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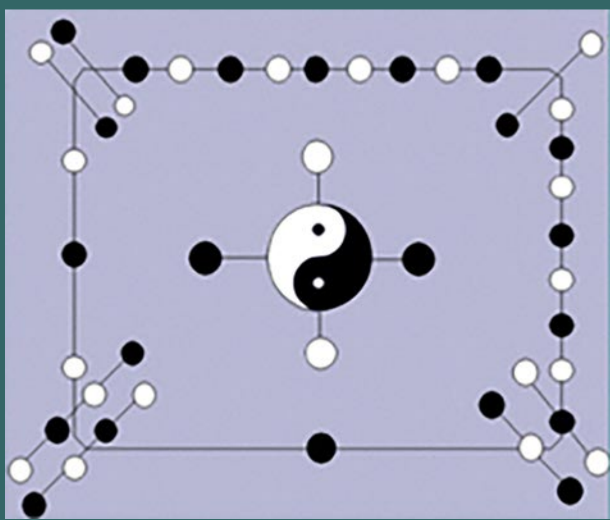

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The science of mathematics presents the most brilliant example of how pure reason may successfully enlarge its domain without the aid of experience.

By Immanuel Kant, a German Philosopher.

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$*$-Ricci Tensor on Generalized Sasakian-Space-Form

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Abstract: The aim of the paper is to study $*$-Ricci tensor in generalized Sasakian space form. We study the generalized Sasakian space form admitting $*$-conformal η- Ricci soliton and analyse the behaviour of the soliton. Also, we prove $*$-Ricci semi-symmetric and Pseudo $*$-Ricci semisymmetric generalized Sasakian space forms are $*$-Ricci flat.

Key Words: Einstein manifold, η-Einstein manifold, $*\eta$-Einstein manifold, Ricci soliton, $*$-Ricci tensor.

AMS(2010): 53E20, 53C25.

§1. Introduction

The generalized Sasakian space forms have been investigated by numerous researchers like Alegre and Carriazo [6-8]. Thereafter generalized Sasakian spaceform [GSSF] have been studied by many authors [10, 16, 19, 36, 37].

An almost contact metric manifold M is a GSSF if there exist three functions f_1, f_2, f_3 on M such that curvature tensor R is given by

$$\begin{aligned} R(X_1, X_2)X_3 = & f_1\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\} + f_2\{g(X_1, \phi X_3)\phi X_2 \\ & - g(X_2, \phi X_3)\phi X_1 + 2g(X_1, \phi X_2)\phi X_3\} \\ & + f_3\{\eta(X_1)\eta(X_3)X_2 - \eta(X_2)\eta(X_3)X_1 \\ & + g(X_1, X_3)\eta(X_2)\xi - g(X_2, X_3)\eta(X_1)\xi\} \end{aligned} \tag{1.1}$$

for any vector fields X_1, X_2, X_3 on M. In such a case we represent the manifold as $M(f_1, f_2, f_3)$.

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If $f_1 = \dfrac{c+3}{4}$, $f_2 = \dfrac{c-1}{4}$ and $f_3 = \dfrac{c-1}{4}$, then a GSSF with Sasakian structure develops Sasakian-space-forms.

A self-similizing elucidation to the Ricci flow [14,15] is said to be a Ricci soliton [42] in case it moves by only a one parameter family of diffeomorphism and scaling. Ricci soliton has been studied by many authors (See [1, 3, 9, 11, 12, 18,31C 34, 38, 39]) and is defined as:

$$L_V g + 2Ric + 2\lambda g = 0. \tag{1.2}$$

The η-Ricci soliton [12] and the Conformal η-Ricci soliton [26] are defined respectively as

$$L_V g + 2Ric = 2\lambda g + 2\mu\eta \otimes \eta, \tag{1.3}$$

$$L_V g + 2Ric + \left[2\lambda - \left(P + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0, \tag{1.4}$$

where L_V is the Lie derivative in the direction of V, Ric is the Ricci tensor, g is the Riemannian metric, V is a vector field, and λ and μ are parameters. The Ricci soliton is said to be shrinking, steady and expanding if λ is negative, zero and positive, respectively. Some related developments can be found in [1, 2, 4, 13, 20C24, 28C35, 38C41].

§2. **Preliminaries**

A $(2n+1)$-dim Riemannian manifold (M, g) is called an almost compact manifold if the following results hold [6]

$$-X_1 + \eta(X_1)\xi = \phi^2(X_1), \tag{2.1}$$

$$1 = \eta(\xi), \tag{2.2}$$

$$g(X_1, \xi) = \eta(X_1), \eta(\phi\xi) = 0, \tag{2.3}$$

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2), \tag{2.4}$$

$$g(X_1, \phi X_2) = -g(\phi X_1, X_2), \tag{2.5}$$

$$g(\phi X_1, X_1) = 0, \tag{2.6}$$

$$(\nabla_{X_1}\eta)(X_2) = g(\nabla_{X_1}\xi, X_2), \tag{2.7}$$

$$\nabla_{X_1}\xi = -\beta\phi X_1, \tag{2.8}$$

for all $X_1 \in TM$ and a function β such that $\xi\beta = 0$.

In view of (2.8), we get

$$(\nabla_{X_1}\eta)(X_2) = g(\nabla_{X_1}\xi, X_2) = -\beta g(\phi X_1, X_2). \tag{2.9}$$

For a $(GSSF)_{2n+1}$, we have

$$R(X_1, X_2)\xi = (f_1 - f_3)[\eta(X_2)X_1 - \eta(X_1)X_2], \tag{2.10}$$

$$R(\xi, X_1)X_2 = (f_1 - f_3)[g(X_1, X_2)\xi - \eta(X_2)X_1], \tag{2.11}$$

$$g(R(\xi, X_1)X_2, \xi) = (f_1 - f_3)g(\phi X_1, \phi X2), \tag{2.12}$$

$$R(\xi, X_1)\xi = (f_1 - f_3)\phi^2 X_1, \tag{2.13}$$

$$S(X_1, X_2) = (2nf_1 + 3f_2 - f_3)g(X_1, X_2) - (3f_2 + (2n-1)f_3)\eta(X_1)\eta(X_2), \tag{2.14}$$

where S is the Ricci tensor and r is the scalar curvature tensor of the space-forms.

§3. ∗-Ricci Tensor in GSSF

Let M be an GSSF with Ricci tensor S. The ∗-Ricci tensor and ∗-scalar curvature of M are defined by

$$S^*(X_1, X_2) = \sum_{i=1}^{2n+1} R(X_1, e_i, \phi e_i, \phi X_2), \qquad r^* = \sum_{i=1}^{2n+1} S^*(e_i, e_i) \tag{3.1}$$

for all $X_1, X_2 \in TM$, where $e_1, \cdots, e_{2n+1}$ is an orthonormal basis of the tangent space TM. By using the first Bianchi identity and (3.1) we get

$$S^*(X_1, X_2) = \frac{1}{2}\sum_{i=1}^{2n+1} g(\phi R(X_1, \phi X_2)e_i, e_i). \tag{3.2}$$

Let M is a GSSF, replace $X_3 = \phi X_3$ in (1.1) and taking inner product with ϕW, and then using (2.1) and (2.2) the resultant equation becomes

$$\begin{aligned} R(X_1, X_2, \phi X_3, \phi W) = {} & f_1\{g(X2, \phi X_3)g(X_1, \phi W) - g(X_1, \phi X_3)g(X_2, \phi W)\} \\ & + f_2\{-g(X_1, X_3)g(\phi X_2, \phi W) + \eta(X_1)\eta(X_3)g(\phi X_2, \phi W) \\ & + g(X_2, X_3)g(\phi X_1, \phi W) - \eta(X_2)\eta(X_3)g(\phi X_1, \phi W) \\ & - 2g(X_1, \phi X_2)g(X_3, \phi W)\}. \end{aligned} \tag{3.3}$$

Let $[e_i]_{i=1}^{2n+1}$ be an orthonormal basis of the tangent space at each point of the manifold. Then setting $X_2 = X_3 = e_i$ in (3.3) and proceeding summation over $1 \leq i \leq 2n+1$ and also by using (2.1) and (2.3), we get

$$R(X_1, e_i, \phi e_i, \phi W) = f_1 g(\phi X_1, \phi W) + f_2(2n+1)g(\phi X_1, \phi W). \tag{3.4}$$

Hence, we have the following result.

Theorem 3.1 *In a GSSF $M(f_1, f_2, f_3)$, the $*$-Ricci tensor is obtained by*

$$S^*(X_1, W) = [f_1 + (2n+1)f_2]g(X_1, W) - [f_1 + (2n+1)f_2]\eta(X_1)\eta(W). \tag{3.5}$$

The following corollary is immediate.

Corollary 3.1 *A GSSF $M(f_1, f_2, f_3)$ is an $*$-η-Einstein manifold.*

§4. $*$-Ricci Semisymmetric GSSF

A GSSF $M(f_1, f_2, f_3)$ is called Ricci semisymmetric if $R(X_1, X_2) \cdot S = 0$ for all $X_1, X_2 \in TM$. Similarly, we define $*$-Ricci semisymmetric by $R(X_1, X_2) \cdot S^* = 0$.

Let us consider a GSSF $M(f_1, f_2, f_3)$ that satisfies

$$R(X_1, X_2) \cdot S^* = 0. \tag{4.1}$$

From (4.1), we have

$$S^*(R(X_1, X2)U_1, V_1) + S^*(U_1, R(X_1, X_2)V_1) = 0. \tag{4.2}$$

Substituting $X_1 = U_1 = \xi$ we get

$$S^*(R(\xi, X_2)\xi, V_1) + S^*(\xi, R(\xi, X_2)V_1) = 0. \tag{4.3}$$

Using $S^* = 0$ and $S^*(X_1, \xi) = 0$ and (2.10) in (4.3) we obtain

$$(f_1 - f_3)S^*(X_2, V_1) = 0, \tag{4.4}$$

which gives either $(f_1 - f_3) \neq 0$ or $S^*(X_2, V_1) = 0$. Hence, we have the following theorem.

Theorem 4.1 *Let M be a GSSF is $*$-Ricci semisymmetric. Then either $f_1 \neq f_3$ or $M(f_1, f_2, f_3)$ is $*$-Ricci flat.*

§5. ϕ-Pseudo $*$-Ricci Symmetric GSSF

Definition 5.1 *A GSSF M is called ϕ-pseudo Ricci symmetric if the $*$-Ricci operator Q^* satisfies*

$$\phi^2((\nabla_{X_1} Q^*)(X_2)) = 2K(X_1)Q^*(X_2) + K(X_2)Q^*X_1 + S^*(X_2, X_1)\rho, \tag{5.1}$$

for any vector field X_1, X_2 where K is a non-zero 1-form.

If, in particular, $K = 0$ then manifold is called ϕ-$*$-Ricci symmetric [27]. Let us take a GSSF M, which is ϕ-pseudo $*$-Ricci symmetric. Then by virtue of (2.1), it follows from (5.1) that

$$-(\nabla_{X_1}Q^*)(X_2) + \eta(\nabla_{X_1}Q^*)(X_2)\xi = 2K(X_1)Q^*(X_2) + K(X_2)Q^*X_1 + S^*(X_2, X_1)\rho, \quad (5.2)$$

from which it follows that

$$\begin{aligned} -g((\nabla_{X_1}Q^*)(X_2), X_3) &+ S^*(Q\nabla_{X_1}X_2, X_3) + \eta((\nabla_{X_1}Q^*)(X_2))\eta(X_3) \\ &= 2K(X_1)S^*(X_2, X_3) + K(X_2)S^*(X_1, X_3). \end{aligned} \quad (5.3)$$

Take $X_2 = \xi$ in (5.3) and use (2.8), (3.5) to get

$$-\beta S^*(\phi X_1, X_3) + \eta((\nabla_{X_1}Q^*)(\xi))\eta(X_3) = K(\xi)S^*(X_1, X_3). \quad (5.4)$$

Put $X_3 = \phi X_3$ in (5.4) and use (3.5)

$$-\beta F g(\phi X_1, \phi X_3) = K(\xi)F g(X_1, X_3). \quad (5.5)$$

By using (2.4) and (3.5), then contracting (5.5) on top of X_1 and X_3, we obtain

$$r^* = \frac{K(\xi)F(2n+1)}{\beta}. \quad (5.6)$$

Hence, we have the following theorem.

Theorem 5.1 *If GSSF M is a ϕ-pseudo $*$ Ricci symmetric, then*

$$r^* = \frac{K(\xi)F(2n+1)}{\beta}.$$

In particular, if $K = 0$, In view of (3.5) and (5.5), we obtain

$$\beta S^*(X_1, X_3) = 0. \quad (5.7)$$

Hence, we have the following corollary.

Corollary 5.1 *A ϕ-$*$-Ricci symmetric GSSF is $*$-Ricci flat provided $\beta \neq 0$.*

§6. $*$-Conformal η-Ricci Soliton in GSSF

Definition 6.1 *The $*$-Conformal η-Ricci soliton is defined as*

$$L_V g + 2Ric^* + \left[2\lambda - \left(P + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0, \quad (6.1)$$

where L_V is the Lie derivative along the vector field V, λ and μ are constants, Ric^* is the $*$-Ricci tensor, P is a scalar non- dynamical field and n is the dimension of manifold.

Let M be a GSSF admiting $*$-conformal η-Ricci soliton (g, v, λ, μ). When $V = \xi$ in (6.1),

$$L_\xi g(X_1, X_2) + 2S^*(X_1, X_2) + \left[2\lambda - \left(P + \frac{2}{n}\right)\right] g(X_1, X_2) + 2\mu\eta(X_1)\eta(X_2) = 0. \tag{6.2}$$

This can be written as

$$\begin{aligned} g(\nabla_{X_1}\xi, X_2) &+ g(X_1, \nabla_{X_2}\xi) + 2S^*(X_1, X_2) \\ &+ \left[2\lambda - \left(P + \frac{2}{n}\right)\right] g(X_1, X_2) + 2\mu\eta(X_1)\eta(X_2) = 0. \end{aligned} \tag{6.3}$$

Using (2.8) in (6.3), we get

$$\begin{aligned} g(-\beta\phi X_1, X_2) &+ g(X_1, -\beta\phi X_2) + 2S^*(X_{1,2}) \\ &+ \left[2\lambda - \left(P + \frac{2}{n}\right)\right] g(X_1, X_2) + 2\mu\eta(X_1)\eta(X_2) = 0. \end{aligned} \tag{6.4}$$

Since M is a GSSF, making use of (2.5) in (6.4), we know

$$S^*(X_1, X_2) = -\left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right] g(X_1, X_2) - \mu\eta(X_1)\eta(X_2). \tag{6.5}$$

From (6.5), we have the $*$-scalar curvature

$$r^* = -\left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right) + \mu\right](2n+1),$$

which is a constant. In view of (3.5) and (6.5), we have

$$\begin{aligned} &\left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right] g(X_1, X_2) + \mu\eta(X_1)\eta(X_2) \\ &+[f_1 + (2n+1)f_2]g(X_1, X_2) - [f_1 + (2n+1)f_2]\eta(X)\eta(X_2) = 0. \end{aligned} \tag{6.6}$$

Using X_1 by ϕX_1 in (6.6), we have

$$\left\{(f_1 + (2n+1)f_2) + \left(\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right)\right\} g(\phi X_1, X_2) = 0. \tag{6.7}$$

Interchanging X_1 and X_2 we obtain

$$\left\{(f_1 + (2n+1)f_2) + \left(\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right)\right\} g(\phi X_2, X_1) = 0. \tag{6.8}$$

Solving (6.7) and (6.8), we get

$$\lambda = \frac{1}{2}\left(P + \frac{2}{n}\right) - F, \tag{6.9}$$

where $F = (f_1 + (2n+1)f_2)$. Thus, we have the following theorem.

Theorem 6.1 *Let M be a GSSF admitting $*$-conformal η-Ricci soliton. Then, the nature of soliton is*

(1) *steady when* $P = 2F - \frac{2}{n}$;

(2) *expanding when* $P > 2F - \frac{2}{n}$;

(3) *shrinking when* $P < 2F - \frac{2}{n}$.

Making use of (6.9) in (6.5), the following corollary is immediate.

Corollary 6.1 *If a GSSF $M(f_1, f_2, f_3)$ admitting a $*$-conformal η-Ricci soliton, then M is Sasaki-$*$-η-Einstein.*

Setting $X_1 = \xi$ in (6.6), we get

$$\left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right) + \mu\right]\eta(X_2) = 0. \tag{6.10}$$

Take $X_2 = \xi$ in (6.10) we obtain

$$\mu = -\left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]. \tag{6.11}$$

Making use of (6.9) and (6.11) in (6.5) we have

$$S^*(X_1, X_2) = Fg(X_1, X_2) - \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\eta(X_1)\eta(X_2). \tag{6.12}$$

In view of (6.12) and (5.7), by putting $X_1 = X_2 = \xi$, we have the following corollary.

Corollary 6.2 *If a $*$-conformal η-Ricci soliton on a ϕ-psuedo $*$-Ricci symmetric GSSF, then the nature of soliton is*

(1) *steady when* $P = 2F - \frac{2}{n}$;

(2) *expanding when* $P > 2F - \frac{2}{n}$;

(3) *shrinking when* $P < 2F - \frac{2}{n}$.

Definition 6.2 *A GSSF is said to be $*$-weakly symmetric if there exists 1-form A, B, C, D, E on M such that the condition*

$$\begin{aligned}(\nabla_{X_1} S^*)(X_3, W_1) =& A(X_1)S^*(X_3, W_1) + B(R(X_1, X_3)W_1) \\ &+ C(X_3)S^*(X_1, W_1) + D(W_1)S^*(X_1, X_3) + E(R(X_1, W_1)X_3),\end{aligned} \tag{6.13}$$

where the 1-form E is defined by $E(X_1) = g(X_1, V_1)$, $\forall x \in \chi(M)$.

Definition 6.3([11]) *A GSSF is said to be $*$-weakly Ricci-symmetric if there exists 1-form*

ε, σ, E on M such that the condition

$$(\nabla_{X_1} S^*)(X_2, X_3) = \varepsilon(X_1) S^*(X_2, X_3) + \sigma(X_2) S^*(X_1, X_3) + E(X_3) S^*(X_1, X_2), \tag{6.14}$$

holds for all vector fields $X_1, X_2, X_3, W \in \chi(M)$. If $\varepsilon = \sigma = E$, then M is said to be pseudo Ricci-symmetric.

Let M be a weakly symmetric GSSF. Then substituting $W = \xi$ in (6.13), we have

$$\begin{aligned}(\nabla_{X_1} S^*)(X_3, \xi) \quad = \quad & A(X_1) S^*(X_3, \xi) + B(R(X_1, X_3)\xi) \\ & + C(X_3) S^*(X_1, \xi) + D(\xi) S^*(X_1, X_3) + E(R(X_1, \xi) X_3).\end{aligned} \tag{6.15}$$

In view of (2.10) and (6.12), equation (6.15) reduces to

$$\begin{aligned}(\nabla_{X_1} S^*)(X_3, \xi) \quad = \quad & A(X_1) \left\{ F\eta(Z) - \left[\lambda - \frac{1}{2} \left(P + \frac{2}{n} \right) \right] \eta(X_3) \right\} \\ & + B(f_1 - f_3) \{ \eta(Z) X - \eta(X_1) X_3 \} \\ & + C(X_3) \left\{ F\eta(X_1) - \left[\lambda - \frac{1}{2} \left(P + \frac{2}{n} \right) \right] \eta(X_1) \right\} \\ & + D(\xi) \left\{ Fg(X_1, X_3) - \left[\lambda - \frac{1}{2} \left(P + \frac{2}{n} \right) \right] \eta(X_1) \eta(X_3) \right\} \\ & + E(R(X_1, \xi) X_3).\end{aligned} \tag{6.16}$$

Considering the covariant derivative of the $*$-Ricci tensor S^* along the vector field X_1, we obtain

$$(\nabla_{X_1} S^*)(X_3, \xi) = \nabla_{X_1} S^*(X_3, \xi) - S^*(\nabla_{X_1} X_3, \xi) - S^*(X_3, \nabla_{X_1} \xi). \tag{6.17}$$

By the use of (2.8) and (6.12) above equation takes the form

$$(\nabla_{X_1} S^*)(X_3, \xi) = -B \left\{ \lambda - \frac{1}{2} \left(P + \frac{2}{n} \right) \right\} g(\phi X_1, X_3). \tag{6.18}$$

In view of (6.16) and (6.18), we obtain

$$\begin{aligned}& A(X_1) \left\{ F\eta(X_3) - \left[\lambda - \frac{1}{2} \left(P + \frac{2}{n} \right) \right] \eta(X_3) \right\} + B(f_1 - f_3) \{ \eta(X_3) X_1 - \eta(X_1) X_3 \} \\ & + C(Z) \left\{ F\eta(X_1) - \left[\lambda - \frac{1}{2} \left(P + \frac{2}{n} \right) \right] \eta(X_1) \right\} \\ & + D(\xi) \left\{ Fg(X_1, X_3) - \left[\lambda - \frac{1}{2} \left(P + \frac{2}{n} \right) \right] \eta(X_1) \eta(X_3) \right\} \\ & + E(R(X_1, \xi) X_3) = -B \left\{ \lambda - \frac{1}{2} \left(P + \frac{2}{n} \right) \right\} g(\phi X_1, X_3).\end{aligned} \tag{6.19}$$

Setting $X_1 = X_3 = \xi$ in (6.19) and on simplification, it yields

$$\left\{ F + \left[\lambda - \frac{1}{2} \left(P + \frac{2}{n} \right) \right] \right\} \{ A(\xi) + C(\xi) + D(\xi) \} = 0 \tag{6.20}$$

which implies that the vanishing of the 1-form $A + C + D$ over the vector field ξ is necessary in order that M is a Ricci soliton on weakly symmetric GSSF. As similar to the previous calculation, it can be easily shown that

$$\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\{A(\xi) + C(\xi) + D(\xi)\} = 0$$

holds for arbitrary vector field X_1 on M. This gives the following theorem.

Theorem 6.2 *If (g, ξ, λ, μ) is a $*$-conformal η-Ricci soliton on a weakly symmetric GSSF, then the sum of 1-form is zero everywhere, provided*

$$F + (\lambda - \frac{1}{2}(P + \frac{2}{n})) \neq 0.$$

Suppose that M is a $*$- weakly Ricci symmetric GSSF. Taking $X_3 = \xi$ in (6.14) and by use of (6.12), we have

$$\begin{aligned}(\nabla_{X_1} S^*)(X_2, \xi) &= \varepsilon(X_1)\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\eta(X_2) \\ &\quad + \sigma(X_2)\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\eta(X_1) \\ &\quad + E(\xi)\left\{Fg(X_1, X_2) + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\eta(X_1)\eta(X_2)\right\}. \qquad (6.21)\end{aligned}$$

Again replacing X_3 by X_2 in (6.18), we get

$$(\nabla_{X_1} S^*)(X_2, \xi) = -B\left\{\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right\} g(\phi X_1, X_2). \qquad (6.22)$$

Comparing equations (6.21) and (6.22), we get

$$\begin{aligned}\varepsilon(X_1)&\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\eta(X_2) + \sigma(X_2)\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\eta(X_1) \\ &+ E(\xi)\left\{Fg(X_1, X_2) + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\eta(X_1)\eta(X_2)\right\} \\ &= -B\left\{\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right\} g(\phi X_1, X_2). \qquad (6.23)\end{aligned}$$

Setting $X_1 = X_2 = \xi$ in (6.23), we have

$$\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\{\varepsilon(\xi) + \sigma(\xi) + E(\xi)\} = 0. \qquad (6.24)$$

Again, putting $X_1 = \xi$ in (6.23), we have

$$\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\sigma(X_2) = \sigma(\xi)\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}. \qquad (6.25)$$

Replacing X_2 with X_1, it yields

$$\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\sigma(X_1) = \sigma(\xi)\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}. \tag{6.26}$$

If we take $X_2 = \xi$ in (6.23), we get

$$\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\varepsilon(X_1) = \varepsilon(\xi)\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}. \tag{6.27}$$

Similarly, we have

$$\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}E(X_1) = E(\xi)\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}. \tag{6.28}$$

Adding (6.26), (6.27) and (6.28) and using (6.24), we get

$$\left\{F + \left[\lambda - \frac{1}{2}\left(P + \frac{2}{n}\right)\right]\right\}\{\sigma(X_1) + \varepsilon(X_1) + \rho(X_1)\} = 0,$$

for all $X_1 \in \chi(M)$. Thus, we have the following result.

Theorem 6.3 *Let M be a $*$-weakly Ricci symmetric GSSF admits $*$-conformal η- Ricci soliton. Then the sum of 1-forms is zero, i.e., $\varepsilon+\sigma+\rho = 0$ everywhere, provided $F+(\lambda-\frac{1}{2}(P+\frac{2}{n})) \neq 0$.*

§7. Conclusions

In this paper, the generalized Sasakian space form admitting $*$-conformal η- Ricci soliton has been studied and the behaviour of the soliton is analysed. Also, it is proved that the $*$-Ricci semi-symmetric and the pseudo $*$-Ricci semisymmetric generalized Sasakian space forms are $*$-Ricci flat.

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References

[1] P. G. Angadi, G. S. Shivaprasanna, G. Somashekhara and P. Siva Kota Reddy, Ricci-Yamabe solitons on submanifolds of some indefinite almost contact manifolds, *Adv. Math., Sci. J.*, 9(11) (2020), 10067-10080.

[2] P. G. Angadi, G. S. Shivaprasanna, R. Rajendra, P. Siva Kota Reddy and G. Somashekhara, η-Yamabe soliton on 3-dimensional α-Para Kenmotsu manifold, *Chinese Journal of Mathematical Sciences*, 1(1) (2021), 29–37.

[3] P. G. Angadi, G. S. Shivaprasanna, G. Somashekhara and P. Siva Kota Reddy, Ricci

solitons on (LCS)-manifolds under D-homothetic deformation, *Ital. J. Pure Appl. Math.*, 46 (2021), 672–683.

[4] P. G. Angadi, P. Siva Kota Reddy, G. S. Shivaprasanna and G. Somashekhara, On weakly symmetric generalized (k, μ)-space forms, *Proc. Jangjeon Math. Soc.*, 25(2) (2022), 133-144.

[5] P. Alegre, D. E. Blair and A. Carriazo, Generalized Sasakian-space-forms, *Israel J. Math.*, 14 (2004), 157C183.

[6] P. Alegre and A. Carriazo, Structures on generalised Sasakian-space-forms, *Differential Geom. Appl.*, 26 (2008), 656C666.

[7] P. Alegre and A. Carriazo, Submanifolds of generalised Sasakian space forms, *Taiwanese J. Math.*, 13 (2009), 923C941.

[8] P. Alegre and A. Carriazo, Generalised Sasakian space forms and conformal changes of the metric, *Results in Math.*, 59 (2011), 485C493.

[9] N. Basu and A. Bhattacharyya, Conformal Ricci soliton in Kenmotsu manifold, *Glob. J. Adv. Res. Class. Mod. Geom.*, 4(1) (2015), 15-21.

[10] A. Carriazo, On generalised Sasakian space forms, *Proceedings of the Ninth International Workshop on Diff. Geo.*, 9 (2005), 31C39.

[11] M. C. Chaki, On pseudo Ricci symmetric manifolds, *Bulgar. J. Phys.*, 15 (1988), 526-531.

[12] J. T. Cho and M. Kimura, Ricci solitons and real hypersurfaces in a complex space form, *Tohoku Math. J. (2)*, 61(2) (2009), 205-212.

[13] S. Girish Babu, R. Rajendra, P. Siva Kota Reddy and N. Pavani, (ϵ)-Kenmotsu Manifold admitting Schouten-Van Kampen Connection, *Journal of AppliedMath*, 1(2) (2023), 1-7.

[14] R. S. Hamilton, Three manifold with positive Ricci curvature, *J. Differential Geom.*, 17(2) (1982), 255-306.

[15] R. S. Hamilton, The Ricci flow on surfaces, *Contemporary Mathematics*, 71 (1988), 237-261.

[16] A. A. Hosseinzadeh, Some curvature properties of generalized Sasakian space forms, *Proc. Nat. Acad. Sci. India Sect. A*, 89(4) (2019), 721-727.

[17] U. K. Kim, Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms, *Note Mat.*, 26 (2006), 55-67.

[18] G. Kaimakamis and K. Panagiotidou, $*$-Ricci solitons of real hypersurface in non-flat complex space forms, *J. Geom. Phys.*, 76 (2014), 408-413.

[19] Y. B. Maralabhavi and G. S. Shivaprasanna, Second order parallel tensor s on generalized Sasakian space forms, *Int. J. Math. Sci. Eng. Appl.*, 1 (2012), 11-21.

[20] H. G. Nagaraja, Dipansha Kumari and P. Siva Kota Reddy, Submanifolds of (k, μ)- Contact Metric Manifold as Ricci Solitons, *Proc. Jangjeon Math. Soc.*, 24(1) (2021), 11-19.

[21] R. T. Naveen Kumar, P. Siva Kota Reddy, Venkatesha and Khaled A. A. Alloush, $(LCS)_n$-Manifold endowed with torseforming vector field and concircular curvature tensor, *Int. J. Math. Comb.*, 3 (2022), 38-47.

[22] R. T. Naveen Kumar, P. Siva Kota Reddy, Venkatesha and M. Sangeetha, Certain results on (k, μ)-contact metric manifold endowed with concircular curvature tensor, *Commun. Math. Appl.*, 14(1) (2023), 215-225.

[23] B. Phalaksha Murthy, R. T. Naveen Kumar, P. Siva Kota Reddy and Venkatesha, On

$N(k)$-contact metric manifold endowed with pseudo-quasi-conformal curvature tensor, *Adv. Math., Sci. J.*, 10(4) (2021), 1969–1982.

[24] B. Phalaksha Murthy, R. T. Naveen Kumar, P. Siva Kota Reddy and Venkatesha, Extended pseudo projective curvature tensor on $(LCS)_n$-manifold, *Proc. Jangjeon Math. Soc.*, 25(3) (2022), 347-354.

[25] Z. Olszak, On the existence of generalized complex space forms, *Israel J. Math.* 65 (1989), 214C218.

[26] M. D. Siddiqi, Conformal η-Ricci solitons in δ-Lorentzian trans Sasakian manifolds, *Int. J. Maps Math.*, 1(1) (2018), 15-34.

[27] S. S. Shukla and M. K. Shukla, On ϕ-Ricci symmetric Kenmotsu manifolds, *Novi Sad J. Math.*, 39(2) (2009), 89-95.

[28] G. Somashekhara, N. Pavani and P. Siva Kota Reddy, Invariant sub-manifolds of LP-Sasakian manifolds with semi-symmetric connection, *Bull. Math. Anal. Appl.*, 12(2) (2020), 35-44.

[29] G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, C-Bochner curvature tensor under D-homothetic deformation in LP-Sasakian manifold, *Bull. Int. Math. Virtual Inst.*, 11(1) (2021), 91-98.

[30] G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, Indefinite Sasakian manifold with quarter-symmetric metric connection, *Proc. Jangjeon Math. Soc.*, 24(1) (2021), 91-98.

[31] G. Somashekhara, P. Siva Kota Reddy, N. Pavani and G. J. Manjula, η-Ricci-Yamabe solitons on submanifolds of some indefinite almost contact manifolds, *J. Math. Comput. Sci.*, 11(3) (2021), 3775-3791.

[32] G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, Conformal Ricci soliton in an indefinite trans-Sasakian manifold, *Vladikavkaz Math. J*, 23(3) (2021), 43-49.

[33] G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, Ricci solitons and generalized weak symmetries under D-homothetically deformed LP-Sasakian manifolds, *Ital. J. Pure Appl. Math.*, 46 (2021), 684-695.

[34] G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, Conformal η-Ricci solitons in Lorentzian para-Sasakian manifold admitting semi-symmetric metric connection, *Ital. J. Pure Appl. Math.*, 46 (2021), 1008-1019.

[35] G. Somashekhara, P. Siva Kota Reddy, K. Shivashankara and N. Pavani, Slant submanifolds of generalized Sasakian-space-forms, *Proc. Jangjeon Math. Soc.*, 25(1) (2022), 83-88.

[36] G. Somashekhara, R. Rajendra, G. S. Shivaprasanna, P. Siva Kota Reddy, Savithri Shashidhar and Khaled A. A. Alloush, Generalised Sasakian spaceform in submanifolds, Int. J. Math. Comb., 3 (2022), 69-81.

[37] G. Somashekhara, K. Bhavya and G. S. Shivaprasanna, Invariant submanifold of generalized Sasakian space form with semi-symmetric metric connection, *Montes Taurus Journal of Pure and Applied Mathematics*, 4(1) (2022), 128C134.

[38] G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, η-Ricci soliton in an indefinite trans-Sasakian manifold admitting semi-symmetric metric connection, *Bol. Soc. Parana. Mat. (3)*, 41 (2023), 1-9.

