory and Discrete Mathematics*, 19(4) (2013), 86-92.

[26] P. Siva Kota Reddy, Radial Signed Graphs, *International J. Math. Combin.*, 4 (2016), 128-134.

[27] Swamy, P. Somashekar, and Khaled A. A. Alloush, Note on Full Signed Graphs and Full Line Signed Graphs, *International J. Math. Combin.*, 1 (2023), 111-115.

[28] T. Sozánsky, Enumeration of weak isomorphism classes of signed graphs, *J. Graph Theory*, 4(2) (1980), 127-144.

[29] T. Zaslavsky, Signed graphs, *Discrete Appl. Math.*, 4 (1982), 47-74.

International J.Math. Combin. Vol.2(2023), 65-73

Pair Difference Cordial Labeling of Some Trees and Some Graphs Derived From Cube Graph

R. Ponraj

Department of Mathematics, Sri Paramakalyani College, Alwarkurichi–627 412, India

A. Gayathri

Research Scholor, Reg.No:20124012092023

Department of Mathematics, Manonmaniam Sundaranar University, Abhishekapati, Tirunelveli–627 012, India

S. Somasundaram

Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli– 627 012, India

E-mail: ponrajmaths@gmail.com, gayugayathria555@gmail.com, somutvl@gmail.com

Abstract: In this paper we study the pair difference cordial labeling behavior of some trees and some graphs derived from cube graph.

Key Words: Smarandachely pair difference cordial labeling, pair difference cordial labeling, tree, star, cube, Y-tree, W-tree.

AMS(2010): 05C78.

§1. Introduction

In this paper we consider only finite, undirected and simple graphs.The concept of pair difference cordial labeling of a graph was introduced and studied some properties of pair difference cordial labeling in [4]. By definition, let $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm \lfloor p/2 \rfloor\}$. Consider a mapping $f : V \longrightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. Such a labeling is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$. Otherwise, it is called a Smarandachely pair difference cordial labeling if $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \geq 2$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the numbers of edges labeled or not labeled with 1.

A graph G for which there exists a pair difference cordial labeling or Smarandachely pair difference cordial labeling is called a pair difference cordial graph or Smarandachely pair difference cordial graph. The pair difference cordial labeling behavior of several graphs have been investigated in [4,5,6,7,8,9,10,11]. In this paper we investigate pair difference cordial labeling behavior of some trees and some graphs derived from cube graph.Terms not defined here are follow from [2,3].

[1]Received March 29, 2023, Accepted June 9, 2023.

§2. Preliminaries

Definition 2.1([2]) *Let P_n be the path $a_1a_2a_3\cdots a_n$. A $Y-$tree Y_n is the tree of order $n+1$ whose vertex set is $V(Y_n)=\{a_1,a_2,a_3,\cdots,a_n,a\}$ and the edge set $E(Y_n)=E(P_n)\cup\{a_{n-1}a\}$. In other words Y_n is obtained by attaching the vertex a to the vertex a_{n-1} of P_n.*

Definition 2.2([2]) *A $W-$ graph $W(n)$ is the graph with vertex set*

$$\{c_1,c_2,b,w,d\}\bigcup\{x^1,x^2,x^3,\cdots,x^n\}\bigcup\{y^1,y^2,y^3,\cdots,y^n\}$$

and the edge set

$$\{c_1x^1,c_1x^2,\cdots,c_1x^n\}\bigcup\{c_2y^1,c_2y^2,\cdots,c_2y^n\}\bigcup\{c_1b,c_1w,c_2w,c_2d\}.$$

Definition 2.3([2]) *A $W-$tree $WT(n,k)$ is a graph obtained by taking $k-$ copies of $W-$ graph $W(n)$ and a new vertex a and joining a which in each copy d where $n\geq 2,k\geq 3$.*

Let $V(WT(n,k))=\{a,c_1^i,c_2^i,d^i,x_1^i,x_2^i,x_3^i,\cdots,x_{n+1}^i,y_1^i,y_2^i,y_3^i,\cdots,y_{n+1}^i:1\leq i\leq k\}$, $E(WT(n,k))=\{ac_1^i,ac_2^i,d^ic_1^i,d^ic_2^i,c_1^ix_j^i,c_2^ix_j^i:1\leq i\leq k,1\leq j\leq n\}$. Obviously $WT(n,k)$ has $nk(k+1)+k(n+1)+1$ vertices and $nk(k+1)+k(n+1)$ edges.

Definition 2.4([3]) *Let G be the graph and $G_1,G_2,G_3,\cdots,G_n;n\geq 2$ be n copies of the graph G. Then the graph obtained by adding an edge from G_i to G_{i+1}, $i=1,2,3,\cdots,n-1)$ is called path union of graph G.*

Definition 2.5([3]) *Let $G_1,G_2,G_3,\cdots,G_n$ be any $n-$ graphs. A graph obtained by replacing each vertex of $K_{1,n}$ except the apex vertex by the graph $G_1,G_2,G_3,\cdots,G_n$ is known as an open star of graphs which is denoted by $OS(G_1,G_2,G_3,\cdots,G_n)$. If $G_1=G_2=G_3=\cdots=G_n=G$ then it is denoted by $OS(n.G)$.*

Definition 2.6([3]) *A hypercube is an $n-$ dimensional analogue of a square $(n=2)$ and a cube $(n=3)$ which is also known as an $n-$ cube or $n-$ dimensional cube which is denoted by Q_n.*

§3. Graphs Obtained From Trees

Theorem 3.1 *A Y-tree is pair difference cordial for all values of $n\geq 3$.*

Proof Take the vertex set and edge set from Definition 2.1. The proof is divided into the following 4 cases.

Case 1. $n\equiv 0(mod4)$.

Assign the labels $1,2,-1,-2$ respectively to the vertices a_1,a_2,a_3,a_4 and allocate the values $3,4,-3,-4$ individually to the vertices a_5,a_6,a_7,a_8. Net we put the labels $5,6,-5,-6$ separately to the vertices a_9,a_{10},a_{11},a_{12} and assign the labels $7,8,-7,-8$ respectively to the vertices $a_{13},a_{14},a_{15},a_{16}$. Proceeding like this process until we reach the vertex a_n. Finally assign the label -1 to the vertex a.

In this case $\Delta_{f_1} = \Delta_{f_1^c} = \frac{n}{2}$.

Case 2. $n \equiv 1(mod4)$.

Assign the labels as in Case 1 to the vertices $a_i, 1 \leq i \leq n-1$). And then, assign the labels $\frac{n+1}{2}, -(\frac{n+1}{2})$ to the vertices a_n, a. Then $\Delta_{f_1} = \frac{n+1}{2}, \Delta_{f_1^c} = \frac{n-1}{2}$.

Case 3. $n \equiv 2(mod4)$.

Assign the labels as in Case 1 to the vertices $a_i, 1 \leq i \leq n-2$). Lastly assign the labels $\frac{n}{2}, -(\frac{n}{2}), \frac{n-2}{2}$ to the vertices a_{n-1}, a_n, a.

In this case $\Delta_{f_1} = \Delta_{f_1^c} = \frac{n}{2}$.

Case 4. $n \equiv 3(mod4)$.

Assign the labels as in Case 1 to the vertices $a_i, 1 \leq i \leq n-3$). Finally assign the labels $\frac{n-1}{2}, \frac{n+1}{2}, -(\frac{n-1}{2}), -(\frac{n+1}{2})$ to the vertices a_{n-2}, a_{n-1}, a_n, a. Then $\Delta_{f_1} = \frac{n-1}{2}, \Delta_{f_1^c} = \frac{n+1}{2}$. □

Theorem 3.2 *The W-tree $WT(2,n)$ is not pair difference cordial for all values of $n \geq 3$.*

Proof A $WT(2,n)$ has $7n+3$ vertices and $7n+2$ edges. Our proof is divided into 2 cases following.

Case 1. n is even.

The maximum possible of $\Delta_{f_1} = 4n$. Then $\Delta_{f_1^c} \geq 7n+2-4n. \Delta_{f_1^c} - \Delta_{f_1} \geq 3n+2 > 1$.

Case 2. n is odd.

The maximum possible of $\Delta_{f_1} = 4n+1$. Then $\Delta_{f_1^c} \geq 7n+2-4n-1$. $\Delta_{f_1^c} - \Delta_{f_1} \geq 3n+1 > 1$.

Therefore, a wheel $WT(2,n)$ is not pair difference cordial. □

§4. Graphs Obtained From Cube

Theorem 4.1 *The path union of $n-$ copies of Q_3 is pair difference cordial for all values of $n \geq 2$.*

Proof Let G be the graph obtained by joining $n-$ copies of the cube Q_3. Let

$$\begin{aligned} V(G) &= \{x_{i1}, y_{i1}, x_{i2}, y_{i2}, x_{i3}, y_{i3}, x_{i4}, y_{i4} : 1 \leq i \leq n\}, \\ E(G) &= \{x_{i1}x_{i2}, x_{i2}x_{i3}, x_{i3}x_{i4}, x_{i1}x_{i4}, y_{i1}y_{i2}, y_{i2}y_{i3}, y_{i3}y_{i4}, y_{i1}y_{i4} : 1 \leq i \leq n\} \\ & \bigcup \{x_{ij}y_{ij} : 1 \leq i \leq n, 1 \leq j \leq 4\}. \end{aligned}$$

Obviously, G has $8n$ vertices and $13n-1$ edges. Our proof is divided into 2 cases following.

Case 1. n is even.

When $n = 2$, Assign the labels $1, 2, 3, 4, -1, -2, -3, -4$ respectively to the vertices x_{11}, x_{12},

$x_{13}, x_{14}, y_{11}, y_{12}, y_{13}, y_{14}$ and assign the labels $5, 6, 7, 8, -5, -6, -7, -8$ respectively to the vertices $x_{21}, x_{22}, x_{23}, x_{24}, y_{21}, y_{22}, y_{23}, y_{24}$.

If $n \geq 4$, define a map ψ from the vertex set $V(G)$ to the set $\{\pm 1, \pm 2, \cdots, \pm 4n\}$ by

$$\begin{aligned}
\psi(x_{i1}) &= 8i-7, && i = 1, 3, 5, \cdots, n-1,\\
\psi(x_{i2}) &= 8i-6, && i = 1, 3, 5, \cdots, n-1,\\
\psi(x_{i3}) &= 8i-5, && i = 1, 3, 5, \cdots, n-1,\\
\psi(x_{i4}) &= 8i-4, && i = 1, 3, 5, \cdots, n-1,\\
\psi(y_{i1}) &= -(8i-7), && i = 1, 3, 5, \cdots, n-1,\\
\psi(y_{i2}) &= -(8i-6), && i = 1, 3, 5, \cdots, n-1,\\
\psi(y_{i3}) &= -(8i-5), && i = 1, 3, 5, \cdots, n-1,\\
\psi(y_{i4}) &= -(8i-4), && i = 1, 3, 5, \cdots, n-1,\\
\psi(x_{i1}) &= 8i-2, && i = 2, 4, 6, \cdots, n,\\
\psi(x_{i2}) &= 8i-3, && i = 2, 4, 6, \cdots, n,\\
\psi(x_{i3}) &= 8i-1, && i = 2, 4, 6, \cdots, n,\\
\psi(x_{i4}) &= 8i, && i = 2, 4, 6, \cdots, n,\\
\psi(y_{i1}) &= -(8i-2), && i = 2, 4, 6, \cdots, n,\\
\psi(y_{i2}) &= -(8i-3), && i = 2, 4, 6, \cdots, n,\\
\psi(y_{i3}) &= -(8i-1), && i = 2, 4, 6, \cdots, n,\\
\psi(y_{i4}) &= -(8i), && i = 2, 4, 6, \cdots, n.
\end{aligned}$$

Case 2. n is odd.

Define a map $\psi : V(G) \to \{\pm 1, \pm 2, \cdots, \pm 4n\}$ by

$$\begin{aligned}
\psi(x_{i1}) &= 8i-7, && i = 1, 3, 5, \cdots, n,\\
\psi(x_{i2}) &= 8i-6, && i = 1, 3, 5, \cdots, n,\\
\psi(x_{i3}) &= 8i-5, && i = 1, 3, 5, \cdots, n,\\
\psi(x_{i4}) &= 8i-4, && i = 1, 3, 5, \cdots, n,\\
\psi(y_{i1}) &= -(8i-7), && i = 1, 3, 5, \cdots, n,\\
\psi(y_{i2}) &= -(8i-6), && i = 1, 3, 5, \cdots, n,\\
\psi(y_{i3}) &= -(8i-5), && i = 1, 3, 5, \cdots, n,\\
\psi(y_{i4}) &= -(8i-4), && i = 1, 3, 5, \cdots, n,\\
\psi(x_{i1}) &= 8i-2, && i = 2, 4, 6, \cdots, n-1,\\
\psi(x_{i2}) &= 8i-3, && i = 2, 4, 6, \cdots, n-1,\\
\psi(x_{i3}) &= 8i-1, && i = 2, 4, 6, \cdots, n-1,
\end{aligned}$$

$$\begin{aligned}
&\psi(x_{i4}) = 8i, && i = 2, 4, 6, \cdots, n-1,\\
&\psi(y_{i1}) = -(8i-2), && i = 2, 4, 6, \cdots, n-1,\\
&\psi(y_{i2}) = -(8i-3), && i = 2, 4, 6, \cdots, n-1,\\
&\psi(y_{i3}) = -(8i-1), && i = 2, 4, 6, \cdots, n-1,\\
&\psi(y_{i4}) = -(8i), && i = 2, 4, 6, \cdots, n-1.
\end{aligned}$$

Table 2 given below establishes that this vertex labeling f is a pair difference cordial.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$\frac{13n-1}{2}$	$\frac{13n-1}{2}$
n is even	$\frac{13n}{2}$	$\frac{13n-2}{2}$

This completes the proof. □

Theorem 4.2 *A graph obtained by joining two copies of Q_3 by path P_n is pair difference cordial for all values of $n \geq 4$.*

Proof Let G be the graph obtained by joining two copies of Q_3 by path P_n with

$$\begin{aligned}
V(G) &= \{x_{i1}, y_{i1}, x_{i2}, y_{i2}, x_{i3}, y_{i3}, x_{i4}, y_{i4} : 1 \leq i \leq 2\} \bigcup \{z_k : 1 \leq k \leq n-2\},\\
E(G) &= E(Q_3) \bigcup \{z_i z_{i+1} : 1 \leq i \leq n-2\} \bigcup \{z_1 y_{14}, z_{n-2} x_{11}\}.
\end{aligned}$$

Obviously, G has $n+14$ vertices and $n+23$ edges.