[39] G. Somashekhara, R. Rajendra, G. S. Shivaprasanna and P. Siva Kota Reddy, Pseudo parallel and generalized Ricci pseudo parallel invariant submanifolds of a generalized Sasakian space form, *Proc. Jangjeon Math. Soc.*, 26(1) (2023), 69-78.

[40] G. Somashekhara, S. Girish Babu, R. Rajendra, P. Siva Kota Reddy and Khaled A. A. Alloush, Lorentzian β-Kenmotsu manifolds admitting semi-symmetric non-metric connection, *Bull. Int. Math. Virtual Inst.*, 13(1) (2023), 51-62.

[41] P. Somashekhara, R. T. Naveen Kumar, P. Siva Kota Reddy, Venkatesha and Khaled A. A. Alloush, Pseudo projective curvature tensor on generalized Sasakian space forms, *Proceedings of the Jangjeon Math. Soc.*, 26(3) (2023), 243-251.

[42] P. Topping, *Lecture on Ricci Flow*, Cambridge University Press, Cambridge, (2006).

International J.Math. Combin. Vol.4(2023), 14-32

Number of Spanning Trees of Sequence of Some Families of Graphs That Have the Same Average Degree and Their Entropies

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Abstract: The number of spanning trees is an important quantity characterizing the reliability of a network (graph). Generally, the number of spanning trees in a network can be obtained directly by calculating a related determinant corresponding to the network. However, for a large network, evaluating the relevant determinant is intractable. In this paper, we investigated the number of spanning trees in three sequences of families of graphs of the same average degree $\frac{16}{3}$. We used the electrically equivalent transformations and rules of weighted generating function which avoids the laborious computation of the determinant for counting the number of spanning trees. Finally, we determined the entropy of our studied graphs.

Key Words: Number of spanning trees, electrically equivalent transformations, entropy.

AMS(2010): 05C30, 05C50, 05C63.

§1. Introduction

The counting spanning trees in networks (graphs) is a fascinating and central issue in statistical physics, because of its inherent relevance to diverse aspects in related fields. For instance, the number of spanning trees is an important measure of reliability of a network [1], [2]. Again, for example, it is exactly the number of recurrent configurations of the Abelian sand-pile models [3],[4], which have been studied extensively in the past two decades as a paradigm of the self-organized criticality [5]. On the other hand, the problem of spanning trees has numerous connections with other interesting problems associated with networks, such as dimer coverings [8], Potts model [7] random walks [8], origin of fractality for fractal scale-free networks [8, 9] and many others.

The number of spanning trees $\tau(G)$ of a finite connected undirected graph G is an acyclic $(n-1)$ - edge spanning sub-graph.

There exist various methods for finding this number. Kirchhoff's matrix tree theorem named after Gustav Kirchhoff[10] is a theorem about the number of spanning trees in a graph,

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showing that this number can be computed in polynomial time from the determinant of a submatrix of the Laplacian matrix of the graph; specifically, the number is equal to any co-factor of the Laplacian matrix.

Another method to count the complexity of a graph is using Laplacian eigenvalues. Let G be a connected graph with k vertices. Kelmans and Chelnoknoy [11] derived the following formula:

$$\tau(G) = \frac{1}{k}\prod_{i=1}^{k-1}\mu_i, \tag{1.1}$$

where $k = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k = 0$ are the eigenvalues of the Kirchhoff matrix L.

Degenerating the graph through successive elimination of contraction of its edges represent the core of another way to compute the complexity of a graph [12, 13, 14]. If $G = (V, E)$ is a multigraph with $e \in E$, then $G.e$ is the graph obtained from G by contracting the degree until its endpoints are a single vertex. The formula for computing the number of spanning trees of a multigraph G is given by:

$$\tau(G) = \tau(G - e) + \tau(G.e). \tag{1.2}$$

This formula is beautiful but not practically useful (grows exponentially with the size of the graph- may be as many as $2^{|E(G)|}$ terms. For a summary of further results for calculating umber of spanning trees of graphs, see [15, 16, 17, 18].

§2. Electrically Equivalent Transformations

To begin with, we briefly review the electrically equivalent transformation technique introduced in [19, 20, 21, 22]. An edge-weighted graph G (with the weight function ω : E(G) $\to [0, \infty)$) can be considered as an electrical network with the weights being the conductances of the corresponding edges. The weighted number of spanning trees in G is defined as follows:

Let G be an edge weighted graph, G' be the corresponding electrically equivalent graph, $\tau(G)$ denotes the weighted number of spanning trees G.

(1) Parallel edges: If two parallel edges with conductances u and v in G are merged into a single edge with conductances $u + v$ in G', then $\tau(G') = \tau(G)$.

(2) Serial edges: If two serial edges with conductances u and v in G are merged into a single edge with conductance $\frac{uv}{u+v}$ in G', then $\tau(G') = \frac{1}{u+v}\tau(G)$.

(3) $\sqcup - Y$ transformation: If a triangle with conductances u, v and w in G is changed into an electrically equivalent star graph with conductances

$$x = \frac{uv + vw + wu}{u}, y = \frac{uv + vw + wu}{v} \text{ and } z = \frac{uv + vw + wu}{w}$$

in H', then $\tau(G') = \frac{(uv+vw+wu)^2}{uvw}\tau(G)$.

(4) $Y - \sqcup$ transformation: If a star graph with conductances u, v and w in G is changed into an electrically equivalent triangle with conductances

$$x = \frac{vw}{u + v + w}, y = \frac{uw}{u + v + w} \text{ and } z = \frac{uv}{u + v + w}$$

in G', then $\tau(G') = \frac{1}{u+v+w}\tau(G)$.

In this work, we compute the number of spanning trees of three sequences of graphs of the same average degree we named it E_n, F_n and H_n respectively.

§3. Number of Spanning Trees in the Sequences of E_n Graph

Consider the sequence of graphs $\mathrm{E}_1, \mathrm{E}_2, \cdots, \mathrm{E}_n$ constructed as shown in Figure 1. According to this construction, the number of total vertices $|V(\mathrm{E}_n)|$ and edges $|E(\mathrm{E}_n)|$ are $|V(\mathrm{E}_n)| = 9n-6$ and $|E(\mathrm{E}_n)| = 24n-21, n = 1, 2, \cdots$. The average degree of E_n is 16/3 in the large n limit.

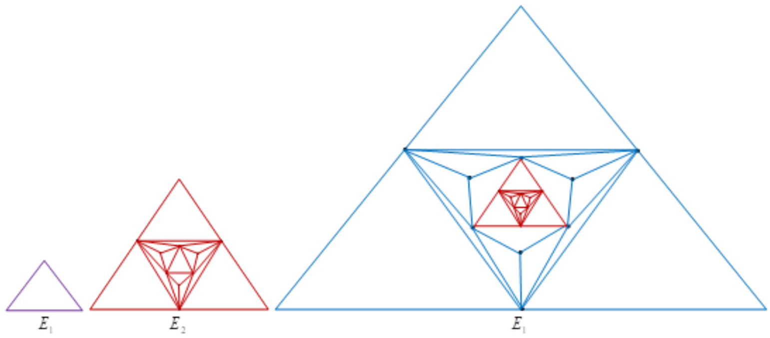

Figure 1. Some sequences of graph E_n

Theorem 3.1 *For $n \geq 1$, the number of spanning trees in sequence of the graph E_n is given by*

$$\frac{1}{27} \times 16^{n-4} \left(256 - 13 \times 64^n\right)^2.$$

Proof We use the electrically equivalent transformation to transform E_i to E_{i-1}. Figures 2-6 illustrate the transformation process from E_2 to E_1.

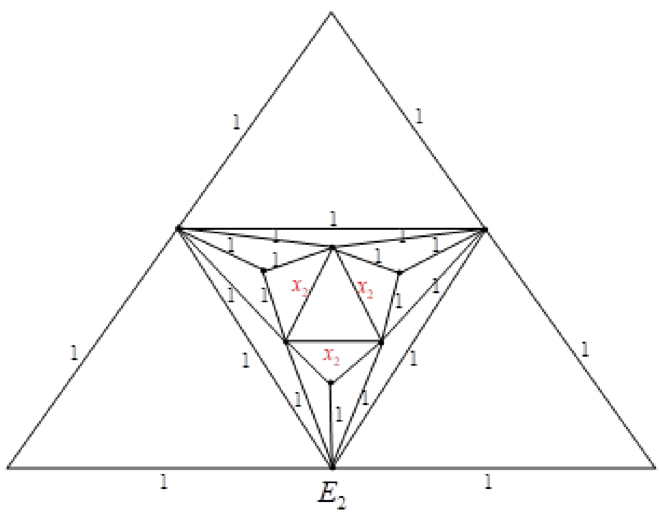

Figure 2

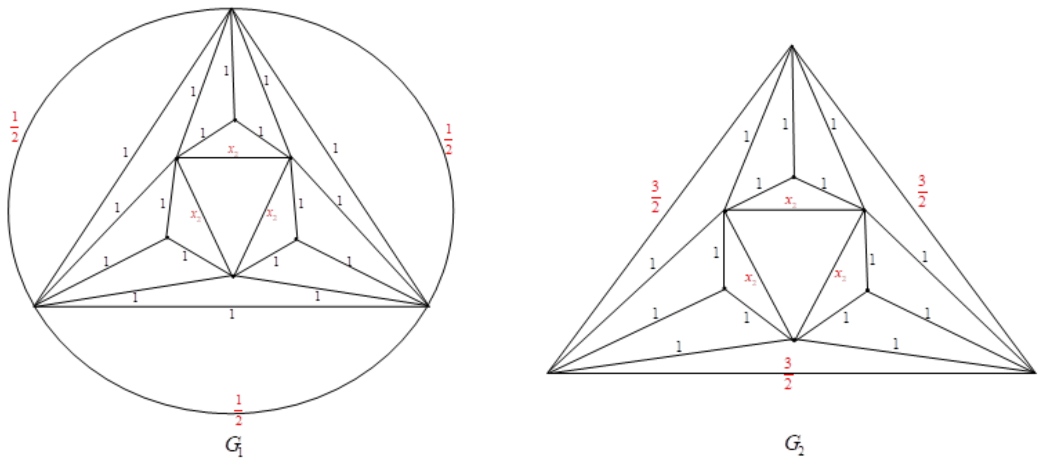

Figure 3

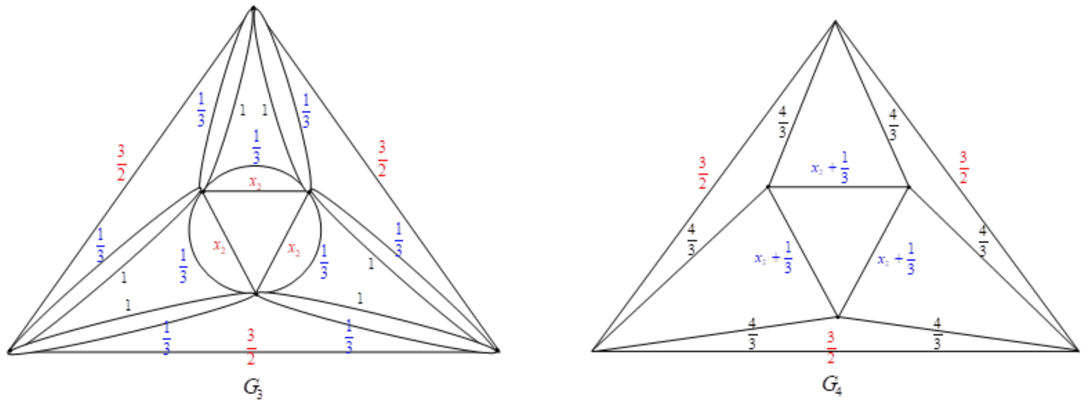

Figure 4

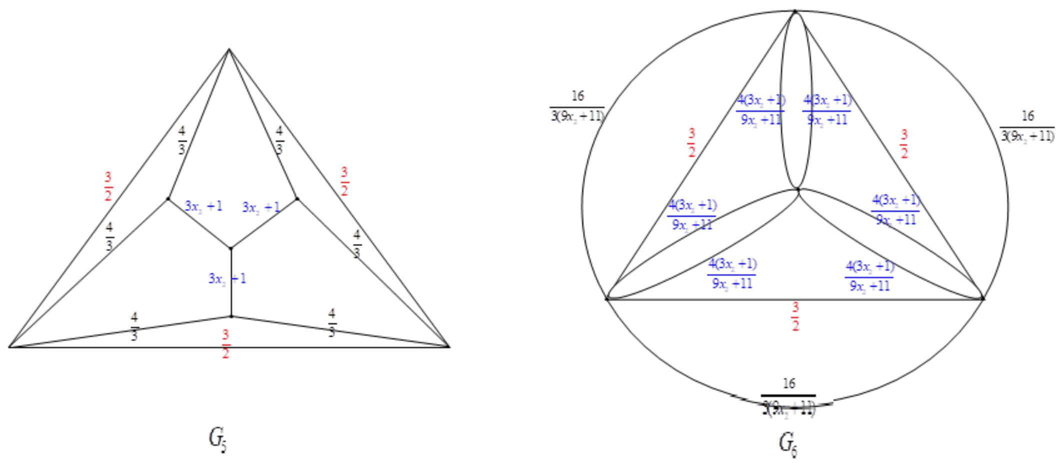

Figure 5

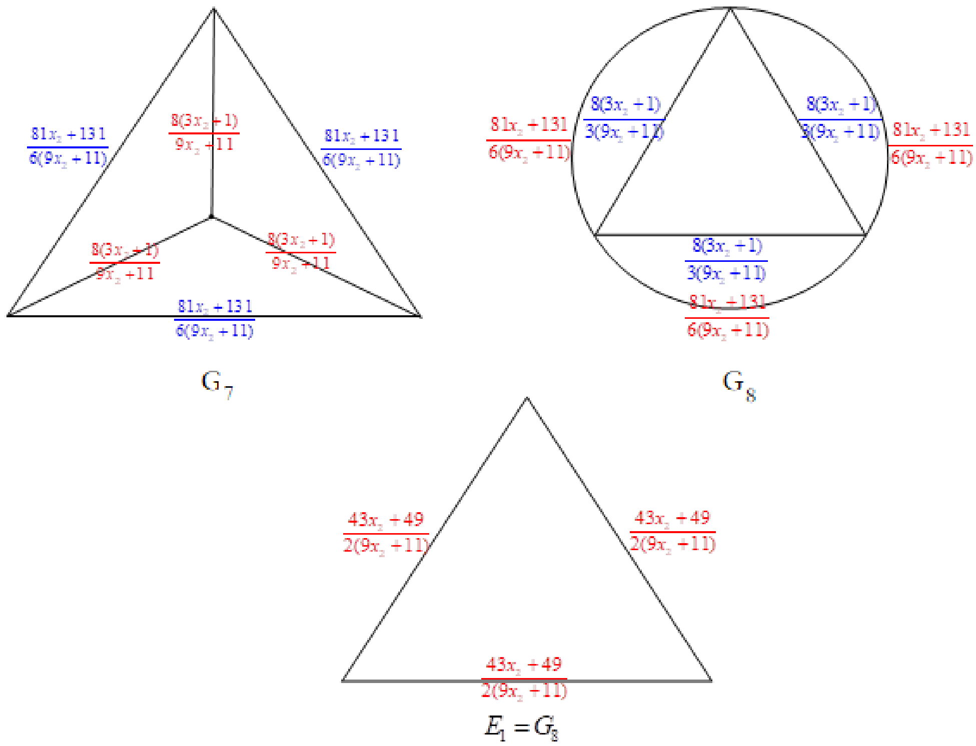

Figure 6

Using the properties given in Section 2, we have the following transformations:

$$
\begin{aligned}
\tau(G_1) &= \left[\frac{1}{2}\right]^3 \tau(\mathrm{E}_2), \quad \tau(G_2) = \tau(G_1), \quad \tau(G_3) = \left[\frac{1}{3}\right]^3 \tau(G_2), \\
\tau(G_4) &= \tau(G_3), \quad \tau(G_5) = (9x_2+3)\,\tau(G_4), \quad \tau(G_6) = \left[\frac{3}{9x_2+11}\right]^3 \tau(G_5), \\
\tau(G_7) &= \tau(G_6), \quad \tau(G_8) = \frac{(9x_2+11)}{24\,(3x_2+1)}\tau(G_7) \text{ and } \tau(\mathrm{E}_1) = \tau(G_8).
\end{aligned}
$$

Combining these nine transformations, we get

$$\tau(\mathrm{E}_2) = 16\,(18x_2+22)^2\,\tau(\mathrm{E}_1), \tag{3.1}$$

where

$$\tau(\mathrm{E}_1) = 3\times(16)^{n-1}x_1^2\left[\prod_{i=2}^{n}(18x_i+22)\right]^2.$$

Further,

$$\tau(\mathrm{E}_n) = \prod_{i=2}^{n} 16\,(18x_2+22)^2, \tag{3.2}$$

where $x_{i-1} = \frac{43x_i+49}{18x_i+22}, i = 2, 3, \cdots, n$. Its characteristic equation is $18\mu^2 - 21\mu - 49 = 0$, which have two roots $\mu_1 = \frac{-7}{6}$ and $\mu_2 = \frac{7}{3}$. Subtracting these two roots into both sides of

$x_{i-1}=\frac{43x_i+49}{18x_i+22}$, we get

$$x_{i-1}+\frac{7}{6}=\frac{43x_i+49}{18x_i+22}+\frac{7}{6}=\frac{64\left(x_i+\frac{7}{6}\right)}{(18x_i+22)}. \tag{3.3}$$

$$x_{i-1}-\frac{7}{3}=\frac{43x_i+49}{18x_i+22}-\frac{7}{3}=\frac{\left(x_i-\frac{7}{3}\right)}{(18x_i+22)}. \tag{3.4}$$

Let $y_i=\frac{x_i+\frac{7}{6}}{x_i-\frac{7}{3}}$. Then by Eqs.(3.3) and (3.4), we get $y_{i-1}=(64)y_i$ and $y_i=(64)^{n-i}y_n$. Therefore,

$$x_i=\frac{(64)^{n-i}\left(\frac{7}{3}\right)y_n+\frac{7}{6}}{(64)^{n-i}y_{n-1}}.$$

Thus

$$x_1=\frac{(64)^{n-1}\left(\frac{7}{3}\right)y_n+\frac{7}{6}}{(64)^{n-1}y_{n-1}}. \tag{3.5}$$

Using the expression $x_{n-1}=\frac{43x_n+49}{18x_n+22}$ and denoting the coefficients of $43x_n+49$ and $18x_n+22$ as σ_n and δ_n we have

$$18x_n+22=\sigma_0\left(43x_n+49\right)+\delta_0\left(18x_n+22\right),$$

$$18x_{n-1}+22=\frac{\sigma_1\left(43x_n+49\right)+\delta_1\left(18x_n+22\right)}{\sigma_0\left(43x_n+49\right)+\delta_0\left(18x_n+22\right)},$$

$$\vdots$$

$$18x_{n-i}+22=\frac{\sigma_i\left(43x_n+49\right)+\delta_i\left(18x_n+22\right)}{\sigma_{i-1}\left(43x_n+49\right)+\delta_{i-1}\left(18x_n+22\right)'}, \tag{3.6}$$

$$18x_{n-(i+1)}+22=\frac{\sigma_{i+1}\left(43x_n+49\right)+\delta_{i+1}\left(18x_n+22\right)}{\sigma_i\left(43x_n+49\right)+\delta_i\left(18x_n+22\right)}, \tag{3.7}$$

$$\vdots$$

$$18x_2+22=\frac{\sigma_{n-2}\left(43x_n+49\right)+\delta_{n-2}\left(18x_n+22\right)}{\sigma_{n-3}\left(43x_n+49\right)+\delta_{n-3}\left(18x_n+22\right)'}.$$

Substituting Eq.(3.6) into Eq.(3.2), we obtain

$$\tau\left(\mathrm{E}_n\right)=3\times(16)^{n-1}x_1^2\left[\sigma_{n-2}\left(43x_n+49\right)+\sigma_{n-2}\left(18x_n+22\right)\right]^2, \tag{3.8}$$

where $\sigma_0=0,\delta_0=1$ and $\sigma_1=18,\delta_1=22$.

By the expression $x_{n-1}=\frac{43x_n+49}{18x_n+22}$ and Eqs.(3.6) and (3.7), we have

$$\sigma_{i+1}=65\sigma_i-64\sigma_{i-1};\delta_{i+1}=65\delta_i-64\delta_{i-1}. \tag{3.9}$$

The characteristic equation of Eq.(3.9) is $\gamma^2-65\gamma+64=0$ which have two roots $\gamma_1=64$ and $\gamma_2=1$. The general solutions of Eq. (3.9) are $\sigma_i=a_1\gamma_1^i+a_2\gamma_2^i;\delta_i=b_1\gamma_1^i+b_2\gamma_2^i$. Using the initial conditions $\sigma_0 = 0, \delta_0 = 1$ and $\sigma_1 = 18, \delta_1 = 22$, yields

$$\sigma_i = \frac{2}{7}(64)^i - \frac{2}{7}; b_i = \frac{1}{3}(64)^i + \frac{2}{3}. \tag{3.10}$$

If $x_n = 1$, it means that E_n without any electrically equivalent transformation. Plugging Eq.(3.10) into Eq.(3.8), we have

$$\tau(\mathrm{E}_n) = 3 \times (16)^{n-1} x_1^2 \left[\frac{832}{21}(64)^{n-2} + \frac{8}{21}\right]^2, n \geq 2. \tag{3.11}$$

When $n = 1, \tau(E) = 3$ which satisfies Eq.(3.11). Therefore, the number of spanning trees in the sequence of the graph E_n is given by

$$\tau(\mathrm{E}_n) = 3 \times (16)^{n-1} x_1^2 \left[\frac{832}{21}(64)^{n-2} + \frac{8}{21}\right]^2, n \geq 1, \tag{3.12}$$

where

$$x_1 = \frac{91(64)^{n-1} - 28}{39(64)^{n-1} + 24}, \qquad n \geq 1. \tag{3.13}$$

Inserting Eq.(3.13) into Eq.(3.12) we obtain the result. □

§4. Number of Spanning Trees in the Sequences of F_n Graph

Consider the sequence of graphs $\mathrm{F}_1, \mathrm{F}_2, \cdots, \mathrm{F}_n$ constructed as shown in Figure 7. According to this construction, the number of total vertices $|V(\mathrm{F}_n)|$ and edges $|E(\mathrm{F}_n)|$ are $|\mathrm{V}(\mathrm{F}_n)| = 9n - 6$ and $|E(\mathrm{F}_n)| = 24n - 21, n = 1, 2, \cdots$. The average degree of F_n is $\frac{16}{3}$ in the large n limit.

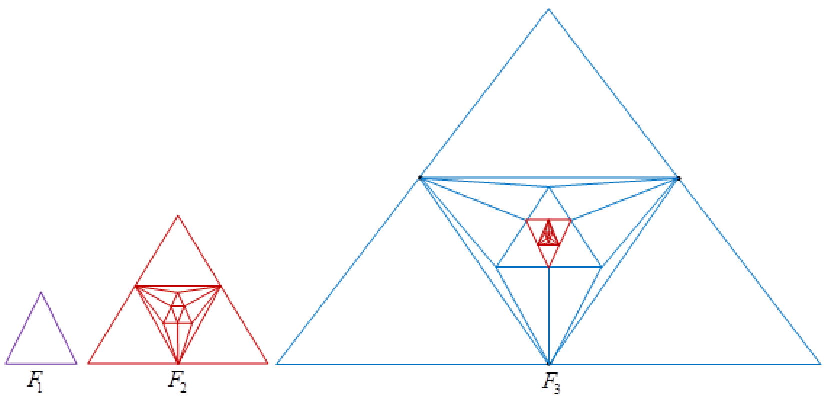

Figure 7. Some sequences of graph F_n

Theorem 4.1 *For $n \geq 1$, the number of spanning trees in the sequence of F_n graph is given by $\frac{A_n}{B_n}$, where*

$$A_n = \left(400^{n-3}\left((85 - 21\sqrt{15})(2(4+\sqrt{15}))^n + (8 - 2\sqrt{15})^n(85 + 21\sqrt{15})\right)^2 \left(-61(321 + 83\sqrt{15})\right.\right.$$

$$
\begin{aligned}
& \quad +(31+8\sqrt{15})^n(951+365\sqrt{15})\Big)^2\Big), \\
B_n &= 3\left(61(31+8\sqrt{15})+(64+5\sqrt{15})(31+8\sqrt{15})^n\right)^2.
\end{aligned}
$$

Proof We use the electrically equivalent transformation to transform F_i to F_{i-1}. Figures 8-13 illustrate the transformation process from F_2 to F_1.

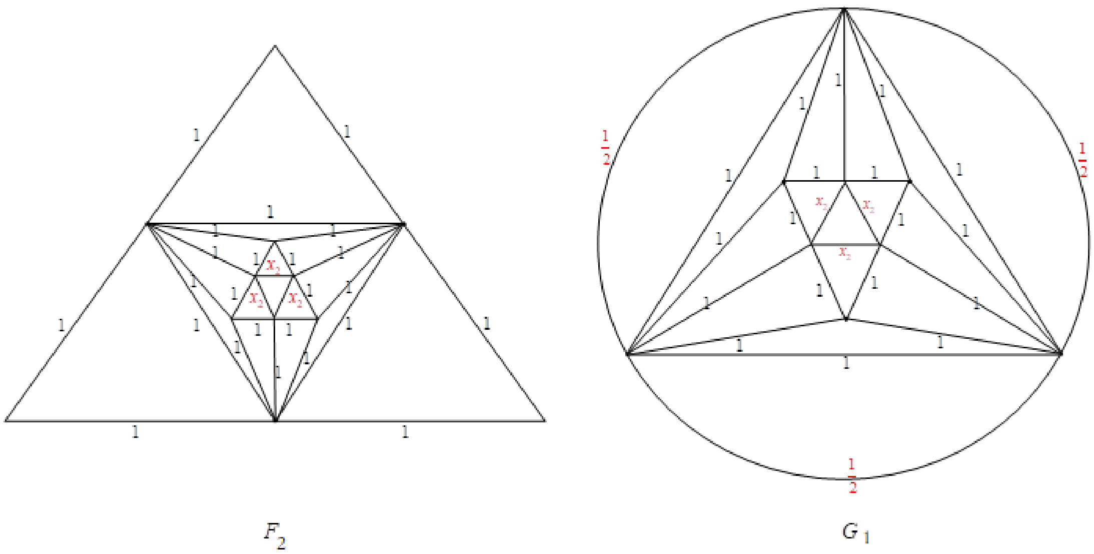

Figure 8

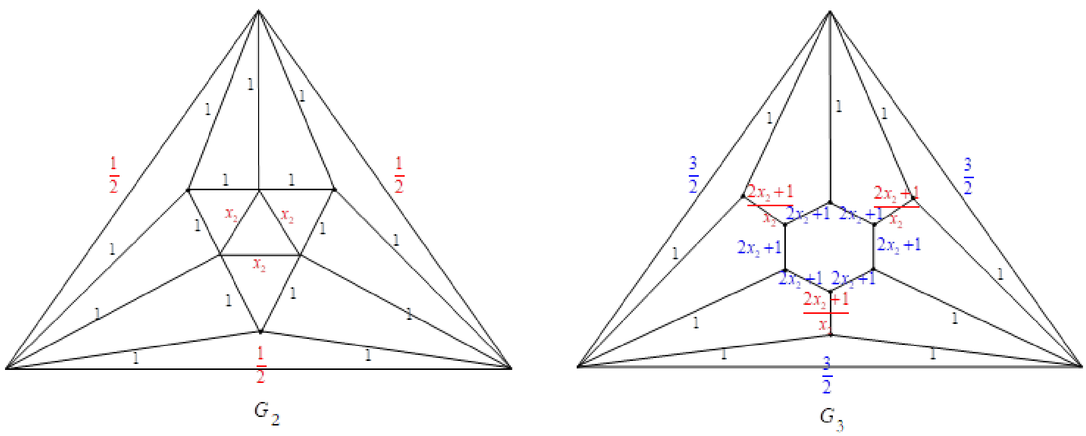

Figure 9

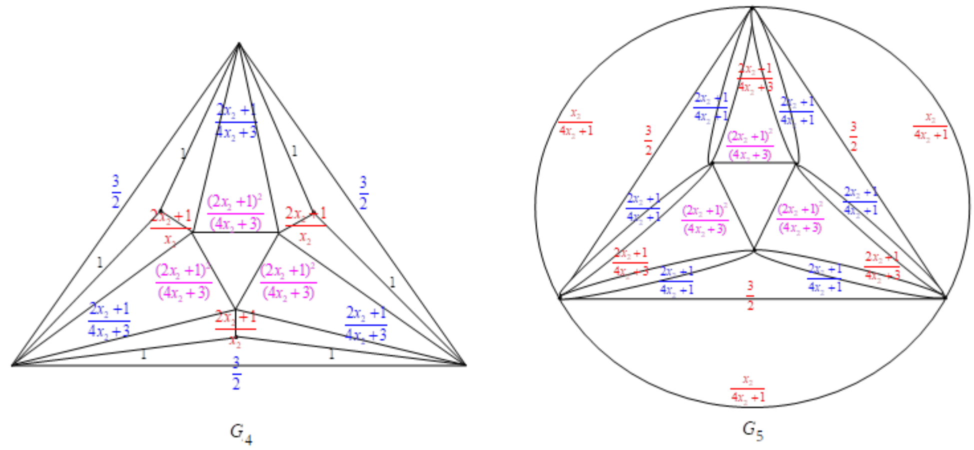

Figure 10

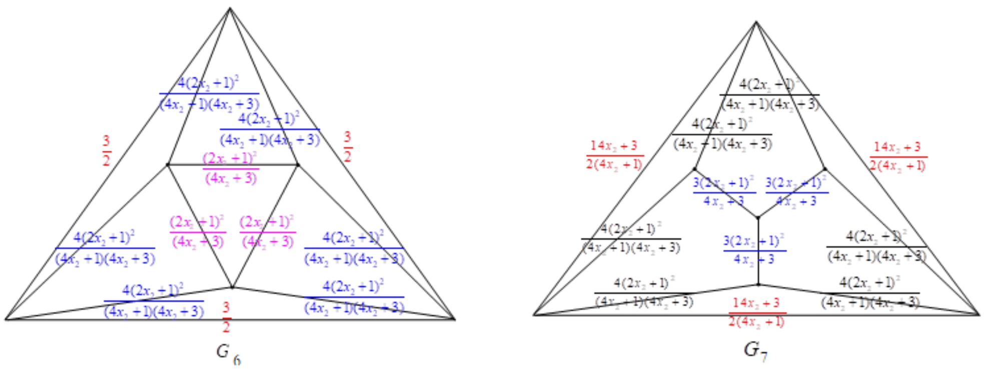

Figure 11

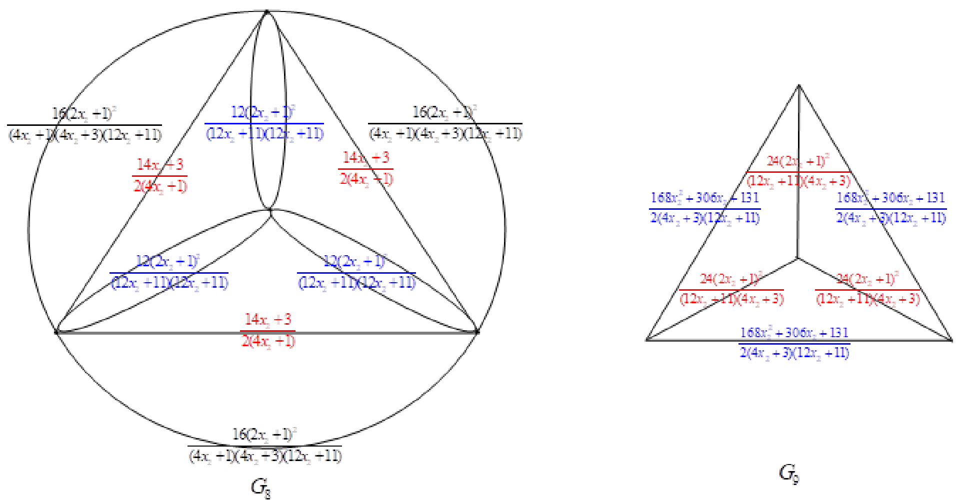

Figure 12

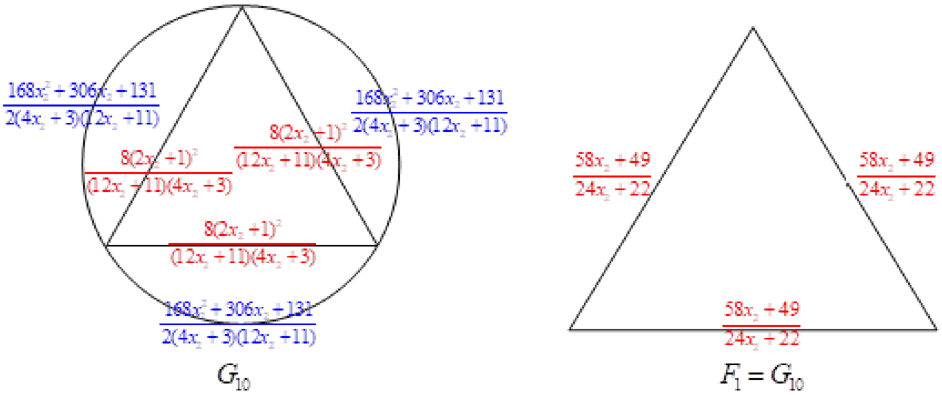

Figure 13

Using the properties given in Section 2, we have the following the transformations:

$$
\begin{aligned}
\tau(G_1) &= \left[\frac{1}{2}\right]^3 \tau(\mathrm{F}_2), \quad \tau(G_2) = \tau(G_1), \quad \tau(G_3) = \left[\frac{(2x_2+1)^2}{x_2}\right]^3 \tau(G_2), \\
\tau(G_4) &= \left[\frac{1}{4x_2+3}\right]^3 \tau(G_3), \quad \tau(G_5) = \left[\frac{x_2}{4x_2+3}\right]^3 \tau(G_4), \quad \tau(G_6) = \tau(G_5), \\
\tau(G_7) &= 9\left[\frac{(2x_2+1)^2}{4x_2+3}\right] \tau(G_6), \quad \tau(G_8) = \left[\frac{(4x_2+1)(4x_2+3)}{(2x_2+1)^2(12x_2+11)}\right]^3 \tau(G_7), \\
\tau(G_9) &= \tau(G_8), \quad \tau(G_{10}) = \frac{(12x_2+11)(4x_2+3)}{72(2x_2+1)^2}\tau(G_9) \text{ and } \tau(\mathrm{F}_1) = \tau(G_{10}).
\end{aligned}
$$

Combining these eleven transformations, we have

$$\tau(\mathrm{F}_2) = 16(24x_2+22)^2 \tau(\mathrm{F}_1). \tag{4.1}$$

Further

$$\tau(\mathrm{F}_n) = \prod_{i=2}^{n} 16(24x_2+22)^2 \tau(\mathrm{F}_1) = 3 \times (16)^{n-1} x_1^2 \left[\prod_{i=2}^{n}(24x_i+22)\right]^2, \tag{4.2}$$

where $x_{i-1} = \frac{58x_i+49}{24x_i+22}, i = 2, 3, \ldots, n$. Its characteristic equation is $24\mu^2 - 36\mu - 49 = 0$, which have two roots $\mu_1 = \frac{9-5\sqrt{15}}{12}$ and $\mu_2 = \frac{9+5\sqrt{15}}{12}$. Subtracting these two roots into both sides of $x_{i-1} = \frac{58x_i+49}{24x_i+22}$, we get

$$x_{i-1} - \frac{9-5\sqrt{15}}{12} = \frac{58x_i+49}{24x_i+22} - \frac{9-5\sqrt{15}}{12} = 10(4+\sqrt{15})\frac{\left(x_i - \frac{9-5\sqrt{15}}{12}\right)}{(24x_i+22)}, \tag{4.3}$$

$$x_{i-1} - \frac{9+5\sqrt{15}}{12} = \frac{58x_i+49}{24x_i+22} - \frac{9+5\sqrt{15}}{12} = 10(4-\sqrt{15})\frac{\left(x_i - \frac{9+5\sqrt{15}}{12}\right)}{(24x_i+22)}. \tag{4.4}$$

Let $y_i = \frac{x_i - \frac{9-5\sqrt{15}}{12}}{x_i - \frac{9+5\sqrt{15}}{12}}$. Then by Eqs.(4.3) and (4.4), we get $y_{i-1} = (31+8\sqrt{15})y_i$ and $y_i = (31+8\sqrt{15})^{n-i}y_n$. Therefore

$$x_i = \frac{(31+8\sqrt{15})^{n-i}\left(\frac{9+5\sqrt{15}}{12}\right)y_n - \frac{9-5\sqrt{15}}{12}}{(31+8\sqrt{15})^{n-i}y_n - 1}.$$

Thus

$$x_1 = \frac{(31+8\sqrt{15})^{n-1}\left(\frac{9+5\sqrt{15}}{12}\right)y_n - \frac{9-5\sqrt{15}}{12}}{(31+8\sqrt{15})^{n-1}y_n - 1}. \tag{4.5}$$

Using the expression $x_{n-1} = \frac{58x_n+49}{24x_n+22}$ and denoting the coefficients of $58x_n + 49$ and $24x_n + 22$ as σ_n and δ_n we have

$$24x_n + 22 = \sigma_0 (58x_n + 49) + \delta_0 (24x_n + 22),$$
$$24x_{n-1} + 22 = \frac{\sigma_1 (58x_n + 49) + \delta_1 (24x_n + 22)}{\sigma_0 (58x_n + 49) + \delta_0 (24x_n + 22)},$$
$$\vdots$$
$$24x_{n-i} + 22 = \frac{\sigma_i (58x_n + 49) + \delta_i (24x_n + 22)}{\sigma_{i-1} (58x_n + 49) + \delta_{i-1} (24x_n + 22)}, \tag{4.6}$$
$$24x_{n-(i+1)} + 22 = \frac{\sigma_{i+1} (58x_n + 49) + \delta_{i+1} (24x_n + 22)}{\sigma_i (58x_n + 49) + \delta_i (24x_n + 22)}, \tag{4.7}$$
$$\vdots$$
$$24x_2 + 22 = \frac{\sigma_{n-2} (58x_n + 49) + \delta_{n-2} (24x_n + 22)}{\sigma_{n-3} (58x_n + 49) + \delta_{n-3} (24x_n + 22)}$$

Substituting Eq.(4.6) into Eq.(4.2), we obtain

$$\tau(\mathrm{F}_n) = 3 \times (16)^{n-1} x_1^2 \left[\sigma_{n-2} (58x_n + 49) + \sigma_{n-2} (24x_n + 22)\right]^2, \tag{4.8}$$

where $\sigma_0 = 0, \delta_0 = 1$ and $\sigma_1 = 24, \delta_1 = 22$.

By the expression $x_{n-1} = \frac{58x_n+49}{24x_n+22}$ and Eqs.(4.6) and (4.7), we have

$$\sigma_{i+1} = 80\sigma_i - 100\sigma_{i-1}; \delta_{i+1} = 80\delta_i - 100\delta_{i-1}. \tag{4.9}$$

The characteristic equation of Eq.(4.9) is $\gamma^2 - 80\gamma + 100 = 0$ which have two roots $\gamma_1 = 10(4+\sqrt{15})$ and $\gamma_2 = 10(4-\sqrt{15})$.

The general solutions of Eq.(4.9) are $\sigma_i = a_1\gamma_1^i + a_2\gamma_2^i; \delta_i = b_1\gamma_1^i + b_2\gamma_2^i$. Using the initial conditions $\sigma_0 = 0, \delta_0 = 1$ and $\sigma_1 = 24, \delta_1 = 22$, yields

$$\sigma_i = \frac{2\sqrt{15}}{25}(10(4+\sqrt{15}))^i - \frac{2\sqrt{15}}{25}(10(4-\sqrt{15}))^i,$$
$$b_i = \left(\frac{25-3\sqrt{15}}{50}\right)(10(4+\sqrt{15}))^i + \left(\frac{25+3\sqrt{15}}{50}\right)(10(4-\sqrt{15}))^i. \tag{4.10}$$

If $x_n = 1$, it means that F_n without any electrically equivalent transformation. Plugging

Eq.(4.10) into Eq.(4.8), we have

$$\tau\left(\mathrm{F}_n\right) = 3\times(16)^{n-1}x_1^2\left[\left(\frac{115+29\sqrt{15}}{5}\right)(40+10\sqrt{15})^{n-2} + \left(\frac{115-29\sqrt{15}}{5}\right)(40-10\sqrt{15})^{n-2}\right]^2 \quad (4.11)$$

for integer $n\geq 2$. When $n=1, \tau\left(F_1\right)=3$ which satisfies Eq.(4.11). Therefore, for $, n\geq 1$, the number of spanning trees in the sequence of the graph F_n is given by

$$\tau\left(\mathrm{F}_n\right) = 3\times(16)^{n-1}x_1^2\left[\left(\frac{115+29\sqrt{15}}{5}\right)(40+10\sqrt{15})^{n-2} + \left(\frac{115-29\sqrt{15}}{5}\right)(40-10\sqrt{15})^{n-2}\right]^2, \quad (4.12)$$

where

$$x_1=\frac{(31+8\sqrt{15})^{n-1}(951+365\sqrt{15})+61(9-5\sqrt{15})}{(31+8\sqrt{15})^{n-1}(64+5\sqrt{15})+732}, n\geq 1. \quad (4.13)$$

Inserting Eq.(4.13) into Eq.(4.12), we obtain the result. □

§5. Number of Spanning Trees in the Sequences of H_n Graph

Consider the sequence of graphs $\mathrm{H}_1, \mathrm{H}_2, \cdots, \mathrm{H}_n$ constructed as shown in Figure 14. According to this construction, the number of total vertices $|V\left(\mathrm{H}_n\right)|$ and edges $|E\left(\mathrm{H}_n\right)|$ are $|\mathrm{V}\left(\mathrm{H}_n\right)|=9n-6$ and $|E\left(\mathrm{H}_n\right)|=24n-21$ for integers $n=1,2,\cdots$. The average degree of H_n is in the large n limit which is $\dfrac{16}{3}$.

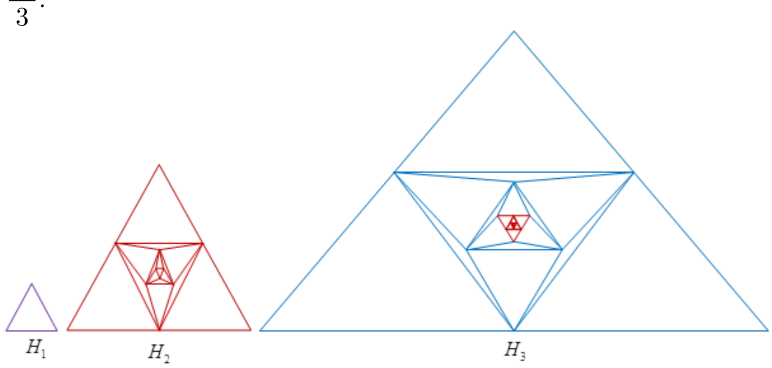

Figure 14. Some sequences of H_n

Theorem 5.1 *For $n\geq 1$, the number of spanning trees in the sequence of H_n is given by*

$$2^{n-15}(115+\sqrt{13209})^{2n} \times \left(-76(-63+\sqrt{13209})+\left(\frac{1}{8}(13217-115\sqrt{13209})\right)^{1-n}(8421+97\sqrt{13209})\right)^2 \times \frac{\left(29563-257\sqrt{13209}+\left(\frac{1}{8}(13217-115\sqrt{13209})\right)^n(29563+257\sqrt{13209})\right)^2}{\left(58159227\left(38+8^{-n}(325+\sqrt{13209})(13217+115\sqrt{13209})^{n-1}\right)^2\right)}.$$

Proof We use the electrically equivalent transformation to transform H_i to H_{i-1}. Figures

15-19 illustrate the transformation process from H_2 to H_1. Using the properties given in Section 2, we have the following the transformations:

$$\tau(G_1)=\left[\frac{1}{2}\right]^3\tau(H_2),\ \tau(G_2)=\tau(G_1),\ \tau(G_3)=9x_2\tau(G_2),\ \tau(G_4)=\left[\frac{1}{3x_2+2}\right]^3\tau(G_3),$$

$$\tau(G_5)=\tau(G_4),\ \tau(G_6)=\left(\frac{3x_2+2}{18x_2}\right)\tau(G_5),\ \tau(G_7)=\tau(G_6),\ \tau(G_8)=9\left(\frac{5x_2+3}{3x_2+2}\right)\tau(G_7),$$

$$\tau(G_9)=\left[\frac{3x_2+2}{21x_2+13}\right]^3\tau(G_8),\ \tau(G_{10})=\tau(G_9),\tau(G_{11})=\left[\frac{21x_2+13}{18(5x_2+3)}\right]\tau(G_{10}),\ \tau(H_1)=\tau(G_{11}).$$

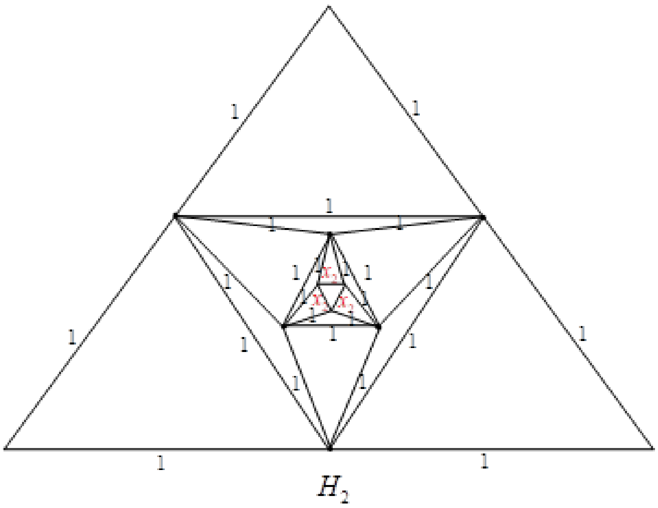

Figure 15

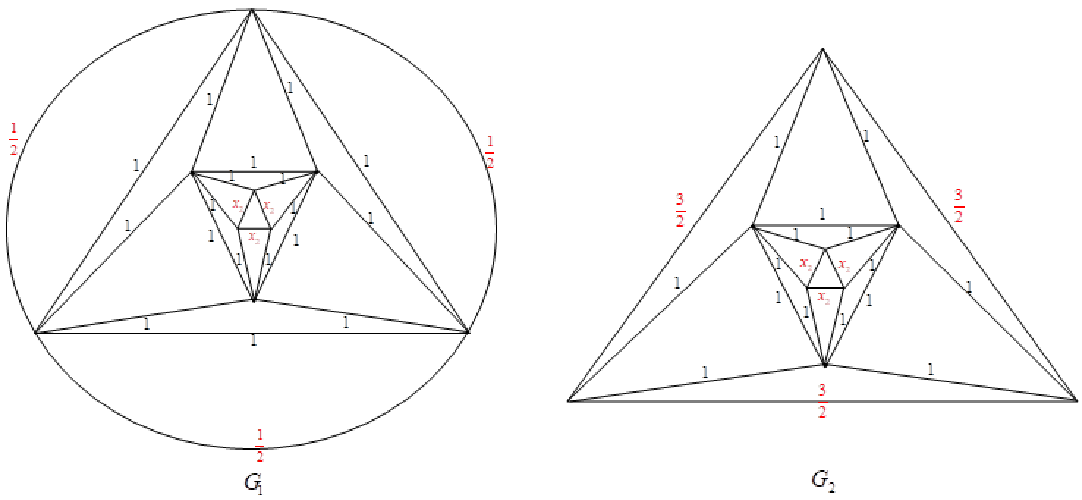

Figure 16

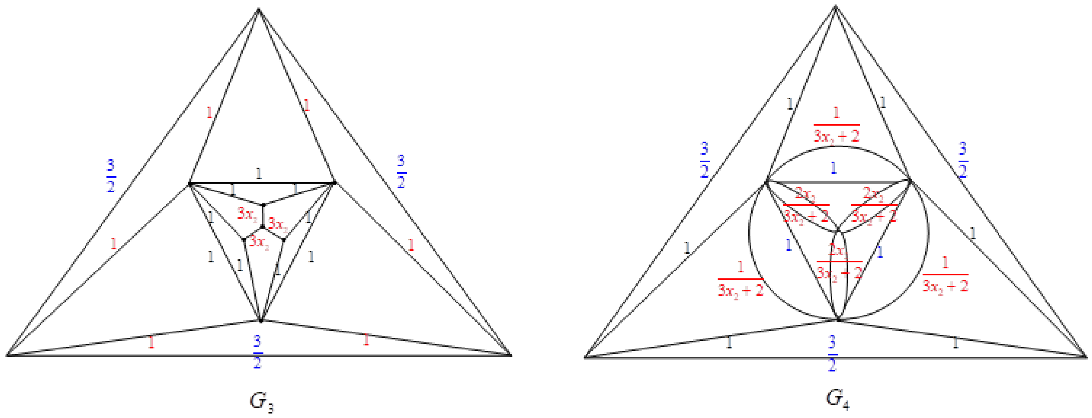

Figure 17

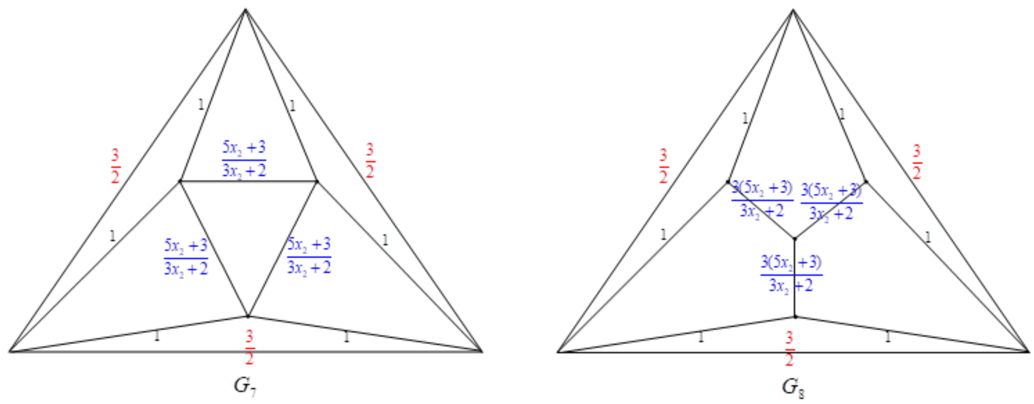

Figure 18

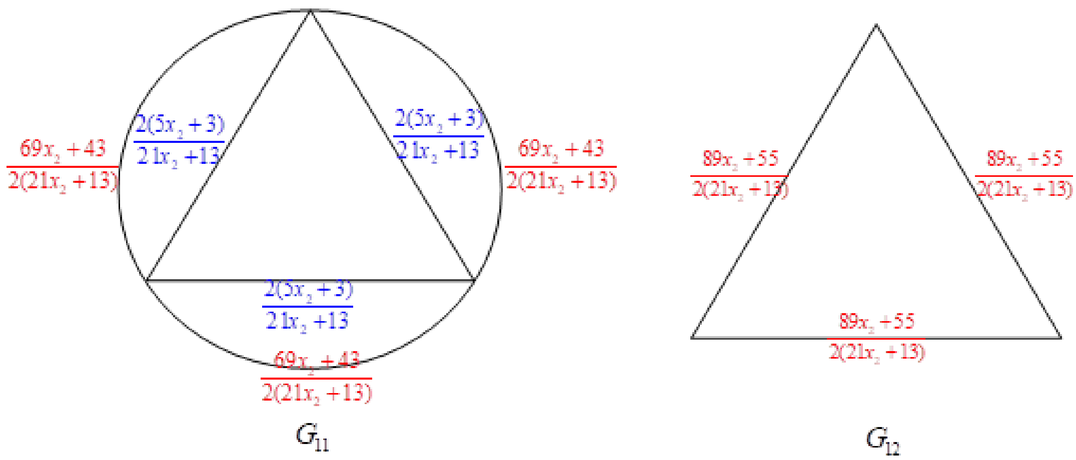

Figure 19

Combining these twelve transformations, we get

$$\tau\left(\mathrm{H}_2\right)=8\left(42x_2+26\right)^2\tau\left(\mathrm{H}_1\right). \tag{5.1}$$

Further

$$\tau\left(\mathrm{H}_n\right)=\prod_{i=2}^{n}8\left(42x_2+26\right)^2\tau\left(\mathrm{H}_1\right)=3\times(8)^{n-1}x_1^2\left[\prod_{i=2}^{n}\left(42x_i+26\right)\right]^2, \tag{5.2}$$

where $x_{i-1}=\frac{89x_i+55}{42x_i+26}, i=2,3,\ldots,n$. Its characteristic equation is $42\mu^2-63\mu-55=0$, which have two roots $\mu_1=\frac{63-\sqrt{13209}}{84}$ and $\mu_2=\frac{63+\sqrt{13209}}{84}$. Subtracting these two roots into both sides of $x_{i-1}=\frac{89x_i+55}{42x_i+26}$, we get

$$x_{i-1}-\frac{63-\sqrt{13209}}{84}=(115+\sqrt{13209})\frac{\left(x_i-\frac{68-\sqrt{13209}}{84}\right)}{2\left(42x_i+26\right)}, \tag{5.3}$$

$$x_{i-1}-\frac{63+\sqrt{13209}}{84}=(115-\sqrt{13209})\frac{\left(x_i-\frac{68+\sqrt{13209}}{84}\right)}{2\left(42x_i+26\right)}. \tag{5.4}$$

Let $y_i=\frac{x_i-\frac{63-\sqrt{18209}}{84}}{x_i-\frac{63+\sqrt{18209}}{84}}$. Then by Eqs.(5.3) and (5.4), we get $y_{i-1}=\left(\frac{13217+115\sqrt{13209}}{8}\right)y_i$ and $y_i=\left(\frac{13217+115\sqrt{13209}}{8}\right)^{n-i}y_n$. Therefore

$$x_i=\frac{\left(\frac{13217+115\sqrt{13209}}{B}\right)^{n-i}\left(\frac{63+\sqrt{13209}}{84}\right)y_n-\frac{63-\sqrt{13209}}{84}}{\left(\frac{13217+115\sqrt{13209}}{8}\right)^{n-i}y_n-1}.$$

Thus

$$x_1=\frac{\left(\frac{13217+115\sqrt{13209}}{8}\right)^{n-1}\left(\frac{63+\sqrt{13209}}{84}\right)y_n-\frac{63-\sqrt{13209}}{84}}{\left(\frac{13217+115\sqrt{13209}}{8}\right)^{n-1}y_n-1}. \tag{5.5}$$

Using the expression $x_{n-1}=\frac{89x_n+55}{42x_n+26}$ and denoting the coefficients of $89x_n+55$ and $42x_n+26$ as σ_n and δ_n we have

$$\begin{aligned}
42x_n+26&=\sigma_0\left(89x_n+55\right)+\delta_0\left(42x_n+26\right),\\
42x_{n-1}+26&=\frac{\sigma_1\left(89x_n+55\right)+\delta_1\left(42x_n+26\right)}{\sigma_0\left(89x_n+55\right)+\delta_0\left(42x_n+26\right)},\\
42x_{n-2}+26&=\frac{\sigma_2\left(89x_n+55\right)+\delta_2\left(42x_n+26\right)}{\sigma_1\left(89x_n+55\right)+\delta_1\left(24x_n+26\right)},\\
&\vdots
\end{aligned}$$

$$42x_{n-i}+26=\frac{\sigma_i\left(89x_n+55\right)+\delta_i\left(42x_n+26\right)}{\sigma_{i-1}\left(89x_n+55\right)+\delta_{i-1}\left(42x_n+26\right)}, \tag{5.6}$$

$$42x_{n-(i+1)}+26=\frac{\sigma_{i+1}\left(89x_n+55\right)+\delta_{i+1}\left(42x_n+26\right)}{\sigma_i\left(89x_n+55\right)+\delta_i\left(42x_n+26\right)}, \tag{5.7}$$

$$\vdots$$

$$42x_2+26=\frac{\sigma_{n-2}\left(89x_n+55\right)+\delta_{n-2}\left(42x_n+26\right)}{\sigma_{n-3}\left(89x_n+55\right)+\delta_{n-3}\left(42x_n+26\right)}. \tag{5.8}$$

Substituting Eq.(5.6) into Eq.(5.2), we obtain

$$\tau\left(\mathrm{H}_n\right)=3\times(8)^{n-1}x_1^2\left[\sigma_{n-2}\left(89x_n+55\right)+\sigma_{n-2}\left(42x_n+26\right)\right]^2, \tag{5.9}$$

where $\sigma_0=0, \delta_0=1$ and $\sigma_1=42, \delta_1=26$.

By the expression $x_{n-1}=\frac{89x_n+55}{42x_n+26}$ and Eqs.(5.6), (5.7), we have

$$\sigma_{i+1}=115\sigma_i-4\sigma_{i-1}; \delta_{i+1}=115\delta_i-4\delta_{i-1}$$

The characteristic equation of Eq.(5.9) is $\gamma^2-115\gamma+4=0$ which have two roots

$$\gamma_1=\left(\frac{115+\sqrt{13209}}{2}\right) \text{ and } \gamma_2=\left(\frac{115-\sqrt{13209}}{2}\right).$$

The general solutions of Eq.(5.9) are $\sigma_i=a_1\gamma_1^i+a_2\gamma_2^i; \delta_i=b_1\gamma_1^i+b_2\gamma_2^i$. Using the initial conditions $\sigma_0=0, \delta_0=1$ and $\sigma_1=42, \delta_1=26$, yields

$$\sigma_i=\frac{2\sqrt{13209}}{629}\left(\frac{115+\sqrt{13209}}{2}\right)^i-\frac{2\sqrt{13209}}{629}\left(\frac{115-\sqrt{13209}}{2}\right)^i,$$

$$\delta_i=\left(\frac{629-3\sqrt{13209}}{1258}\right)\left(\frac{115+\sqrt{13209}}{2}\right)^i+\left(\frac{629+3\sqrt{13209}}{1258}\right)\left(\frac{115-\sqrt{13209}}{2}\right)^i. \tag{5.10}$$

If $x_n=1$, it means that H_n without any electrically equivalent transformation. Plugging Eq.(5.10) into Eq.(5.8), we have

$$\begin{aligned}\tau\left(\mathrm{H}_n\right) &= 3\times(8)^{n-1}x_1^2\times\left[\left(\frac{21386+186\sqrt{13209}}{629}\right)\left(\frac{115+\sqrt{13209}}{2}\right)^{n-2}\right.\\ &\quad\left.+\left(\frac{21386-186\sqrt{13209}}{629}\right)\left(\frac{115-\sqrt{13209}}{2}\right)^{n-2}\right]^2\end{aligned} \tag{5.11}$$

for integers $n\geq 2$. When $n=1, \tau\left(H_1\right)=3$ which satisfies Eq.(5.11). Therefore, the number of spanning trees in the sequence of the graph H_n is given by

$$\begin{aligned}\tau\left(\mathrm{H}_n\right) &= 3\times(8)^{n-1}x_1^2\left[\left(\frac{21386+186\sqrt{13209}}{629}\right)\left(\frac{115+\sqrt{13209}}{2}\right)^{n-2}\right.\\ &\quad\left.+\left(\frac{21386-186\sqrt{13209}}{629}\right)\left(\frac{115-\sqrt{13209}}{2}\right)^{n-2}\right]^2\end{aligned} \tag{5.12}$$

for integers $n \geq 1$, where

$$x_1 = \frac{\left(\frac{13217+115\sqrt{13209}}{B}\right)^{n-1}(8421+97\sqrt{13209})+76(63-\sqrt{13209})}{21\left(\frac{13217+115\sqrt{13209}}{8}\right)^{n-1}(325+\sqrt{13209})+6384}, \quad n \geq 1. \tag{5.13}$$

Inserting Eq.(5.13) into Eq.(5.12) we obtain the result. □

§6. Numerical Results

An illustration on the numbers of spanning trees in graphs E_n, F_n and H_n are listed in Table 1 following.

n	$\tau(E_n)$	$\tau(F_n)$	$\tau(H_n)$
1	3	3	3
2	406272	549552	497664
3	26879275008	54966988800	52627418112
4	1761820718333952	5452053012480000	5564612377337856
5	115462949411396517888	540704118669312000000	588379800446293966848
6	7566980125843657045573632	53623893196800000000000000	62212920881826474870964224

Table 1

§7. Spanning Tree Entropy

After having explicit Formulas for the number of spanning trees of the sequence of the three families of graphs E_n, F_n and H_n, we can calculate its spanning tree entropy Z which is a finite number and a very interesting quantity characterizing the network structure, defined as in [23, 24] as

$$Z(G) = \lim_{n\to\infty} \frac{\ln \tau(G)}{|V(G)|}. \tag{7.1}$$

for a graph G. Particularly, we know that

$$Z(E_n) = \frac{16}{9}(\ln 2) = 1.232261654$$

$$Z(F_n) = \frac{1}{9}(\ln[1600] + 2\ln[4+\sqrt{15}]) = 1.278292561,$$

$$Z(H_n) = \ln[2] - \frac{2}{9}(\ln[115-\sqrt{13209}]) = 1.285411179.$$

Now we compare the value of entropy in our graphs with other graphs. The entropy of the graph H_n is larger than the entropy of the graph E_n and the graph F_n. In addition the entropy of the families E_n, F_n and H_n which have average degree 16/3 is larger than the entropy of fractal scale free lattice [25] which has the entropy1.040 and 3-prism graph of average degree 4 which has entropy1.0445 [26] and two dimensional Sierpinski gasket [27] which has the entropy

1.166 of the same average degree 4 but the entropy of the families E_n, E_n and H_n is smaller than the entropy of Apollonian graph [28] which has the entropy 1.3540 of average degree 6.

§8. Conclusions

In this work, we enumerated the number of spanning trees in the sequences of three sequences of graphs of average degree 16/3 using electrically equivalent transformations. An advantage of this method lies in the avoidance of laborious computation of Laplacian spectra that is needed for a generic method for determining spanning trees.

References

[1] W.J. Tzeng and F. Y. Wu, Spanning trees on hypercubic lattices and nonorientable surfaces, *Journal Applied Mathematics Letters*, Vol.13, Issue 7, October (2000),19-25.

[2] F.Y. Wu, Number of spanning trees on a Lattice, *J. Phys.* A 10, (1977) 113-115.

[3] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez , D.-U. Hwang: A Complex networks: Structure and dynamics, *Physics Reports*, Vol.424, Issues 4-5, February (2006) 175-308.

[4] A. Dhar and D. Roy, Heat transport in harmonic lattices, *Journal of Statistical Physics*, Vol.125, (2006), 801-820.

[5] F. T. Boesch, On unreliability polynomials and graph connectivity in reliable network synthesis, *J. Graph Theory*, 10 (1986) 339-352.

[6] Gábor J. Szabó, Mikko Alava and János Kertész, Geometry of minimum spanning trees on scalefree networks, *Physica A: Statistical Mechanics and its Applications*, Vol.330, Issues 1, December (2003), 31-36.

[7] S.N. Majumdar and Deepak Dhar, Equivalence between the Abelian sandpile model and the $q \to 0$ limit of the Potts model, *Physica A: Statistical Mechanics and its Applications*, Vol.185, Issues 1-4, 15 June (1992), 129-145.

[8] L. Grady, Random Walks for Image Segmentation, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, Vol.28, Issue 11, November (2006), 1768 - 1783.

[9] J. S. Kim, K-I Goh, B. Kahng and D. Kim, Fractality and self-similarity in scale-free networks, *New Journal of Physics*, Vol. 9(2007), 177.

[10] G. G. Kirchhoff, Über die Auflösung der Gieichungen auf welche man bei der Untersucher der linearen Verteilung galuanischer Strome gefhrt wird, *Ann. Phg. Chem.*, 72 (1847), 497-508.

[11] A. K. Kelmans andV, M. Chelnokox, A certain polynomial of a graph and graphs with an extremal number of trees, *Journal of Combinatorial Theory* B, Vol. 16, (1974), 197-214.

[12] N.L. Biggs, *Algebraic Graph Theory*(2nd Edn.), Cambridge Univ. Press, Cambridge, (1993) pp: 205.

[13] S. N. Daoud, The deletion-contraction method for counting the number of spanning trees of graphs, *European Journal of Physical Plus*, Vol.130. No. 10, Oct.(2015) 1-14.

[14] S. N. Daoud, Complexity of graphs generated by wheel graph and their asymptotic limits, *Journal of the Egyptian Math. Soc.*, Vol.25, Issue 4, October (2017), 424-433.

[15] S. N. Daoud, Generating formulas of the number of spanning trees of some special graphs, *Eur. Phys. J. Plus*, Vol.129, (2014), 1-14.

[16] S. N. Daoud, Number of spanning trees in different product of complete and complete tripartite graphs, *Ars Combinatoria*, Vol. 139 (2018), 85-103.

[17] Jia-Bao Liu and S. N. Daoud, Complexity of some of Pyramid Graphs Created from a Gear Graph, *Symmetry*, 2018, 10, 689; doi:10.3390/sym10120689.

[18] S. N. Daoud, Number of spanning trees of Cartesian and composition products of graphs and Chebyshev polynomials, *IEEE Access*, Vol.7 (2019), 71142 - 71157.

[19] Teufl E., Wagner S., Determinant identities for Laplace matrices, *Linear Algebra Appl.*, 432, (2010), 441-457.

[20] S. N. Daoud, Number of spanning trees in the sequence of some Nonahedral graphs, *Utilitas Mathematica*, Vol. 115 (2020), 1-18.