Case 1. $n \equiv 0(mod 4)$.

Assign labels $1, 2, 3, 4, 5, 6, 7, 8$ respectively to vertices $x_{11}, x_{12}, x_{13}, x_{14}, y_{11}, y_{12}, y_{13}, y_{14}$ and assign the labels $-1, -2, -3, -4, -5, -6, -7, -8$ respectively to the vertices $x_{21}, x_{22}, x_{23}, x_{24}, y_{21}, y_{22}, y_{23}, y_{24}$.

Assign the labels $9, 10, -9, -10$ respectively to the vertices z_1, z_2, z_3, z_4 and allocate the values $11, 12, -11, -12$ individually to the vertices z_5, z_6, z_7, z_8. Net we put the labels $5, 6, -5, -6$ separately to the vertices $z_9, z_{10}, z_{11}, z_{12}$ and assign the labels $7, 8, -7, -8$ respectively to the vertices $z_{13}, z_{14}, z_{15}, z_{16}$. Proceeding like this process until we reach the vertex z_{n-4}. Finally assign the labels $\frac{n+14}{2}, -(\frac{n+14}{2})$ to the vertex z_{n-3}, z_{n-2}.

Case 2. $n \equiv 1(mod 4)$.

Assign the labels as in Case 1 to the vertices $x_{ij}, y_{ij}, 1 \leq i \leq 2, 1 \leq j \leq 4,)$ and $z_k, 1 \leq k \leq n-5$. And then, assign the labels $\frac{n+13}{2}, -(\frac{n+13}{2}), -(\frac{n+11}{2})$ to the vertices $z_{n-4}, z_{n-3}, z_{n-2}$.

Case 3. $n \equiv 2(mod 4)$.

Assign the labels as in case 1 to the vertices $x_{ij}, y_{ij}, 1 \leq i \leq 2, 1 \leq j \leq 4,)$ and $z_k, 1 \leq k \leq n-6$. Lastly assign the labels $\frac{n+12}{2}, \frac{n+14}{2}, -(\frac{n+12}{2}), -(\frac{n+14}{2})$ to the vertices $z_{n-5}, z_{n-4}, z_{n-3}, z_{n-2}$.

Case 4. $n \equiv 3(mod4)$.

Assign the labels as in case 1 to the vertices $x_{ij}, y_{ij}, 1 \leq i \leq 2, 1 \leq j \leq 4,)$ and $z_k, 1 \leq k \leq n-7$. Finally assign the labels $\frac{n+12}{2}, \frac{n+14}{2}, -(\frac{n+12}{2}), -(\frac{n+14}{2}), -(\frac{n+12}{2})$ to the vertices $z_{n-6}, z_{n-5}, z_{n-4}, z_{n-3}, z_{n-2}$.

The Table 3 given below establishes that this vertex labeling f is a pair difference cordial.

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
$n \equiv 0 \pmod 4$	$\frac{n+24}{2}$	$\frac{n+22}{2}$
$n \equiv 1 \pmod 4$	$\frac{n+23}{2}$	$\frac{n+23}{2}$
$n \equiv 2 \pmod 4$	$\frac{n+24}{2}$	$\frac{n+22}{2}$
$n \equiv 3 \pmod 4$	$\frac{n+23}{2}$	$\frac{n+23}{2}$

This completes the proof. □

Theorem 4.3 *An $S(n.Q_3)$ is pair difference cordial for all even n.*

Proof Our proof is divided into 2 cases following.

Case 1. $n \equiv 0(mod4)$.

Define a map $\psi : V(G) \to \{\pm 1, \pm 2, \cdots, \pm 4n\}$ by

$$\begin{aligned}
\psi(x) &= 1, && \\
\psi(x_{i1}) &= 4i-3, && 1 \leq i \leq \frac{n}{2}, \\
\psi(x_{i2}) &= 4i-2, && 1 \leq i \leq \frac{n}{2}, \\
\psi(x_{i3}) &= 4i-1, && 1 \leq i \leq \frac{n}{2}, \\
\psi(x_{i4}) &= 4i, && 1 \leq i \leq \frac{n}{2}, \\
\psi(y_{i1}) &= -(4i-3), && 1 \leq i \leq \frac{n}{2}, \\
\psi(y_{i2}) &= -(4i-2), && 1 \leq i \leq \frac{n}{2}, \\
\psi(y_{i3}) &= -(4i-1), && 1 \leq i \leq \frac{n}{2}, \\
\psi(y_{i4}) &= -4i, && 1 \leq i \leq \frac{n}{2}, \\
\psi(x_{(\frac{n}{2}+2i-1)1}) &= 2n+4i-3, && 1 \leq i \leq \frac{n}{4}, \\
\psi(x_{(\frac{n}{2}+2i-1)2}) &= 2n+4i-2, && 1 \leq i \leq \frac{n}{4}, \\
\psi(x_{(\frac{n}{2}+2i-1)3}) &= 2n+4i-1, && 1 \leq i \leq \frac{n}{4}, \\
\psi(x_{(\frac{n}{2}+2i-1)4}) &= 2n+4i, && 1 \leq i \leq \frac{n}{4}, \\
\psi(y_{(\frac{n}{2}+2i-1)1}) &= 2n+4i+4, && 1 \leq i \leq \frac{n}{4},
\end{aligned}$$

$$\psi(y_{(\frac{n}{2}+2i-1)2}) = 2n+4i+3, \qquad 1 \le i \le \frac{n}{4},$$
$$\psi(y_{(\frac{n}{2}+2i-1)3}) = 2n+4i+2, \qquad 1 \le i \le \frac{n}{4},$$
$$\psi(y_{(\frac{n}{2}+2i-1)4}) = 2n+4i+1, \qquad 1 \le i \le \frac{n}{4},$$
$$\psi(x_{(\frac{n}{2}+2i)1}) = -(2n+4i-3), \qquad 1 \le i \le \frac{n}{4},$$
$$\psi(x_{(\frac{n}{2}+2i)2}) = -(2n+4i-2), \qquad 1 \le i \le \frac{n}{4},$$
$$\psi(x_{(\frac{n}{2}+2i)3}) = -(2n+4i-1), \qquad 1 \le i \le \frac{n}{4},$$
$$\psi(x_{(\frac{n}{2}+2i)4}) = -(2n+4i), \qquad 1 \le i \le \frac{n}{4},$$
$$\psi(y_{(\frac{n}{2}+2i)1}) = -(2n+4i+4), \qquad 1 \le i \le \frac{n}{4},$$
$$\psi(y_{(\frac{n}{2}+2i)2}) = -(2n+4i+3), \qquad 1 \le i \le \frac{n}{4},$$
$$\psi(y_{(\frac{n}{2}+2i)3}) = -(2n+4i+2), \qquad 1 \le i \le \frac{n}{4},$$
$$\psi(y_{(\frac{n}{2}+2i)4}) = -(2n+4i+1), \qquad 1 \le i \le \frac{n}{4},$$

Case 2. $n \equiv 1(mod4)$.

Define a map ψ from the vertex set $V(G)$ to the set $\{\pm1, \pm2, \cdots, \pm4n\}$ by

$$\psi(x) = 3,$$
$$\psi(x_{i1}) = 4i-3, \qquad 1 \le i \le \frac{n+2}{2},$$
$$\psi(x_{i2}) = 4i-2, \qquad 1 \le i \le \frac{n+2}{2},$$
$$\psi(x_{i3}) = 4i-1, \qquad 1 \le i \le \frac{n+2}{2},$$
$$\psi(x_{i4}) = 4i, \qquad 1 \le i \le \frac{n+2}{2},$$
$$\psi(y_{i1}) = -(4i-3), \qquad 1 \le i \le \frac{n+2}{2},$$
$$\psi(y_{i2}) = -(4i-2), \qquad 1 \le i \le \frac{n+2}{2},$$
$$\psi(y_{i3}) = -(4i-1), \qquad 1 \le i \le \frac{n+2}{2},$$
$$\psi(y_{i4}) = -4i, \qquad 1 \le i \le \frac{n+2}{2},$$
$$\psi(x_{(\frac{n}{2}+2i-1)1}) = 2n+4i+1, \qquad 1 \le i \le \frac{n-2}{4},$$
$$\psi(x_{(\frac{n}{2}+2i-1)2}) = 2n+4i+2, \qquad 1 \le i \le \frac{n-2}{4},$$
$$\psi(x_{(\frac{n}{2}+2i-1)3}) = 2n+4i+3, \qquad 1 \le i \le \frac{n-2}{4},$$

$$\psi(x_{(\frac{n}{2}+2i-1)4}) = 2n+4i+4, \qquad 1 \leq i \leq \frac{n-2}{4},$$
$$\psi(y_{(\frac{n}{2}+2i-1)1}) = 2n+4i+5, \qquad 1 \leq i \leq \frac{n-2}{4},$$
$$\psi(y_{(\frac{n}{2}+2i-1)2}) = 2n+4i+6, \qquad 1 \leq i \leq \frac{n-2}{4},$$
$$\psi(y_{(\frac{n}{2}+2i-1)3}) = 2n+4i+7, \qquad 1 \leq i \leq \frac{n-2}{4},$$
$$\psi(y_{(\frac{n}{2}+2i-1)4}) = 2n+4i+8, \qquad 1 \leq i \leq \frac{n-2}{4},$$
$$\psi(x_{(\frac{n}{2}+2i)1}) = -(2n+4i+1), \qquad 1 \leq i \leq \frac{n-2}{4},$$
$$\psi(x_{(\frac{n}{2}+2i)2}) = -(2n+4i+2), \qquad 1 \leq i \leq \frac{n-2}{4},$$
$$\psi(x_{(\frac{n}{2}+2i)3}) = -(2n+4i+3), \qquad 1 \leq i \leq \frac{n-2}{4},$$
$$\psi(x_{(\frac{n}{2}+2i)4}) = -(2n+4i+4), \qquad 1 \leq i \leq \frac{n-2}{4},$$
$$\psi(y_{(\frac{n}{2}+2i)1}) = -(2n+4i+5), \qquad 1 \leq i \leq \frac{n-2}{4},$$
$$\psi(y_{(\frac{n}{2}+2i)2}) = -(2n+4i+6), \qquad 1 \leq i \leq \frac{n-2}{4},$$
$$\psi(y_{(\frac{n}{2}+2i)3}) = -(2n+4i+7), \qquad 1 \leq i \leq \frac{n-2}{4},$$
$$\psi(y_{(\frac{n}{2}+2i)4}) = -(2n+4i+8), \qquad 1 \leq i \leq \frac{n-2}{4},$$

In both cases $\Delta_{f_1} = \Delta_{f_1^c} = \frac{13n}{2}$. □

A pair difference cordial labeling of $S(6.Q_3)$ is shown in Figure 1.

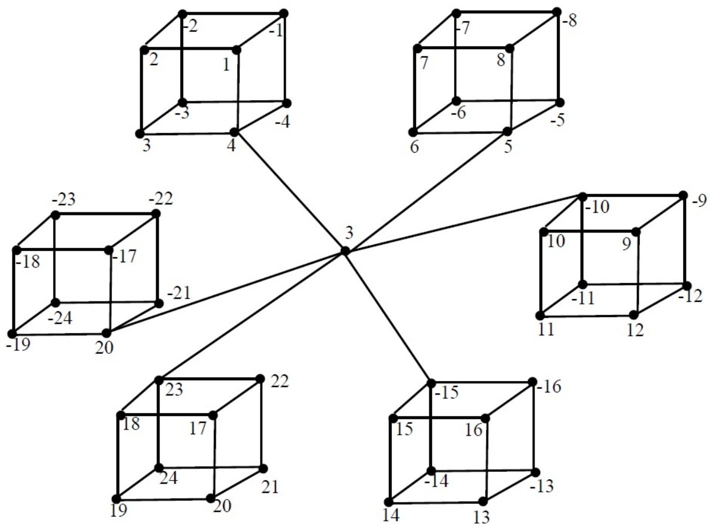


Figure 1

References

[1] Gallian J.A, A Dynamic survey of graph labeling, *The Electronic Journal of Combinatorics.*, 19, (2016).

[2] Harary F, *Graph Theory*, Addision Wesley, New Delhi, 1969.

[3] Jekil A. Gadhiya, Shanti S. Khunti and Mehul P. Rupani, Product cordial labeling of hyper cube related graphs, *Malaya Journal of Matematik*, Vol.8(4), (2020), 1991–1994.

[4] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordial labeling of graphs, *J.Math. Comp.Sci.*, Vol.11(3), (2021), 2551–2567.

[5] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordiality of some snake and butterfly graphs, *Journal of Algorithms and Computation*, Vol.53(1), (2021), 149–163.

[6] Ponraj,R., Gayathri,A., and Somasundaram,S., Pair difference cordial graphs obtained from the wheels and the paths, *J. Appl. and Pure Math.*, Vol.3 No. 3-4, (2021), pp. 97–114.

[7] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordiality of some graphs derived from ladder graph, *J.Math. Comp.Sci.*, Vol.11 No 5, (2021), 6105–6124.

[8] Ponraj R., Gayathri A. and Somasundaram S., Some pair difference cordial graphs, *Ikonion Journal of Mathematics*, Vol.3(2), (2021), 17–26.

[9] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordial labeling of planar grid and mangolian tent, *Journal of Algorithms and Computation*, Vol.53(2) (2021), 47–56.

[10] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordiality of some special graphs, *J. Appl. and Pure Math.*, Vol.3 No. 5-6, (2021), pp. 263–274.

[11] Ponraj R., Gayathri A. and Soma Sundaram S., Pair difference cordiality of mirror graph, shadow graph and splitting graph of certain graphs, *Maltepe Journal of Mathematics*, Vol.4, 1(2022), 24–32.

International J.Math. Combin. Vol.2(2023), 74-77

A Note on Product Irregularity Strength of Graphs

Umme Salma and H. M. Nagesh

(Department of Science and Humanities, PES University, Bangalore, India)

E-mail: ummesalma@pesu.pes.edu, nageshhm@pes.edu

Abstract: For a graph G without isolated vertices and without isolated edges, a *product irregular labeling* $w : E(G) \to \{1, 2, \cdots .m\}$ is a labeling of the edges of G such that for any two distinct vertices u and v of G the product of labels of the edges incident with u is different from the product of labels of the edges incident with v. The minimal m for which there exist a product irregular labeling is called the *product irregularity strength* of G and is denoted by $ps(G)$. In this note, we find the product irregularity strength of block graph of cycle-star graph and sunlet graph.

Key Words: Smarandachely H product-irregular labeling, product-irregular labeling, product irregularity strength, block graph, cycle-star graph, sunlet graph.

AMS(2010): 05C05, 05C15, 05C78.

§1. Introduction

Throughout this paper let G be a simple graph, i.e., a graph without loops or multiple edges, without isolated vertices and without isolated edges. Let the vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. Let $w : E(G) \to \{1, 2, \cdots .m\}$ be an integer labeling of the edges of G. Then the *product degree* $pd_G(v)$ of a vertex $v \in V(G)$ in the graph G with respect to the labeling w is defined by

$$pd_G(v) = \prod_{v \in e} w(e).$$

A labeling w is said to be *product-irregular* if for every pair of vertices $u, v \in V(G)$, $u \neq v$,

$$pd_G(u) \neq pd_G(v).$$

Generally, for a typical subgraph $H \prec G$, a labeling w is said to be *Smarandachely H product-irregular* if for every pair of vertices $u, v \in V(G)$, $u \neq v$, there are $pd_G(u) \neq pd_G(v)$ for $u, v \in V(G) \backslash V(H)$ but $pd_G(u) = pd_G(v)$ for $u, v \in V(H)$. Clearly, if $H = \emptyset$, such a Smarandachely H product-irregular property is nothing else but the product-irregular property.

The *product irregularity strength* $ps(G)$ of G is the smallest value of m for which there exists a product-irregular labeling $w : E(G) \to \{1, 2, \cdots .m\}$.

[1]Received March 3, 2023, Accepted June 10, 2023.

This concept was first introduced by Anholcer in [1] as a multiplicative version of the well studied concept of irregularity strength of graphs introduced by Chartrand et al. in [3].

The corona product of two graphs G and H, denoted by $G \bigodot H$, is a graph obtained by taking one copy of G (which has n vertices) and n copies $H_1, H_2, \cdots, H_n$ of H, and then joining the *ith* vertex of G to every vertex in H_i. The corona product $C_n \bigodot K_1$ is called the *sunlet graph*.

A graph G is *connected* if between any two distinct vertices there is a path. A maximal connected subgraph of G is called a *component* of G. A *cut-vertex* of a graph is one whose removal increases the number of components. A *non-separable graph* is connected, nontrivial, and has no cut-vertices. A *block* of a graph is a maximal non-separable subgraph. If two distinct blocks B_1 and B_2 are incident with a common cut-vertex, then they are called *adjacent blocks*.

There are many graph operators (or graph valued functions) with which one can construct a new graph from a given graph, such as the line graphs, the block graphs, and their generalizations.

The *block graph* of a graph G, written $B(G)$, is the graph whose vertices are the blocks of G and in which two vertices are adjacent whenever the corresponding blocks have a cut-vertex in common.

Jelena Sedlar [5] introduced the concept of cycle-star graph as follows: The *cycle-star graph*, written $CS_{k,n-k}$, is a graph with n vertices consisting of the cycle graph of length k and $n-k$ leafs appended to the same vertex of the cycle.

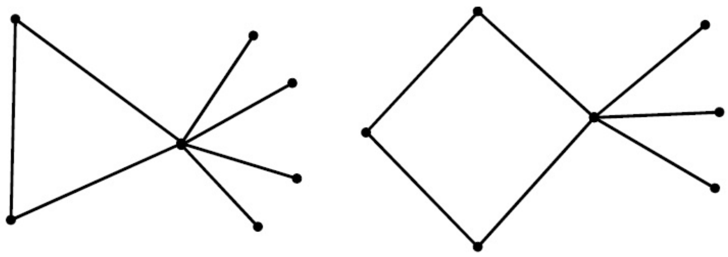


Figure 1 The cycle-star graphs $CS_{3,4}$ and $CS_{4,3}$

§2. Preliminary Results

Let n_d denote the number of vertices of degree d, where $\delta(G) \leq d \leq \Delta(G)$. Anholcer in [1] showed that

$$ps(G) \geq max_{\delta(G) \leq d \leq \Delta(G)} \left\{ \left\lceil \frac{d}{e} n_d^{\frac{1}{d}} - d + 1 \right\rceil \right\}. \quad (1)$$

If the graph G is $r-$regular, then the expression (1) reduces to

$$ps(G) \geq \left\lceil \frac{r}{e} n^{\frac{1}{r}} - r + 1 \right\rceil. \quad (2)$$

Also, for a cycle C_n on $n \geq 3$ vertices, the bounds on $ps(C_n)$ is given in [1]. That is, for $n \geq 3$,

$$ps(C_n) \geq \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil$$

for $n > 17$,

$$ps(C_n) \geq \left\lceil \left(\frac{n}{1 - log_e 2} \right)^{\frac{1}{2}} \right\rceil$$

and that for every $\epsilon > 0$ there exists n_0 such that for every $n \geq n_0$,

$$ps(C_n) \leq \left\lceil (1+\epsilon)\sqrt{2n}\; log_e n \right\rceil.$$

Anholcer in [2] considered the product irregularity strength of complete bipartite graphs $K_{m,n}$ and proved that for two integers m and n such that $2 \geq m \geq n$, $ps(K_{m,n}) = 3$ if and only if $n \geq \binom{m+2}{2}$.

However, the studies on the product irregularity strength of the intersection graph on the vertex set of a graph was not attempted. In this paper we have made an attempt to fill this gap and study the the product irregularity strength of the block graph of cycle-star graph and sunlet graph.

§3. Product Irregularity Strength of Block Graph of Cycle-Star Graph $CS_{k,n-k}$

The following result in [4] determines the exact value of product irregularity strength of a complete graph K_n on $n \geq 3$ vertices.

Theorem 3.1 *For every complete graph K_n on $n \geq 3$ vertices, $ps(K_n) = 3$.*

We now use Theorem 3.1 to find the exact value of product irregularity strength of block graph of cycle-star graph $CS_{k,n-k}$ for $k \geq 3$ and $n - k \geq 2$.

Theorem 3.2 *Let $G = CS_{k,n-k}$ be a cycle-star graph, where $k \geq 3$ and $n - k \geq 2$. Then $ps(B(G)) = 3$.*

Proof Since the block graph a cycle-star graph $CS_{k,n-k}$ with $k \geq 3$ and $n - k \geq 2$ leafs is a complete graph K_n on $n \geq 3$ vertices, from Theorem 3.1, it follows that $ps(B(G)) = 3$. This completes the proof. □

§4. Product Irregularity Strength of Block Graph of Sunlet Graph $C_n \bigodot K_1$

In this section we find the exact value of product irregularity strength of block graph of sunlet graph $C_n \bigodot K_1, n \geq 3$.

Theorem 4.1 *Let $G = C_n \bigodot K_1$, $n \geq 3$, be a sunlet graph. Then $ps(B(G)) = n$.*

Proof Let $G = C_n \bigodot K_1$, $n \geq 3$, be a sunlet graph. By definition, the block graph of sunlet graph is a star graph $K_{1,n}$ on $n \geq 3$ vertices. Let $v_1, v_2, \cdots, v_n$ be pendant vertices and v_0 be the central vertex of $K_{1,n}$. At first, let us weight all the edges consecutively starting from 1 to n. Then the product degree of vertices $v \in B(G)$ is $pd_{B(G)}(v_i) = i$ for $1 \leq i \leq n$ and $pd_{B(G)}(v_i) = n!$. Clearly, product degrees of all vertices are distinct. Hence $ps(B(G)) = n$. This completes the proof. □

§5. Conclusion

In this note, we have found the exact values of product irregularity strength of block graph of cycle-star graph and sunlet graph. However, to find the exact values of product irregularity strength of different graph operators still remain open.

References

[1] Anholcer M. (2009), Product irregularity strength of graphs, *Discrete Math*, 309, 6434-6439.

[2] Anholcer M. (2014), Product irregularity strength of certain graphs, *ARS Math. Contemp*, 7, 23C29.

[3] Chartrand G., Jacobson M.S., Lehel J., Oellermann O.R., & Saba F. (1988), Irregular networks, *Congr. Numer*, 64, 187-192.

[4] Skowronek-Kaziow J. (2012), Multiplicative vertex-colouring weightings of graphs, *Inform. Process. Lett*, 112, 191-194.

[5] Sedlar J. (2013), Extremal unicyclic graphs with respect to additively weighted Harary index, *Miskolic mathematical Notes*, 16(2), 1-16.

International J.Math. Combin. Vol.2(2023), 78-86

Cube Sum Labeling (Taxi-Cab Labeling) of Graphs

Mitesh J. Patel

(Tolani College of Arts and Science, Adipur-Kachchh, Gujarat, India)

G. V. Ghodasara

(H. & H. B. Kotak Institute of Science, Rajkot, Gujarat, India)

E-mail: miteshmaths1984@gmail.com, gaurang_enjoy@yahoo.co.in

Abstract: Let $G(V,E)$ be a graph with order p and size q. A bijection $f: V(G) \rightarrow \{0,1,2,\cdots,p-1\}$ is said to be a cube sum labeling if the induced function $f^*: E(G) \rightarrow \mathbb{N}$ defined by $f^*(uv) = [f(u)]^3 + [f(v)]^3$ is injective. Such a function f is said to be a cube sum labeling and the graph G is a cube sum graph. In this paper we discuss some algebraic properties and evaluate some families of cube sum graph.

Key Words: Smarandachely cube sum H labeling, cube sum graph, complete graph, tree, wheel graph, helm graph, Fermat's Last Theorem.

AMS(2010): 05C78.

§1. Introduction

Labeling of graphs is one of the emerging topics in graph theory. The credit goes to Rosa [1] to explore this innovative idea. If the vertices or edges or both of the graph are assigned values subject to certain condition(s) then it is known as graph labeling. The idea of graph labeling was originated in 1967. Till then graph labeling has attracted many researchers and due to the wholehearted efforts for research in this field, more than 200 graph labeling techniques and more than 2500 research papers are available. A dynamic survey on graph labeling is regularly updated by Gallian [6] and it is published by *The Electronic Journal of Combinatorics*.

In this paper we consider simple, finite, undirected and connected graph. A graph $G(V,E)$ with p vertices and q edges is also denoted as $G(p,q)$ graph. We refer to Bondy and Murty [5] for the standard terminology and notations related to graph theory and Burton [2] for the terms related to number theory. We denote an edge with end vertices u and v by uv.

A square sum labeling is one of the graph labeling techniques, where edge label is obtained by sum of squares of labels of end vertices of the corresponding edge. The square sum labeling was introduced by Ajitha, Arumugam and Germina.

Definition 1.1(Ajitha et al., [10]) *A graph $G=(V,E)$ with p vertices and q edges is said to be a square sum graph, if there exists a bijection $f: V(G) \rightarrow \{0,1,2,\cdots,p-1\}$ such that the induced function $f^*: E(G) \rightarrow \mathbb{N}$, defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, is injective.*

[1]Received February 28, 2023, Accepted June 12, 2023.

[2]Correspondence: miteshmaths1984@gmail.com

Many interesting results are carried out for square sum labeling of graphs. The literature on square sum labeling is accessible in electronic form in different research papers such as [3], [4], [7], [10] etc.

A cube sum labeling was introduced by Vediyappan Govindan, Sandra Pinelas and S.Dhivya [9] as follow and they proved that paths, cycle, stars, wheel graph, fan graphs are cube sum graph.

Definition 1.2(Vediyappan Govindan et al., [9]) *A graph $G = (V, E)$ with p vertices and q edges is said to be a cube sum graph, if there exists a bijection $f : V(G) \to \{0, 1, 2, \cdots, p-1\}$ such that the induced function $f^* : E(G) \to \mathbb{N}$, defined by $f^*(uv) = (f(u))^3 + (f(v))^3$, is injective.*

Notice that 1729 is the smallest natural number expressible as a sum of two cubes in two different ways as $12^3 + 1^3$ and $9^3 + 10^3$. From the story of G.H. Hardy and Srinivasa Ramanujan, 1729 is known as *Ramanujan number* or *Taxi-cab number* [2]. Other numbers which can expressed as sum of two cubes in two different ways are $4104 = 2^3 + 16^3 = 9^3 + 15^3, 13832 = 2^3 + 24^3 = 18^3 + 20^3, 20683 = 10^3 + 27^3 = 19^3 + 24^3$, etc. Taxi-cab number is related with sum of cube of two numbers. So, we also refer cube sum labeling as *Taxi-cab labeling* as well.

In this paper we have used the Fermat's Last Theorem [2] which states that *No three positive integers a, b and c satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than* 2.

§2. Cube Sum Labeling

Definition 2.1 *A bijective function $f : V(G) \to \{0, 1, 2, \cdots, p-1\}$ is said to be a cube sum labeling if the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = [f(u)]^3 + [f(v)]^3$ is injective.*

Generally, let $H \prec G$ be a typical subgraph of G such as those of path, cycle. If such an induced function f^* is injective on $E(G) \setminus E(H)$ but not on $E(G)$, such a labeling f is said to be a Smarandachely cube sum H labeling. Particularly, if $H = \emptyset$, then such a Smarandachely cube sum H labeling is nothing else but a cube sum labeling.

A graph G with cube sum labeling is called a cube sum graph.

Lemma 2.2(Burton, [2]) *The cube of any integer is one of the form $9k, 9k + 1$ or $9k + 8$.*

Theorem 2.3 *Let G be a cube sum graph with cube sum labeling f. Then, for any edge $e \in E(G)$, $f^*(e) \not\equiv 3, 4, 5, 6(mod\ 9)$.*

Proof Let $u, v \in V(G)$, $f(u) = a$ and $f(v) = b$. Then, for edge $e = uv \in E(G)$, $f^*(uv) = a^3 + b^3$.