[21] S. N. Daoud and Wedad Saleha, Complexity trees of the sequence of some nonahedral graphs generated by triangle Helixon, *Heliyon Journal*, 6(9), Sep (2020).

[22] Jia-Bao Liu and S. N. Daoud, Number of spanning trees in the sequence of some graphs, *Complexity*, Volume 2019— Article ID 4271783 — https://doi.org/10.1155/2019/4271783.

[23] F.Y. Wu, Number of spanning trees on a lattice, *J. Phys. A: Math. Gen.*, 10 (1977), 113-115.

[24] R. Lyons, Asymptotic enumeration of spanning trees, *Combin. Probab. Comput.*, 14 (2005), 491-522.

[25] Z. Zhang, H. Liu., Wu B., Zou T., Spanning trees in a fractal scale -free lattice, *Phys. Rev. E*, 83, 016116 (2011).

[26] W. Sun, S. Wang, and J. Zhang, Counting spanning trees in prism and anti-prism graphs, *Journal of Applied Analysis and Computation*, Vol.6(2016), 65-75.

[27] S. Chang, L. Chen, W. Yang, Spanning trees on the Sierpinski gasket, *J. Stat. Phys.*, 126 (2007), 649-667.

[28] J. Zhang, W. Sun and G. Xu, Enumeration of spanning trees on Apollonian networks, *Journal of Statistical Mechanics: Theory and Experiment*, Vol. 2013, No. 9, Article IDP09015, 2013.

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A Study on Constant and Regular Hesitancy Fuzzy Soft Graphs

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Abstract: This article discusses about the degree of vertex, edge in hesitancy fuzzy soft graphs (HFSG). A new kind of graph called constant HFSG and totally constant HFSG are established. The order and size of such graphs are also dealt. The regular and totally regular concepts are introduced over the HFSG, and its properties discussed.

Key Words: Constant HFSG, totally constant HFSG, regular HFSG, size and order, totally regular HFSG, neutrosophic fuzzy graph.

AMS(2010): 05C12, 03E72, 05C72.

§1. Introduction

Molodtstov dealt with uncertainity and unclear objects using notions of soft set theory [2]. A new set combining the soft sets and fuzzy sets, was then developed and proved to be more effective by Maji.et al [2]. Akram and Nawaz dealed with properties of fuzzy soft graphs [3]. Torra.V in [4] defined the hesitancy fuzzy sets. The order and size in fuzzy graphs was found by Nagoor Gani and Basheer [11]. Gani and Radha [10] worked on regular fuzzy graphs. The concept of constant intuitionistic Fuzzy graph dealt by Karunambigai et.al [5]. Santhi Maheswari and Sekar worked on regular FG in [15], [16]. [9] introduced constant hesitancy fuzzy graph and established some concepts. Pathinathan et.al developed Hesitancy fuzzy graphs [7], and also defined regular hesitancy fuzzy graph [8]. Hesitancy fuzzy soft graphs notions were given by Rajeswari [6].

This article deals with degree of vertex and edge in HFSG. The concept of regular and constant are observed over these HFSG and its characteristics are dealt with.

§2. Preliminaries

Definition 2.1 *A fuzzy graph G, contains a nonempty set V with functions $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$: $\forall\ u, v \in V$, $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$, where σ and μ are fuzzy vertex set and*

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edge set respectively in G.

Generally, let $N_G = (A, B)$ be neutrosophic fuzzy graph, i.e., let $A = (T_A, I_A, F_A)$ be a neutrosophic fuzzy relation on $B = (T_B, I_B, F_B)$, which is a neutrosophic fuzzy set on set V. If $F_A = 0$ and $F_B = 0$, then such a neutrosophic fuzzy graph is nothing else but a fuzzy graph.

Definition 2.2 *The pair (F, A) is soft set over the universal set, where $A \subseteq E$ and $F : a \to \mathscr{P}(U)$. That is a soft set over U is parametered collection of subsets of U.*

Definition 2.3 *An FSG $\widetilde{G}$ is a 4-tuple, such that*

(1) $\mathscr{G}^*$ *is crisp graph;*
(2) $\mathscr{A}$ *is the parameter set;*
(3) $(\widetilde{\mathscr{F}}, \mathscr{A})$ *is fuzzy soft set over vertex set V;*
(4) $(\widetilde{\mathscr{K}}, \mathscr{A})$ *is fuzzy soft set over edge set E.*

Then, $\left(\widetilde{\mathscr{F}}(a), \widetilde{\mathscr{K}}(a)\right)$ *is fuzzy (sub)graph of* $\mathscr{G}^*, \forall\ a \in \mathscr{A}$ *and can be denoted as* $\widetilde{\mathscr{H}}(a)$.

The membership value of the edge in an FSG is given as

$$\widetilde{K}(a)(xy) \leq \min\left\{\widetilde{F}(a)(x), \widetilde{F}(a)(y)\right\}.$$

Definition 2.4 *If $\widetilde{G}$ is an FSG, then the vetex degree is*

$$d_{\widetilde{G}}(u) = \sum_{a_i \in A} (\sum_{u \neq v} \widetilde{\mathscr{K}}(a_i)(uv)).$$

Definition 2.5 *If $\widetilde{G}$ is an FSG, then edge degree of uv is given as*

$$d_{\widetilde{G}}(uv) = d_{\widetilde{G}}(u) + d_{\widetilde{G}}(v) - 2(\sum_{a_i \in A} \widetilde{\mathscr{K}}(a_i)(uv)).$$

Definition 2.6 *Let U be the universal set and E be set of parameters, then HF(U) is set of all hesitant fuzzy sets over U. Then, the pair (F, E) is hesitant fuzzy soft set if $F(e) \in HF(U)$, for every $e \in E$.*

Definition 2.7 *A hesitancy fuzzy graph $\tilde{G} = (\tilde{V}, E)$ such that $\mu_1 : \tilde{V} \to [0, 1]$ (membership), $\gamma_1 : \tilde{V} \to [0, 1]$ (non membership), $\beta_1 : \tilde{V} \to [0, 1]$ (hesitancy membership), also $\mu_1 + \gamma_1 + \beta_1 = 1$ for all vertices.*

Also $E \subseteq \tilde{V} \times \tilde{V}$, where $\mu_1 : \tilde{V} \times \tilde{V} \to [0, 1]$, $\gamma_1 : \tilde{V} \times \tilde{V} \to [0, 1]$, $\beta_1 : \tilde{V} \times \tilde{V} \to [0, 1]$ such that

$$\mu_2(uv) \leq \wedge[\mu_1(u), \mu_1(v)],$$

$$\gamma_2(uv) \leq \vee[\gamma_1(u), \gamma_1(v)],$$

$$\beta_2(uv) \leq \wedge[\beta_1(u), \beta_1(v)]$$

and $0 \leq \mu_2(uv) + \gamma_2(uv) + \beta_2(uv) \leq 1$ for all edges.

Definition 2.8 *For a HFSG, its order is*

$$o(\tilde{G}) = \left(\sum_{a_i \in A, v_i \in V} \mu_1(v_i), \sum_{a_i \in A, v_i \in V} \gamma_1(v_i), \sum_{a_i \in A, v_i \in V} \beta_1(v_i) \right).$$

Definition 2.9 *The size of a HFSG is*

$$s(\tilde{G}) = \left(\sum_{a_i \in A, v_i v_j \in E} \mu_2(v_i v_j), \sum_{a_i \in A, v_i v_j \in E} \gamma_2(v_i v_j), \sum_{a_i \in A, v_i v_j \in E} \beta_2(v_i v_j) \right).$$

§3. Degree in HFSG

Definition 3.1 *Let $\tilde{G}$ be a hesitancy fuzzy soft graph (HFSG). Then,*

$$\begin{aligned} d_\mu(u) &= \sum_{a_i \in A} (\sum_{u \neq v} \tilde{K}_{(a_i)} \mu_2(uv)), \\ d_\gamma(u) &= \sum_{a_i \in A} (\sum_{u \neq v} \tilde{K}_{(a_i)} \gamma_2(uv)), \\ d_\beta(u) &= \sum_{a_i \in A} (\sum_{u \neq v} \tilde{K}_{(a_i)} \beta_2(uv)). \end{aligned}$$

Therefore, $$d_{\tilde{G}}(u) = (d_\mu(u), d_\gamma(u), d_\beta(u)).$$

Definition 3.2 *Let $\tilde{G}$ be a HFSG, then total degree of the vertex $v \in V$ is given as*

$$td_{\tilde{G}}(v) = \left(d_\mu(v) + \sum_{a_i \in A} (\mu_1(v)), d_\gamma(v) + \sum_{a_i \in A} (\gamma_1(v)), d_\beta(v) + \sum_{a_i \in A} (\beta_1(v)) \right).$$

Example 3.3 Consider the following HFSG, we demonstrate the above definition.

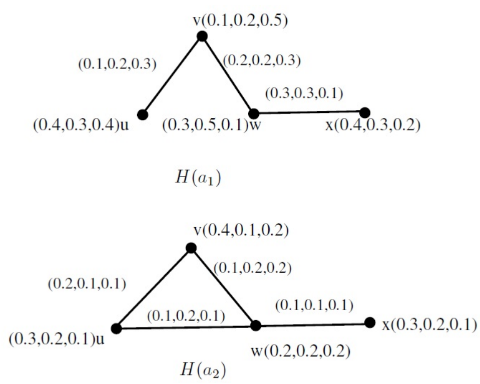

Figure 3.1

The degree of the vertices are found as $d_{\tilde{G}}(u) = (0.4, 0.5, 0.5)$, $d_{\tilde{G}}(v) = (0.6, 0.7, 0.9)$, $d_{\tilde{G}}(w) = (0.7, 0.8, 0.7)$, $d_{\tilde{G}}(x) = (0.4, 0.4, 0.2)$.

The total degree is found as $td_{\tilde{G}}(u) = (1.1, 1.0, 1.0)$, $td_{\tilde{G}}(v) = (1.1, 1.0, 1.6)$, $td_{\tilde{G}}(w) = (1.2, 1.5, 1.0)$, $td_{\tilde{G}}(x) = (1.1, 0.9, 0.5)$.

§4. Constant HFSG

Definition 4.1 *If degree of all the vertices are same, then the HFSG is called constant HFSG (c-HFSG). That is, if, $\tilde{G}$ is a HFSG and if $d_\mu(x_i) = k_1$, $d_\gamma(x_i) = k_2$ and $d_\beta(x_i) = k_3$, $\forall x_i \in V$. Then $\tilde{G}$ is said to be (k_1, k_2, k_3)- HFSG or c-HFSG of degree (k_1, k_2, k_3).*

Example 4.2 The following is a constant-HFSG.

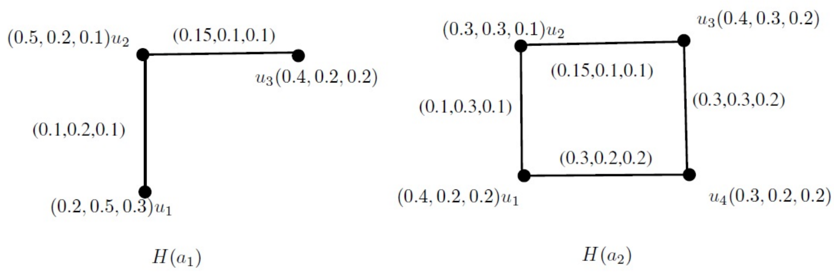

Figure 4.1

The degree of the vetices are $d_{\tilde{G}}(u_1) = (0.6, 0.5, 0.4)$, $d_{\tilde{G}}(u_2) = (0.6, 0.5, 0.4)$, $d_{\tilde{G}}(u_3) = (0.6, 0.5, 0.4)$, $d_{\tilde{G}}(u_4) = (0.6, 0.5, 0.4)$. Here $d_\mu(u_i) = 0.6$, $d_\gamma(u_i) = 0.5$, $d_\beta(u_i) = 0.4$, for all $u_i \in V$. Therefore, it is c-HFSG.

Definition 4.3 *Let $\tilde{G}$ be a HFSG, it is said to be totally constant HFSG (tc-HFSG), if the total degree of all the vertices are same. That is, if a HFSG, having total degree of all its vertices as (l_1, l_2, l_3), then it is (l_1, l_2, l_3)- totally constant HFSG.*

Example 4.4 The following graph illustrates a totally constant HFSG.

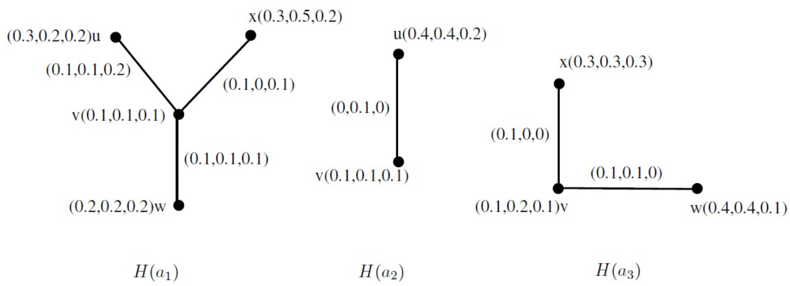

Figure 4.2

The total degree of all the vertices are found to be $(0.8, 0.8, 0.6)$. That is $td_\mu(v_i) = 0.8$, $td_\gamma(v_i = 0.8)$, $td_\beta(v_i) = 0.6$, for all $v_i \in V$. Therefore it is tc-HFSG.

Remark 4.5 A c-HFSG need not be tc-HFSG and vice versa.

Example 4.6 Consider the example 4.2, which is (0.6,0.5,0.4)-constant HFSG. But while finding the total degree of all the vertices, we have $td_{\tilde{G}}(u_1) = (1.2, 1.2, 0.9)$, $td_{\tilde{G}}(u_2) = (1.4, 1.0, 0.6)$, $td_{\tilde{G}}(u_3) = (1.4, 1.0, 0.6)$, $td_{\tilde{G}}(u_4) = (0.9, 0.7, 0.6)$. It is not same, hence it is not totally constant-HFSG.

While taking the example 4.4, which is totally constant-HFSG. But the degree of its vertices are $d_{\tilde{G}}(u) = (0.5, 0.4, 0.3)$, $d_{\tilde{G}}(v) = (0.1, 0.2, 0.2)$, $d_{\tilde{G}}(w) = (0.2, 0.2, 0.1)$, $d_{\tilde{G}}(x) = (0.2, 0, 0.1)$, which are not same, thus it is not constant-HFSG.

Theorem 4.7 *Let $\tilde{G}$ be a c-HFSG. And if $\sum\limits_{a_i \in A, v_i \in V} \tilde{F}(a_i)(v_i)$ is a constant function for all vertices, then $\tilde{G}$ is totally constant-HFSG.*

Proof Suppose $\tilde{G}$ is constant-HFSG, also given that $\sum\limits_{a_i \in A, v_i \in V} \tilde{F}(a_i)(v_i)$ is a constant function. Then

$$\sum_{a_i \in A, u_i \in V} \tilde{F}(a_i)(\mu_1(u_i)) = m_1, \sum_{a_i \in A} \tilde{F}(a_i)(\gamma_1(u_i)) = m_2, \sum_{a_i \in A} \tilde{F}(a_i)(\beta_1(u_i)) = m_3,$$

for $\forall u_i \in V$.

Since $\tilde{G}$ is c-HFSG, let it be $(t_1, t_2, t_3)-$ constant HFSG. This implies that $d_{\tilde{G}}(\mu)(u_i) = t_1$, $d_{\tilde{G}}(\gamma)(u_i) = t_2$, $d_{\tilde{G}}(\beta)(u_i) = t_3$, $\forall u_i \in V$.

Then, the total degree of the vertices are

$$\begin{aligned} td_{\tilde{G}}(u_i) &= d_{\tilde{G}}(\mu)(u_i) + \sum_{a_i \in A} \tilde{F}(a_i)(\mu_1(u_i)), d_{\tilde{G}}(\gamma)(u_i) \\ &\quad + \sum_{a_i \in A} \tilde{F}(a_i)(\gamma_1(u_i)), d_{\tilde{G}}(\beta)(u_i) + \sum_{a_i \in A} \tilde{F}(a_i)(\beta_1(u_i)) \\ &\Rightarrow td_{\tilde{G}}(u_i) = (t_1 + m_1, t_2 + m_2, t_3 + m_3), \forall u_i \in V. \end{aligned}$$

Therefore, it is totally constant HFSG. □

Note 4.8 For a HFSG $\tilde{G}$, its order is given by

$$o(\tilde{G}) = \sum_{a_i \in A} o(H(a_i)).$$

Note 4.9 For a HFSG $\tilde{G}$, its size is

$$s(\tilde{G}) = \sum_{a_i \in A} \sum_{uv \in E} (\mu_2, \gamma_2, \beta_2)(uv).$$

Result 4.10 *The size of a c-HFSG or a (k_1, k_2, k_3) c-HFSG is given by*

$$\left[\frac{hk_1}{2}, \frac{hk_2}{2}, \frac{hk_3}{2}\right],$$

where $h = \left|\tilde{G}\right|$.

Observation 4.11 The following are observed using the above defined graphs.

(1) Let $\tilde{G}$ be a (l_1, l_2, l_3) totally constant-HFSG, then

$$2s(\tilde{G}) + o(\tilde{G}) = (hl_1, hl_2, hl_3),$$

where $h = |V|$.

(2) For $\tilde{G}$ be a (t_1, t_2, t_3) c-HFSG and (l_1, l_2, l_3) tc-HFSG, then the order is given as

$$o(\tilde{G}) = \left(h(l_1 - t_1), h(l_2 - t_2), h(l_3 - t_3)\right),$$

where $h = |V|$.

§5. Regular HFSG

Definition 5.1 *A HFSG $\tilde{G}$ is regular, when $(d_\mu, d_\gamma, d_\beta)$ (degree) of all the vertices are the same constant. That is, if $\tilde{G}$ is a $((\mu_{1i}, \gamma_{1i}, \beta_{1i}), (\mu_{2i}, \gamma_{2i}, \beta_{2i}))$ HFSG and if $d_\mu(v_i) = d_\beta(v_i) = d_\beta(v_i) = m$, $\forall v \in V$ and $ainA$, then $\tilde{G}$ is m-regular HFSG.*

Example 5.2 Examine the following example.

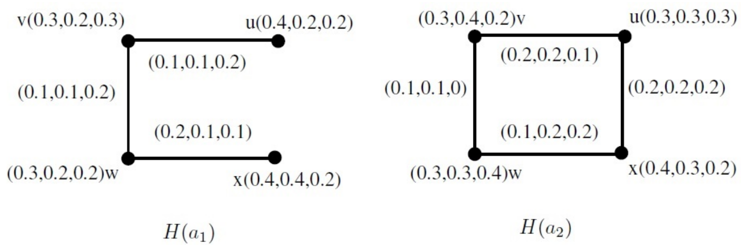

Figure 5.1

In this the degree of all the vertices are found to be $d_{\tilde{G}}(u) = (0.5, 0.5, 0.5)$, $d_{\tilde{G}}(v) = (0.5, 0.5, 0.5)$, $d_{\tilde{G}}(w) = (0.5, 0.5, 0.5)$, $d_{\tilde{G}}(x) = (0.5, 0.5, 0.5)$. Here $d_\mu(v_i) = 0.5$, $d_\gamma(v_i) = 0.5$, $d_\beta(v_i) = 0.5$, for all $v_i \in V$. Therefore it is regular HFSG or 0.5-regular HFSG.

Definition 5.3 *A HFSG $\tilde{G}$ is totally regular, when total degree of all vertices are the alike. That is if $td_\mu(v_i) = td_\beta(v_i) = td_\beta(v_i) = k$, $\forall v \in V$ and $a \in A$, $\Rightarrow \tilde{G}$ is k-totally regular HFSG.*

Example 5.4 Consider the graph in Figure 5.2 following.

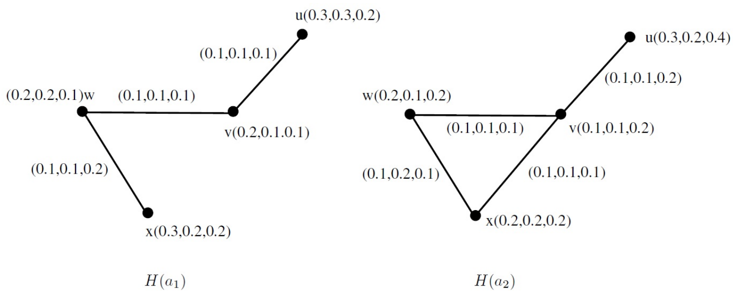

Figure 5.2

In this the total degree of all the vertices are found as $td_{\tilde{G}}(u) = (0.8, 0.8, 0.8)$, $td_{\tilde{G}}(v) = (0.8, 0.8, 0.8)$, $td_{\tilde{G}}(w) = (0.8, 0.8, 0.8)$, $td_{\tilde{G}}(x) = (0.8, 0.8, 0.8)$. Here $td_{\mu}(v_i) = 0.8$, $td_{\gamma}(v_i) = 0.8$, $td_{\beta}(v_i) = 0.8$, for all $v_i \in V$. Therefore it is totally regular HFSG or 0.8-totally regular HFSG.

Definition 5.5 *Let $\tilde{G}$ be a hesitancy fuzzy soft graph. The degree of the edge uv in E is defined as*

$$deg_{\tilde{G}}(uv) = d_{\tilde{G}}(u) + d_{\tilde{G}}(v) - 2((\mu_2, \gamma_2, \beta_2)(uv)).$$

Definition 5.6 *Let $\tilde{G}$ be a HFSG. The total degree of the edge uv in E is defined as*

$$tdeg_{\tilde{G}}(uv) = d_{\tilde{G}}(u) + d_{\tilde{G}}(v) - ((\mu_2, \gamma_2, \beta_2)(uv)).$$

Example 5.7 We consider the below hesitancy fuzzy soft graph.

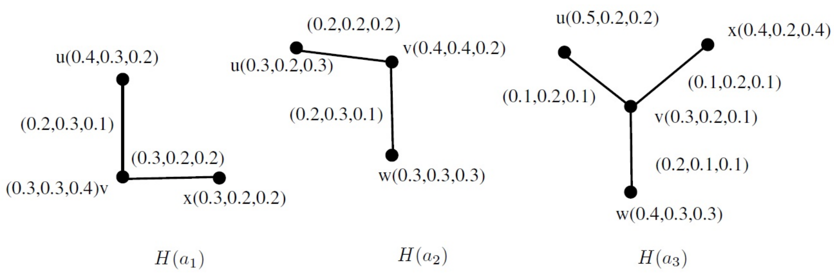

Figure 5.3

The degree of the edges are $deg(uv) = (0.8, 0.8, 0.5)$, $deg(vw) = (0.9, 1.1, 0.7)$, $deg(vx) = (0.9, 1.1, 0.6)$. The total degree of the edges are $tdeg(uv) = (1.3, 1.5, 0.9)$, $tdeg(vw) = (1.3, 1.5, 0.9)$, $tdeg(vx) = (1.3, 1.5, 0.9)$.

Definition 5.8 *A HFSG $\tilde{G}$ is edge regular, if the edge degree of all the edges are alike. That is*

$$deg_{\tilde{G}}\mu_2(v_iv_j) = deg_{\tilde{G}}\gamma_2(v_iv_j) = deg\tilde{G}_{\beta_2}(v_iv_j) = p.$$

Then, $\tilde{G}$ is called p-edge regular HFSG.

Definition 5.9 *A HFSG $\tilde{G}$ is edge totally regular, if the total edge degree of all the edges are alike. That is,*

$$tdeg_{\tilde{G}}\mu_2(v_iv_j) = tdeg\tilde{G}_{\gamma_2}(v_iv_j) = tdeg\tilde{G}_{\beta_2}(v_iv_j) = r.$$

Then, $\tilde{G}$ is called r- totally edge regular hesitancy fuzzy soft graph.

Example 5.10 We use below graph to explain the definition.

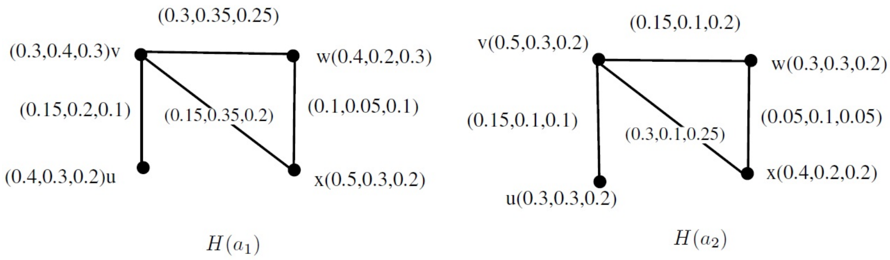

Figure 5.4

The edge degree are given as $deg_{\tilde{G}}(uv) = (0.9, 0.9, 0.9)$, $deg_{\tilde{G}}(vw) = (0.9, 0.9, 0.9)$, $deg_{\tilde{G}}(wx) = (0.9, 0.9, 0.9)$, $deg_{\tilde{G}}(vx) = (0.9, 0.9, 0.9)$. Therefore the graph is 0.9-edge regular HFSG.

Example 5.11 The following graph demonstrates the above definition.

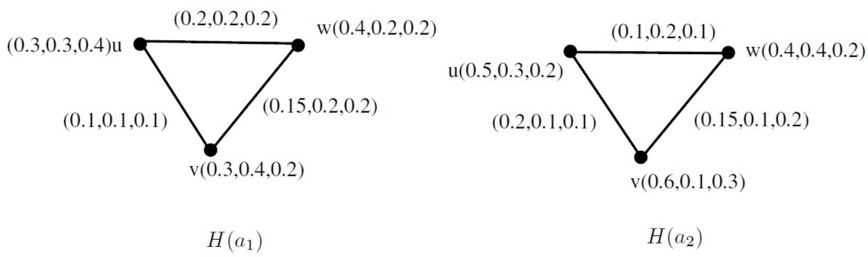

Figure 5.5

The total edge degree are found as $tdeg_{\tilde{G}}(uv) = (0.9, 0.9, 0.9)$, $tdeg_{\tilde{G}}(vw) = (0.9, 0.9, 0.9)$, $tdeg_{\tilde{G}}(wu) = (0.9, 0.9, 0.9)$. Thus the graph is 0.9-totally edge regular HFSG.

Remark 5.12 A HFSG which is edge regular, not necessarily be total edge regular and vice versa.

Remark 5.13 A regular HFSG can be constant HFSG, but the converse not necessarily true.

Remark 5.114 A totally regular HFSG can be totally constant HFSG, but converse may not be true.

Theorem 5.15 *Suppose $\tilde{G}$ is a HFSG and if its subgraphs $H(a_i), a_i \in A$ are fuzzy cycles of even length, with membership values of alternate edges alike, then $\tilde{G}$ is constant HFSG.*

Proof Consider the subgraphs of $\tilde{G}$, $H(a_i), a_i \in A$. Let us take only two parameters a_1 and a_2, such that the membership value of edges in $H(a_1)$ and $H(a_2)$ are alternatively same.

Let the membership value of the edges, e_i in $H(a_1)$ is (l_1, m_1, n_1) and (l_2, m_2, n_2), when i is odd and even respectively. And for edges e_j in $H(a_2)$, the membership value is (p_1, q_1, r_1) and (p_2, q_2, r_2), when j is odd and even respectively. Then we have, the degree of vertices as

$$d_{\tilde{G}}(v_i) = (l_1 + p_1 + l_2 + p_2, m_1 + q_1 + m_2 + q_2, n_1 + r_1 + n_2 + r_2)$$

for all $v_i \in V$, which

$$\Rightarrow (d_\mu(\tilde{G}), d_\gamma(\tilde{G}), d_\beta(\tilde{G}))(v_i) = \text{constant}$$

for all $v_i \in V$. Thus, it is c-HFSG. □

Theorem 5.16 *If $\tilde{G}$, a c-HFSG satisfying the conditions of above theorem, then it is totally constant HFSG, when $(\mu_1, \gamma_1, \beta_1)$ is constant for all the vertices.*

Proof Suppose $\tilde{G}$ is c-HFSG, then we have $(d_\mu(\tilde{G}), d_\gamma(\tilde{G}), d_\beta(\tilde{G}))(v_i) = constant$ for all $v_i \in V$. Also given that $(\mu_1, \gamma_1, \beta_1)$ is constant for all vertices, then the total degree of all the vertices is also constant, since

$$td_{\tilde{G}}(v) = \left(d_\mu(v) + \sum_{a_i \in A} (\mu_1(v)), d_\gamma(v) + \sum_{a_i \in A} (\gamma_1(v)), d_\beta(v) + \sum_{a_i \in A} (\beta_1(v)) \right),$$

which $\Rightarrow \tilde{G}$ is totally constant HFSG. □

Theorem 5.17 *Suppose $\tilde{G}$ is a HFSG and if its subgraphs $H(a_i), a_i \in A$ are fuzzy cycles of any length and if $\sum_{a_i \in A, e_i \in E} K(a_i)(e_i)$, are alike and same constant for all the edges, then $\tilde{G}$ is regular HFSG.*

Proof Given $\tilde{G}$ is a HFSG and also $\sum_{a_i \in A, e_i \in E} K(a_i)(e_i)$ are alike and same constant for all edges. Let us consider any two subgraphs of $\tilde{G}$ with parameters set a_1 and a_2, then we have

$$\sum_{a_i \in A, e_i \in E} K(a_i)(e_i) = (m, m, m).$$

Then, the degree of the vertices are $d_{\tilde{G}}(v_i) = (2m, 2m, 2m)$ for all $v_i \in V$. This implies that $\tilde{G}$ is regular-HFSG. □

Theorem 5.18 *Suppose $\tilde{G}$ is a HFSG and its subgraphs are fuzzy cylcles of any length and if*

$$\sum_{a_i \in A, e_i \in E} K(a_i)(e_i), \quad \sum_{a_i \in A, v_i \in V} F(a_i)(v_i)$$

are alike and same constant for all edges and vertices respectively, then $\tilde{G}$ is both regular and totally regular hesitancy fuzzy soft graph.

Proof Let us consider $\tilde{G}$ such that

$$\sum_{a_i \in A, e_i \in E} K(a_i)(e_i)$$

are alike and same constant for all vertices, then by above theorem, $\tilde{G}$ is regular HFSG. Let it be (m, m, m) regular HFSG.

Let

$$\sum_{a_i \in A, v_i \in V} F(a_i)(v_i) = (k, k, k)$$

for all vertices. Considering the total degree of all the vertices, it is found that

$$td_{\tilde{G}}(v_i) = (m + k, m + k, m + k) \Rightarrow \tilde{G}$$

is totally regular HFSG. □

References

[1] P. K. Maji, A. R. Roy, R. Biswas, An application of soft sets in a decision making problem, *Computers and Mathematics with Applications*, 2002, 44(89), 1077-1083.

[2] D. Molodtsov, Soft set theory-First results, *Computers and Mathematics with Applications*, 1999, 34(4-5), 19-31.

[3] Muhammad Akram and Saira Nawaz, On fuzzy soft graphs, *Italian Journal of Pure and Applied Mathematics*, 2015, 34, 497-514.

[4] Torra.V, Hesitant fuzzy sets, *International Journal of Intelligent System*, 2010, 25(6), 529-539.

[5] Karunambigai, Parvathi, Bhuvaneswari, Constant intuitionistic fuzzy graphs, *NIFS*, 2011, 17(1), 37-47.

[6] R.Rajeswari, Hesitant fuzzy soft Graph, *International Journal of Current Research and Modern Education*, Special Issue, 2017, July,104-107.

[7] Pathinathan, Jon Arockiaraj and Jesintha Roselin, Hesitancy fuzzy graphs, *Indian Journal of Science and Technology*, 2015, 8(35), 1-5.

[8] Pathinathan, Jon Arockiaraj, On regular hesitancy fuzzy graph, *Global Journal of Pure and Applied Mathematics*, 2016, 12(3), 512-518.

[10] Pathinathan, Jon Arockiaraj and Mike Dison, Constant hesitancy fuzzy graphs, *Global Journal of Pure and Applied Mathematics*, 2016, 12(2), 287-292.

[11] Nagoor Gani and Radha, On regular fuzzy graphs, *Journal of Physical Sciences*, 2008, 12, 33-40.

[13] Nagoor Gani and Basheer Ahmed, Order and size in fuzzy graphs, *Bulletin of Pure and Applied Sciences*, 2003, 22(1), 145-148.

[14] N.R. Santhi Maheswari and C. Sekar, $(r, 2, r(r-1))$-Regular graphs, *International Journal of Mathematics and Soft Computing*, 2012, Vol. 2, No. 2, 25-33.

[15] N.R. Santhi Maheswari and C. Sekar, $(r, 2, (r-1)(r-1))$-Regular graphs, *International Journal of Mathematical Combinatorics*, Vol. 4, 40-51.

[16] N.R. Santhi Maheswari and C. Sekar 2012, Some $(r, 2, k)$-regular graphs containing a given graph, *International Journal of Engineering Research and Technology*, Vol. 1, No. 10, 1-9.