Since a and b are integers, from Lemma 3.1, $a^3 \equiv 0, 1$ or $8(mod\ 9)$ and $b^3 \equiv 0, 1$ or $8(mod\ 9)$. But then $a^3 + b^3 \equiv 0, 1, 2, 7$ or $8(mod\ 9)$. Hence, the result is proved. $\square$

Lemma 2.4(Burton, [2]) *The cube of any integer is one of the form $7k$ or $7k \pm 1$.*

Theorem 2.5 *Let G be a cube sum graph with cube sum labeling f. Then for any edge $e \in E(G)$, $f^*(e) \not\equiv 3, 4(mod\ 7)$.*

Proof Let $f(u) = a$ and $f(v) = b$. Then, for edge $e = uv \in E(G), f^*(uv) = a^3 + b^3$. From lemma 3.2, since a and b are integers, $a^3 \equiv 0, 1$ or $6(mod\ 7)$, and $b^3 \equiv 0, 1$ or $6(mod\ 7)$. But then $a^3 + b^3 \equiv 0, 1, 2, 5$ *or* $6(mod\ 7)$. This completes the proof. □

Theorem 2.6 *If $G(p, q)$ is cube sum graph with cube sum labeling f, then*

$$\sum_{uv \in E(G)} f^*(uv) = \sum_{v \in V(G)} [f(v)]^3 d(v)$$

where $d(v)$ is the degree of vertex v in G.

Proof Let $f : V(G) \to \{0, 1, 2, \cdots, p-1\}$ be a cube sum labeling of a graph G with each edge uv is assigned the label $f^*(uv) = [f(u)]^3 + [f(v)]^3$.

Now every edge is incident with exactly two vertices and degree of a vertex is the number of edges incident with that vertex. Then, while counting the total sum of edge labels, the number of times of repetition (occurrence) of each vertex label is equal to the number of edges incident to the corresponding vertex. Then the sum of $f^*(e)$ count $[f(v)]^3$ at total number of times an edge is incident with a vertex v. So

$$\sum_{uv \in E(G)} f^*(uv) = \sum_{v \in V(G)} [f(v)]^3 d(v).$$ □

Corollary 2.7 *If $G(p, q)$ is an r-regular cube sum graph, then*

$$\sum_{uv \in E(G)} f^*(uv) = \frac{r(p-1)^2 p^2}{4}.$$

Proof From Theorem 3.3, we have

$$\sum_{uv \in E(G)} f^*(uv) = \sum_{v \in V(G)} [f(v)]^3 d(v). \tag{1}$$

Here, $G(p, q)$ is an r-regular cube sum graph, i.e. $d(v) = r,\ \forall v \in V(G)$.

$$\begin{aligned} \sum_{uv \in E(G)} f^*(uv) &= r \sum_{v \in V(G)} [f(v)]^3 \quad \{\text{from } (1)\} \\ &= r\big(0^3 + 1^3 + \cdots + (p-1)^3\big) = \frac{r(p-1)^2 p^2}{4}. \end{aligned}$$ □

§3. Some Cube Sum Graphs

Theorem 3.1 *A complete graph K_n is a cube sum graph if and only if $n \leq 11$.*

Proof Let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(K_n) = \{v_iv_j \mid 1 \leq i,j \leq n, i \neq j\}$. Here, $|V(K_n)| = n$ and $|E(K_n)| = \frac{n(n-1)}{2}$.

Case 1. $n \leq 11$.

Let us define a function $f : V(K_n) \to \{0, 1, 2, \cdots, n-1\}$ as

$$f(v_i) = i - 1\ ;\ 1 \leq i \leq n.$$

It is obvious that f is bijective and the induced function $f^* : E(K_n) \to \mathbb{N}$ defined by

$$f^*(uv) = (f(u))^3 + (f(v))^3,$$

for every $uv \in E(K_n)$ is injective. Hence, K_n is a cube sum graph for $n \leq 11$.

Case 2. $n > 11$.

Notice that every two vertices are adjacent to each other in a complete graph. So, defining a mapping $f : V(K_n) \to \{0, 1, 2, \cdots, n-1\}$ in any form, we have two edges e_i and e_j such that $f^*(e_i) = 12^3 + 1^3 = 1729$ and $f^*(e_j) = 9^3 + 10^3 = 1729$. Thus, the induced function f^* is not injective. Hence, K_n is not a cube sum graph for $n > 11$. □

Theorem 3.2 *A complete bipartite graph $K_{2,n}$ is a cube sum graph for any integer $n \geq 1$.*

Proof Let $V_1 = \{v_1, v_{n+2}\}$ and $V_2 = \{v_2, v_3, \cdots, v_{n+1}\}$ be bipartition of $V(K_{1,n}) = \{v_1, v_2, v_3, \cdots, v_{n+1}, v_{n+2}\}$ and $E(K_{1,n}) = \{v_iv_j \mid i = 1, n+2 \text{ and } j = 2, 3, \cdots, n+1\}$. Here, $|V(K_{2,n})| = n+2$ and $|E(K_{2,n})| = 2n$. Let us define a function $f : V(K_{2,n}) \to \{0, 1, 2, \cdots, n+1\}$ as

$$f(v_i) = i - 1\ ;\ 1 \leq i \leq n+2.$$

It is obvious that f is bijective.

Furthermore, one can observe that

$$\begin{aligned} f^*(v_1v_2)(=1) &< f^*(v_1v_3)(=2^3) < f^*(v_1v_4)(=3^3) \\ &< \cdots < f^*(v_1v_{n+1})(=n^3) < f^*(v_{n+2}v_2)(=(n+1)^3+1) \\ &< f^*(v_{n+2}v_3)(=(n+1)^3+2^3) < \cdots < f^*(v_{n+2}v_{n+1})(=(n+1)^3+n^3). \end{aligned}$$

Then, the induced function $f^* : E(K_{2,n}) \to \mathbb{N}$ defined by

$$f^*(uv) = (f(u))^3 + (f(v))^3,$$

for every $uv \in E(K_{2,n})$ is injective. Hence, $K_{2,n}$ is a cube sum graph. □

Theorem 3.3 *Every tree is a cube sum graph.*

Proof Let $v_{0,0}$ be a vertex with maximum degree in a tree T. Choose $v_{0,0}$ as a root vertex of T (say zero level vertex). Let l be the height of T. Consider $n_0 = 0$.

Let n_1 be the number of vertices at distance one from $v_{0,0}$ and let us denote these vertices by $v_{1,1}, v_{1,2}, \cdots v_{1,n_1}$. These vertices are first level vertices. Let n_2 be the number of vertices at distance two from $v_{0,0}$ which are denoted by $v_{2,1}, v_{2,2}, \cdots v_{2,n_2}$. These vertices are second level vertices. We give priority as in ascending order.

Repeating this way, let n_l be the number of vertices at distance l from $v_{0,0}$ which are denoted by $v_{l,1}, v_{l,2}, \cdots v_{l,n_l}$. These are l^{th} level vertices.

The above process is possible because there is one and only one path between any pair of vertices in any tree. Here, $|V(T)| = \sum\limits_{i=1}^{l}(n_i) + 1 = n$ and $|E(T)| = \sum\limits_{i=1}^{l}(n_i) = n - 1$.

Let us define a function $f : V(T) \to \{0, 1, 2, 3, \cdots, n-1\}$ as

$$f(v_{i,j}) = \begin{cases} 0; & i = 0, j = 0. \\ f(v_{i-1,n_{i-1}}) + j; & 1 \leq i \leq l, 1 \leq j \leq \ n_i. \end{cases}$$

Here, vertex labels are in ascending order from zero level vertex to l level vertices. So, it is obvious that f is bijective and for edge labels we have following arguments. We have following two cases for edge labels. Without loss of generality, let e_1 and e_2 be any two arbitrary edges of tree T.

Case 1. Let e_1 and e_2 be two incident edges. Then obviously $f^*(e_1) \neq f^*(e_2)$.

Case 2. Let $e_1 = v_1v_2$ and $e_2 = v_3v_4$ be the edges such that e_1 and e_2 have no common vertex. Here $\{f(v_1), f(v_2), f(v_3), f(v_4)\}$ is non-empty subset of $\mathbb{N}$. So, by well ordering principle, $\{f(v_1), f(v_2), f(v_3), f(v_4)\}$ has a least element, say $f(v_1)$.

Since T is a tree, at least one of the vertex v_3 or v_4 is not adjacent to v_1. If v_4 is not adjacent to v_1, then $f(v_4) > f(v_2)$ and if v_3 is not adjacent to v_1 then $f(v_3) > f(v_2)$. So, $f^*(e_1) \neq f^*(e_2)$. Thus, the induced function $f^* : E(G) \to \mathbb{N}$ defined by

$$f^*(uv) = (f(u))^3 + (f(v))^3,$$

for every $uv \in E(G)$ is injective. Hence, tree T is a cube sum graph. $\square$

Theorem 3.4 *A cycle C_n is a cube sum graph.*

Proof Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(C_n) = \{v_iv_{i+1} \mid 1 \leq i \leq n-1\} \bigcup \{v_nv_1\}$. Here, $|V(C_n)| = n$ and $|E(C_n)| = n$.

Let us define a function $f : V(C_n) \to \{0, 1, 2, \cdots, n-1\}$ as per subsequent two cases.

Case 1. n is even.

In this case, define

$$f(v_i) = \begin{cases} 0; & i = 1. \\ 2i - 3; & 2 \leq i \leq \ \frac{n+2}{2}. \\ n - 2(i - \frac{n+2}{2}); & \frac{n+2}{2} < i \leq \ n. \end{cases}$$

It is obvious that f is bijective and we can observe that

$$\begin{aligned} f^*(v_1v_2)(=1) &< f^*(v_1v_n)(=2^3) < f^*(v_2v_3)(=1^3+3^3) \\ &< \cdots < f^*(v_{\frac{n}{2}}v_{\frac{n+2}{2}}) < f^*(v_{\frac{n+4}{2}}v_{\frac{n+2}{2}}). \end{aligned}$$

Case 2. n is odd.

In this case, define

$$f(v_i) = \begin{cases} 0; & i = 1. \\ 2i-3; & 2 \le i \le \frac{n+1}{2}. \\ n+1-2(i-\frac{n+1}{2}); & \frac{n+1}{2} < i \le n. \end{cases}$$

It is obvious that f is bijective and we can observe that

$$\begin{aligned} f^*(v_1v_2)(=1) &< f^*(v_1v_n)(=2^3) < f^*(v_2v_3)(=1^3+3^3) \\ &< \cdots < f^*(v_{\frac{n}{2}}v_{\frac{n+2}{2}}) < f^*(v_{\frac{n+4}{2}}v_{\frac{n+2}{2}}). \end{aligned}$$

So, in both the cases the induced function $f^* : E(C_n) \to \mathbb{N}$ defined by

$$f^*(uv) = (f(u))^3 + (f(v))^3,$$

is injective. Hence, C_n is a cube sum graph. □

Theorem 3.5 *A wheel W_n is a cube sum graph.*

Proof Let $V(W_n) = \{v_0, v_1, \cdots, v_n\}$ and $E(W_n) = \{v_0v_i \mid 1 \le i \le n\} \bigcup \{v_iv_{i+1} \mid 1 \le i \le n-1\} \bigcup \{v_nv_1\}$, where v_0 is apex and $v_1, v_2, \cdots, v_n$ are rim vertices of W_n. Clearly, $|V(W_n)| = n+1$ and $|E(W_n)| = 2n$.

Let us define a function $f : V(W_n) \to \{0, 1 \cdots, n\}$ as follows.

$$f(v_i) = \begin{cases} 0 \ ; i = 0. \\ 1 \ ; i = 1. \\ 2(i-1) \ ; 1 < i \le \lfloor \frac{n+2}{2} \rfloor. \\ 2n-2i+3 \ ; \lfloor \frac{n+2}{2} \rfloor < i \le n. \end{cases}$$

It is obvious that f is bijective. We consider the following two cases for the edge labels.

Case 1. n is odd.

From above vertex labels, one can observe that labels of rim edges are in ascending order as

$$\begin{aligned} f^*(v_1v_2)(=1^3+2^3) &< f^*(v_1v_n)(=1^3+3^3) < f^*(v_2v_3)(=2^3+4^3) \\ &< f^*(v_nv_{n-1})(=3^3+5^3) < \cdots < f^*(v_{\frac{n+1}{2}}v_{\frac{n+3}{2}})(=(n-1)^3+n^3). \end{aligned}$$

From Fermat's Last Theorem, $f^*(v_0v_i)$ is never equal to any of above edge labels for integers $1 \leq i \leq n$.

Case 2. n is even.

From above vertex labels, one can observe that labels of rim edges are in ascending order as

$$\begin{aligned} f^*(v_1v_2)(=1^3+2^3) &< f^*(v_1v_n)(=1^3+3^3) < f^*(v_2v_3)(=2^3+4^3) \\ &< f^*(v_nv_{n-1})(=3^3+5^3) < \cdots < f^*(v_{\frac{n+2}{2}}v_{\frac{n+4}{2}})(=(n-1)^3+n^3). \end{aligned}$$

From Fermat's Last Theorem, $f^*(v_0v_i)$ $(1 \leq i \leq n)$ is never equal to any one of above edge labels. So, in both the cases the induced function $f^* : E(W_n) \to \mathbb{N}$ defined by

$$f^*(uv) = (f(u))^3 + (f(v))^3,$$

for every $uv \in E(W_n)$ is injective. Hence, W_n is a cube sum graph. $\square$

Corollary 3.6 *A gear G_n is a cube sum graph.*

Corollary 3.7 *A shell S_n is a cube sum graph.*

Theorem 3.8 *A helm H_n is a cube sum graph.*

Proof Let $V(H_n) = \{v_0, v_i, u_i \mid 1 \leq i \leq n\}$ and $E(H_n) = \{v_0v_i \mid 1 \leq i \leq n\} \bigcup \{v_iu_i \mid 1 \leq i \leq n\} \bigcup \{v_iv_{i+1} \mid 1 \leq i \leq n-1\} \bigcup \{v_nv_1\}$, where v_0 is apex, $v_1, v_2, \cdots, v_n$ are rim vertices and $u_1, u_2 \cdots, u_n$ are pendant vertices of helm H_n. Obviously, $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$.

Let us define a function $f : V(H_n) \to \{0, 1 \cdots, 2n\}$ as follows.

Case 1. n is even.