[17] N.R. Santhi Maheswari and C. Sekar, On $(2, k)$-regular fuzzy graphs and totally $(2, k)$-regular fuzzy graphs, *International Journal of Mathematics and Soft Computing*, 2014, Vol. 4, No. 2, 59-69.

[18] N.R. Santhi Maheswari and C. Sekar, On $(r, 2, k)$-regular fuzzy graphs, *JCMCC*, 97(2016), 11-21 (April).

[19] L.Subha Lakshmi and N.R.Santhi Maheswari, On pseudo highly and pseudo strongly edge irregular graphs, *Proceedings of International Conference ICRRMMCSA22* (2022), ISBN: 978-81-9078-935-6.

[20] L.Subha Lakshmi and N.R.Santhi Maheswari, Support highly edge irregular FSGs, *Neuroquantology*, November 2022,20(17), 897-900.

[21] L.Subha Lakshmi and N.R.Santhi Maheswari, A novel approach to solving fuzzy assignment problem, *International Journal of Mathematical Archive*, 3(10), October 2022, 16-20.

[22] N.R.Santhi Maheswari and L.Subha Lakshmi, Support strongly edge irregular FSGs, *International Journal of Mathematical Combinatorics*, December 2022, Vol.4, 41-49.

International J.Math. Combin. Vol.4(2023), 44-53

On Infinitesimal Transformation in a Finsler Space

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Abstract: In the present communication studies have been carried out with special reference to infinitesimal projective and special projective transformations in a Finsler space and accordingly results have been derived in the form of theorems in a projective symmetric and non-symmetric Finsler space.

Key Words: Finsler spaces, Projective transformation, affine and non-affine infinitesimal projective transformation, Lie-derivative.

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§1. Introduction

Berwald introduced a connection coefficient $C^i_{jk}(x,\dot{x})$ defined by

$$C^i_{jk}(x,\dot{x}) \stackrel{\text{def}}{=} \frac{\partial^2 G^i}{\partial x^j \partial x^k} \tag{1.1}$$

and accordingly the covariant derivative of an arbitrary covariant vector i X in the sense of Berwald is given by Rund [4]

$$X^i_{(j)} = \frac{\partial X^i}{\partial x^j} - \frac{\partial X^i}{\partial \dot{x}^h}\frac{\partial G^h}{\partial \dot{x}^j} + G^i_{jh}X^h. \tag{1.2}$$

The functions G^i appearing in (1.2) are positively homogeneous of degree two in its directional arguments $\dot{x}^j$ and satisfies the following identities

$$G^i_{hkr}\dot{x}^r = G^i_{hkr}\dot{x}^k = G^i_{kkr}\dot{x}^h, \;\; G^i_{hk}\dot{x}^h = 0 \;\text{ and }\; G^i_k\dot{x}^k = 2G^i. \tag{1.3}$$

The geodesic deviation has been defined in the following form

$$\frac{\partial^2 Z^j}{\partial u^2} + H^j_k(x,\dot{x})x^k = 0, \tag{1.4}$$

where the vector Z^i is called the variation vector and the tensor $H^i_k(x,\dot{x})$ is being defined by

$$H^j i_k = 2\partial_k G^i - \partial_h \dot{\partial}_k G^i \dot{x}^h + 2G^i_{kl}G^l - \dot{\partial}_l G^i \dot{\partial}_k G^l. \tag{1.5}$$

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The tensors defined by

$$H^i_{jk}(x,\dot{x}) = \frac{1}{3}\left(\frac{\partial H^i_k}{\partial \dot{x}^j} - \frac{\partial H^i_j}{\partial \dot{x}^k}\right) \quad \text{and} \quad H^i_{jkl} = \frac{\partial H^i_{jl}}{\partial \dot{x}^k} \tag{1.6}$$

are respectively termed as Berwalds deviation tensor and Berwalds curvature tensor and they satisfy the following

$$H^k_{khj} = H_{jh} - H_{hj}, \quad H_i \dot{x}^i = (n-1)H, \quad H^j_{ki}\dot{x}^k = H^j_i = H^j_{ik}\dot{x}^k. \tag{1.7}$$

The projective covariant derivative of an arbitrary tensor $T^i_j(x,\dot{x})$ is given by Misra [2] as

$$T^i_{j((k))} = \partial_k T^i_j - \dot{\partial}_s T^s_j \Pi^i_{rk}\dot{x}^r + T^h_j \Pi^i_{hk} - T^i_h \Pi^h_{jk}, \tag{1.8}$$

where

$$\Pi^i_{jk}(x,\dot{x}) \stackrel{\text{def}}{=} G^i_{jk} - \frac{1}{n+1}\left(2\delta^i_{(j} G^r_{<r>k)} + \dot{x}^i G^i_{rkg}\right) \tag{1.9}$$

are called projective connection coefficient and these coefficients are symmetric in its lower indices. Involving the projective covariant derivative ,we have the following commutation formulae

$$\begin{aligned} &\partial_h \left(T^i_{j((k))}\right) - \left(\partial_h T^i_j\right)_{((k))} = T^s_j \Pi^i_{shk} - T^i_s \Pi^s_{jhk}, \\ &2T^i_{[((h))((k))]} = -\dot{\partial}_r T^i_j Q^r_{shk}\dot{x}^s + T^s_j Q^i_{shk} - T^i_s Q^s_{jhk}. \end{aligned} \tag{1.10}$$

where,

$$Q^i_{hjk} \stackrel{\text{def}}{=} 2\left\{\partial_{[k}\Pi^i_{j]h} - \Pi^i_{rh[j}\Pi^r_{k]} + \Pi^r_{h[j}\Pi^i_{k]r}\right\}. \tag{1.11}$$

is called the projective entity and satisfies the following relations

$$\begin{aligned} &Q^i_{hjk} + Q^i_{jkh} + Q^i_{khj} = 0, \\ &Q^i_{hjk((s))} + Q^i_{hks((j))} + Q^i_{hsj((k))} = 0, \\ &Q^i_{ijk} = Q_{jk}, \qquad Q^i_{jk} = \frac{2}{3}\dot{\partial}_{[j} Q_{k]^i}, \\ &Q^i_{hjk} = \dot{\partial}_h Q^i_{jk}, \quad Q^i_{ijk} = Q^i_{jk}, \quad Q^i_k \dot{x}_k = 0, \\ &Q^i_{jk} = -Q^i_{kj} \ \text{ and } \ Q^i_{hk}\dot{x}^h = Q^i_k. \end{aligned} \tag{1.12}$$

The projective connection coefficient $\Pi^i_{jk}(x,\dot{x})$ satisfies the following relations

$$\begin{aligned} &\Pi^i_{hkr} - \dot{\partial}_h \Pi^i_{kr}, \quad \Pi^i_{hk} = \dot{\partial}_h \Pi^i_k, \\ &\Pi^i_{hkr}\dot{x}^h = 0 \ \text{ and } \ \Pi^i_{hk}\dot{x}^h = \Pi^i_k. \end{aligned} \tag{1.13}$$

§2. NonCAffine Infinitesimal Projective Transformation

In view of the Berwalds covariant derivative [4], the Lie-derivative of a tensor field $T^i_j(x,\dot{x})$ and the connection parameter $G^i_{jk}(x,\dot{x})$ are given as under [7] following

$$\mathcal{L}_\nu T^i_j(x,\dot{x}) = T^i_{j(h)}\nu^h + \left(\dot{\partial}_h T^i_j\right)\nu^h_{(s)}\dot{x}^s + T^i_h \nu^h_{(j)}, \tag{2.1}$$

$$\mathcal{L}_\nu G^i_{jk}(x,\dot{x}) = \nu^i_{(j)(k)} H^i_{jkh}\nu^h + G^i_{sjk} v^s_{(r)}\dot{x}^r. \tag{2.2}$$

where $H^i_{jkh}(x,\dot{x})$ has been defined by (1.6).

We also have the following communication formula from [7]

$$\dot{\partial}_l\left(\mathcal{L}_\nu T^i_j\right) - \mathcal{L}_\nu\left(\dot{\partial}_l T^i_j\right) = 0, \tag{2.3}$$

$$\mathcal{L}_\nu T^i_{j(k)} - \left(\mathcal{L}_\nu T^i_j\right)_{(k)} = T^i_j \mathcal{L}_\nu G^i_{kh} - \left(\dot{\partial}_h T^i_j\right)\mathcal{L}_\nu G^h_{ks}\dot{x}^s, \tag{2.4}$$

$$\left(\mathcal{L}_\nu G^i_{jh}\right)_{(k)} - \left(\mathcal{L}_\nu G^i_{kj}\right)_{(j)} = \mathcal{L}_\nu H^i_{hjk} + \left(\mathcal{L}_\nu G^r_{kl}\right) G^i_{rjh}\dot{x}^l - \left(\mathcal{L}_\nu G^r_{jl}\right) G^i_{rkh}\dot{x}^l. \tag{2.5}$$

Now, we give the following definitions which will be used in the later discussions.

Definition 2.1 *A Finsler space F_n is said to admit an affine motion [3] provided there exists a vector $v^i(x)$ such that*

$$(L)_\nu G^i_{jk}(x,\dot{x}) \;=\; 0. \tag{2.6}$$

Definition 2.2 *A Finsler space is said to be symmetric [1] if the Berwalds curvature tensor field $H^i_{hjk}(x,\dot{x})$ satisfies the relation*

$$H^i_{hjk(m)} \;=\; 0 \tag{2.7}$$

The following relations also hold good in such a symmetric Finsler space

$$H^i_{jk(m)} = 0, \quad H^i_{j(m)} = 0 \;\text{ and }\; H_{(m)} = 0. \tag{2.8}$$

We now consider an infinitesimal point transformation

$$\overline{x}^i = x^i + v^i(x)dt \tag{2.9}$$

where, $v^i(x)$ stands for a non-zero contravariant vector field defined over the domain of the space and dt is an infinitesimal point constant. If such a transformation transforms the system of geodesics into the same system then such a transformation in F_n is termed as infinitesimal projective transformation. It has been mentioned in [3] that the necessary and sufficient condition in order that the infinitesimal point transformation given by (2.9) be an infinitesimal projective transformation is given by the following equation

$$\mathcal{L}_\rho G^i_{jk} = \overline{G}^i_{jk} - G^i_{jk} = \delta^i_k p_k + \delta^i_k p_j - g_{jk}g^{il}d_l, \tag{2.10}$$

where, $p_k(x,\dot{x})$ and $d_l(x,\dot{x})$ are covariant vectors and satisfy the following identities

$$\dot{\partial}_j p = p_j, \quad p_{hk} = \dot{\partial}_h \dot{\partial}_k p, \quad p_{hk}\dot{x}^h = p_k,$$
$$p_{hk}\dot{x}^h\dot{x}^k = p, \quad \dot{\partial}_j d = d_j, \quad d_{hk} = \dot{\partial}_h \dot{\partial}_k d,$$
$$d_{hk}\dot{x}^h = d_k \quad \text{and} \quad d_{hk}\dot{x}^h\dot{x}^k = d, \tag{2.11}$$

Keeping in mind the formula (2.5), the Lie-derivative of H^i_{hjk} can be expressed in the following form

$$\mathcal{L}_\rho H^i_{hjk} = \left(\mathcal{L}_\rho G^i_{jh}\right)_{(k)} - \left(\mathcal{L}_\rho G^i_{kh}\right)_{(j)} + \left(\mathcal{L}_\rho G^r_{il}\right)\dot{x}^l G^i_{rhk}. \tag{2.12}$$

Using (2.10) and (1.3) in (2.12), we get

$$\begin{aligned}\mathcal{L}_\rho H^i_{hjk} \;&=\; \delta^i_j p_{h(k)} - \delta^i_k p_{h(j)} + \delta^i_h p_{j(k)} - \delta^i_h p_{k(j)} - g_{jh} g^{il} d_{l(k)} + g_{kh} g^{il} d_{l(j)} \\ &\quad + g_{kl} g^{rm} G^i_{rjh} d_m \dot{x}^l - g_{ij} g^{rm} G^i_{rkh} d_m \dot{x}^l.\end{aligned} \tag{2.13}$$

We multiply (2.13) by $\dot{x}^h\dot{x}^j$ and thereafter note (2.11) and the homogeneity property of $H^i_{hjk}(x,\dot{x})$ and get

$$\mathcal{L}_\rho H^i_k = 2\dot{x}^i p_{(k)} - \delta^i_k p_{(j)}\dot{x}^j - \dot{x}^i p_{k(j)}\dot{x}^j - g_{jh} g^{il} d_{l(k)}\dot{x}^h\dot{x}^j + g_{kh} g^{il} d_{l(j)}\dot{x}^h\dot{x}^j. \tag{2.14}$$

Now, allow a contraction in (2.14) with respect to the indices i, k and thereafter use equations (1.7), (2.11) and get

$$\mathcal{L}_\rho H = -p_{(j)}\dot{x}^j + \frac{1}{n-1}\left(d_{(j)}\dot{x}^j - g_{jh} g^{il} d_{l(i)}\dot{x}^h\dot{x}^j\right). \tag{2.15}$$

With the help of (2.15) and (2.14), we get

$$\begin{aligned}\left(\mathcal{L}_\rho H^i_k - \mathcal{L}_\rho H \delta^i_k\right) \;&=\; 3\dot{x}^i p_{(k)} - \delta^i_k p_{k(j)}\dot{x}^j + g_{kh} g^{il} d_{l(j)}\dot{x}^h\dot{x}^j \\ &\quad - \frac{1}{n-1}\left\{d_k\dot{x}^i + (2-n) g_{jh} g^{il} d_{l(k)}\dot{x}h\dot{x}^j\right\}.\end{aligned} \tag{2.16}$$

Differentiate (2.16) partially with respect to $\dot{x}^r$ and thereafter allow a contraction in the resulting equation with respect to the indices i and r, we get the following

$$\begin{aligned}\mathcal{L}_\rho \dot{\partial}_r H^r_k - \mathcal{L}_\rho \dot{\partial}_k H \;&=\; (3n+2)p_k - (n+3)p_{k(j)} + d_{k(j)}\dot{x}^j + g_{kh} g^{rl}\dot{x}^h\{d_{rl(j)} + d_{lr}\} \\ &\quad + \frac{5-n}{n-1} \times d_k + 2\dot{x}^h\dot{x}^j \times \frac{C^l_{sl}}{g_{rs}} \times \left\{\frac{2-n}{n-1} \times g_{rh} d_{l(k)} - g_{kh} d_{l(j)}\right\}\end{aligned} \tag{2.17}$$

after making use of (1.7) and (2.11).

The underlined equation

$$\overline{G}^i(x,\dot{x}) = G^i(a,\dot{x}) - P(s,\dot{x})\dot{x}^i \tag{2.18}$$

represents the most general modification of the function i G which will leave (2.18) unchanged. Thus, we say that the equation (2.18) defines the projective change [4] of the function $G^i(x,\dot{x})$. The tensor defined by

$$W^j_k(x,\dot{x}) = H^j_k - H\delta^j_k - \frac{1}{n+1}\left(\dot{\partial}_l H^j_k - \delta^j_k H\right)\dot{x}^l \tag{2.19}$$

is invariant under the projective change (2.18) and therefore it is regarded as projective deviation tensor. This deviation tensor also satisfies the following identities

$$W^j_j = 0, \quad \dot{\partial}_k W^j_h \dot{x}^h = -W^j_k \quad \text{and} \quad \dot{\partial}_i W^i_k = 0. \tag{2.20}$$

The Lie-derivative of the projective deviation tensor $W^i_j(x,\dot{x})$ in view of (2.16) and (2.17) can be written in the following form

$$\begin{aligned} \mathcal{L}_\rho W^i_k &= \frac{1}{n+1}\Bigg\{p_{(k)}\dot{x}^i + 2p_{k(j)}\dot{x}^i\dot{x}^j + \frac{4-n}{n-1}d_{(k)}\dot{x}^i \\ &\quad -\dot{x}^i\left[d_{k(j)}\dot{x}^j + g_{kh}g^{rl}(d_{rl(j)} + d_{l(r)}) + 2\dot{x}^h\dot{x}^j\frac{C^l_{sr}}{g_{rs}}\left(\frac{2-n}{n-1}g_{rh}d_{l(k)} - g_{kh}d_{l(j)}\right)\right]\Bigg\} \\ &\quad -\delta^i_k p_{(j)}\dot{x}^j + g_{kh}g^{il}d_{l(j)}\dot{x}^h\dot{x}^j + \frac{2-n}{n-1}g_{jh}g^{il}d_{l(k)}\dot{x}^h\dot{x}^j. \end{aligned} \tag{2.21}$$

We now apply the commutation formula given by (2.4) to the projective deviation tensor $W^i_j(x,\dot{x})$ and get

$$\mathcal{L}_\rho W^i_{j(r)} - \left(\mathcal{L}_\rho W^i_j\right)_{(r)} = W^h_j \mathcal{L}_\rho G^i_{rh} - W^i_h \mathcal{L}_\rho G^h_{jr} - \left(\dot{\partial}_h W^i_j\right)\left(\mathcal{L}_\rho G^h_{rs}\right)\dot{x}^s. \tag{2.22}$$

Using (2.2) and (2.3) in (2.22), we get

$$\begin{aligned} \mathcal{L}_\rho W^i_{j(r)} - \left(\mathcal{L}_\rho W^i_j\right)_{(r)} &= W^i_j\left(\delta^i_r p_r - g_{rh}g^{il}d_l\right) - W^i_r p_j - 2W^i_j p_r \\ &\quad + g^{hl}d_l\left[W^i_h g_{jr} + \left(\dot{\partial}_h W^i_j\right)g_{rs}\dot{x}^s - \left(\dot{\partial}_r W^i_j\right)p\right]. \end{aligned} \tag{2.23}$$

We now allow a contraction in (2.23) with respect to the indices i and r and thereafter use (2.20) and get

$$\mathcal{L}_\rho W^i_{j(i)} - \left(\mathcal{L}_\rho W^i_j\right)_{(i)} = (n-2)W^h_j p_h - W^h_j d_h + g^{hl}d_l\left\{W^i_h g_{ji} + \left(\dot{\partial}_h W^i_j\right)g_{is}\dot{x}^s\right\}. \tag{2.24}$$

Now, transvect $\dot{x}^r$ in (2.23) and thereafter use (2.3) and (2.20), we get

$$\begin{aligned} \left[\mathcal{L}_\rho W^i_{j(r)} - \left(\mathcal{L}_\rho W^i_j\right)_{(r)}\right]\dot{x}^r &= W^h_j \dot{x}^i p_h - 4W^i_j p - W^h_j g_{rh}g^{il}d_l\dot{x}^r + g^{hl}d_l\dot{x}^r \\ &\quad + g^{hl}d_l\dot{x}^r\left[W^i_h g_{jr} + (\dot{\partial}_h W^i_j)g_{rs}\dot{x}^s\right]). \end{aligned} \tag{2.25}$$

We now make an assumption that the space under consideration is symmetric one, i.e., $W^i_{j(r)} = 0$ and as such under this assumption the equations (2.24) and (2.25) can alternatively be written in the following forms

$$\left(\mathcal{L}_\rho W^i_j\right)_{(r)} = (2-n)W^i_j p_r + W^i_j d_r - g^{hl} d_l \left[W^i_h d_{ri} + \left(\dot{\partial}_h W^i_j\right) g_{rs}\dot{x}^s\right] \tag{2.26}$$

and

$$\left(\mathcal{L}_\rho W^i_j\right)_{(r)} \dot{x}^r = W^i_j p - W^h_j \dot{x}^i p_h + W^h_j g_{rh} g^{il} d_l \dot{x}^r - g_{hl} d^l \dot{x}^r \left[W^i_h g_{jr} + \left(\dot{\partial}_h W^i_j\right) g_{rs}\dot{x}^s\right]. \tag{2.27}$$

We propose to eliminate the term $W^h_j p_h$ with the help of (2.26) and (2.27) and the result of elimination will give the following

$$M^i_j = \left\{W^h_j d_h - g^{hl} d_l \left[W^r_h g_{jr} + \left(\dot{\partial} W^i_j\right) g_{rs}\dot{x}^s\right]\right\} \dot{x}^i, \tag{2.28}$$

where,

$$M^i_j = \left(\mathcal{L}_\rho W^i_j\right)_{(r)} \dot{x}^r + (2-n)\left(\mathcal{L}_\rho W^i_j\right)_{(r)} \dot{x}^r. \tag{2.29}$$

At this stage, if we assume that the Finsler space F_n admits a projective motion which will be characterized by

$$\mathcal{L}_\rho G^i_{jk} \ = \ 0. \tag{2.30}$$

Therefore, in such a case, with the help of (2.10) and (2.30) we shall easily arrive at the conclusion that the vectors $p(x,\dot{x})$ and $d(x,\dot{x})$ should separately vanish.

With the help of all these observations, we can therefore state the following conclusions.

Theorem 2.1 *In a Finsler space F_n, the equation (2.28) always holds provided the space under consideration admits a nonCaffine infinitesimal transformation such that the Berwalds covariant derivative of W^i_j remains an invariant.*

Theorem 2.2 *In a Finsler space F_n, $M^i_j = 0$ (where M^i_j has been given by (2.29)) provided the space under consideration admits an affine infinitesimal transformation such that the Berwalds covariant derivative of W^i_j remains an invariant.*

Theorem 2.3 *In a Finsler space F_n, the equation (2.28) necessarily holds provided the space under consideration is symmetric one and it admits a non-affine infinitesimal transformation.*

Theorem 2.4 *In a Finsler space F_n, the equation (2.26) necessarily holds provided the space under consideration is symmetric.*

§3. Infinitesimal Special Projective Transformation

In view of the projective covariant derivative as has been given by (1.8) and the projective connection coefficient $\Pi^i_{jk}(x,\dot{x})$ as has been given by (1.9), the Lie-derivatives of an arbitrary tensor $T^i_j(x,\dot{x})$ and the projective connection coefficient are respectively given by

$$\mathcal{L}_\rho T^i_j(x,\dot{x}) = T^i_{j((r))} v^r + \left(\dot{\partial}_s T^i_j\right) v^s_{((r))}\dot{x}^r - T^r_j v^i_{((r))} + T^i_r v^r_{((j))} \tag{3.1}$$

and

$$\mathcal{L}_v \Pi^i_{mk}(x,\dot{x}) = v^i_{((m))((k))} + Q^i_{mkr} v^r + \left(\dot{\partial}_r \Pi^i_{mk}\right) v^r_{((s))} \dot{x}^s. \tag{3.2}$$

In the operators $\mathcal{L}_v$, $\dot{\partial}$ and (( )), we have the following commutation formulae

$$\dot{\partial}_\rho \left(\mathcal{L}_\nu T^i_j\right) - \mathcal{L}_\nu \left(\dot{\partial}_\rho T^i_j\right) = 0,$$

$$\left(\mathcal{L}_\nu T^i_j\right)_{((r))} - \mathcal{L}_\nu T^i_{j((r))} = T^i_j \mathcal{L}_\nu \Pi^l_{lr} - T^i_l \mathcal{L}_\nu \Pi^l_{rj} - \left(\dot{\partial}_l T^i_j\right) \mathcal{L}_\nu \Pi^l_{rm} \dot{x}^m \quad \text{and}$$

$$\left(\mathcal{L}_\nu \Pi^i_{hj}\right)_{((k))} - \left(\mathcal{L}_\nu \Pi^i_{hk}\right)_{((j))} = \mathcal{L}_\nu Q^i_{hkj} + \left(\mathcal{L}_\nu \Pi^l_{jb}\right) \Pi^i_{hkl} \dot{x}^b + \left(\mathcal{L}_\nu \Pi^l_{kb}\right) \Pi^i_{jhl} \dot{x}^b. \tag{3.3}$$

In order that the infinitesimal point transformation given by (2.9) may define an infinitesimal special projective transformation, it is necessary and sufficient that [3]

$$\mathcal{L}_\nu \Pi^i_{jk} = \overline{\Pi}^i_{jk} - \Pi^i_{jk} = \delta^i_j b_k + \delta^i_k b_j - g_{jk} g^{il} c_l, \tag{3.4}$$

where, $b_k(x,\dot{x})$ and $c_l(x,\dot{x})$ are covariant vectors and they satisfy the following relations

$$\begin{aligned} &\dot{\partial}_j b = b_j, \quad b_{hk} = \dot{\partial}_h \dot{\partial}_k b, \quad b_{hk} \dot{x}^h = b_k, \\ &b_{hk} \dot{x}^h \dot{x}^k = b, \quad \dot{\partial}_j = c_j, \quad c_{jk} = \dot{\partial}_j \dot{\partial}_k c, \\ &c_{hk} \dot{x}^h = c_k, \quad \text{and} \quad c_{hk} \dot{x}^h \dot{x}^k = c. \end{aligned} \tag{3.5}$$

Using (3.4), (3.5) and the commutation formula given by (3.3), the Lie-derivative of the projective entity $Q^i_{hjk}(x, \dot{x}0$ can be written in the following form

$$\begin{aligned} \mathcal{L}_\nu Q^i_{hjk} = \; & \delta^i_j b_{h((k))} + \delta^i_h b_{j((k))} - g_{jh} g^{il} c_{l((k))} - g_{jh((k))} g^{il} c_l \\ & - g_{jh} g^{il}_{((k))} c_l - \delta^i_k b_{h((j))} - \delta^i_h b_{k((j))} + g_{kh} g^{il} c_{l((j))} \\ & + g_{kh((j))} g^{il} c_l + g_{kh} g^{il}_{((j))} c_l - \delta^r_k b \Pi^i_{rjh} + g_{kl} g^{rm} c_m \Pi^i_{rjh} \dot{x}^l \\ & + b \delta^r_j \Pi^i_{rkh} - g_{jl} g^{rm} c_m \dot{x}^i \Pi^i_{rhk}. \end{aligned} \tag{3.6}$$

Now, transvect $\dot{x}^h \dot{x}^j$ in (3.6) and therefore use (1.12) and (1.13) together, we get

$$\begin{aligned} \mathcal{L}_\nu Q^i_k = \; & 2\dot{x}^i b_{((k))} - \delta^i_k b_{((j))} \dot{x}^j + \dot{x}^h \dot{x}^j \Big[g_{kh} \left(g^{il}_{((j))} c_l + g^{il} c_{l((j))} \right) \\ & - g_{jh} \left(g^{il}_{((k))} c_l + g^{il} c_{l((k))} \right) - g_{jh((k))} g^{il} c_l \Big]. \end{aligned} \tag{3.7}$$

We allow a contraction in (3.6) with respect to the indices i and k and thereafter transvecting the equation thus obtained by $\dot{x}^h \dot{x}^j$, we get

$$\begin{aligned} \mathcal{L}_\nu Q_{hj} \dot{x}^h \dot{x}^j = \; & (1-n) b_{((j))} \dot{x}^j + c_{((j))} \dot{x}^j + g^{il} c_l \dot{x}^h \dot{x}^j \left(g_{ih((j))} - g_{jh((i))} \right) \\ & - g_{jh} \dot{x}^h \dot{x}^j \left(g^{il} c_{l((i))} + g^{il}_{((i))} c_l \right) + g_{ih} g^{il}_{((j))} c_l \dot{x}^h \dot{x}^j. \end{aligned} \tag{3.8}$$

where we have taken into account (1.12).

We now eliminate $b_{((j))}\dot{x}^j$ using (3.7), (3.8) and get

$$\begin{aligned} L^i_k(x,\dot{x}) &= 2(1-n)b_{((k))}\dot{x}^i - b_{k((j))}\dot{x}^i\dot{x}^j \\ &+g_{kh}\dot{x}^h\dot{x}^j\left(g^{il}_{((j))}c_l + g^{il}c_{l((j))}\right) \\ &-g^{il}_{((k))}c_l - g^{il}c_{l((k))} + c_{((j))}\dot{x}^j\delta^i_k + g^{il}c_l\dot{x}^h\dot{x}^j\delta^i_k\left(g_{ih((j))} - g_{jh((i))}\right) \\ &-g_{jh}\dot{x}^h\dot{x}^j\delta^i_k\left(g^{il}c_{l((i))} - g^{il}_{((i))}c_l\right) + g_{ih}g^{il}_{((j))}c_l\dot{x}^h\dot{x}^j, \end{aligned} \tag{3.9}$$

where,

$$L^i_k \stackrel{\text{def}}{=} \mathcal{L}_\nu Q^i_k + \delta^i_k\mathcal{L}_\nu Q_{hj}\dot{x}^h\dot{x}^j. \tag{3.10}$$

We apply the commutation formula (3.36) to the projective deviation tensor $W^i_j(x,\dot{x})$ and thereafter use (3.4) and (3.5) to get

$$\begin{aligned} \left(\mathcal{L}_\nu W^i_j\right)_{((r))} - \mathcal{L}_\nu W^i_{j((r))} &= W^l_j\delta^i_r b_l - W^l_j g_{rl}g^{ip}c_p - W^i_r b_j + W^i_l g_{rj}g^{lp}c_p \\ &- \left(\dot{\partial}_r W^i_j\right)b - 2W^i_j b_r - \left(\dot{\partial}_r W^i_j\right)g_{lm}g^{lp}c_p\dot{x}^m. \end{aligned} \tag{3.11}$$

Allow a contraction in (3.11) with respect to the indices i and r, we get

$$\left(\mathcal{L}_\nu W^i_j\right)_{((i))} - \mathcal{L}_\nu W^i_{j((i))} = (n-2)W^l_j b_l - W^l_j c_l + g^{lp}c_p\left(W^i_l g_{ij} - \left(\dot{\partial}W^i_j\right)g_{im}\dot{x}^m\right). \tag{3.12}$$

Now, transvect (3.11) by $\dot{x}^r$ and thereafter use (3.5), we get

$$\begin{aligned} \left(\left(\mathcal{L}_\nu W^i_j\right)_{((i))} - \mathcal{L}_\nu W^i_{j((r))}\right)\dot{x}^r &= W^l_j b_l\dot{x}^i - 4W^i_j b - W^l_j g_{rl}g^{ip}c_p\dot{x}^r \\ &+W^i_l g_{rj}g^{lp}c_p\dot{x}^r - \left(\dot{\partial}_l W^i_j\right)g_{rm}g^{lp}c_p\dot{x}^r\dot{x}^m. \end{aligned} \tag{3.13}$$

We make the supposition that the infinitesimal special projective transformation given by (3.4) leaves invariant the projective covariant derivative of the projective deviation tensor, i.e.,

$$\mathcal{L}_\nu W^i_{j((r))} = 0. \tag{3.14}$$

As a result of this supposition, the equations (3.12) and (3.13) can respectively be expressed in the following alternative form

$$\left(\mathcal{L}_\nu W^i_j\right)_{((i))} = (n-2)W^l_j b_l - W^l_j c_l + g^{lp}c_p\left(g_{ij}W^i_l - \left(\dot{\partial}_l W^i_j\right)g_{im}\dot{x}^m\right) \tag{3.15}$$

and

$$\left(\mathcal{L}_\nu W^i_j\right)_{((r))} = W^l_j b_l\dot{x}^i - 4W^i_j b - W^l_j g_{rl}g^{ip}c_p\dot{x}^r + W^i_l g_{rj}g^{ip}c_p\dot{x}^r - \left(\dot{\partial}W^i_j\right)g_{rm}g^{lp}c_p\dot{x}^r\dot{x}^m. \tag{3.16}$$

We now propose to eliminate $W^l_j b_l$ with the help of (3.15) and (3.16), the result of elimination gives the following

$$B^i_j(x,\dot{x}) = \dot{x}^j \left\{-W^l_j c_l + g^{lp} c_p \left[W^k_l g_{kj} - \left(\dot{\partial}_l W^r_j\right) g_{rm}\dot{x}^m\right]\right\} + (n-2) \times \left[4W^i_j b + W^l_j g_{rl} g^{ip} c_p \dot{x}^r - W^i_l g_{rj} g^{lp} c_p \dot{x}^r + \left(\dot{\partial} W^i_j\right) g_{rm} g^{lp} c_p \dot{x}^r \dot{x}^m\right], \quad (3.17)$$

where,

$$B^i_j(x,\dot{x}) \overset{\text{def}}{=} \left(\mathcal{L}_\nu W^i_j\right)_{((r))} \dot{x}^r - (n-2)\left(\mathcal{L}_\nu W^i_j\right)_{((r))} \dot{x}^r. \quad (3.18)$$

In order that the space under consideration may admit a special projective affine motion, we always have

$$\mathcal{L}_\nu \Pi^i_{jh} = 0. \quad (3.19)$$

Using (3.4) and (3.19), we easily arrive at the conclusion that the vectors $b(x,\dot{x})$ and $c(x,\dot{x})$ must separately vanish.

In the light of all these observations, we can therefore state results following.

Theorem 3.1 *In a Finsler space F_n, the equation (3.17) always holds provided the space under consideration admits a non-affine infinitesimal special projective transformation such that the projective covariant derivative of projective deviation tensor W^i_j remains an invariant.*

Theorem 3.2 *In a Finsler space F_n, $B^i_j(x,\dot{x})$ given by (3.18) always vanishes provided the space under consideration admits an affine infinitesimal special projective transformation such that the projective covariant derivative of the projective deviation tensor W^i_j remains an invariant.*

If the Finsler space F_n under consideration be assumed to be symmetric one i.e., $W^i_{j((r))} = 0$, then under such an assumption the equation (3.14) will always hold. Therefore, we can state the result following.