In this case, define

$$\begin{aligned} f(v_i) &= \begin{cases} 0 \ ; i = 0. \\ 1 \ ; i = 1. \\ 4i-5 \ ; 2 \leq i \leq \frac{n+2}{2}. \\ 2n-3-4\big(i-\frac{n+4}{2}\big) \ ; \frac{n+2}{2} < i \leq n. \end{cases} \\ f(u_i) &= \begin{cases} 4i-4 \ ; 2 \leq i \leq \frac{n+2}{2}. \\ 2n-2-4\big(i-\frac{n+4}{2}\big) \ ; \frac{n+2}{2} < i \leq n. \end{cases} \end{aligned}$$

It is obvious that f is bijective and for the edge labels in the graph there are three possibilities as follows:

(1) Edge labels on rim edges are

$$\begin{aligned} f^*(v_1v_2)(=1^3+3^3) &< f^*(v_1v_n)(=1^3+5^3) \\ &< f^*(v_2v_3)(=3^3+7^3) < f^*(v_nv_{n-1})(=5^3+9^3) \\ &< \cdots < f^*(v_{\frac{n+2}{2}}v_{\frac{n+4}{2}})(=(2n-1)^3+(2n-3)^3). \end{aligned}$$

They are in ascending order of the form $2k$ ($k \in \mathbb{N}$) because the common end vertices of these edges are labeled by odd numbers (naturally distinct).

(2) Edge labels on edges incident to pendant vertices are

$$\begin{aligned} f^*(v_1u_1)(=1^3+2^3) &< f^*(v_2u_2)(=3^3+4^3) \\ &< f^*(v_nu_n)(=5^3+6^3) \\ &< \cdots < f^*(v_{\frac{n+2}{2}}u_{\frac{n+2}{2}})(=(2n-1)^3+(2n)^3). \end{aligned}$$

They are in ascending order of the form $2k+1$ ($k \in \mathbb{N}$) because common end vertices of these edges are labeled by consecutive numbers (naturally distinct).

(3) Edge labels on edges incident to apex are

$$\begin{aligned} f^*(v_0v_1)(=1^3) &< f^*(v_0v_2)(=3^3) < f^*(v_0v_n)(=5^3) \\ &< \cdots < f^*(v_0v_{\frac{n+2}{2}})(=(2n-1)^3). \end{aligned}$$

They are in ascending order of the form $2k+1$ ($k \in \mathbb{N}$) because common end vertices of these edges are labeled by 0 and other end vertices by odd numbers (naturally distinct).

It is clear that the labels of possibilities (1) and (2) are distinct.

From Fermat's Last Theorem, the edge labels in the possibilities (3) are distinct from the edge labels in the possibilities (1) and (2). So, the labels of above all possibilities are internally as well as externally distinct.

Case 2. n is odd.

In this case, define

$$\begin{aligned} f(v_i) &= \begin{cases} 0 \ ; i=0. \\ 1 \ ; i=1. \\ 4i-5 \ ; 2 \le i \le \frac{n+1}{2}. \\ 2n-1-4\left(i-\frac{n+3}{2}\right) \ ; \frac{n+1}{2} < i \le n. \end{cases} \\ f(u_i) &= \begin{cases} 4i-4 \ ; 2 \le i \le \frac{n+1}{2}. \\ 2n-4\left(i-\frac{n+3}{2}\right) \ ; \frac{n+1}{2} < i \le n. \end{cases} \end{aligned}$$

Using the arguments similar to the case 1, one can observe that in this case the function f is bijective and for every $uv \in E(G)$ the induced edge labels $f^*(uv) = (f(u))^3 + (f(v))^3$ are all distinct.

So, in both the cases the induced function $f^* : E(H_n) \to \mathbb{N}$ defined by

$$f^*(uv) = (f(u))^3 + (f(v))^3,$$

is injective. Hence H_n is a cube sum graph. $\square$

§4. Concluding Remarks

Labeling of discrete structure is a potential area of research. We have discussed some algebraic properties of cube sum graph. We have also proved that the following graphs are cube sum graphs: a complete graph K_n if and only if $n \leq 11$, a complete bipartite graph $K_{2,n}$ for $n \geq 1$, every tree, cycle graph, wheel graph, gear graph, shell graph and helm graph. To investigate more results for various graphs as well as in the context of different graph operations is an open area of research.

References

[1] A. Rosa, On certain valuations of the vertices of theory of graphs, *International Symposium*, Rome, Gordon and Breach, N. Y. and Dunod Paris, (1966), 349-355.

[2] David M. Burton, *Elementary Number Theory* (Sixth edition), Tata McGraw-Hill, 2006.

[3] G. V. Ghodasara and Mitesh J. Patel, Innovative results on square sum labeling of graphs, *Journal of Graph Labeling*, Volume 4(1)(2018), 15-24.

[4] G. V. Ghodasara and Mitesh J. Patel, Some bistar related square sum graphs, *International Journal of Mathematics Trend and Technology*, 47(3)(2017), 172-177.

[5] J. A. Bondy and U.S. Murty, *Graph Theory with Applications*, Elsevier Science Publication, 1982.

[6] J. A. Gallian, A dynamic survey of graph labeling, *The Electronics Journal of Combinatorics*, (2019), 1-535.

[7] J. Shiama, Square sum labeling for some middle and total graphs, *International Journal of Computer Applications*, 37 (4), (2012).

[8] L. W. Beineke and S. M. Hegde, *Discuss. Math. Graph Theory*, 21, 63 (2001).

[9] Vediyappan Govindan, Sandra Pinelas And S.Dhivya, Cube sum labeling of graphs, *Annals Of Communications in Mathematics*, 3(2) (2020), 171-176.

[10] Ajitha, S. Arumugam and K. A. Germina, On square sum graphs, *AKCE International Journal of Graphs and Combinatorics*, 6(1)(2009), 1-10.

International J.Math. Combin. Vol.2(2023), 87-104

An Atlas of Roman Domination Polynomials of Graphs of Order at Most Six

Deepak G.[1], Manjunath N.[2], Manjunatha R.[3], Shashidhara J.M.[4] and Akram Alqesmah[5]

1. Department of Mathematics, South East Asian College of Engineering and Technology KR Puram, Bangalore, India
2. Department of Sciences and Humanities, School of Engineering and Technology, CHRIST (Deemed to be University) Begaluru, Karnataka, India
3. Department of Mathematics, School of Science, Jain (Deemed to be University), Bangalore, Karnataka, India
4. Department of Mathematics, Sri Venkateshwara College of Engg, Bangalore, Karnataka, India
5. Department of Mathematics, Faculty of Education of Tour Albaha, University of Lahij, Lahij, Yemen

E-mail: deepak1873@gmail.com

Abstract: The Roman domination polynomial of a graph G of order p is defined as $R(G,x) = \sum\limits_{j=\gamma_R(G)}^{2n} r(G,j)x^j$, where $r(G,j)$ is the number of Roman dominating functions of G of weight j [5]. The roots of a Roman domination polynomial of a graph are called the Roman domination roots of that graph. In this article, the Roman domination polynomials of all the connected graphs of order less than or equal to six are obtained and their roots are computed. Furthermore, all these graphs and their Roman domination polynomials and roots are illustrated in a table.

Key Words: Atlas, Roman domination polynomial, Roman domination roots, graphs of order at most 6.

AMS(2010): 05C31, 05C69, 05C76.

§1. Introduction

Throughout this paper all the considered graph are finite simple graphs i.e. all the graphs here are finite, undirected and have no self-loops or multiple edges. Let $G = (V,E)$ be a graph. The order and the size of G are denoted respectively by $|V(G)| = n$ and $|E(G)| = m$.

The Roman domination number of a graph $G = (V,E)$, $\gamma_R(G)$, has been defined in [7] as the smallest weight, $W(f(V))$, of a function $f : V(G) \to \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$, where $W(f(V)) = \sum\limits_{u\in V(G)} f(u)$. A function $f : V(G) \to \{0,1,2\}$ with this condition is called a Roman dominating function of the graph $G = (V,E)$ or in brief an RDF of G. For more details about Roman domination and its properties, the reader is referred to [6].

In [5], Deepak *et al.* introduced the Roman domination polynomial of a graph G as

[1]Received March 30, 2023, Accepted June 14, 2023.

$R(G,x) = \sum_{j=\gamma_R(G)}^{2n} r(G,j)x^j$, where $r(G,j)$ is the number of Roman dominating functions of G of weight j and studied some of its properties. The roots of the Roman domination polynomial of a graph G are called the Roman domination roots of G. In addition, the Roman domination polynomials of paths and cycles are studied in details in [4] and [3], respectively.

As with all the types of graph polynomials, the analysis of the Roman domination polynomial of graphs can give us some informations about graphs. Similar to the domination polynomial of graphs [2, 1], an atlas for the Roman domination polynomials of graphs of order at most six is presented in this article. Moreover, the Roman domination polynomials of all the connected graphs of order less than or equal to six and their roots are illustrated in a table. Furthermore, for computing the Roman domination polynomials of the disconnected graphs of order less than or equal to six the following lemma can be used.

Lemma 1.1([5]) *If a graph G consists of m components $G_1, \cdots, G_m$, then $R(G,x) = R(G_1,x) \times R(G_2,x) \times \cdots \times R(G_m,x)$.*

Some coefficients of the polynomials are computed by using the following theorem.

Theorem 1.2([5]) *Let G be a graph on n vertices with i isolated vertices, t vertices of degree one and l vertices of degree two. Suppose $R(G,x) = \sum_{j=\gamma_R(G)}^{2n} r(G,j)x^j$ is the Roman domination polynomial of G. Then the following hold:*

(i) $r(G, 2n-1) = n$;

(ii) $i = \frac{n(n+1)}{2} - r(G, 2n-2)$;

(iii) $r(G, 2n-3) = 2\binom{n}{2} + \binom{n}{3} - i(n-1) - t$;

(iv) *If G has s K_2-components, then*

$$r(G, 2n-4) = \binom{n}{2} + 3\binom{n}{3} + \binom{n}{4} - i(n-1) + \binom{i}{2} - t(n-1) + s - l.$$

(v) *If $G \neq K_2$, then $r(G,2) = |\{v \in V(G) : |deg(v) = n-1\}|$*

and the other coefficients are computed by determining all the possible functions $f : V(G) \to \{0,1,2\}$ of some size and reduce the cases when $f : V(G) \to \{0,1,2\}$ is not an RDF function of the graph G. For instance, all the possible functions of size $2n-5$, $2n-6$, $2n-7$ and $2n-8$ are given as:

(i) *For size $2n-5$ there is $3\binom{n}{3} + 4\binom{n}{4} + \binom{n}{5}$ possible function;*

(ii) *For size $2n-6$ there is $\binom{n}{3} + 6\binom{n}{4} + 5\binom{n}{5} + \binom{n}{6}$ possible function;*

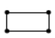
(iii) *For size $2n-7$ there is $4\binom{n}{4} + 10\binom{n}{5} + 6\binom{n}{6} + \binom{n}{7}$ possible function;*

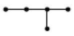
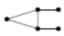
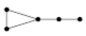
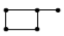
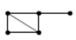
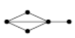
(iv) *For size $2n-8$ there is $\binom{n}{4} + 10\binom{n}{5} + 15\binom{n}{6} + 7\binom{n}{7} + \binom{n}{8}$ possible function.*

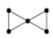
All the roots of the polynomials are found by using the Matlab program. On the other hand, the repetition of any root is expressed as an exponent on that root. For example, the three times repetition of the zero root is expressed as $(0)^3$.

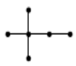
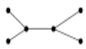
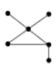
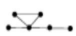
§2. Roman Domination Polynomials of All Connected Graphs of order ≤ 6

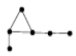
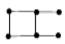
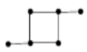
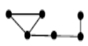
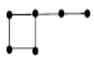
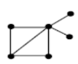
In the following, a table illustrates all the connected graphs of order less than of equal to six with their Roman domination polynomials and roots.

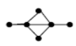
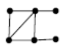
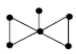
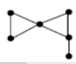
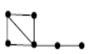
Graph	Roman Domination Polynomial	Roman Domination Roots
	x^2+x	$0,\ -1$
	$x^4+2x^3+3x^2$	$(0)^2,\ -1\pm\sqrt{2}i$
	$x^6+3x^5+6x^4+5x^3+x^2$	$(0)^2,\ -1,\ -0.2848,\ -0.8576\pm 1.6662i$
	$x^6+3x^5+6x^4+7x^3+3x^2$	$(0)^2,\ (-1)^2,\ \frac{-1\pm\sqrt{11}i}{2}$
	$x^8+4x^7+10x^6+13x^5+10x^4+3x^3+x^2$	$(0)^2,\ -0.1062\pm 0.3824i,\ -0.9827\pm 0.8465i,$ $-0.9111\pm 1.7159i$
	$x^8+4x^7+10x^6+14x^5+11x^4+2x^3$	$(0)^3,\ -0.2471,\ -1.2146\pm 0.8713i,$ $-0.6618\pm 1.7846i$
	$x^8+4x^7+10x^6+15x^5+16x^4+5x^3$	$(0)^3,\ -0.4599,\ -0.2992\pm 1.7264i,$ $-1.4708\pm 1.174i$
	$x^8+4x^7+10x^6+16x^5+15x^4+4x^3$	$(0)^3,\ -0.402,\ -1.4178\pm 0.8204i,$ $-0.3812\pm 1.8877i$
	$x^8+4x^7+10x^6+16x^5+17x^4+8x^3+2x^2$	$(0)^2,\ -0.3296\pm 0.3569i,\ -1.352\pm 0.9293i,$ $-0.3184\pm 1.7455i$