Theorem 3.3 *In a symmetric Finsler space F_n, the equation (3.17) always holds provided the space under consideration admits a non-affine infinitesimal special projective transformation characterized by (3.4).*

Theorem 3.4 *In a symmetric Finsler space F_n, B^i_j characterized by (3.18) always vanishes provided the space under consideration admits an affine infinitesimal special projective transformation.*

§4. Conclusion

The present communication has been divided into three sections of which the first section is introductory, the second section deals with non-affine infinitesimal transformations, and in this section, we have derived conditions which will hold when the space under consideration admits non-affine as well as an affine infinitesimal transformation and in the sequel have established the conditions which will hold when the space is symmetric and it admits an affine as well as non-affine infinitesimal transformation. The third section deals with infinitesimal special projective transformation. Like the previous section, in this section we have established the conditions which will hold when the space under consideration is symmetric and it admits a non-affine as well as an affine infinitesimal special projective transformation too.

References

[1] Misra R.B., A symmetric Finsler space, *Tensor (N.S.)*, 24(1972), 346-350.

[2] Misra R.B., The projective transformation in a Finsler space, *Ann.De La Soc.Sci.De Bruxelles*, 80(1966), 227-239.

[3] Pande H.D. and Kumar A., Special infinitesimal projective transformation in a Finsler space, *Accad.Naz.Dei.Lincei Rend*, 58(3-4)(1974), 190-193.

[4] Rund H., *The Differential Geometry of Finsler of Finsler Spaces*, Springer Verlag, Berlin 1959.

[5] Sinha R.S., On projective motion in a Finsler space with recurrent curvature, *Tensor (N.S.)*, 1,21(1970), 124-126.

[6] Takano K., On projective motion in Finsler space with bi-recurrent curvature, *Tensor (N.S.)*, 12(1962), 28-32.

[7] Yano K., *The Theory of Lie-Derivatives and Its Applications*, North Holand Publ.Co.(1957), Amesterdam, Holland.

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An Explicit Formula for the Number of Subgroup Chains of the Group $Z_{p^n} \times A_3$

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Abstract: In this paper, we establish a recursive formula for the number of chains of subgroups in the subgroup lattice of the groups formed by the Cartesian products of cyclic groups of prime power order with alternating groups of degree 3. The subgroup chains were characterised by an enumerative technique derived from the set of representatives of isomorphism classes of subgroups with their sizes.

Key Words: Subgroup, alternating group, chains of subgroup, fuzzy subgroup.

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§1. Introduction

The study of chains of subgroups describes the set containing all chains of subgroups of G, which ends with G.A formula for the lattice of a finite cyclic group, for several chains of subgroups, was given by Tărnăuceanu and Bentea [5] by giving its one variable generating function. J.M. Oh in his paper [3] determined the number of subgroups of a finite cyclic group of 4n by giving its multivariable generating function. The problem of counting chains of subgroups in the lattice of subgroups for any given group G got the attention of researchers, especially classifying fuzzy subgroups of finite groups under a natural equivalence relation (see [7], [2]).

In this paper, we follow to obtain the number of chains subgroups of the group $Z_{p^n} \times A_3$. In this regard, in Section 2, we present some preliminary definitions and necessary results on subgroup chains and fuzzy subgroups, which we will need in the next sections. In Sections 3, 4, and 5, we deal with the explicit formula for the number of subgroup chains of the group $Z_{p^n} \times A_3$, for any prime number, by generating recurrence relation with constant coefficients.

§2. Preliminaries

The chain of subgroups method describes the set of all chains of subgroups of that end in G. Suppose that the group G is finite, and let $\mu\colon G \mapsto [0,1]$ be a fuzzy subgroup of G. Put

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$\mu(G) = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ and assume that $\alpha_1<\alpha_2<\cdots<\alpha_r$. Then μ determines the following chain of subgroups of G which ends in G:

$$\mu G\alpha_1 \subset \mu G\alpha_2 \subset \cdots \subset \mu G\alpha_m = G$$

Moreover, for any $x \in G$ and $i = \overline{1, r}$, we have

$$\mu(x) = \alpha_i \Leftrightarrow i = max\{j|x \in \mu G\alpha_j\} \Leftrightarrow x \in \mu G_{\alpha_i} \setminus \mu G\alpha_{1-r}$$

A necessary and sufficient condition for two fuzzy subgroups μ, η of G to be equivalent to $\sim$ has been identified in Volf [2004], i.e., $\mu \sim \eta$ if and only if μ and η have the same set of level subgroups, that is, they determine the same chain of subgroups. This result shows that there exists a bijection between the equivalence classes of fuzzy subgroups of G and the set of chains of subgroups of G that end in G. Clearly, any group with at least two elements has more distinct fuzzy subgroups than subgroups.

Also, the problem of counting all distinct fuzzy subgroups of G can be translated into a combinatorial problem on the subgroup lattice $L(G)$ of G, that is computing the number of all chains of subgroups of G that terminate in G.If $\delta(G)$ denotes the number of all chains of subgroups of G that terminate in G, then $\delta(G)$ is the number of chains of subgroups of length one of G ending in G plus the number of chains of subgroups of length more than one of the group G, which end in G. Hence,

$$\begin{aligned}\delta(G) &= n(H_1) * \delta(H_1) + n(H_1) * \delta(H_2) + n(H_3)\delta(H_3) + \cdots + n(H_\alpha)\delta(H_\alpha)\\ &= \sum_{H_i \in Iso(G)} \delta(H_i) \times n(H_i)\\ &= 2 + \sum_{distict\ \ H_i \in Iso(G)} \delta(H_i) \times n(H_i),\end{aligned}$$

where $Iso(G)$ is the set of representatives of isomorphism classes of subgroups of G and $n(H)$ denotes the size of the isomorphism class with representative H.

Let fixes $\delta(H_1) = \delta(H_\alpha) = 1$, because $\delta(H_1) = \delta(H_\alpha)$ in general , for which H_1 is the trivial group of G and H_α is the improper subgroup of G. For any $H_i \in Iso(G)$ and $i = \overline{1, \alpha}$,

$$\delta(G) = \sum_{distictH \in Iso(G)} \delta(H) \times n(H). \tag{2.1}$$

In this paper, (2.1) is used to obtain the number of subgroup chains of G because the number of all distinct fuzzy subgroup of G under the natural equivalence relation $\sim$ is equal to the number of subgroup chains of G that terminates in G (see Ogiugo and Amit, 2020).

§3. The Number of Chains of Subgroups of $Z_{2^n} \times A_3$ with $n \geq 1$

Proposition 3.1 *The number of subgroup chains of the group $Z_2 \times A_3$ is 6.*

Proof Let G be $Z_2 \times A_3$, it has the following set of representatives of isomorphism classes of subgroups with sizes $[e, 1]$, $[Z_2, 1]$, $[Z_3, 1]$ and $[Z_2 \times A_3, 1]$. So

$$\delta(G) = \delta(H_e) + \delta(Z_2) + \delta(Z_3) + 1 = 1 + 2 + 2 + 1 = 6. \qquad \square$$

Proposition 3.2 *The number of chains of subgroups of $Z_4 \times A_3$ is 16.*

Proof Let G be $Z_4 \times A_3$, It has the following set of representatives of isomorphism classes of subgroups with respective sizes $[e, 1]$, $[Z_2, 1]$, $[Z_3, 1]$, $[Z_4, 1]$, $[Z_6, 1]$ and $[Z_4 \times A_3, 1]$. So,

$$\begin{aligned}\delta(Z_4 \times A_3) &= \delta(H_e) + \delta(Z_2) + \delta(Z_3) + \delta(Z_4) + \delta(Z_6) + 1 \\ &= 1 + 2 + 2 + 4 + 6 + 1 = 16.\end{aligned}$$

This completes the proof. $\square$

Proposition 3.3 *The number of chains of subgroups of $Z_8 \times A_3$ is 40.*

Proof $Z_8 \times A_3$ has the following set of representatives of isomorphism classes of subgroups with respective sizes $[e, 1]$, $[Z_2, 1]$,$[Z_3, 1]$, $[Z_4, 1]$, $[Z_6, 1]$, $[Z_8, 1]$, $[Z_{12}, 1]$ and $[Z_8 \times A_3, 1]$. So,

$$\delta(Z_8 \times A_3) = \delta(H_e) + \delta(Z_2) + \delta(Z_3) + \delta(Z_4) + \delta(Z_6) + \delta(Z_8) + \delta(Z_{12}) + 1 = 40. \qquad \square$$

Proposition 3.4 *The number of chains of subgroups of $Z_{16} \times A_3$ is 96.*

Proof $Z_{16} \times A_3$ has the following set of representatives of isomorphism classes of subgroups with respective sizes $[e, 1]$, $[Z_2, 1]$,$[Z_3, 1]$, $[Z_4, 1]$, $[Z_6, 1]$, $[Z_8, 1]$, $[Z_{12}, 1]$, $[Z_{16}, 1]$, $[Z_{24}, 1]$ and $[(Z_{16} \times A_3), 1]$. So,

$$\begin{aligned}\delta(Z_{16} \times A_3) &= \delta(H_e) + \delta(Z_2) + \delta(Z_3) + \delta(Z_4) + \delta(Z_6) + \delta(Z_8) \\ &\quad + \delta(Z_{12}) + \delta(Z_{16}) + \delta(Z_{24}) + 1 = 96.\end{aligned}$$

This completes the proof. $\square$

Theorem 3.5 *Let G be $Z_{2^n} \times A_3$,where $n \geq 1$ the number of chains of subgroups of G is $2^n(2 + n)$.*

Proof $Z_{2^n} \times A_3$ as 2^n and 3 are relatively prime , the divisors of $2^n \times 3$ are 1, 3, 2^i and $2^i \times 3$, where $i = 1, 2, \cdots, n$. These, then generate the list of cyclic groups of orders 1, 3, 2^i, $2^i \times 3$, respectively for where $i = 1, 2, \cdots, n$. Thus, we have the following set of representatives of isomorphism classes of subgroups with respective sizes:

$[e, 1]$,

$[Z_2, 1]$, $[Z_{2^2}, 1]$, $[Z_{2^2}, 1]$ $\cdots$ $[Z_{2^n}, 1]$,

$[Z_3, 1]$, $[Z_{2.3}, 1]$, $[Z_{4.3}, 1]$, $[Z_{8.3}, 1]$, $[Z_{16.3}, 1]$ $\cdots$ $[Z_{2^n.3}, 1]$.

So,

$$\begin{aligned}\delta(Z_{2^n} \times A_3) &= \delta(H_e) + \delta(Z_2) + \delta(Z_4) + \delta(Z_8) + \delta(Z_{16}) + \cdots + \delta(Z_{2^n}) + \delta(Z_3) \\ &\quad + \delta(Z_6) + \delta(Z_{12}) + \delta(Z_{24}) + \delta(Z_{48}) + \cdots + \delta(Z_{2^n.3}) + 1.\end{aligned}$$

We establish recurrence relation for $Z_{2^n} \times A_3$, where $n \geq 1$, i.e.,

$$\delta(Z_{2^n} \times A_3) = 1 + \sum_{j=1}^{n} \delta(Z_{2^j}) + \sum_{j=1}^{n-1} \delta(Z_{2^j} \times A_3) + 1. \tag{3.1}$$

Change n to $n-1$ in (3.1), we get

$$\delta(Z_{2^{n-1}} \times A_3) = 1 + \sum_{j=1}^{n-1} \delta(Z_{2^j}) + \sum_{j=1}^{n-2} \delta(Z_{2^j} \times A_3) + 1. \tag{3.2}$$

From (3.1) and (3.2), we get

$$\begin{aligned}\delta(Z_{2^n} \times A_3) - 2\delta(Z_{2^{n-1}} \times A_3) &= \delta(Z_{2^n}), \\ \delta(Z_{2^n} \times A_3) - 2\delta(Z_{2^{n-1}} \times A_3) &= 2^n. \end{aligned} \tag{3.3}$$

To find the solution of recurrence relation in (3.3), let $\delta(Z_{2^n} \times A_3) = X_n$. Then,

$$X_n - 2X_{n-1} = 2^n.$$

We find the solution of recurrence relation in (3.3). Its characteristic solution (C.S) is $X_n = A2^n$ and its particular solution (P.S) is $X_n = nB2^n$ of recurrence relation in (3.3). From (3.3), we have

$$nB2^n - 2B(n-1)2^{n-1} = 2^n$$

$$nB2^n - B(n-1)2^n = 2^n$$

$$nB - (n-1)B = 1$$

$$\cancel{nB} - \cancel{nB} + B = 1 \Rightarrow B = 1$$

Therefore, the general solution of recurrence relation in (3.3) is

$$X_n = A2^n + n2^n. \tag{3.4}$$

Consider the case of $n = 1$ and $X_1 = 6$. In this case we get, $6 = 2A + 2$, $A = 2$ and finally,

$$\delta(Z_{2^n} \times A_3) = 2.2^n + n2^n.$$

Using a recurrence relation solution, we therefore obtain

$$\delta(Z_{2^n} \times A_3) = 2^n(2 + n). \qquad \square$$

Corollary 3.6 *Let n be the positive integer defined in Theorem 3.5 and let $\delta(G)$ be the number of subgroup chains of G. Then, if $n = 2$, $G = Z_4 \times A_3)$, then $\delta(G) = 2^2(2 + 2)$, $\delta(G) = 16$ and if $n = 3$, $G = Z_8 \times A_3)$, then $\delta(G) = 2^3(3 + 2)$, $\delta(G) = 40$.*

§4. The Number of Chains of Subgroups of $Z_{3^n} \times A_3$ with $n \geq 1$

Proposition 4.1 *The number of subgroup chains of $Z_3 \times A_3$ is 10.*

Proof Let G be $Z_3 \times A_3$, it has the following set of representatives of isomorphism classes of subgroups with respective sizes $[e, 1]$, $[Z_3, 4]$ and $[(Z_3 \times A_3), 1]$. Then,

$$\delta(G) = \delta(H_e) + 4\delta(Z_3) + 1 = 10. \qquad \Box$$

Proposition 4.2 *The number of Subgroup chains of $Z_9 \times A_3$ is 32.*

Proof Let G be $Z_9 \times A_3$, it has the following set of representatives of isomorphism classes of subgroups with respective sizes $[e, 1]$, $[Z_3, 4]$, $[Z_3 \times A_3, 1]$, $[Z_9, 3]$ and $[Z_9 \times A_3, 1]$. Then,

$$\delta(G) = \delta(H_e) + 4\delta(Z_3) + \delta(Z_3 \times A_3) + 3\delta(Z_9) + 1 = 32. \qquad \Box$$

Proposition 4.3 *The number of subgroup chains of $Z_{27} \times A_3$ is 88.*

Proof Let G be $Z_{27} \times A_3$, it has the following set of representatives of isomorphism classes of subgroups with respective sizes $[e, 1]$, $[Z_3, 4]$, $[Z_3 \times A_3, 1]$, $[Z_9 \times A_3, 1]$, $[Z_9, 3]$, $[Z_{27}, 3]$ and $[(Z_{27} \times A_3), 1]$. Then,

$$\delta(G) = \delta(H_e) + 4\delta(Z_3) + \delta(Z_3 \times A_3) + \delta(Z_9 \times A_3) + 3\delta(Z_9) + 3\delta(Z_{27}) + 1 = 88. \qquad \Box$$

Theorem 4.4 *Let G be $Z_{3^n} \times A_3$ where $n \geq 1$, then the number of Chains of Subgroups of G is $(3n + 2)2^n$.*

Proof The order of the group $Z_{3^n} \times A_3$ is not relatively prime for any n, we have the divisors of $3^n \times 3$ are 1, 3, 3^j and $3^j \times 3$, where $j = 1, 2, \cdots n$.So that , we obtain cyclic subgroups of $Z_{3^n} \times A_3$ of order 1 and order 3^j respectively, that is identity group and Z_{3^j}, $j = 1, 2, \cdots n$ and non-cyclic Abelian subgroups$Z_{3^j} \times Z_3$ of $3^j \times 3$,$j = 1, 2, \cdots$. Thus, we have the following set of representatives of isomorphism classes of subgroups with respective sizes

$[e, 1]$, $[Z_3, 4]$,

$[Z_3 \times A_3, 1]$, $[Z_9 \times A_3, 1]$, $\cdots$, $[Z_{3^{n-1}} \times A_3, 1]$,

$[Z_9, 3]$, $[Z_{27}, 3]$, $[Z_{81}, 3]$, $\cdots$, $[Z_{3^n}, 3]$.

$$\delta(Z_{3^n} \times A_3) = 1 + 4\delta(Z_3) + 3\sum_{j=1}^{n} \delta(Z_{3^j}) + \sum_{j=1}^{n-1} \delta(Z_{3^j} \times A_3) + 1. \tag{4.1}$$

We establish recurrence relation for $Z_{3^n} \times A_3$ where $n \geq 2$, Let's rewrite equation (4.1) as

$$\delta(Z_{3^n} \times A_3) = 1 + 4 * \delta(Z_3) + 3 * \sum_{j=2}^{n} \delta(Z_{3^j}) + \sum_{j=1}^{n-1} \delta(Z_{3^j} \times A_3) + 1. \tag{4.2}$$

Change n to $n-1$ in (4.1), we get

$$\delta(Z_{3^{n-1}} \times A_3) = 1 + 4 * \delta(Z_3) + 3 * \sum_{j=2}^{n-1} \delta(Z_{3^j}) + \sum_{j=1}^{n-2} \delta(Z_{3^j} \times A_3) + 1. \tag{4.3}$$

From (4.1) and (4.2), we get

$$\delta(Z_{3^n} \times A_3) - 2\delta(Z_{3^{n-1}} \times A_3) = 3\delta(Z_{3^n}).$$

To find the solution of recurrence relation in (4.3), let $\delta(Z_{3^n} \times A_3) = X_n$. Then,

$$X_n - 2X_{n-1} = 3(2^n).$$

Its characteristic solution (C.S) is $X_n = 3C2^n$ and its particular solution (P.S) is $X_n = 3nD2^n$. So that

$$3nD2^n - 3(n-1)D2^n = 3(2^n)$$

$$3nD - 3(n-1)D = 3$$

$$\cancel{3nD} - \cancel{3nD} + 3D = 3$$

$$3D = 3 \Rightarrow D = 1.$$

The general solution in this case is with the C.S given by $3C2^n$. So that we obtain for the general solution:

$$X_n = 3C2^n + 3n2^n.$$

Consider the case of $n = 1$, $X_1 = 10$. In this case we get $10 = 6C + 6$, i.e., $C = \frac{4}{6} = \frac{2}{3}$. So that the general solution becomes

$$\begin{aligned} X_n &= 3 \times \frac{2}{3} \times 2^n + 3n2^n \\ &= 2^{n+1} + 3n2^n = 2^n(2 + 3n). \end{aligned}$$

Therefore,

$$\delta(Z_{3^n} \times A_3) = (3n + 2)2^n.$$

Here, we obtained the formula in case of $p = 3$ because 3^n and 3 are not relatively prime. □

Corollary 4.5 *Let n be the positive integer defined in Theorem 4.4 and let $\delta(G)$ be the number of subgroup chains of G. If $n = 2$ $G = Z_9 \times A_3$), then $\delta(G) = 2^2(3.2 + 2)$, $\delta(G) = 32$ and if $n = 3$ $G = Z_{27} \times A_3$), then $\delta(G) = 2^3(3 \times 3 + 2) = 88$.*

§5. The number of Chains of Subgroups of $Z_{p^n} \times A_3$ with $n \geq 1$ and $p \geq 5$

Proposition 5.1 *The number of subgroup chains of $Z_5 \times A_3$ is 6.*

Proof Let G be $Z_5 \times A_3$. It has the following set of representatives of isomorphism classes of subgroups with respective sizes $[e, 1]$,$[Z_3, 1]$ and $[Z_5, 1]$ and $[Z_5 \times A_3, 1]$. Then,

$$\delta(G) = 1 + \delta(Z_3) + \delta(Z_5) + 1 = 6. \qquad \square$$

Proposition 5.2 *The number of subgroup chains of $Z_{25} \times A_3$ is 16.*

Proof Let G be $Z_{25} \times A_3$. It has the following set of representatives of isomorphism classes of subgroups with respective sizes $[e, 1]$, $[Z_3, 1]$, $[Z_5, 1]$, $[Z_5 \times A_3, 1]$, $[Z_{25}, 1]$ and $[Z_{25} \times A_3, 1]$. Then,

$$\delta(G) = \delta(H_e) + \delta(Z_3) + \delta(Z_5) + \delta(Z_5 \times A_3) + \delta(Z_{25}) + 1 = 16. \qquad \square$$

Proposition 5.3 *The number of subgroup chains of $Z_7 \times A_3$ is 6.*

Proof Let G be $Z_7 \times A_3$. It has the following set of representatives of isomorphism classes of subgroups with respective sizes $[e, 1]$, $[Z_3, 1]$ and $[Z_7, 1]$ and $[Z_7 \times A_3, 1]$. Then,

$$\delta(G) = 1 + \delta(Z_3) + \delta(Z_5) + 1 = 6. \qquad \square$$

Proposition 5.4 *The number of subgroup chains of $Z_{49} \times A_3$ is 6.*

Proof Let G be $Z_{49} \times A_3$. It has the following set of representatives of isomorphism classes of subgroups with respective sizes $[e, 1]$, $[Z_3, 1]$, $[Z_7, 1]$, $[[Z_7 \times A_3, 1]$, $[Z_{49}, 1]$ and $[Z_{49} \times A_3, 1]$. Then,

$$\delta(G) = \delta(H_e) + \delta(Z_3) + \delta(Z_7) + \delta(Z_7 \times A_3) + \delta(Z_{49}) + 1 = 16. \qquad \square$$

Theorem 5.5 *Let G be $Z_{p^n} \times A_3$ where $n \geq 1$ and $p \geq 5$, then the number of chains of subgroups of G is $(n + 2)2^n$.*

Proof As 5 and 3 are relatively prime, the order of $Z_{p^n} \times A_3$ is 5×3, which is cyclic with cyclic subgroups of order $1, 3, 5, 5. \times 3$ where

$$n(\{1\}) = n(Z_3) = n(Z_5) = n(Z_{15}) = 1$$

and

$$\delta(\{1\}) = 1, \delta(Z_3) = 2, \delta(Z_5) = 2, \delta(Z_{15}) = 6.$$

Similarly, as 5^2 and 3 are relatively prime, the order of the group $Z_{5^2} \times A_3)$ is $5^2 \times 3$, which is cyclic with cyclic subgroups $1, 3, 5, 5^2, 5 \times 3, 5^2 \times 3$. Thus, we have the following set of representatives of isomorphism classes of subgroups with respective sizes

$[e, 1]$, $[Z_3, 1]$,
$[Z_p \times A_3, 1]$, $[Z_{p^2} \times A_3, 1]$, $\cdots$ $[Z_{p^{n-1}} \times A_3, 1]$
$[Z_p, 1]$, $[Z_{p^2}, 1]$, $\cdots$ $[Z_{p^n}, 1]$.

We establish recurrence relation for $Z_{p^n} \times A_3$ where $n \geq 1$ and $p \geq 5$,

$$\delta(Z_{p^n} \times A_3) = 1 + \delta(Z_3) + \sum_{j=1}^{n} \delta(Z_{p^j}) + \sum_{j=1}^{n-1} \delta(Z_{p^j} \times A_3) + 1. \tag{5.1}$$

Change n to $n-1$ in (5.1), we get

$$\delta(Z_{p^{n-1}} \times A_3) = 1 + \delta(Z_3) + \sum_{j=1}^{n-1} \delta(Z_{p^j}) + \sum_{j=1}^{n-2} \delta(Z_{p^j} \times A_3) + 1. \tag{5.2}$$

From (5.1) and (5.2), we get

$$\delta(Z_{p^n} \times A_3) - 2\delta(Z_{p^{n-1}} \times A_3) = \delta(Z_{p^n}).$$

Since $\delta(Z_{p^n}) = 2^n$ for all p and n in the literature, let $\delta(Z_{p^n} \times A_3) = X_n$. Then,

$$X_n - 2X_{n-1} = 2^n. \tag{5.3}$$

We find the solution of recurrence relation in (5.3). Its characteristic solution (C.S) is $X_n = E2^n$ and particular solution (P.S) is $X_n = nF2^n$. So that (5.3) becomes

$$nF2^n - (n-1)F2^n = 2^n$$

$$nF - (n-1)F = 1$$

$$\cancel{nF} - \cancel{nF} + F = 1 \Rightarrow F = 1$$

The general solution is with the C.S given by $E2^n$ in this case. So that we obtain for the general solution

$$X_n = E2^n + n2^n.$$

Consider the case when $n = 1$, $X_1 = 6$. In this case, we get

$$6 = 2E + 2, i.e., \quad E = 2.$$

So that the general solution becomes

$$\begin{aligned} X_n &= 2 \times 2^n + n2^n \\ &= 2^{n+1} + n2^n = 2^n(2+n). \end{aligned}$$

Therefore,

$$\delta(Z_{p^n} \times A_3) = (n+2)2^n.$$

Here we obtained this formula for the case $p = 5, 7, 11 \cdots$ because for $p \geq 5$, $n \geq 1.p^n$ and 3 are relatively prime. □

Corollary 5.6 *Let n be the positive integer defined in Theorem 5.5 and let $\delta(G)$ be the number of subgroup chains of G. For $n = 2$, if $G = Z_(25 \times A_3)$ then $\delta(G) = 2^2(2+2) = 16$, if $G = Z_(49 \times A_3)$ then $\delta(G) = 2^2(2+2) = 16$ and for $n = 3$, if $G = Z_(125 \times A_3)$ then $\delta(G) = 2^3(3+2) = 40$, if $G = Z_(343 \times A_3)$ then $\delta(G) = 2^3(3+2) = 40$.*

§6. Conclusion

We have determined explicit formulas for the number of the subgroup chains in the lattice of subgroups of the group $Z_{p^n} \times A_3$, p is any prime number and it is also the number of distinct fuzzy subgroups of $Z_{p^n} \times A_3$ concerning the natural equivalence relation (see e.g [7]).

References

[1] Amit Sehgal, Sarita Sehgal, P. K. Sharma and Manjeet Jakhar, Fuzzy subgroups of a finite abelian group. *Advances in Fuzzy Sets and Systems* Vol. 21(4)(2016), 291-302.

[2] A.C. Volf, Counting fuzzy subgroups and chains of subgroups. *Fuzzy Systems & Artificial Intelligence*Vol. 10(2004), 191C200.

[3] J.M. Oh, The number of chains of subgroups of a finite cycle group, *European Journal of Combinatorics*, Vol. 33, No. 2(2012), 259C266.

[4] Leili Kamali Ardekania, and Bijan Davvaz, Classifying fuzzy (normal) subgroups of the group $D_{2p} \times Z_q$ and finite groups of order $n \leq 20$, *Journal of Intelligent & Fuzzy Systems*, Vol. 33(2017), 3615C3627 DOI:10.3233/JIFS-17301.

[5] M. Tărnăuceanu and L. Bentea, On the number of fuzzy subgroups of finite abelian groups, *Fuzzy Sets and Systems*, Vol. 159(2008), 1084C1096.

[6] M.Ogiugo, Amit Sehgal, S. A. Adebisi and M. EniOluwafe, The number of chains of subgroups in the lattice of subgroups of the group $Z_m \times A_n, n \leq 6, m \leq 3$, *International J.Math. Combin.*, Vol.4(2022), 32-40.

[7] M.Ogiugo and M.EniOluwafe, Classifying a class of the fuzzy subgroups of the alternating group A_n, *Imhotep Mathematical Proceedings* Vol.4(2017), 27- 33.

[8] Ogiugo M. and Amit Sehgal, The number of chains of subgroups for certain finite alternating groups, *Annals of Pure and Applied Mathematics*, Vol 22(1) (2020), 65-70.

[9] M. Ogiugo, Amit Sehgal and M.EniOluwafe, the number of chains of subgroups for certain finite symmetric groups (Submitted).

International J.Math. Combin. Vol.4(2023), 63-74

Classification of the Defining Equations of Flag Varieties $\mathcal{F}\ell_n(\mathbb{C})$

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Abstract: Using only combinatorial technique, we give a formula for classification of the defining equation of flag variety $\mathcal{F}\ell_n(\mathbb{C})$. The formula uses the theory of complete geometric graph based on the indexing set of the monomials of the ideal. In particular, we give a generating function to count the number of classes. The size of each class is also determined. We describe the procedure of obtaining the equations using a complete geometric graph and lastly, we give a formula to count these equations.

Key Words: Flag variety, Plücker coordinate, geometric graph, spanning tree.

AMS(2010): 14M15, 14N15, 05C20, 05C30.

§1. Introduction

Let V be an n-dimension vector space over the field of complex numbers. By a flag F in V, we mean a sequence of subspaces:

$$F_\bullet : \{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_n = V \ \text{ such that } \dim F_i = i.$$

The set of all such flags in V is called the flag variety and denoted by $\mathcal{F}\ell_n(\mathbb{C})$. By fixing a basis $e_1, e_2, \ldots, e_n$, we let $E_\bullet$ to denote the standard flag spanned by

$$E_\bullet = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \ldots, e_n \rangle.$$

The variety can also be described by considering the general linear group $GL(n, \mathbb{C})$ consisting of all non-singular $n \times n$ matrices and let $\mathcal{B}$ be the subset of all invertible upper triangular matrices. A flag $F_\bullet$ can be constructed by allowing F_i be the span of the first i columns of a given matrix Z in $GL(n, \mathbb{C})$. The matrices Z_1 and Z_2 are equivalent, that is, give the same flag if and only if there is an upper triangular matrix Y in $\mathcal{B}$ such that $Z_2 = YZ_1$. This defines an equivalence relation on $GL(n, \mathbb{C})$. Thus $\mathcal{F}\ell_n(\mathbb{C}) = GL(n, \mathbb{C})/B$. The precise implication is that the general linear group $GL(n, \mathbb{C})$ acts transitively on $\mathcal{F}\ell_n(\mathbb{C})$ and the stabilizer of standard flag is the Borel subgroup and hence the identification of $\mathcal{F}(n)$ with G/B. Therefore, $\mathcal{F}\ell_n(\mathbb{C})$ is viewed as a homogeneous space. More is true $\mathcal{F}\ell_n(\mathbb{C})$ is a smooth projective variety being a closed subvariety of the product of Grassmanians $\prod_{k=1}^{n-1} Gr(k, n)$. This gives rise to the Plücker

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embedding

$$\mathcal{F}\ell_n(\mathbb{C}) \hookrightarrow \mathbb{P}^{\binom{n}{1}-1} \times \mathbb{P}^{\binom{n}{2}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}.$$

The image of $\mathcal{F}\ell_n(\mathbb{C})$ via the embedding is cut out by Plücker relations (see [1]). These relations generate the homogeneous ideal of $\mathcal{F}\ell_n(\mathbb{C})$ which we denote by $\mathcal{I}$, indeed $\mathcal{I}$ is minimally generated by these quadrics. It is well known that each flag $F_\bullet$ can be represented by $n \times n$-matrix $A = (a_{ij})$ in which the subspace F_i is spanned by the first i rows. The relations that a point must satisfy in order to lie in the image of $\mathcal{F}\ell_n(\mathbb{C})$ via the embedding are called the Plücker relations. This is achieved by defining the map

$$\phi_n : \mathbb{K}[p_\alpha : \emptyset \neq \alpha \subseteq \{1, \ldots, n\}] \longrightarrow \mathbb{K}[a_{ij} : 1 \leq i \leq n-1, 1 \leq j \leq n]$$

sending each variable p_α to the determinant submatrix of A with row indices $1, \ldots, \mid \alpha \mid$ and column indices in α. It turns out that the ideal I_n of $\mathcal{F}\ell_n(\mathbb{C})$ is the kernel of ϕ_n. This homogeneous ideal is minimally generated by the Plücker relations. These relations which are quadrics are the equations defining the variety $\mathcal{F}\ell_n(\mathbb{C})$ (See [9],[1]).

Our interest is in the classification of these equations using complete geometric graphs. Specifically, we give a formula that partitions the equations by exploiting some similar properties shared by them. This ultimately allows us to know the number of equations in each subdivision thereby counts the generators for each ideal I_n. We plan a sequel paper to exploit this technique to give the degeneration of flag variety $\mathcal{F}\ell_n(\mathbb{C})$. Let $\mathbb{T}$ be a collection of points in the plane in general position. By geometric graph on $\mathbb{T}$, we mean a graph G whose vertices are the elements of $\mathbb{T}$ in which two are said to be adjacent if they are joined by a line segment. Our interest is in a graph where every pair of vertices is adjacent. This is called a complete geometric graph and is denoted by $\mathcal{K}_n$, n is the number of vertices. The number of edges of $\mathcal{K}_n$ is $\frac{n(n-1)}{2}$ which turns out to be the dimension of flag variety $\mathcal{F}\ell_n(\mathbb{C})$. In section 2, we give some background and results relevant to our discussion. In section 3, we describe the procedure to obtain the relations in the complete geometric digraph, $\mathcal{K}_n$ and also compute the relations in $\mathcal{K}_3$ and $\mathcal{K}_4$. In section 4, we give the classifications of relations in $\mathcal{K}_n$ and the class size. We also give generating functions on the classifications and the number of classes in any $\mathcal{K}_n$. This gives the classification of the equations defining flag varieties $\mathcal{F}\ell_n(\mathbb{C})$.

§2. Complete Geometric Directed Graphs

In this section we give some definitions on geometric graphs and trees (see [5], [4], [2], [3], [6], [8], [7], [10] for details).

Definition 2.1 *Let $\mathcal{K}_n$ be a complete geometric digraph with a n points and let $\sigma \subset [n]$. x_σ is said to be a point if $|\sigma| = 1$, a line if $|\sigma| = 2$, a triangle if $|\sigma| = 3$ and so on.*

Remark 2.2 All the x_σ's for which $|\sigma| \geq 3$ are empty, that is, they have no interior points.

Example 2.3 (i) For $n = 3$, the complete geometric digraph is

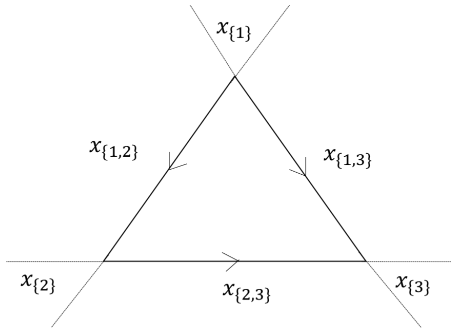

Figure 1

(ii) For $n = 4$, the complete geometric digraph is

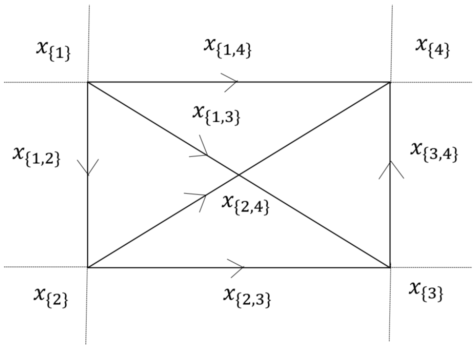

Figure 2

Given a complete geometric digraph $\mathcal{K}_n$, let $F_m = \{x_\sigma : |\sigma| = m, \sigma \subseteq [n]\}$, F_1 set of points, F_2 set of lines and so on. Let $f_m = \#F_m$.

Definition 2.4 (i) *A walk in $\mathcal{K}_n$ is a sequence of vertices $v_0, v_1, \cdots, v_k$ and sequence of edges $(v_i, v_{i+1}) \in F_2$. If v_i are distinct, then we have a path and if $(v_0, v_k) \in F_2$, then $v_0, v_1, \cdots, v_k, v_0$ is a cycle. The length of a path or cycle is the number of edges in it.*

(ii) *A tree is a connected graph without any cycles. The edges of a tree are called branches and the degree 1(number of edges incident with the vertex) vertex are called leaves.*

(iii) *A spanning tree T of a connected graph $\mathcal{K}_n$ is the subgraph of $\mathcal{K}_n$ containing all the vertices of $\mathcal{K}_n$. A chord is an edge of a graph that is not in a given spanning tree.*

(iv) *A rooted tree T with the vertex set V is the tree that has a specially designated vertex $v_1 \in V$. The root of any spanning tree is defined as the vertex with highest degree.*

Remark 2.5 (i) For any spanning tree T of $\mathcal{K}_n$, the number of branches is called the rank, r and the number of chords is called the nullity, μ (cyclomatic number or first Betti number). $r = n - 1$ and $\mu = \frac{(n-1)(n-2)}{2}$.

(ii) There are n^{n-2} spanning tree in a complete graph and n $(n-1)$-valent spanning trees since there are only n vertices with degree $n - 1$.

Lemma 2.6 *Let $\mathcal{C}$ be the set of flag varieties and $\mathcal{B}$ be the set of complete geometric digraphs, there is a bijection*

$$\alpha : \mathcal{C} \longrightarrow \mathcal{B}$$

$$\mathcal{F}l_n(\mathbb{C}) \longmapsto \mathcal{K}_n.$$

Theorem 2.7 *Given a complete geometric digraph $\mathcal{K}_n$, then f_m is given by the coefficient of*

$$P_n(t) = \sum_{|\sigma|=1}^{n} \binom{n}{|\sigma|} t^{|\sigma|}.$$

Proof Given a complete geometric digraph, $\mathcal{K}_n$ with points indexed by $[n]$, let $\sigma \subseteq [n]$ and $|\sigma| = r$. For $r = 1$, we have a point and the number of choice of selection is $\binom{n}{1}$ and for $r = 2$ we have a line and the number of choice of selection is $\binom{n}{2}$. Continuing until $r = n$, we have $\binom{n}{n}$. Then this can be generalised as

$$\binom{n}{1}t + \binom{n}{2}t^2 + \cdots + \binom{n}{r}t^r + \cdots + \binom{n}{n}t^n$$

where the power of t is $|\sigma|$ and the coefficient of t is the number of such σ. $\square$

Theorem 2.7 gives the size of F_m for $1 \leq m \leq n$ in $\mathcal{K}_n$.

n	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
1	1									
2	2	1								
3	3	3	1							
4	4	6	4	1						
5	5	10	10	5	1					
6	6	15	20	15	6	1				
7	7	21	35	35	21	7	1			
8	8	28	56	70	56	28	8	1		
9	9	36	84	126	126	84	36	9	1	
10	10	45	120	210	252	210	120	45	10	1

Table 1. Statistics of f_m in $\mathcal{K}_n$

Let Ω be the union of all F_m, we defined an ordering on Ω as follows:

Given $\sigma, \rho \subseteq [n]$ such that $\sigma = \{a_1 < \cdots < a_m\}$ and $\rho = \{b_1 < \cdots < b_r\}$. Let $x_\sigma \leq x_\rho$ in the poset $\mathcal{P}$ if $m \geq r$ and $\sigma_i \leq \rho_i$ for all $i = 1, \cdots, r$.

Let $\nabla = \{x_\sigma x_\tau + \textit{lower terms} : 1 \leq |\sigma| \leq n-2 \textit{ and } 2 \leq |\tau| \leq n-1\}$ be the set of relations between x_σ's and x_τ's.

Theorem 2.8 *Given x_σ in $\mathcal{K}_n$ such that $|\sigma| = 3$ (i.e x_σ is a triangle), then x_σ can be expressed as a linear combination of x_{τ_i} which sum to zero for $|\tau_i| = 2$, $\tau_i \subset \sigma$ and $\bigcap \tau_i = \emptyset$. Moreover, the number of summands is $|\sigma|$.*

Proof Given a complete geometric digraph, $\mathcal{K}_n$ with points indexed by $[n]$. Suppose $\sigma \subset [n]$ with $|\sigma| > 2$, x_σ is a subgraph of $\mathcal{K}_n$, there is a closed path in x_σ (x_σ are line segments), which is the sum of x_{τ_i} and $\bigcap \tau_i = \emptyset$ and the number of such τ_i is $|\sigma|$. $\square$

Remark 2.9 (i) The sign of x_{τ_i} in Theorem 2.8 is negative if the distance of τ is $|\sigma| - 1$, otherwise positive.

(ii) Theorem 2.8 gives the relation of the paths in x_σ.

Example 2.10 Given the complete geometric digraph $\mathcal{K}_3$. Then triangle, $x_{\{1,2,3\}}$ with lines $x_{\{1,2\}}$, $x_{\{2,3\}}$ and $x_{\{1,3\}}$, can be expressed as

$$x_{\{1,2,3\}} = x_{\{1,2\}} + x_{\{2,3\}} - x_{\{1,3\}} = 0.$$

Remark 2.11 From Example 2.10, $x_{\{1,3\}}$ is called the equivalent path and can be expressed as $x_{\{1,3\}} = x_{\{1,2\}} + x_{\{2,3\}}$.

Corollary 2.12 *Every x_τ such that $|\tau| > 3$ can be expressed as a linear combination of x_{α_i} such that $|\alpha_i| = 3$ and $\alpha_i \subset \tau$.*

Example 2.13 Given the complete geometric digraph $\mathcal{K}_4$. Then $x_{\{1,2,3,4\}}$ with lines $x_{\{1,2\}}$, $x_{\{2,3\}}$, $x_{\{3,4\}}$ and $x_{\{1,4\}}$. Then we have $x_{[4]} = x_{\{1,2\}} + x_{\{2,3\}} + x_{\{3,4\}} - x_{\{1,4\}} = 0$.
$x_{[4]}$ can be decompose into triangles as follows:

$$x_{[4]} = x_{\{1,2,3\}} - x_{\{1,2,4\}} + x_{\{1,3,4\}} - x_{\{2,3,4\}}.$$

The branches (for $|\alpha_i| = 2$) in the spanning trees of $\mathcal{K}_n$ are related. The relation is given by the theorem below which generalizes for $|\alpha_i| \geq 2$.

Theorem 2.14 *Given $\mathcal{K}_n$ and $\sigma \subset [n]$ such that $|\sigma| \geq 3$, then x_σ^τ, the linear combination of x_{α_i} such that $\tau \subset \alpha_i \subset \sigma$ is given by*

$$x_\sigma^\tau = \sum_{i=1}^{|\sigma|-1} (-1)^{i+1} x_{\alpha_i}$$

for $2 \leq |\alpha_i| \leq |\sigma|$ and $1 \leq |\tau| \leq |\sigma| - 1$.

Proof Given a complete geometric digraph, $\mathcal{K}_n$ with vertices indexed by $[n]$. Since $\sigma \bigcap \alpha_i = \tau$, then x_σ^τ is the sum of all subgraphs of x_σ containing the subgraph x_τ. $\square$

Remark 2.15 Theorem 2.14 gives the relation of the branches in the spanning trees of $\mathcal{K}_n$.

Example 2.16 In a complete geometric digraph $\mathcal{K}_n$ with points indexed $[4] = \{1, 2, 3, 4\}$, then $x_{[3]}^{\{1\}} = x_{\{1,2\}} - x_{\{1,3\}}$.

§3. Computation of the Relations in $\mathcal{K}_n$

In this section we give the procedure for computing the relations in a complete geometric digraph $\mathcal{K}_n$ for $n \leq 6$. Given a complete geometric digraph $\mathcal{K}_n$, the order is n and size is $\frac{rn}{2}$, where r is the rank of $\mathcal{K}_n$. The relations in $\mathcal{K}_n$ is defined by its complete subgraphs, that is, the cycle, C_3 and the spanning trees in the complete subgraphs, $K_n(n \geq 4)$ of $\mathcal{K}_n$. Since $\mathcal{K}_1$ and $\mathcal{K}_2$ has no cycles, they have no relation. So $3 \leq n \leq 6$, given $\mathcal{K}_n$ and $\Lambda_{\sigma,\tau} \in \nabla$ as follows:

(1) For $|\sigma| = 2$ and $|\tau| = 1$, any cycle C_3 in $\mathcal{K}_n$ contains three binary spanning trees and each has exactly one chord. These chords are the paths in C_3 which are linearly related as defined by Theorem 2.8 and each chord in the relation is multiplied by the root of its tree.

(2) For $|\sigma| = 2$ and $|\tau| = 2$, any complete geometric subgraph K_4 of $\mathcal{K}_n$ contains four 3-valent spanning trees and each has three chords. The branches are linearly related as defined by Theorem 2.14 and each branch in the relation is multiplied by the chord not adjacent to it. Any of the four 3-valent spanning tree of a K_4 gives the same relation.

(3) For $|\sigma| = 3$ and $|\tau| = 1$, any complete geometric subgraph K_4 of $\mathcal{K}_n$ contains four 3-valent spanning trees and each has three chords which formed a triangle. These triangle are linearly related as defined by Theorem 2.8 and is multiplied by the root of its spanning tree.

(4) For $|\sigma| = 3$, $|\tau| = 2$ and any branch in any 3-valent spanning tree of K_4, there exist a C_3 formed by a chord and the other branches. The branches are linearly related as defined by Theorem 2.14 and each is multiplied by its corresponding C_3. This is repeated for each 3-valent spanning tree.

(5) For integers $n \geq 5$, consider all the complete geometric subgraphs, K_5 of $\mathcal{K}_n$. For any branch in any 4-valent spanning tree of K_5, there is exactly one C_3 formed by the chords not adjacent to the branch. Applying to Theorem 2.14 to the branches and multiplying each branch by these C_3, we realize the graph relation.

(6) For $|\sigma| = 3$ and $|\tau| = 3$, consider all the complete geometric subgraphs, K_5 of $\mathcal{K}_n$. There are six chords in any 4-valent spanning tree of K_5 with three pairs of non-adjacent chords. For any pair, we have two C_3 formed by the chords with the branches. Applying to Theorem 2.14 to the C_3 in each pair containing the branch highest leave (label-wise), we multiply each C_3 in the relation with it corresponding pair.

(7) For integers $n \geq 6$, consider all the complete geometric subgraphs, K_6 of $\mathcal{K}_n$. For any two branches in any 5-valent spanning tree of K_6, there is exactly one C_3(non-adjacent C_3) formed by the chords not adjacent to these branches. There are exactly five of such relations in any of the 5-valent spanning tree, applying to Theorem 2.14 to the C_3 formed by a chord with these branches and multiplying each by the non-adjacent C_3, we realize the graph relation.

(8) For $|\sigma| = 4$ and $|\tau| = 1$, consider all the complete geometric subgraphs, K_5 of $\mathcal{K}_n$. Taking root of each 4-valent spanning tree of K_5 to multiply the C_4 formed by the chords with the leaves. The C_4 in each 4-valent spanning tree are linearly related as defined by Theorem 2.8.

(9) For $|\sigma| = 4$ and $|\tau| = 2$, consider all the complete geometric subgraphs, K_5 of $\mathcal{K}_n$. For any branch in any 4-valent spanning tree of K_5, there is a C_4 formed by the chords with the leaves. These branches are linearly related as defined by Theorem 2.14 and each is multiplied

it corresponding C_4. This is repeated for each 4-valent spanning tree.

(10) For integers $n \geq 6$, consider all the complete geometric subgraphs, K_6 of $\mathcal{K}_n$. For any branch in any 5-valent spanning tree of K_5, there is exactly one C_4 formed by the chords not adjacent to the branch. Applying to Theorem 2.14 to the branches and multiplying each branch by this C_4, we realize the graph relation.

(11) For $|\sigma| = 4$ and $|\tau| = 3$, consider all the complete geometric subgraphs, K_5 of $\mathcal{K}_n$. For any branch in any 4-valent spanning tree of K_5, we have three C_3 containing that branch, a chord and one other branch. For each C_3, there is a C_4 containing that branch, two chords and one other branch These C_3 are linearly related as defined by Theorem 2.14 and each C_3 is multiplied by the corresponding C_4. This is repeated for each branch in all the 4-valent spanning tree.

(12) For integers $n \geq 6$, consider all the complete geometric subgraphs, K_6 of $\mathcal{K}_n$. For any branch in any 5-valent spanning tree of K_6, we have four C_3 containing that branch, a chord and one other branch. For each C_3, there is a C_4 formed by four chords in the leave of that branch and other three branches not in the C_3. These triangles are linearly related as defined by Theorem 2.14 and each C_3 is multiplied by the corresponding C_4. Also for each C_3, there is a C_4 formed by two branches and two chords in the leave of other branches not in the triangle. These C_3 are linearly related as defined by Theorem 2.14 and each C_3 is multiplied by the corresponding C_4. This is repeated for each branch in all the 5-valent spanning tree.

We also consider all the complete geometric subgraphs, K_7 of $\mathcal{K}_n$. For any branch in any 6-valent spanning tree of K_7, we have five C_3 containing that branch, a chord and one other branch. For each C_3, there is a C_4 formed by four chords not adjacent to any of the branches in the C_3. These C_3 are linearly related as defined by Theorem 2.14 and each C_3 is multiplied by the corresponding C_4. This is repeated for each branch in all the 6-valent spanning tree.

Example 3.1 For $n = 3$, the relation is define by the points and lines of the graph in Figure 1. The relation is derived as follows:

Since $\mathcal{K}_3$ is a C_3, then it contains three binary spanning trees and each has exactly one chord. These chords are the paths in C_3 which are linearly related as defined by Theorem 2.8.

$$x_{\{1,3\}} = x_{\{1,2\}} + x_{\{2,3\}}$$

each chord in the relation is multiplied by the root of its tree, we have

$$x_{\{1,2\}}x_{\{3\}} - x_{\{1,3\}}x_{\{2\}} + x_{\{2,3\}}x_{\{1\}} = 0$$

The equations above give the relation for $\mathcal{K}_3$.

Example 3.2 For $n = 4$, the relations are define by the points, lines and triangles of the graph in Figure 2. The set of points, F_1 is $\{x_{\{1\}}, x_{\{2\}}, x_{\{3\}}, x_{\{4\}}\}$, the set of lines, F_2 is $\{x_{\{1,2\}}, x_{\{1,3\}}, x_{\{1,4\}}, x_{\{2,3\}}, x_{\{2,4\}}, x_{\{3,4\}}\}$ and the set of C_3, F_3 is

$$\{x_{\{1,2,3\}}, x_{\{1,2,4\}}, x_{\{1,3,4\}}, x_{\{2,3,4\}}\}.$$

The relations are given below

$$
\begin{aligned}
&x_{\{1,2\}}x_{\{3\}} - x_{\{1,3\}}x_{\{2\}} + x_{\{2,3\}}x_{\{1\}} = 0,\\
&x_{\{1,2\}}x_{\{4\}} - x_{\{1,4\}}x_{\{2\}} + x_{\{2,4\}}x_{\{1\}} = 0,\\
&x_{\{1,3\}}x_{\{4\}} - x_{\{1,4\}}x_{\{3\}} + x_{\{3,4\}}x_{\{1\}} = 0,\\
&x_{\{2,3\}}x_{\{4\}} - x_{\{2,4\}}x_{\{3\}} + x_{\{3,4\}}x_{\{2\}} = 0,\\
&x_{\{2,3\}}x_{\{1,4\}} - x_{\{2,4\}}x_{\{1,3\}} + x_{\{3,4\}}x_{\{1,2\}} = 0,\\
&x_{\{2,3,4\}}x_{\{1\}} - x_{\{1,3,4\}}x_{\{2\}} + x_{\{1,2,4\}}x_{\{3\}} - x_{\{1,2,3\}}x_{\{4\}} = 0,\\
&x_{\{1,3,4\}}x_{\{1,2\}} - x_{\{1,2,4\}}x_{\{1,3\}} + x_{\{1,2,3\}}x_{\{1,4\}} = 0,\\
&x_{\{2,3,4\}}x_{\{1,2\}} - x_{\{1,3,4\}}x_{\{2,3\}} + x_{\{1,2,3\}}x_{\{2,4\}} = 0,\\
&x_{\{2,3,4\}}x_{\{1,3\}} - x_{\{1,3,4\}}x_{\{2,3\}} + x_{\{1,2,3\}}x_{\{3,4\}} = 0,\\
&x_{\{2,3,4\}}x_{\{1,4\}} - x_{\{1,3,4\}}x_{\{2,4\}} + x_{\{1,2,4\}}x_{\{3,4\}} = 0.
\end{aligned}
$$

§4. Classifications of the Equations Defining Flag Varieties

In this section, we give the classifications of the relations in a complete geometric graphs.

Theorem 4.1 *Given $\Lambda_{\sigma,\tau} \in \nabla$, if $\sigma \cap \tau \neq \emptyset$, then σ and τ have at most $n-3$ points of intersection and* $3 \leq |\sigma| + |\tau| \leq 2n-3$.

Proof Given any relation in $\Lambda_{\sigma,\tau}$ such $\alpha, \tau \subset [n]$ then $1 \leq |\sigma| \leq n-2$ and $2 \leq |\tau| \leq n-1$. If $\alpha \cap \tau \neq \emptyset$ and $\tau \nsubseteq \sigma$, then there is at least one point in σ not in τ. Therefore $n-3$ possible points of intersection. It also follows from the bound on $|\sigma|$ and $|\tau|$ that $3 \leq |\sigma| + |\tau| \leq 2n-3$. □

The number of terms in any relation in $\mathcal{K}_n$ is bounded below by the size of C_3 and above by n, which is capture in Theorem 4.2 following.

Theorem 4.2 *In a complete geometric digraph $\mathcal{K}_n$, there are at least three terms and at most n terms in any relations.*

Proof This follows from Theorems 2.8 and 2.14. □

$\mathcal{K}_3$ has one relation which contain three terms, $\mathcal{K}_4$ has ten relations out of which nine relations have three terms each and one relation has four terms and $\mathcal{K}_5$ has sixty-six relations out of which forty-five relations have three terms each, fifteen relation have four terms each and one relation has five terms.

Theorem 4.3 *In a complete geometric digraph $\mathcal{K}_n$, if the elements of F_m form a relation then*

$$f_m \geq \binom{n}{n-2}.$$

Proof Given a complete geometric digraph, $\mathcal{K}_n$. Suppose the elements of F_m form relations then by Theorem 4.1, $m \leq n-2$, which implies that

$$f_m \geq \binom{n}{n-2}. \qquad \square$$

Consider the complete geometric digraph, $\mathcal{K}_3$, it contains no relations between lines and lines since $f_2 = 3$ but $\mathcal{K}_4$ contains one relation between lines and lines since $f_2 = 6$.

Suppose we wish to classify the relations in $\mathcal{K}_n$ as points and lines relations, lines and lines relations, points and C_3 relations, lines and C_3 relations, and so on. For any $\mathcal{K}_n$, the number of relations in any of such classification is given by the following theorem.

Theorem 4.4 *In a complete geometric digraph $\mathcal{K}_n$, for any $E_{\sigma,\tau} \subset \nabla$, the cardinality of $E_{\sigma,\tau}$ ($E_{i,j} = \#E_{\sigma,\tau}$) is given by*

$$E_{i,j} = \begin{cases} \binom{n}{i-1}\binom{n}{j+1}, & \textit{if } i < j \\ \\ \binom{n}{i-2}\binom{n}{j+2}, & \textit{if } i = j \end{cases}$$

for $|\sigma| = i$ and $|\tau| = j$.

Proof Given $E_{\sigma,\tau} \subset \nabla$ in $\mathcal{K}_n$ such that $\sigma = \{\sigma_1, \cdots, \sigma_i\}$ and $\tau = \{\tau_1, \cdots, \tau_i\}$ for $\sigma, \tau \subseteq [n]$ and $\sigma \nsubseteq \tau$. Let $E_{i,j} = \#E_{\sigma,\tau}$, there exist two cases for $E_{i,j}$.

Case 1. If $i < j$, then either $\sigma \cap \tau \neq \emptyset$ or $\sigma \cap \tau = \emptyset$. Suppose $\sigma \cap \tau \neq \emptyset$, then $i + j \geq n$. Since $\sigma \nsubseteq \tau$, then there is a distinct element in σ not in τ. This element is moved to τ, thereby increasing $|\tau|$ by 1 and reducing $|\sigma|$ by 1. Then the choice of selection of $\sigma\tau$ is $\binom{n}{i-1}\binom{n}{j+1}$. But if $\sigma \cap \tau = \emptyset$, then $i + j \leq n$. So a distinct element of σ is moved to τ, thereby increasing $|\tau|$ by 1 and reducing $|\sigma|$ by 1. Hence the choice of selection of $\sigma\tau$ is $\binom{n}{i-1}\binom{n}{j+1}$.

Case 2. If $i = j$, then either $\sigma \cap \tau \neq \emptyset$ or $\sigma \cap \tau = \emptyset$. Suppose $\sigma \cap \tau \neq \emptyset$, then $i + j \geq n$. Since $\sigma \nsubseteq \tau$, then there are two distinct elements in σ not in τ. These elements are moved to τ, thereby increasing $|\tau|$ by 2 and reducing $|\sigma|$ by 2. Then the choice of selection of $\sigma\tau$ is $\binom{n}{i-2}\binom{n}{j+2}$. But if $\sigma \cap \tau = \emptyset$, then $i + j \leq n$. So, the two distinct elements of σ are moved to τ, thereby increasing $|\tau|$ by 2 and reducing $|\sigma|$ by 2. Hence the choice of selection of $\sigma\tau$ is $\binom{n}{i-2}\binom{n}{j+2}$. This completes the proof. $\square$

Example 4.5 Consider relations of $\mathcal{K}_4$, $E_{\{1,2\}} = \binom{4}{0}\binom{4}{3} = 4$, $E_{\{2,2\}} = \binom{4}{0}\binom{4}{4} = 1$, $E_{\{1,3\}} = \binom{4}{0}\binom{4}{4} = 1$ and $E_{\{2,3\}} = \binom{4}{1}\binom{4}{4} = 4$.

Theorem 4.4 gives the number of relations in any class ($E_{\sigma,\tau}$). The following theorem gives a generating functions classifying the relations in $\mathcal{K}_n$.

Theorem 4.6 *In a complete geometric digraph $\mathcal{K}_n$, for any $E_{\sigma,\tau} \subset \nabla$ such that $|\sigma| = r$ and*

$|\tau| = m$, then the cardinality of $E_{\sigma,\tau}$ in ∇ for a fixed m is given by

$$\gamma_n^{\{m\}}(q) = \sum_{r=1}^{m-1} \binom{n}{m+1}\binom{n}{r-1} q^{(r,m)} + \binom{n}{m+2}\binom{n}{m-2} q^{(m,m)}$$

for $2 \leq m \leq n-1$, $n \geq 3$.

Proof Given a complete geometric digraph, $\mathcal{K}_n$. For $E_{\sigma,\tau} \subset \nabla$ such that $|\sigma| = r$ and $|\tau| = m$. Then, either $r < m$ or $r = m$ in ∇. So, from Theorem 4.4, by fixing m and $1 \leq r \leq m$ we can express the number of relation $q^{(r,m)}$ as a generating function $\gamma_n^{\{m\}}(q)$ for integers $2 \leq m \leq n-1$. □

Example 4.7 In $\mathcal{K}_3$, $n = 3$, $m = 2$, then we have

$$\gamma_3^{\{2\}}(q) = q^{(1,2)}.$$

In $\mathcal{K}_4$, $n = 4$, $m = 2, 3$, then we have

$$\begin{aligned} \gamma_4^{\{2\}}(q) &= 4q^{(1,2)} + q^{(2,2)}, \\ \gamma_4^{\{3\}}(q) &= q^{(1,3)} + 4q^{(2,3)}. \end{aligned}$$

In $\mathcal{K}_5$, $n = 5$, $m = 2, 3, 4$, then we have

$$\begin{aligned} \gamma_5^{\{2\}}(q) &= 10q^{(1,2)} + 5q^{(2,2)}, \\ \gamma_5^{\{3\}}(q) &= 5q^{(1,3)} + 25q^{(2,3)} + 5q^{(3,3)}, \\ \gamma_5^{\{4\}}(q), &= q^{(1,4)} + 5q^{(2,4)} + 10q^{(3,4)}. \end{aligned}$$

Total number of relations in $\mathcal{K}_n$, for $n = 3, 4$ and 5 are $1, 10$ and 66 respectively.

Theorem 4.8 *In a complete geometric digraph $\mathcal{K}_n$, for any $E_{\sigma,\tau} \subset \nabla$ such that $|\sigma| = i$ and $|\tau| = j$, then the cardinality of $E_{\sigma,\tau}$ in ∇ is given by*

$$M_n(q) = \sum_{j=2}^{n-1}\sum_{i=1}^{j-1} \binom{n}{i-1}\binom{n}{j+1} q^{(i,j)} + \sum_{r=2}^{n-2} \binom{n}{r-2}\binom{n}{r+2} q^{(r,r)}$$

for $n \geq 3$.

Proof Given a complete geometric digraph, $\mathcal{K}_n$, for any $E_{\sigma,\tau} \subset \nabla$ and $n \geq 3$. By Theorem 4.6, the sum over all possible $\gamma_n^{\{i\}}(q)$ equals $M_n(q)$ for $n \geq 3$. □

Example 4.9 In $\mathcal{K}_3$, $n = 3$,

$$M_3(q) = q^{(1,2)}.$$

In $\mathcal{K}_4$, $n = 4$,

$$M_4(q) = 4q^{(1,2)} + q^{(1,3)} + 4q^{(2,3)} + q^{(2,2)}.$$

In $\mathcal{K}_5$, $n = 5$,

$$M_5(q) = 10q^{(1,2)} + 5q^{(1,3)} + 25q^{(2,3)} + q^{(1,4)} + 5q^{(2,4)} + 10q^{(3,4)} + 5q^{(2,2)} + 5q^{(3,3)}.$$

Theorem 4.10 *In a complete geometric digraph $\mathcal{K}_n$, the number of classes in $\mathcal{K}_n$ is two less than the size of $\mathcal{K}_n$ for $n \geq 3$.*

Proof Given $\mathcal{K}_n$, from Theorem 4.8 the number of terms in $M_n(q)$ gives the number of classes in $\mathcal{K}_n$. The number of terms is $\frac{n(n-1)}{2} - 2$ which is less than the size of $\mathcal{K}_n$. □

Remark 4.11 (i) The coefficient of $q^{(k,k)}$ equals $q^{(k-1,k+1)}$ for $k \geq 2$. Also $q^{(i+r,i+m)}$ and $q^{(r,m)}$ have equal coefficient for $m + r < n$ and $1 \leq i \leq n - 3$.

(ii) The number of equations defining flag varieties $\mathcal{F}\ell_n(\mathbb{C})$ is given by

$$M_n = \sum_{j=2}^{n-1}\sum_{i=1}^{j-1}\binom{n}{i-1}\binom{n}{j+1} + \sum_{r=2}^{n-2}\binom{n}{r-2}\binom{n}{r+2}$$

with values for small number n in Table 2.

Order(n)	**Size**	**Number of relations(M_n)**	**Number of Classes**
3	3	1	1
4	6	10	4
5	10	66	8
6	15	365	13
7	21	1835	19
8	28	8705	26
9	36	39748	34
10	45	176740	43
11	55	770914	53
12	66	3314601	64
13	78	14094822	76
14	91	59418623	89
15	105	248756927	103
16	120	1035577973	118
17	136	4291186292	134
18	153	17713099208	151
19	171	72878464142	169
20	190	299021980928	188

Table 2. Statistics of a complete geometric digraph

References

[1] Miller Ezra and Sturmfels Bernd, *Combinatorial Commutative Algebra*, Springer-Verlag, New York, 2005.

[2] Duque Frank and Fabila-Monroy Ruy, Non-crossing monotone paths and binary trees in edge-ordered complete geometric graphs, *arXiv*, math.CO 1703.05378v2, 2017.

[3] Li Jianxi, Chee Shiu Wai and An Chang, The number of spanning trees in a graph, *Applied Mathematics Letter*, 23 (2010), 286–290.

[4] Z. Abu-Sbeih Moh'd, On the number of spanning trees of K_n and $K_{m,n}$, *Discrete Mathematics*, 84 (1990), 205–207.

[5] Aichholzer Oswin and Cabello Sergio, Edge-removal and non-crossing configurations in geometric graphs, *Discrete Mathematics and Theoretical Computer Science*, 12 (2010), No. 1, 75–86.

[6] Aichholzer Oswin and Hackl Thomas, Packing plane spanning trees and paths in complete geometric graphs, *arXiv*, cs.CG 1707.05440v1, 2017.

[7] A. Rado P, A note on paths in complete directed graphs, *Bulletin on the London Mathematical Society*, 2 (1970), No. 1, 66–68.

[8] Bosem Prosenjit, Hurtado Ferran, Rivera-Campo Eduardo and R.Wood David, Partitions of complete geometric graphs into plane trees, *Computational Geometry*, 34 (2006), 116–124.

[9] Fulton William, Young tableaux: with application to representation theory and geometry, *London Mathematical Society Student Texts* 35, Cambridge University Press, Cambridge, 1997.

[10] Lu Xiaoyun, Wang Da-Wei, Pan Jiaofeng and C. K. Wong, Rooted spanning tree in tournaments, *Graphs and Combinatorics*, 16 (2000), 411–427.

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Further Results on Super (a, d) Edge-Antimagic Graceful Labeling of Graphs

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Abstract: An (a, d)-edge-antimagic graceful labeling is a bijection g from $V(G) \cup E(G)$ into $\{1, 2, \cdots, |V(G)| + |E(G)|\}$ such that for each edge $xy \in E(G)$, $|g(x) + g(y) - g(xy)|$ form an arithmetic progression starting from a and having a common difference d. An (a, d)-edge-antimagic graceful labeling is called super (a, d)-edge-antimagic graceful if $g(V(G)) = \{1, 2, \cdots, |V(G)|\}$. A graph that admits an super (a, d)-edge-antimagic graceful labeling is called a super (a, d)-edge-antimagic graceful graph. In this paper, we prove the super (a, d) edge antimagic gracefulness of regular graphs. Later, we study the non-regular graph is super $(a, 1)$-edge-antimagic graceful graph. Finally, we find super edge-antimagic graceful labeling of some classes of graphs.