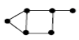
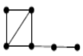
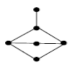
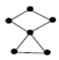
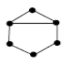
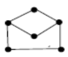
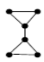
Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^8+4x^7+10x^6+16x^5+19x^4+12x^3+4x^2$	$(0)^2,\ -0.5274\pm0.5087i,\ -1.3019\pm1.0899i,\ -0.1708\pm1.5986i$
	$x^{10}+5x^9+15x^8+26x^7+29x^6+21x^5+10x^4+4x^3+x^2$	$(0)^2,\ -1,\ -0.575,\ 0.0259\pm0.5197i,\ -0.69\pm0.9483i,\ -1.0485\pm1.89i$
	$x^{10}+5x^9+15x^8+27x^7+32x^6+21x^5+6x^4+x^3$	$(0)^3,\ -1,\ -0.1803\pm0.2468i,\ -0.6458\pm1.7634i,\ -1.1739\pm1.2873i$
	$x^{10}+5x^9+15x^8+28x^7+34x^6+23x^5+6x^4$	$(0)^4,\ -1,\ -0.5973,\ -1.1535\pm1.1497i,\ -0.5479\pm1.8674i$
	$x^{10}+5x^9+15x^8+28x^7+35x^6+27x^5+12x^4+4x^3+x^2$	$(0)^2,\ -1,\ -0.6222,\ -0.0191\pm0.4241i,\ -1.0795\pm1.1832i,\ -0.5904\pm1.7686i$
	$x^{10}+5x^9+15x^8+28x^7+36x^6+27x^5+10x^4+2x^3$	$(0)^3,\ -1,\ -0.2632\pm0.3289i,\ -1.2977\pm1.2965i,\ -0.4391\pm1.7768i$
	$x^{10}+5x^9+15x^8+29x^7+38x^6+31x^5+12x^4+x^3$	$(0)^3,\ (-1)^2,\ -0.1113,\ -0.3914\pm1.7864i,\ -1.053\pm1.2558i$
	$x^{10}+5x^9+15x^8+29x^7+38x^6+29x^5+10x^4+x^3$	$(0)^3,\ -1,\ -0.1613,\ -0.574,\ -1.2552\pm1.1673i,\ -0.3771\pm1.8796i$
	$x^{10}+5x^9+15x^8+30x^7+40x^6+31x^5+10x^4$	$(0)^4,\ (-1)^2,\ -0.3193\pm1.9689i,\ -1.1807\pm1.058i$
	$x^{10}+5x^9+15x^8+29x^7+40x^6+35x^5+16x^4+3x^3$	$(0)^3,\ -1,\ -0.4934\pm0.2123i,\ -1.2496\pm1.3342i,\ -0.2571\pm1.7452i$
	$x^{10}+5x^9+15x^8+29x^7+39x^6+33x^5+16x^4+5x^3+x^2$	$(0)^2,\ -1,\ -0.6085,\ -0.11\pm0.3958i,\ -1.2109\pm1.2087i,\ -0.3749\pm1.785i$

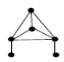
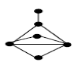
Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{10}+5x^9+15x^8+30x^7+41x^6+37x^5+18x^4+4x^3+x^2$	$(0)^2,\ -1,\ -1.3489,\ -0.0581\pm0.2926i,\ -0.9508\pm1.2585i,\ -0.3167\pm1.8022i$
	$x^{10}+5x^9+15x^8+30x^7+42x^6+37x^5+16x^4+2x^3$	$(0)^3,\ (-1)^2,\ -0.1977,\ -1.1768\pm1.2281i,\ -0.2244\pm1.8564i$
	$x^{10}+5x^9+15x^8+30x^7+42x^6+37x^5+16x^4+2x^3$	$(0)^3,\ (-1)^2,\ -0.1977,\ -1.1768\pm1.2281i,\ -0.2244\pm1.8564i$
	$x^{10}+5x^9+15x^8+29x^7+41x^6+39x^5+22x^4+7x^3+x^2$	$(0)^2,\ -1,\ -0.4598,\ -0.3386\pm0.3367i,\ -0.2004\pm1.6457i,\ -1.2311\pm1.398i$
	$x^{10}+5x^9+15x^8+30x^7+42x^6+39x^5+22x^4+8x^3+2x^2$	$(0)^2,\ (-1)^2,\ -0.1147\pm0.4777i,\ -1.0835\pm1.1503i,\ -0.3018\pm1.7963i$
	$x^{10}+5x^9+15x^8+30x^7+43x^6+41x^5+22x^4+6x^3+x^2$	$(0)^2,\ (-1)^2,\ -0.1639\pm0.28i,\ -1.1513\pm1.291i,\ -0.1848\pm1.7724i$
	$x^{10}+5x^9+15x^8+30x^7+44x^6+43x^5+22x^4+4x^3$	$(0)^3,\ (-1)^2,\ -0.3665,\ -1.2308\pm1.3839i,\ -0.0859\pm1.7817i$
	$x^{10}+5x^9+15x^8+30x^7+44x^6+45x^5+28x^4+10x^3+2x^2$	$(0)^2,\ (-1)^2,\ -0.2436\pm0.4032i,\ -1.1373\pm1.3645i,\ -0.1191\pm1.686i$
	$x^{10}+5x^9+15x^8+30x^7+45x^6+47x^5+28x^4+8x^3+x^2$	$(0)^2,\ (-1)^2,\ -0.2484\pm0.1789i,\ -1.2315\pm1.4418i,\ -0.0201\pm1.7228i$
	$x^{10}+5x^9+15x^8+30x^7+45x^6+49x^5+34x^4+14x^3+3x^2$	$(0)^2,\ (-1)^2,\ -0.3355\pm0.477i,\ -0.0235\pm1.6156i,\ -1.141\pm1.4413i$
	$x^{10}+5x^9+15x^8+30x^7+45x^6+51x^5+40x^4+20x^3+5x^2$	$(0)^2,\ (-1)^2,\ -0.5\pm0.6887i,\ 0.0198\pm1.469i,\ -1.0198\pm1.469i$

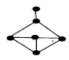
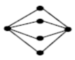
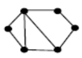
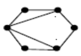
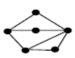
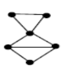
Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{12}+6x^{11}+21x^{10}+45x^9+65x^8+66x^7+51x^6+30x^5+15x^4+5x^3+x^2$	$(0)^2,\ 0.1235\pm0.63i,\ -0.3468\pm0.3391i,\ -0.5\pm0.866i,\ -1.1904\pm0.6991i,\ -1.0862\pm2.0572i$
	$x^{12}+6x^{11}+21x^{10}+46x^9+69x^8+69x^7+45x^6+18x^5+6x^4+x^3$	$(0)^3,\ -0.3121,\ -0.0323\pm0.4384i,\ -1.0811\pm0.6413i,\ -0.8774\pm1.3707i,\ -0.8532\pm1.7983i$
	$x^{12}+6x^{11}+21x^{10}+46x^9+70x^8+70x^7+43x^6+14x^5+3x^4$	$(0)^4,\ -0.1845\pm0.3661i,\ -1.0193\pm0.5608i,\ -0.569\pm1.7459i,\ -1.2272\pm1.5512i$
	$x^{12}+6x^{11}+21x^{10}+47x^9+73x^8+75x^7+46x^6+12x^5+x^4$	$(0)^4,\ -0.1569,\ -0.2961,\ -1.1689\pm0.632i,\ -0.4916\pm1.8189i,\ -1.1129\pm1.4817i$
	$x^{12}+6x^{11}+21x^{10}+47x^9+73x^8+75x^7+48x^6+16x^5+3x^4$	$(0)^4,\ -0.2402\pm0.3241i,\ -1.0707\pm0.6568i,\ -0.5512\pm1.855i,\ -1.1379\pm1.351i$
	$x^{12}+6x^{11}+21x^{10}+48x^9+76x^8+80x^7+51x^6+14x^5+x^4$	$(0)^4,\ -0.1065,\ -0.4101,\ -1.289\pm0.6654i,\ -0.466\pm1.9189i,\ -0.9867\pm1.348i$
	$x^{12}+6x^{11}+21x^{10}+47x^9+73x^8+78x^7+59x^6+32x^5+15x^4+5x^3+x^2$	$(0)^2,\ 0.1005\pm0.5357i,\ -0.3634\pm0.3076i,\ -0.9047\pm1.0954i,\ -1.0993\pm0.7781i,\ -0.733\pm1.8761i$
	$x^{12}+6x^{11}+21x^{10}+47x^9+74x^8+77x^7+55x^6+23x^5+7x^4+x^3$	$(0)^3,\ -0.2808,\ -0.1229\pm0.4157i,\ -0.6917\pm0.9552i,\ -0.5851\pm1.8882i,\ -1.46\pm1.1647i$
	$x^{12}+6x^{11}+21x^{10}+47x^9+75x^8+81x^7+54x^6+30x^5+3x^4$	$(0)^4,\ -0.1226,\ -1.7056,\ -0.1608\pm0.8745i,\ -0.4263\pm1.8985i,\ -1.4987\pm1.5959i$
	$x^{12}+6x^{11}+21x^{10}+48x^9+77x^8+85x^7+61x^6+24x^5+6x^4+x^3$	$(0)^3,\ -0.4477,\ -0.0846\pm0.3091i,\ -1.2938\pm0.6559i,\ -0.4577\pm1.7908i,\ -0.9401\pm1.4635i$

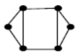
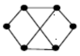
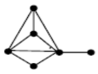
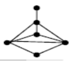
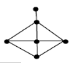
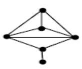
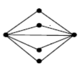
Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{12}+6x^{11}+21x^{10}+48x^9+77x^8+$ $84x^7+60x^6+25x^5+7x^4+x^3$	$(0)^3,\ -0.333,\ -0.135\pm 0.3698i,$ $-1.1933\pm 0.7199i,\ -0.4782\pm 1.8756i,$ $-1.027\pm 1.2682i$
	$x^{12}+6x^{11}+21x^{10}+48x^9+78x^8+$ $87x^7+62x^6+22x^5+4x^4$	$(0)^4,\ -0.2759\pm 0.2872i,\ -1.302\pm 0.7156i,$ $-0.3364\pm 1.8063i,\ -1.0857\pm 1.4852i$
	$x^{12}+6x^{11}+21x^{10}+48x^9+78x^8+$ $86x^7+59x^6+20x^5+3x^4$	$(0)^4,\ -0.2987\pm 0.2091i,\ -1.1757\pm 0.6497i,$ $-0.3324\pm 1.8639i,\ -1.1932\pm 1.4373i$
	$x^{12}+6x^{11}+21x^{10}+48x^9+78x^8+$ $86x^7+61x^6+24x^5+5x^4$	$(0)^4,\ -0.3289\pm 0.392i,\ -1.0706\pm 0.6814i,$ $-1.2184\pm 1.3314i,\ -0.382\pm 1.8693i$
	$x^{12}+6x^{11}+21x^{10}+48x^9+78x^8+$ $88x^7+65x^6+26x^5+5x^4$	$(0)^4,\ -0.3725\pm 0.3183i,\ -1.2292\pm 0.6886i,$ $-0.3643\pm 1.7517i,\ -1.034\pm 1.4858i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+81x^8+$ $93x^7+70x^6+28x^5+5x^4$	$(0)^4,\ -0.3903\pm 0.2555i,\ -0.8845\pm 1.4059i,$ $-1.3704\pm 0.678i,\ -0.3548\pm 1.8539i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+81x^8+$ $92x^7+65x^6+20x^5+2x^4$	$(0)^4,\ -0.193,\ -0.3455,\ -1.444\pm 0.647i,$ $-0.2776\pm 1.8986i,\ -1.0092\pm 1.4952i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+81x^8+$ $91x^7+64x^6+22x^5+3x^4$	$(0)^4,\ -0.3236\pm 0.1375i,\ -1.3071\pm 0.6816i,$ $-1.0564\pm 1.3279i,\ -0.3129\pm 1.9443i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+84x^8+$ $96x^7+69x^6+24x^5+3x^4$	$(0)^4,\ -0.2931,\ -0.3912,\ -0.9289\pm 1.2843i,$ $-1.4442\pm 0.6431i,\ -0.2848\pm 2.0214i$
	$x^{12}+6x^{11}+21x^{10}+48x^9+78x^8+$ $88x^7+69x^6+37x^5+16x^4+5x^3+x^2$	$(0)^2,\ 0.0483\pm 0.4866i,\ -0.3807\pm 0.322i,$ $-1.0655\pm 0.7386i,\ -0.4789\pm 1.8345i,$ $-1.1231\pm 1.2337i$

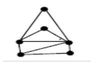
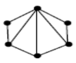
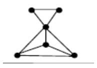
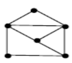
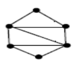
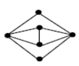
Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{12}+6x^{11}+21x^{10}+48x^9+78x^8+88x^7+67x^6+32x^5+11x^4+2x^3$	$(0)^3,\ -0.4201,\ -0.1104\pm0.4858i,\ -1.1744\pm0.6852i,\ -0.4308\pm1.7973i,\ -1.0744\pm1.3725i$
	$x^{12}+6x^{11}+21x^{10}+48x^9+79x^8+91x^7+69x^6+30x^5+8x^4+x^3$	$(0)^3,\ -0.3093,\ -0.1986\pm0.3323i,\ -1.1854\pm0.6472i,\ -0.2844\pm1.7695i,\ -1.177\pm1.516i$
	$x^{12}+6x^{11}+21x^{10}+48x^9+79x^8+92x^7+72x^6+33x^5+9x^4+x^3$	$(0)^3,\ -0.2376,\ -0.2756\pm0.3559i,\ -1.2094\pm0.6657i,\ -0.2836\pm1.7083i,\ -1.1126\pm1.5483i$
	$x^{12}+6x^{11}+21x^{10}+48x^9+80x^8+94x^7+73x^6+32x^5+7x^4$	$(0)^4,\ -0.418\pm0.4055i,\ -1.117\pm0.6553i,\ -0.2117\pm1.7592i,\ -1.2534\pm1.5328i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+81x^8+94x^7+75x^6+38x^5+15x^4+5x^3+x^2$	$(0)^2,\ 0.0702\pm0.4345i,\ -0.4437\pm0.2135i,\ -0.8269\pm1.3436i,\ -1.3747\pm0.6844i,\ -0.425\pm1.8568i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+82x^8+96x^7+74x^6+32x^5+8x^4+x^3$	$(0)^3,\ -0.3786,\ -0.1866\pm0.2785i,\ -1.3112\pm0.6703i,\ -0.2771\pm1.8538i,\ -1.0359\pm1.4185i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+82x^8+97x^7+77x^6+35x^5+9x^4+x^3$	$(0)^3,\ -0.2893,\ -0.2676\pm0.2811i,\ -1.3411\pm0.6708i,\ -0.2853\pm1.7984i,\ -0.9614\pm1.4678i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+82x^8+97x^7+77x^6+35x^5+9x^4+x^3$	$(0)^3,\ -0.2893,\ -0.2676\pm0.2811i,\ -1.3411\pm0.6708i,\ -0.2853\pm1.7984i,\ -0.9614\pm1.4678i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+82x^8+96x^7+76x^6+36x^5+10x^4+x^3$	$(0)^3,\ -0.184,\ -0.3372\pm0.4009i,\ -1.2633\pm0.7621i,\ -0.332\pm1.8505i,\ -0.9755\pm1.2738$

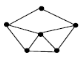
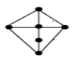
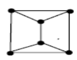
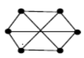
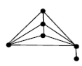
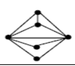
Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{12}+6x^{11}+21x^{10}+49x^9+83x^8+$ $99x^7+78x^6+34x^5+7x^4$	$(0)^4,\ -0.4344\pm0.3425,\ -1.2443\pm0.7076i,$ $-1.1133\pm1.4237i,\ -0.208\pm1.8371i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+83x^8+$ $100x^7+81x^6+37x^5+8x^4$	$(0)^4,\ -0.4721\pm0.3624i,\ -1.2814\pm0.7156i,$ $-0.2047\pm1.7874i,\ -1.0418\pm1.4677i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+83x^8+$ $99x^7+76x^6+30x^5+5x^4$	$(0)^4,\ -0.3847\pm0.2078i,\ -1.2966\pm0.6407i,$ $-0.1699\pm1.856i,\ -1.1487\pm1.5097i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+83x^8+$ $99x^7+76x^6+30x^5+5x^4$	$(0)^4,\ -0.3847\pm0.2078i,\ -1.2966\pm0.6407i,$ $-0.1699\pm1.856i,\ -1.1487\pm1.5097i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+85x^8+$ $102x^7+81x^6+37x^5+9x^4+x^3$	$(0)^3,\ -0.4793,\ -0.2359\pm0.2063i,$ $-1.34\pm0.5136i,\ -0.2629\pm1.9138i,$ $-0.9216\pm1.3839i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+86x^8+$ $104x^7+83x^6+37x^5+7x^4$	$(0)^4,\ -0.5422\pm0.2074i,\ -1.2695\pm0.6782i,$ $-0.983\pm1.3109i,\ -0.2053\pm1.9216i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+86x^8+$ $104x^7+81x^6+32x^5+5x^4$	$(0)^4,\ -0.3956\pm0.1422i,\ -1.4133\pm0.6206i,$ $-1.0285\pm1.4517i,\ -0.1626\pm1.9301i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+86x^8+$ $104x^7+81x^6+32x^5+5x^4$	$(0)^4,\ -0.3956\pm0.1422i,\ -1.4133\pm0.6206i,$ $-1.0285\pm1.4517i,\ -0.1626\pm1.9301i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+86x^8+$ $106x^7+91x^6+46x^5+11x^4$	$(0)^4,\ -0.5786\pm0.39i,\ -0.7173\pm1.4495i,$ $-1.4467\pm0.734i,\ -0.2575\pm1.7935i$
	$x^{12}+6x^{11}+21x^{10}+48x^9+80x^8+$ $96x^7+81x^6+45x^5+18x^4+5x^3+x^2$	$(0)^2,\ -0.0184\pm0.4127i,\ -0.4318\pm0.3685i,$ $-1.129\pm0.6906i,\ -0.2618\pm1.6586i,$ $-1.159\pm1.5295i$

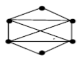
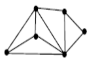
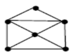
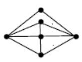
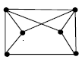
Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{12}+6x^{11}+21x^{10}+48x^9+80x^8+96x^7+81x^6+41x^5+13x^4+2x^3$	$(0)^3,\ -0.3915,\ -0.2307\pm0.4047i,\ -0.1986\pm1.6219i,\ -1.3076\pm0.8313i,\ -1.0673\pm1.5915i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+82x^8+97x^7+81x^6+46x^5+20x^4+6x^3+x^2$	$(0)^2,\ -0.0193\pm0.5142i,\ -0.3809\pm0.2499i,\ -0.9361\pm1.1908i,\ -1.2662\pm0.7793i,\ -0.3976\pm1.8522i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+83x^8+100x^7+83x^6+44x^5+17x^4+5x^3+x^2$	$(0)^2,\ 0.0212\pm0.4112i,\ -0.4442\pm0.2853i,\ -1.2421\pm0.7185i,\ -0.2605\pm1.8109i,\ -1.0744\pm1.3845i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+83x^8+101x^7+84x^6+42x^5+13x^4+2x^3$	$(0)^3,\ -0.4247,\ -0.2232\pm0.3935i,\ -1.3145\pm0.6601i,\ -0.2104\pm1.7613i,\ -1.0396\pm1.5166i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+83x^8+101x^7+84x^6+42x^5+13x^4+2x^3$	$(0)^3,\ -0.4247,\ -0.2232\pm0.3935i,\ -1.3145\pm0.6601i,\ -0.2104\pm1.7613i,\ -1.0396\pm1.5166i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+84x^8+104x^7+88x^6+44x^5+12x^4+x^3$	$(0)^3,\ -0.1341,\ -0.4273\pm0.3811i,\ -1.2532\pm0.6966i,\ -0.1351\pm1.7632i,\ -1.1174\pm1.5131i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+84x^8+104x^7+86x^6+40x^5+10x^4+x^3$	$(0)^3,\ -0.2416,\ -0.3148\pm0.2456i,\ -1.3003\pm0.6331i,\ -0.1038\pm1.7952i,\ -1.1603\pm1.5787i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+82x^8+96x^7+72x^6+32x^5+8x^4+x^3$	$(0)^3,\ (-1)^2,\ -0.6702,\ -0.2146\pm0.2651i,\ -1.1889\pm1.4508i,\ -0.2615\pm1.8914i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+84x^8+105x^7+93x^6+51x^5+15x^4+x^3$	$(0)^3,\ -0.0902,\ -0.584\pm0.4736i,\ -1.2491\pm0.8035i,\ -0.1482\pm1.6641i,\ -0.9736\pm1.4954i$

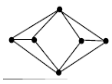
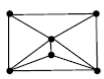
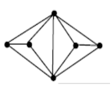
Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{12}+6x^{11}+21x^{10}+49x^9+85x^8+107x^7+92x^6+46x^5+11x^4$	$(0)^4,\ -0.5542\pm 0.4259i,\ -1.1816\pm 0.7297i,\ -0.0686\pm 1.7718i,\ -1.1956\pm 1.5114i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+86x^8+106x^7+91x^6+49x^5+18x^4+5x^3+x^2$	$(0)^2,\ 0.0065\pm 0.377i,\ -0.4814\pm 0.2683i,\ -0.8424\pm 1.3964i,\ -0.2552\pm 1.8481i,\ -1.4275\pm 0.6819i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+86x^8+106x^7+89x^6+44x^5+13x^4+2x^3$	$(0)^3,\ -0.4773,\ -0.2069\pm 0.35i,\ -1.4357\pm 0.626i,\ -0.2084\pm 1.8498i,\ -0.9105\pm 1.4676i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+86x^8+106x^7+93x^6+52x^5+17x^4+2x^3$	$(0)^3,\ -0.2189,\ -0.5046\pm 0.4894i,\ -0.6196\pm 1.2791i,\ -0.3357\pm 1.8302i,\ -1.4307\pm 0.7725i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+87x^8+109x^7+93x^6+46x^5+12x^4+x^3$	$(0)^3,\ -0.1411,\ -0.4254\pm 0.3143i,\ -1.3917\pm 0.6763i,\ -0.1365\pm 1.8433i,\ -0.9758\pm 1.4644i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+87x^8+109x^7+93x^6+46x^5+12x^4+x^3$	$(0)^3,\ -0.1411,\ -0.4254\pm 0.3143i,\ -1.3917\pm 0.6763i,\ -0.1365\pm 1.8433i,\ -0.9758\pm 1.4644i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+87x^8+109x^7+93x^6+46x^5+12x^4+x^3$	$(0)^3,\ -0.1411,\ -0.4254\pm 0.3143i,\ -1.3917\pm 0.6763i,\ -0.1365\pm 1.8433i,\ -0.9758\pm 1.4644i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+88x^8+112x^7+95x^6+44x^5+9x^4$	$(0)^4,\ -0.485\pm 0.2888i,\ -1.3653\pm 0.6551i,\ -1.1019\pm 1.5178i,\ -0.0479\pm 1.8706i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+87x^8+110x^7+98x^6+53x^5+15x^4+x^3$	$(0)^3,\ -0.092,\ -0.5575\pm 0.4075i,\ -0.8202\pm 1.4852i,\ -0.1641\pm 1.7577i,\ -1.4123\pm 0.7401i$

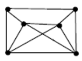
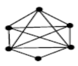
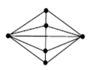
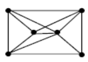
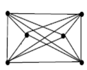
Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{12}+6x^{11}+21x^{10}+50x^9+88x^8+112x^7+99x^6+52x^5+13x^4$	$(0)^4,\ -0.6273\pm 0.4267i,\ -0.9408\pm 1.3728i,\ -1.3306\pm 0.8333i,\ -0.1013\pm 1.8161i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+88x^8+112x^7+97x^6+48x^5+11x^4$	$(0)^4,\ -0.5453\pm 0.3684i,\ -1.34\pm 0.7294i,\ -1.0427\pm 1.4531i,\ -0.072\pm 1.8455i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+88x^8+112x^7+95x^6+44x^5+9x^4$	$(0)^4,\ -0.485\pm 0.2888i,\ -1.3653\pm 0.6551i,\ -1.1019\pm 1.5178i,\ -0.0479\pm 1.8706i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+84x^8+105x^7+93x^6+54x^5+22x^4+6x^3+x^2$	$(0)^2,\ -0.0786\pm 0.4225i,\ -0.4153\pm 0.2981i,\ -1.252\pm 0.7062i,\ -0.182\pm 1.7154i,\ -1.0722\pm 1.4899i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+84x^8+106x^7+94x^6+52x^5+18x^4+3x^3$	$(0)^3,\ -0.4495,\ -0.2757\pm 0.4629i,\ -1.3173\pm 0.6507i,\ -0.1142\pm 1.6899i,\ -1.0681\pm 1.6039i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+85x^8+108x^7+95x^6+52x^5+19x^4+5x^3$	$(0)^3,\ -0.8284,\ -0.0878\pm 0.4909i,\ -1.2316\pm 0.7055i,\ -0.0743\pm 1.763i,\ -1.1922\pm 1.5644i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+85x^8+109x^7+98x^6+54x^5+17x^4+2x^3$	$(0)^3,\ -0.2333,\ -0.4386\pm 0.4226i,\ -1.2605\pm 0.6864i,\ -0.0447\pm 1.7103i,\ -1.1397\pm 1.592i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+85x^8+109x^7+100x^6+58x^5+19x^4+2x^3$	$(0)^3,\ -0.1758,\ -0.5658\pm 0.5017i,\ -1.2057\pm 0.7781i,\ -0.064\pm 1.6618i,\ -1.0767\pm 1.5277i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+86x^8+106x^7+95x^6+60x^5+29x^4+10x^3+2x^2$	$(0)^2,\ -0.0012\pm 0.581i,\ -0.4883\pm 0.315i,\ -0.7487\pm 1.114i,\ -0.3524\pm 1.9001i,\ -1.4095\pm 0.7885i$

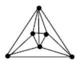
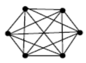
Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{12}+6x^{11}+21x^{10}+50x^{9}+87x^{8}+$ $110x^{7}+98x^{6}+56x^{5}+22x^{4}+6x^{3}+x^{2}$	$(0)^2,\ -0.0566\pm0.4029i,\ -0.4434\pm0.2502i,$ $-1.3964\pm0.682i,\ -0.1827\pm1.8053i,$ $-0.9209\pm1.4429i$
	$x^{12}+6x^{11}+21x^{10}+50x^{9}+87x^{8}+$ $111x^{7}+99x^{6}+54x^{5}+18x^{4}+3x^{3}$	$(0)^3,\ -0.493,\ -0.2562\pm0.4156i,$ $-1.4321\pm0.6182i,\ -0.1255\pm1.7715i,$ $-0.9397\pm1.5633i$
	$x^{12}+6x^{11}+21x^{10}+50x^{9}+88x^{8}+$ $113x^{7}+102x^{6}+58x^{5}+21x^{4}+5x^{3}+x^{2}$	$(0)^2,\ -0.027\pm0.3248i,\ -0.5297\pm0.3493i,$ $-1.345\pm0.7385i,\ -0.1065\pm1.8048i,$ $-0.9918\pm1.4336i$
	$x^{12}+6x^{11}+21x^{10}+50x^{9}+88x^{8}+$ $114x^{7}+107x^{6}+65x^{5}+24x^{4}+5x^{3}+x^{2}$	$(0)^2,\ -0.0431\pm0.2804i,\ -0.6378\pm0.4173i,$ $-0.8158\pm1.4693i,\ -0.1205\pm1.7096i,$ $-1.3828\pm0.8164i$
	$x^{12}+6x^{11}+21x^{10}+50x^{9}+88x^{8}+$ $114x^{7}+103x^{6}+56x^{5}+17x^{4}+2x^{3}$	$(0)^3,\ -0.2542,\ -0.422\pm0.3556i,$ $-1.3894\pm0.6657i,\ -0.0554\pm1.7847i,$ $-1.006\pm1.5497i$
	$x^{12}+6x^{11}+21x^{10}+50x^{9}+88x^{8}+$ $114x^{7}+105x^{6}+60x^{5}+19x^{4}+2x^{3}$	$(0)^3,\ -0.1831,\ -0.5358\pm0.4304i,$ $-0.9159\pm1.5054i,\ -0.0806\pm1.7439i,$ $-1.3761\pm0.7416i$
	$x^{12}+6x^{11}+21x^{10}+50x^{9}+88x^{8}+$ $114x^{7}+103x^{6}+56x^{5}+17x^{4}+2x^{3}$	$(0)^3,\ -0.2542,\ -0.422\pm0.3556i,$ $-1.3894\pm0.6657i,\ -0.0554\pm1.7847i,$ $-1.006\pm1.5497i$
	$x^{12}+6x^{11}+21x^{10}+50x^{9}+88x^{8}+$ $114x^{7}+105x^{6}+60x^{5}+19x^{4}+2x^{3}$	$(0)^3,\ -0.1831,\ -0.5358\pm0.4304i,$ $-0.9159\pm1.5054i,\ -0.0806\pm1.7439i,$ $-1.3761\pm0.7416i$
	$x^{12}+6x^{11}+21x^{10}+50x^{9}+88x^{8}+$ $114x^{7}+105x^{6}+60x^{5}+19x^{4}+2x^{3}$	$(0)^3,\ -0.1831,\ -0.5358\pm0.4304i,$ $-0.9159\pm1.5054i,\ -0.0806\pm1.7439i,$ $-1.3761\pm0.7416i$

Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{12}+6x^{11}+21x^{10}+50x^9+89x^8+117x^7+107x^6+58x^5+16x^4+x^3$	$(0)^3,\ -0.0849,\ -0.5511\pm0.3827i,\ -1.3403\pm0.715i,\ 0.0048\pm1.8007i,\ -1.071\pm1.5332i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+89x^8+118x^7+109x^6+62x^5+18x^4+x^3$	$(0)^3,\ -0.0709,\ -0.6957\pm0.5181i,\ -1.1718\pm0.5179i,\ 0.0102\pm1.7809i,\ -1.1072\pm1.5419i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+89x^8+117x^7+107x^6+58x^5+16x^4+x^3$	$(0)^3,\ -0.0849,\ -0.5511\pm0.3827i,\ -1.3403\pm0.715i,\ 0.0048\pm1.8007i,\ -1.071\pm1.5332i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+120x^7+111x^6+60x^5+15x^4$	$(0)^4,\ -0.6364\pm0.4i,\ -1.283\pm0.7667i,\ -1.1381\pm1.5159i,\ 0.0575\pm1.8177i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+120x^7+111x^6+60x^5+15x^4$	$(0)^4,\ -0.6364\pm0.4i,\ -1.283\pm0.7667i,\ -1.1381\pm1.5159i,\ 0.0575\pm1.8177i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+85x^8+110x^7+103x^6+64x^5+27x^4+7x^3+x^2$	$(0)^2,\ -0.2107\pm0.4016i,\ -0.3587\pm0.3219i,\ -1.2598\pm0.695i,\ -0.0725\pm1.6492i,\ -1.0983\pm1.5824i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+85x^8+111x^7+106x^6+66x^5+25x^4+4x^3$	$(0)^3,\ -0.3743,\ -0.4699\pm0.5431i,\ -1.2836\pm0.712i,\ -0.014\pm1.596i,\ -1.0454\pm1.6378i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+88x^8+114x^7+107x^6+68x^5+31x^4+10x^3+2x^2$	$(0)^2,\ -0.0443\pm0.4978i,\ -0.5229\pm0.3243i,\ -0.9295\pm1.4096i,\ -0.1523\pm1.7575i,\ -1.3511\pm0.7477i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+88x^8+115x^7+108x^6+66x^5+27x^4+7x^3+x^2$	$(0)^2,\ -0.1532\pm0.3941i,\ -0.4109\pm0.2588i,\ -1.3937\pm0.6711i,\ -0.0866\pm1.7314i,\ -0.9557\pm1.5443i$

Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{12}+6x^{11}+21x^{10}+50x^9+88x^8+116x^7+111x^6+68x^5+25x^4+4x^3$	$(0)^3$, -0.4065, $-0.4318\pm0.4781i$, $-1.4156\pm0.6723i$, $-0.0405\pm1.6731i$, $-0.9089\pm1.6189i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+89x^8+118x^7+112x^6+68x^5+26x^4+6x^3+x^2$	$(0)^2$, $-0.0835\pm0.3041i$, $-0.5301\pm0.3671i$, $-1.3446\pm0.7231i$, $-0.0164\pm1.7521i$, $-1.0255\pm1.5257i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+89x^8+118x^7+114x^6+72x^5+28x^4+6x^3+x^2$	$(0)^2$, $-0.0847\pm0.2693i$, $-0.6226\pm0.4356i$, $-0.9221\pm1.4781i$, $-0.0345\pm1.7066i$, $-1.3362\pm0.8198i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+89x^8+119x^7+115x^6+70x^5+24x^4+3x^3$	$(0)^3$, -0.243, $-0.5545\pm0.4425i$, $-1.3725\pm0.7272i$, $0.0242\pm1.7089i$, $-0.9757\pm1.5901i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+89x^8+119x^7+115x^6+70x^5+24x^4+3x^3$	$(0)^3$, -0.243, $-0.5545\pm0.4425i$, $-1.3725\pm0.7272i$, $0.0242\pm1.7089i$, $-0.9757\pm1.5901i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+89x^8+119x^7+115x^6+70x^5+24x^4+3x^3$	$(0)^3$, -0.243, $-0.5545\pm0.4425i$, $-1.3725\pm0.7272i$, $0.0242\pm1.7089i$, $-0.9757\pm1.5901i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+121x^7+116x^6+70x^5+25x^4+5x^3+x^2$	$(0)^2$, $-0.0363\pm0.2693i$, $-0.6238\pm0.3994i$, $-1.2869\pm0.779i$, $0.043\pm1.7737i$, $-1.096\pm1.5049i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+122x^7+119x^6+72x^5+23x^4+2x^3$	$(0)^3$, -0.1297, $-0.6492\pm0.4324i$, $-1.324\pm0.7875i$, $0.0788\pm1.74i$, $-1.0408\pm1.5613i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+122x^7+119x^6+72x^5+23x^4+2x^3$	$(0)^3$, -0.1297, $-0.6492\pm0.4324i$, $-1.324\pm0.7875i$, $0.0788\pm1.74i$, $-1.0408\pm1.5613i$

Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+122x^7+121x^6+76x^5+25x^4+2x^3$	$(0)^3,\ -0.1121,\ -0.736\pm0.4347i,\ 0.0763\pm1.7018i,\ -1.3382\pm0.8915i,\ -0.946\pm1.5361i$
	$x^{12}+6x^{11}+21x^{10}+49x^9+85x^8+112x^7+111x^6+76x^5+35x^4+9x^3+x^2$	$(0)^2,\ -0.2865\pm0.1324i,\ -0.4153\pm0.6139i,\ -0.0178\pm1.5072i,\ -1.2854\pm0.7218i,\ -0.995\pm1.6462i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+89x^8+119x^7+117x^6+78x^5+36x^4+11x^3+2x^2$	$(0)^2,\ -0.1224\pm0.4731i,\ -0.5155\pm0.3399i,\ -1.3496\pm0.7314i,\ -0.0412\pm1.6919i,\ -0.9713\pm1.5201i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+87x^8+110x^7+98x^6+56x^5+23x^4+6x^3+x^2$	$(0)^2,\ -0.095\pm0.4168i,\ -0.3314\pm0.3237i,\ -1.448\pm0.67i,\ -0.1814\pm1.8122i,\ -0.9443\pm1.459i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+89x^8+121x^7+123x^6+82x^5+32x^4+5x^3$	$(0)^3,\ -0.3499,\ -0.5934\pm0.4954i,\ 0.0784\pm1.6146i,\ -1.4059\pm0.7324i,\ -0.9041\pm1.6807i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+123x^7+124x^6+82x^5+33x^4+7x^3+x^2$	$(0)^2,\ -0.1152\pm0.2405i,\ -0.6359\pm0.439i,\ -1.3318\pm0.7978i,\ 0.074\pm1.685i,\ -0.9911\pm1.5662i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+122x^7+123x^6+84x^5+37x^4+10x^3+2x^2$	$(0)^2,\ -0.0913\pm0.3631i,\ -0.7025\pm0.4349i,\ -0.9207\pm1.4437i,\ 0.0185\pm1.6731i,\ -1.3039\pm0.9196i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+123x^7+124x^6+82x^5+33x^4+7x^3+x^2$	$(0)^2,\ -0.1152\pm0.2405i,\ -0.6359\pm0.439i,\ -1.3318\pm0.7978i,\ 0.074\pm1.685i,\ -0.9911\pm1.5662i$

Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+124x^7+127x^6+84x^5+31x^4+4x^3$	$(0)^3,\ -0.2388,\ -0.6756\pm 0.4587i,\ -1.3675\pm 0.7958i,\ 0.1242\pm 1.6615i,\ -0.9617\pm 1.6404i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+124x^7+127x^6+84x^5+31x^4+4x^3$	$(0)^3,\ -0.2388,\ -0.6756\pm 0.4587i,\ -1.3675\pm 0.7958i,\ 0.1242\pm 1.6615i,\ -0.9617\pm 1.6404i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+89x^8+121x^7+125x^6+90x^5+44x^4+13x^3+2x^2$	$(0)^2,\ -0.3369\pm 0.3633i,\ -0.4276\pm 0.4667i,\ 0.0164\pm 1.5594i,\ -1.3856\pm 0.7407i,\ -0.8664\pm 1.6238i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+123x^7+126x^6+90x^5+45x^4+15x^3+3x^2$	$(0)^2,\ -0.1346\pm 0.524i,\ -0.5926\pm 0.389i,\ 0.0122\pm 1.655i,\ -1.299\pm 0.8058i,\ -0.986\pm 1.4882i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+124x^7+129x^6+92x^5+43x^4+12x^3+2x^2$	$(0)^2,\ -0.1808\pm 0.3545i,\ -0.618\pm 0.4469i,\ 0.0759\pm 1.6192i,\ -1.341\pm 0.8078i,\ -0.936\pm 1.5796i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+125x^7+132x^6+94x^5+41x^4+9x^3+x^2$	$(0)^2,\ -0.1737\pm 0.1812i,\ -0.6682\pm 0.4706i,\ 0.1377\pm 1.6032i,\ -1.377\pm 0.8028i,\ -0.9187\pm 1.6633i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+125x^7+134x^6+102x^5+53x^4+17x^3+3x^2$	$(0)^2,\ -0.2721\pm 0.4424i,\ -0.5909\pm 0.4568i,\ 0.0924\pm 1.542i,\ -1.3515\pm 0.8172i,\ -0.8779\pm 1.606i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+126x^7+135x^6+96x^5+39x^4+6x^3$	$(0)^3,\ -0.3189,\ -0.7075\pm 0.4716i,\ 0.1916\pm 1.5988i,\ -1.4076\pm 0.7946i,\ -0.917\pm 1.7322i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+126x^7+139x^6+112x^5+63x^4+22x^3+4x^2$	$(0)^2,\ -0.4209\pm 0.5088i,\ -0.527\pm 0.4644i,\ 0.1355\pm 1.4616i,\ -1.3631\pm 0.8255i,\ -0.8245\pm 1.6489i$

Graph	Roman Domination Polynomial	Roman Domination Roots
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+126x^7+137x^6+104x^5+51x^4+14x^3+2x^2$	$(0)^2$, $-0.2378\pm 0.2739i$, $-0.6605\pm 0.4862i$, $0.164\pm 1.5421i$, $-1.3872\pm 0.8092i$, $-0.8785\pm 1.6945i$
	$x^{12}+6x^{11}+21x^{10}+50x^9+90x^8+126x^7+141x^6+120x^5+75x^4+30x^3+6x^2$	$(0)^2$, $-0.5662\pm 0.3387i$, $-0.5\pm 0.866i$, $0.1321\pm 1.3332i$, $-0.7327\pm 1.5959i$, $-1.3332\pm 0.8446i$

Now, all connected graphs of order$\leq$ 6 with their Roman domination polynomials and roots are listed in the table.

References

[1] S. Alikhani, *Dominating Sets and Domination Polynomials of Graphs-Domination Polynomial: A New Graph Polynomial*, LAMBERT Academic Publishing, 2012.

[2] S. Alikhania and Yee-hock Pengb, An atlas of domination polynomials of graphs of order at most six, *arXiv:1401.3141v1 [math.CO].* (2014).

[3] Deepak G., Anwar Alwardi and Indiramma M. H., Roman Domination Polynomial of Cycles, *International J. Math. Combin.*, 3 (2021) 86–97.

[4] Deepak G., Anwar Alwardi and Indiramma M. H., Roman Domination Polynomial of Paths, *Palestine Journal of Mathematics*, 11 (2) (2022) 439–448.

[5] Deepak G., Indiramma M. H., N. D. Soner and Anwar Alwardi, On the roman domination polynomial of graphs, *Bulletin of the International Mathematical Virtual Institute*, 11 (2) (2021) 355–365.

[6] Ernie J. Cockayne, Paul A. Dreyer Jr., Sandra M. Hedetniemi and Stephen T. Hedetniemi, Roman domination in graphs, *Discrete Mathematics*, 278 (2004) 11–22.

[7] I. Stewart, Defend the roman empire, *Sci. Amer.*, 281 (6) (1999) 136–139.

Famous Words

A contradictory behavior is out of synchronization or logical inconsistency in developing of things. However it is a universal existence in human's eyes. Thus, a mathematical system should follows the principle of logical consistency and can not contradict itself on one hand. But on the other hand, the contradiction exists everywhere among things and the logical consistency of mathematical system must lead to the limitation of characterizing things, including the incompleteness of formal logic of itself such as the Godel's incompleteness theorem, etc. So, the mathematical reality is definitely less than or different from the natural reality.

— Extracted from *Combinatorial Theory on the Universe*, a book of Dr.Linfan Mao on mathematics with philosophy of science, which systematically discusses the recognition of humans from the local to the whole, published by Global Knowledge-Publishing House in 2023.

Author Information

Submission: Papers only in electronic form are considered for possible publication. Papers prepared in formats latex, eps, pdf may be submitted electronically to one member of the Editorial Board for consideration in the **International Journal of Mathematical Combinatorics** (*ISSN 1937-1055*). An effort is made to publish a paper duly recommended by a referee within a period of $2-4$ months. Articles received are immediately put the referees/members of the Editorial Board for their opinion who generally pass on the same in six week's time or less. In case of clear recommendation for publication, the paper is accommodated in an issue to appear next. Each submitted paper is not returned, hence we advise the authors to keep a copy of their submitted papers for further processing.

Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

Books

[4]Linfan Mao, *Combinatorial Theory on the Universe*, Global Knowledge-Publishing House, USA, 2023.
[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

Figures: Figures should be drawn by TEXCAD in text directly, or as JPG, EPS file. In addition, all figures and tables should be numbered and the appropriate space reserved in the text, with the insertion point clearly indicated.

Copyright: It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the publisher, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

Proofs: One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections after the paper is accepted for publication on the basis of the recommendation of referees. Corrections should be restricted to typesetting errors. Authors are advised to check their proofs very carefully before return.

June 2023

Contents

An International Journal on Mathematical Combinatorics

ISSN 1937-1055