Key Words: Labelling, (a, d)-edge-antimagic total labeling, Smarandachely edge-antimagic total labeling, (a, d)-edge-antimagic graceful labeling, super(a, d)-edge-antimagic graceful labeling.

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§1. Introduction

Throughout this paper, we only concern with connected, undirected simple graphs of order p and size q. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively.

Let $|V(G)| = p$ and $|E(G)| = q$ be the number of vertices and the number of edges of G, respectively. General references for graph-theoretic notions are [1,10].

A labeling of a graph is any map that carries some set of graph elements to numbers. Hartsfield and Ringel [4] introduced the concept of an antimagic labeling and they defined an antimagic labeling of a (p, q) graph G as a bijection f from $E(G)$ to the set $\{1, 2, \cdots, q\}$ such that the sums of label of the edges incident with each vertex $v \in V(G)$ are distinct.

An (a, d)-edge-antimagic total labeling was introduced by Simanjuntak, Bertault and Miller in [9]. This labeling is the extension of the notions of edge-magic labeling, see [5,6].

For a graph $G = (V, E)$, a bijection g from $V(G) \cup E(G)$ into $\{1, 2, \cdots, |V(G)| + |E(G)|\}$ is called an (a, d)-edge-antimagic total labeling of G if the edge-weights $w(xy) = g(x) + g(y) + g(xy), xy \in E(G)$, form an arithmetic progression starting from a and having a common differ-

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ence d. Generally, let $H \prec G$ be a typical subgraph of G with $|V(G-H)| = a',\ |E(G-H)| = b'$. If there is an (a', d')-edge-antimagic total labeling g' on $G - H$, such a labeling g' is called a Smarandachely edge-antimagic total labeling. Particularly, let $H = \emptyset$ or a typical graph in K_2, P_3, C_3 or $S_{1,3}$. We get the (a, d)-edge-antimagic total labeling or nearly (a, d)-edge-antimagic total labeling of G.

The $(a, 0)$-edge-antimagic total labelings are usually called edge-magic in the literature. An (a, d)-edge antimagic total labeling is called super if the smallest possible labels appear on the vertices.

In [7] Marimuthu et al. introduced an edge magic graceful labeling of a graph. They presented some properties of super edge magic graceful graphs and proved some classes of graphs are super edge magic graceful. In [8] Marimuthu and Krishnaveni introduced super edge antimagic graceful labeling.

An (a, d)-edge-antimagic graceful labeling is defined as a one-to-one mapping from $V(G) \cup E(G)$ into the set $\{1, 2, 3, \cdots, p + q\}$ so that the set of edge-weights of all edges in G is equal to $\{a, a + d, a + 2d, \cdots, a + (q - 1)d\}$, for two integers $a \geq 0$ and $d > 0$.

An (a, d)-edge-antimagic graceful labeling g is called super (a, d)-edge-antimagic graceful if $g(V(G)) = \{1, 2, \cdots, p\}$ and $g(E(G)) = \{p+1, p+2, \cdots, p+q\}$. A graph G is called (a, d)-edge-antimagic graceful or super (a, d)-edge-antimagic graceful if there exists an (a, d)-edge-antimagic graceful or a super (a, d)-edge-antimagic graceful labeling of G.

Baca et al. [2] proved super $(a, 1)$-edge-antimagic total labeling of regular graphs. In [3] Baca et.al proved some classes of graphs like Frienship graphs, Fan graphs and Wheel graphs has super edge-antimagic graceful labeling. In this paper, we study super (a, d)-edge-antimagic graceful labeling of regular graphs. We also prove some classes of graphs, including friendship graphs, cycles and fan graphs has super (a, d)-edge-antimagic graceful labeling.

§2. Main Results

Theorem 2.1 *If G is a connected super (a, d)-edge-antimagic graceful graph, then $d \leq 2$.*

Proof Let G be a connected super (a, d)-edge-antimagic graceful graph. Suppose that $d \geq 3$. There exists a bijection $g : V(G) \cup E(G) \to \{1, 2, \cdots, p+q\}$ which is a super (a, d)-edge-antimagic graceful labeling with the set of edge-weights.

$$\begin{aligned} W &= \{w(xy) : w(xy) = |g(x) + g(y) - g(xy)|, xy \in E(G)\} \\ &= \{a, a + d, a + 2d, \cdots, a + (q - 1)d\}. \end{aligned}$$

It is easy to see that the minimum possible edge-weight in a super (a, d)-edge-antimagic graceful labeling is at least $|1 + p - (p + 1)| = 0$.

We observe that $a \geq 0$. On the other hand, the maximum edge-weight is no more than $|1+2-(p+q)| = p+q-3$. Therefore, $a+(q-1)d \leq p+q-3$. This shows that $(q-1)d \leq p+q-3$.

Hence,

$$
\begin{aligned}
& d \leq \frac{p+q-3}{q-1} \Rightarrow 3 \leq d \leq \frac{p+q-3}{q-1} \\
& \Rightarrow 3 \leq \frac{p+q-3}{q-1} \Rightarrow 3 \leq \frac{p-2}{q-1}+1 \Rightarrow 2 \leq \frac{p-2}{q-1} \\
& \Rightarrow 2 \leq \frac{p-2}{p-1-1} \text{ (since the size of every connected graph of order } p \text{ is at least } p-1) \\
& \Rightarrow 2 \leq 1,
\end{aligned}
$$

a contradiction. Hence, $d \leq 2$. □

Theorem 2.2 *Let G be a connected $(p, q)-$graph which is not a tree. If G has a super (a, d)-edge-antimagic graceful labeling then $d = 1$.*

Proof Assume that G has a super (a, d)-edge-antimagic graceful labeling $f : V(G) \cup E(G) \longrightarrow \{1, 2, \cdots, p + q\}$ and $\{w(uv) : uv \in E(G)\} = \{a, a + d, a + 2d, \cdots, a + (q - 1)d\}$ is the set of edge-weights. The minimum possible edge-weight $a \geq 0$. The maximum edge-weight is no more than $p + q - 3$. Thus $a + (q - 1)d \leq p + q - 3$. and

$$d \leq \frac{p+q-3}{q-1}. \tag{2.1}$$

But, $p \leq q$ (Since G is not a tree T). Then, (2.1) gives $d < 2$. □

§3. Super (a, d)-Edge-Antimagic Graceful Labeling of Regular Graphs

Proposition 3.1(Petersen theorem) *Let G be a $2r-$regular graph. Then there exists a $2-factor$ in G.*

Notice that after removing edges of the $2 - factor$ guaranteed by the Petersen theorem we have again an even regular graph. Thus, by induction, an even regular graph has a $2-$factorization.

The construction in the following theorem allows us to find a super $(a, 1)-$ edge-antimagic graceful labeling of any even regular graph. Notice that the construction does not require the graph to be connected. In the following theorem we denote $[a, b]$ is the set of consecutive integers $\{a, a + 1, \cdots, b\}$.

Theorem 3.2 *Let G be a graph on p vertices that can be decomposed into two factors G_1 and G_2. If G_1 is edge-empty or if G_1 is a super (0,1)-edge-antimagic graceful graph and G_2 is a 2r-regular graph then G is super (0,1)-edge-antimagic graceful.*

Proof First we start with the case when G_1 is not edge-empty. Since G_1 is a super (0,1)-edge-antimagic graceful graph with p vertices and q edges, there exists a total labeling $f : V(G_1) \cup E(G_1) \longrightarrow [1, p+q]$ such that $\{\left|f(x)+f(y)-f(xy)\right| : xy \in E(G)\} = \{0, 1, 2 \cdots, q-1\}$.

By the Petersen theorem there exists a $2-$factorization of G_2. We denote the $2-$factors by F_j, $j = 1, 2, \cdots, r$. Let $V(G) = V(G_1) = V(F_j)$ for all j and $E(G)= U_{j=1}^{r} E(F_j) \bigcup E(G_1)$. Each factor F_j is a collection of cycles. We order and orient the cycles arbitrarily. Now by the symbol $e_j^{out}(v_i)$ we denote the unique outgoing arc from the vertex v_i in the factor F_j. We define a total labeling g of G in the way that $g(v) = f(v)$ for $v \in V(G)$, $g(e) = f(e)$ for $e \in E(G_1)$ and $g(e) = q + jp + f(v_i)$ for $e = e_j^{out}(v_i)$. Then, the vertices are labeled by the first p integers. The edges of G_1 by the next q labels and the edges of G_2 by consecutive integers starting at $p+q+1$. Thus g is a bijection $V(G) \cup E(G) \longrightarrow \{1, 2 \cdots, p + q + pr\}$ Since $|E(G)|= q + pr$. It is not difficult to verify that g is a super $(0, 1)$-edge-antimagic graceful labeling of G. The weights of the edges e in $E(G_1)$ is $w_g(e)= w_f(e)$. The weights form the progression $0, 1, 2, \cdots, q - 1$. For convenience, we denote by v_k the unique vertex such that $v_i v_k = e_j^{out}(v_i)$ in F_j. The weights of the edges in F_j, $j = 1, 2, \cdots, r$ are

$$\begin{aligned} w_g(e_j^{out}(v_i)) &= w_g(v_i v_k) = |g(v_i) - (q + jp + f(v_i)) + g(v_k)| \\ &= |f(v_i) - (q + jp + f(v_i)) + f(v_k)| = | - q - jp + f(v_k)| \\ &= | - (q + jp - f(v_k))| \end{aligned}$$

for all $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, r$. Since F_j is a factor, the set $\{f(v_k) : v_k \in F_j\} = [1, p]$. Hence we have that the set of edge-weights in the factor F_j is $[q + (j - 1)p, q + jp - 1]$ and thus the set of all edge-weights in G is $[0, q + rp - 1]$. If G_1 is edge-empty it is enough to take $q = 0$. and proceed with the labeling of factors F_j. □

By taking an edge-empty graph G_1 we have the following theorem.

Theorem 3.3 *All even regular graphs of order p with at least one edge are super* $(0, 1)$*-edge-antimagic graceful.*

The disjoint union of $m \geq 1$ copies of a graph G is denoted by mG.

Theorem 3.4 *Let* k, m *be positive integers. Then the graph* $kP_2 \cup mK_1$ *is super* $(0, 1)$*-edge-antimagic graceful.*

Proof We denote the vertices of the graph $G \cong kP_2 \cup mK_1$ by the symbols $v_1, v_2, \cdots, v_{2k+m}$ in such a way that $E(G)= \{v_i v_{k+m+i} : i = 1, 2, \cdots, k\}$ and the remaining vertices are denoted arbitrarily by the unused symbols. We define the labeling $f : V(G) \cup E(G) \longrightarrow \{1, 2, \cdots, 3k+m\}$ in the following way $f(v_j) = j$ for $j = 1, 2, \cdots, 2k + m$, $f(v_i v_{k+m+i}) = 2k + m + i$ for $i = 1, 2, \cdots, k$. It is easy to see that f is a bijection and that the vertices of G are labeled by the smallest possible numbers. For the edge-weights we get $w_f(v_i v_{k+m+i}) = |f(v_i) + f(v_{k+m+i}) - f(v_i v_{k+m+i})| =$ k-i for $i = 1, 2, \cdots, k$. Thus, f is a super $(0, 1)$-edge-antimagic graceful labeling of G. □

Now by taking $m = 0$ and observing that the number of vertices in kP_2 is $2k$, then we immediately obtain the following corollary.

Corollary 3.5 *If G is an odd regular graph on p vertices that has a* $1 - factor$ *then G is super*

$(0, 1)$*-edge-antimagic graceful.*

§4. Friendship Graphs

The friendship graph $\mathbf{F_n}$ is a set of n triangles having a common center vertex and otherwise disjoint. Let c denote the center vertex. For the i^{th} triangle, let x_i and y_i denote the other two vertices.

Theorem 4.1 *Every friendship graph* $\mathbf{F_n}, n \geq 1$, *has super* $(a, 1)$*-edge-antimagic graceful labeling.*

Proof Label the vertices and edges of $\mathbf{F_n}$ by the following functions g_1 and g_2 respectively.

$$g_1(c) = n + 1, \quad g_1(x_i) = i, \quad g_1(y_i) = 2n + 2 - i \text{ for } 1 \leq i \leq n,$$
$$g_2(x_i c) = 3n + 2i, \quad g_2(y_i c) = 5n + 3 - 2i, \quad g_2(x_i y_i) = 2n + 1 + i.$$

Notice that, in this labeling $a = 0$. It is easy to verify that the set of edge-weights consists of the consecutive integers $\{0, 1, 2, \cdots, 3n - 1\}$ and we arrive at the desired result. □

Figure 1 illustrates the proof of Theorem 4.1.

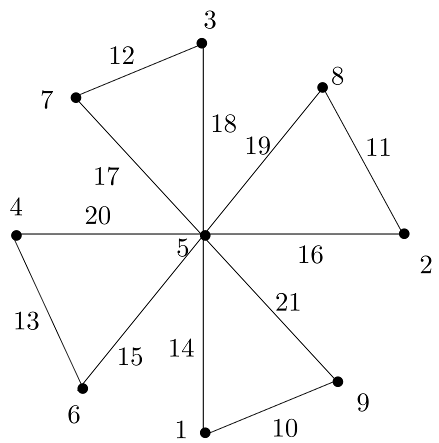

Figure 1. A $(0, 1)$-super edge-antimagic graceful labeling of $\mathbf{F_4}$.

§5. Cycles

Theorem 5.1 *For* $n \geq 3$, *the cycle* C_n *has super* $(a, 1)$*-edge-antimagic graceful labeling.*

Proof Let a cycle C_n be defined as follows:

$$V(C_n) = \{p_1, p_2, \cdots, p_n\} \text{ and}$$
$$E(C_n) = \{p_i p_{i+1} : i = 1, 2, \cdots, n - 1\} \cup \{p_n p_1\}.$$

Also, define the vertex labeling $f_1 : V(C_n) \to \{1, 2, \cdots, n\}$ and the edge labeling $f_2 : E(Cn) \to \{n+1, n+2, \cdots, n+n\}$ in the following way.

$$\begin{aligned} f_1(v_i) &= i, \quad 1 \le i \le n; \\ f_2(v_i v_{i+1}) &= n+1+i \text{ for } 1 \le i \le n-1; \\ f_2(v_n v_1) &= n+1. \end{aligned}$$

Combining the vertex labeling f_1 and the edge labeling f_2 given above, we obtain a total labeling. The set of edge-weights consists of the consecutive integers $\{0, 1, 2, \cdots, n-1\}$. $\square$

An illustration of Theorem 5.1 is given in Figure 2.

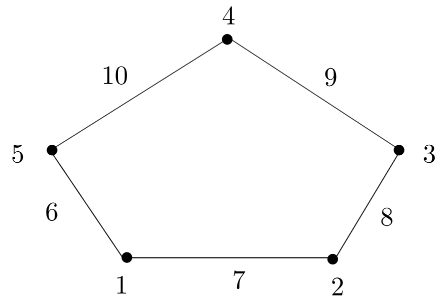

Figure 2. A $(0,1)$-super edge-antimagic graceful labeling of C_5.

§6. Fans

A fan $\mathcal{F}_n, n \ge 2$ is a graph obtained by joining all vertices of path P_n to a further vertex called the center. Thus $\mathcal{F}_n$ contains $n+1$ vertices, say, $c, x_1, x_2, \cdots, x_n$ and $2n-1$ edges say $cx_i, 1 \le i \le n$ and $x_i x_{i+1}, 1 \le i \le n-1$.

Theorem 6.1 *The fan $\mathcal{F}_n$ is super $(a,1)$-edge-antimagic graceful if $2 \le n \le 6$ and $d = 1$.*

Proof Label the vertices of $\mathcal{F}_n$ by $g : V(\mathcal{F}_n) \to \{1, 2, \cdots, n+1\}$ as follows:

If $n = 2$, let the labels of vertices be $g(x_1) = 1, g(x_2) = 2$ and $g(c) = 3$; If $n = 3$, let the labels be $g(x_1) = 1, g(x_2) = 2, g(x_3) = 4$ and $g(c) = 3$; If $n = 4$, let the labels be $g(x_1) = 1, g(x_2) = 2, g(x_3) = 4, g(x_4) = 5$ and $g(c) = 3$; If $n = 5$, let the labels be $g(x_1) = 2, g(x_2) = 1, g(x_3) = 3, g(x_4) = 5$, $g(x_5) = 6$ and $g(c) = 4$, and if $n = 6$, let the labels be $g(x_1) = 2, g(x_2) = 1, g(x_3) = 3, g(x_4) = 5$, $g(x_5) = 7, g(x_6) = 6$ and $g(c) = 4$. Generally, let $W_g = \{w_g(q_i) = 2 + i : 1 \le i \le 2n-1\}$ be the set of edge-weights of edges $q_i \in \mathcal{F}_n$ and label the edges of $\mathcal{F}_n$ by $g_1 : E(\mathcal{F}_n) \to \{n+2, n+3, \cdots, 3n\}$ where

$$g_1(q_i) = \begin{cases} n+1+\frac{i+1}{2} & \text{if } i \text{ is odd,} \\ 2n+1+\frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

Combining the vertex labeling g and the edge labeling g_1 gives a super $(a,1)$-edge-antimagic graceful labeling where

$$W = \{|w_g(q_i) - g_1(q_i)| : 1 \leq i \leq 2n - 1\}$$

is the set of edge-weights. $\square$

A $(0, 1)$-super edge-antimagic graceful labeling of the fan $\mathcal{F}_3$ is given in Figure 3.

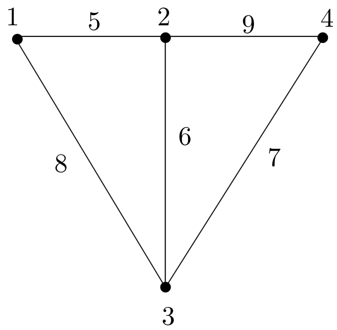

Figure 3. A $(0, 1)$-super edge-antimagic graceful labeling of $\mathcal{F}_3$.

References

[1] M. Baca and M. Miller, *Super Edge-Antimgic Graphs: A Wealth of Problems and Some Solutions*, Brown Walker Press, Boca Raton, Fla, USA, 2008.

[2] M. Baca, P. Kovar, A. Semanicova-Fenovcikova and M. K. Shafiq, On super (a,1)-edge-antimagic total labeling of regular graphs, *Discrete Math.*, 310 (2010), 1408-1412.

[3] M. Baca, Y. Lin, M. Miller and M. Z. Youssef, Edge-antimagic graphs, *Discrete Math.*, 307 (2007), 1232–1244.

[4] N. Hartsfield and G. Ringel, *Pearls in Graph Theory*, Academic Press, Boston, San Diego, New York, London, 1990.

[5] A. Kotzig and A.Rosa, *Magic Valuations of Complete Graphs*, CRM Publisher, 1972.

[6] A. Kotzig and A.Rosa, Magic valuations of finite graphs, *Canadian Mathematical Bulletin*, 13 (1970), 451–461.

[7] G. Marimuthu and M. Balakrishnan, Super edge magic Graceful labeling of Graphs, *Inf. Sci.*, 287 (2014), 140-151.

[8] G. Marimuthu and P.Krishnaveni, Super Edge-antimagic graceful graphs, *Malaya J.Mat.*, 3(3) (2015), 312-317.

[9] R. Simanjuntak, F. Bertault and M. Miller, Two new (a, d)-antimagic graph labelings, in: *Proc. of 11th Australian Workshop of Combinatorial Algorithm*, (2000), 179–189.

[10] D. B. West, *An Introduction to Graph Theory*, Prentice Hall, Engelwood Cliffs, NJ, 1996.

International J.Math. Combin. Vol.4(2023), 82-85

A Note on
Congruences for the Sum of Squares and Triangular Numbers

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Abstract: We derive some congruences for $r_k(n)$ and $t_k(n)$ using their generating functions, where $r_k(n)$ and $t_k(n)$ denote the number of representations of n as a sum of k squares and number of representations of n as a sum of k triangular numbers, respectively.

Key Words: Divisor sum, triangular numbers, sum of squares.

AMS(2010): 05C78.

§1. Introduction

The *sum of squares function*, denoted by $r_k(n)$, gives the number of representations of n as a sum of k squares, where zeros and distinguishing signs and order are allowed. For example, 5 can be written as a sum of two squares in the following ways

$$\begin{aligned} 5 &= (-2)^2 + (-1)^2 = (-2)^2 + (1)^2 \\ &= (2)^2 + (-1)^2 = (2)^2 + (1)^2 \\ &= (-1)^2 + (-2)^2 = (-1)^2 + (2)^2 \\ &= (1)^2 + (-2)^2 = (1)^2 + (2)^2. \end{aligned}$$

So, $r_2(5) = 8$.

The *generating function* for $r_k(n)$ is given by

$$\theta(q)^k = \sum_{n=0}^{\infty} (-1)^n \, r_k(n) \, q^n, \tag{1.1}$$

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where

$$\theta(q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \qquad (|q| < 1).$$

By Gauss's formula [1, formula 7.324], we know that

$$\theta(q) = \prod_{j=1}^{\infty} \frac{1-q^j}{1+q^j} = \prod_{n\geq 1}(1-q^{2n})(1-q^{2n-1})^2 \qquad (|q| < 1). \tag{1.2}$$

For any positive integer n, the numbers $n(n+1)/2$ are the *triangular numbers*. The *sum of triangular numbers function*, denoted by $t_k(n)$, gives the number of representations of n as a sum of r triangular numbers where representations with different orders are counted as unique. For instance, $t_2(7) = 2$ since $7 = 1 + 6 = 6 + 1$.

The generating function for $t_k(n)$ is given by

$$\Psi^k(q) = \sum_{n=0}^{\infty} t_k(n)\, q^n, \tag{1.3}$$

where

$$\Psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = 1 + q + q^3 + q^6 + \cdots \qquad (|q| < 1).$$

By Gauss's formula [1, Eq.7. 321 on p.6], we have

$$\Psi(q) = \prod_{j=1}^{\infty} \frac{(1-q^{2j})^2}{(1-q^j)} = \prod_{j=1}^{\infty}(1+q^j)^2\,(1-q^j) \qquad |q| < 1. \tag{1.4}$$

2. Some Congruences for $r_k(n)$ and $t_k(n)$

Lemma 2.1 *Let $S_1(n) = \sum_{odd\ d|n} \frac{2}{d}$, where $\sum_{odd\ d|n}$ denotes the sum over all odd divisors d of n. Then*

$$r_k(n) = \frac{-k}{n} \sum_{j=1}^{n} (-1)^j\, j\, S_1(j)\, r_k(n-j) \qquad (k, n \geq 1). \tag{2.1}$$

Proof Taking logarithm on both sides of equation (1.2), we have

$$\begin{aligned}
\log\theta(q) &= \sum_{j=1}^{\infty} \log(1-q^j) - \sum_{j=1}^{\infty} \log(1+q^j) \\
&= -\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} \frac{q^{lj}}{l} + \sum_{j'=1}^{\infty}\sum_{l'=1}^{\infty} \frac{q^{l'j'}(-1)^{l'}}{l'} \\
&= -\sum_{n=1}^{\infty} q^n \left(\sum_{d|n} \frac{1-(-1)^d}{d} \right).
\end{aligned}$$

From the equation (1.1), we get

$$\log\left\{\sum_{n=0}^{\infty}(-1)^n\, r_k(n)\, q^n\right\} = -k\sum_{n=1}^{\infty} S_1(n)\, q^n.$$

Differentiating the preceding equation with respect to q gives

$$\sum_{n=1}^{\infty}(-1)^n\, r_k(n)\, n\, q^{n-1} = -k\sum_{n=1}^{\infty} S_1(n)\, n\, q^{n-1}\sum_{n=0}^{\infty}(-1)^n\, r_k(n)\, q^n.$$

Comparing coefficients of q^n on both sides of the above equation we get equation (2.1). □

Lemma 2.2 *Let* $S_2(n) = \sum_{d|n}\frac{1+2\,(-1)^d}{d}$, *where* $\sum_{d|n}$ *denotes sum over all divisors* d *of* n. *Then*

$$t_k(n) = \frac{-k}{n}\sum_{j=1}^{n} j\, S_2(j)\, t_k(n-j) \qquad (k, n \geq 1). \tag{2.2}$$

Proof Taking logarithm on both sides of equation (1.4), we have

$$\begin{aligned}\log(\Psi(q)) &= \sum_{j=1}^{\infty} 2\,\log(1+q^j) + \sum_{j=1}^{\infty}\log(1-q^j)\\ &= -\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} 2\,\frac{(-1)^l\, q^{lj}}{l} - \sum_{j'=1}^{\infty}\sum_{l'=1}^{\infty}\frac{q^{l'j'}}{l'}\\ &= -\sum_{n=1}^{\infty} q^n \sum_{d|n}\frac{1+2\,(-1)^d}{d}.\end{aligned}$$

Then, we proceed as in the proof of the preceding lemma to arrive at equation (2.2). □

From equations (2.1) and (2.2), we deduce the following theorem.

Theorem 2.3 *Let* n *and* k *be integers such that* $(n, k) = 1$. *Then*

$$r_k(n) \equiv 0 \pmod{k}, \tag{2.3}$$

and

$$t_k(n) \equiv 0 \pmod{k}. \tag{2.4}$$

From equations (1.2) and (1.4), we deduce the following theorem.

Theorem 2.4 *For all primes* p, *we have*

$$r_{kp}(np) \equiv r_k(n) \pmod{p}, \tag{2.5}$$

and

$$t_{kp}(np) \equiv t_k(n) \pmod{p}. \tag{2.6}$$

Theorem 2.5 *If p is a prime number, then*

$$r_{p+1}(n) \equiv \sum_j r_1(t) \pmod{p}, \tag{2.7}$$

where j is an integer and $t = (n - j^2)/p$ is integer.

Proof Using

$$\theta(q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \qquad (|q| < 1),$$

we have

$$\sum_{n=0}^{\infty}(-1)^n \, r_{p+1}(n) \, q^n = \sum_{n=0}^{\infty}(-1)^n \, r_p(n) \, q^n \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}$$

Comparing coefficients of q^n on both sides in the above equation, we have

$$r_{p+1}(n) = \sum_j r_p(n - j^2).$$

Now from equation (2.3), we know that $r_p(n-j^2) \equiv 0 \pmod{p}$ if p and $n-j^2$ are co-prime. Also, from equation (2.5), when $(n - j^2)$ is divisible by p, we have

$$r_p(n - j^2) \equiv r_1(t) \pmod{p},$$

where $t = (n - j^2)/p$. □

Proceeding as in the proof of above theorem, we deduce the following theorem.

Theorem 2.6 *If p is a prime number, then*

$$t_{p+1}(n) \equiv \sum_j t_1(t) \pmod{p}, \tag{2.8}$$

where j is positive integer and $t = \left(n - \frac{j(j+1)}{2}\right)/p$ is integer.

References

[1] N. J. Fine, *Basic Hypergeometric Series and Applications*, American Mathematical Soc., 1988.

[2] J. M. Gandhi, Congruences for $p_r(n)$ and Ramanujan's τ-function, *Amer. Math. Monthly* 70, No. 3 (1963) 265–274.

International J.Math. Combin. Vol.4(2023), 86-94

Pair Difference Cordial Labeling of Double Cone, Double Step Grid, Double Arrow and Shell Related Graphs

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Abstract: In this paper we investigate the pair difference cordial labeling behaviour of double cone,double step grid,sun flower, shell graph and double arrow graph.

Key Words: Pair difference cordial labeling, double cone, double step grid, sun flower, shell graph, double arrow graph, Smarandachely pair difference cordial labeling, Smarandachely pair difference cordial labeling graph.

AMS(2010): 05C78.

§1. Introduction

In this paper we consider only finite, undirected and simple graphs. Cordial labeling was introduced by Cachit [1] in the year 1987. Also cordial related labeling technique was studied in [1,2,3,4,5,6,7,8,9,10,11]. In this sequal the notion of pair difference cordial labeling of a graph was introduced in [14], which is defined as follows:

Let $G = (V, E)$ be a (p, q) graph and let

$$\rho = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm\rho\}$ be the set of labels. Consider a mapping $f : V \longrightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. Such a labeling is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1. A graph G for which there exists a pair difference cordial labeling is called a pair difference cordial graph.

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Generally, let $H \prec G$ be a typical subgraph of G. If there is a pair difference cordial labeling on graph $G - H$. Then, we say G is Smarandachely pair difference cordial labeling on H and G is called a Smarandachely pair difference cordial labeling graph on H. Particularly, if $H = \emptyset$ such a Smarandachely pair difference cordial labeling is nothing else but a pair difference cordial labeling on G.

The pair difference cordial labeling behavior of several graphs like path, cycle, star, wheel, triangular snake, alternate triangular snake, butterfly etc have been investigated in [14-24]. In this paper we investigate the pair difference cordial labeling behavior of double cone,double step grid,sun flower, shell graph and double arrow graph. Terms not defined here are follow from Gallian [12] and Harary [13].

§2. Preliminaries

Definition 2.1([13]) *The subdivision graph $S(G)$ of a graph G is obtained by replacing each edge uv by a path uvw.*

Definition 2.2([12]) *Take the tha paths $P_n, P_n, P_{n-1}, \cdots, P_2$ on $n, n, n-2, n-4, \cdots, 4, 2$ vertices and arrange them centrally horizontal where n is even and $n \neq 2$. A graph obtained by joining vertical vertices of given successive paths is known as a double step grid of size n. It is denoted by DSt_n.*

For illustration, DSt_n is shown in Figure 1.

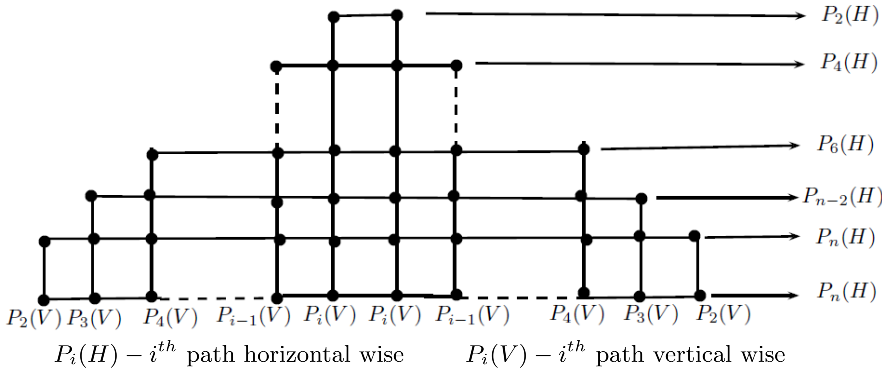

Figure 1

Definition 2.3([13]) *Double arrow graphs obtained from $P_n \times P_n$ by joinin two vertices u, v with first and last copy of the path P_n.Let $a_{i,j}$ be the vertices of prism $P_n \times P_n$.*

Definition 2.4([12]) *The graph $C_n + 2K_1$ is called the double cone graph.*

Definition 2.5([12]) *The sunflower graph SF_n is obtained by taking a wheel $W_n = C_n + K_1$ where C_n is the cycle $a_1a_2a_3 \cdots a_na_1$, $V(K_1) = \{a\}$ and the new vertices $b_1, b_2, b_3, \cdots, b_n$ where b_i is join by the vertices $b_ib_{i+1}(modn)$.*

Definition 2.6([12]) *A shell graph is defined as a cycle $C_n : a_1a_2a_3 \cdots a_na_1$ with $(n-3)$ chords* *sharing a common end point called the apex. Shell graph are denoted as $C_{(n,n-3)}$. A shell S_n is also called fan F_{n-1}.*

Definition 2.7([25]) *An ice cream graph is obtained by combining a shell graph and a path P_2 graaph keeping a_1 and a_n common where $n > 3$ sharing common end point called the apex vertex a_0. It is denoted by IC_n.*

§3. Main Results

Theorem 3.1 *A double step grid DSt_n is pair difference cordial for all even values of $n \geq 4$.*

Proof First we consider the paths $P_2, P_3, P_4, \cdots, P_{\frac{n}{2}}$ from left to right. Assign the labels $1, 2$ respectively to the vertices of the path P_2 from top to bottom and assign the labels $3, 4, 5$ respectively to the vertices of the path P_3 from bottom to top. Now assign the labels $6, 7, 8, 9$ to the vertices of the path P_4 from top to bottom and assign the labels $10, 11, 12, 13, 14$ to the vertices of the path P_5 from bottom to top. Proceeding like this until we reach the path $P_{\frac{n}{2}}$.

Next, we consider the paths $P_2, P_3, P_4, \cdots, P_{\frac{n}{2}}$ from right to left. Assign the labels $-1, -2$ respectively to the vertices of the path P_2 from top to bottom and assign the label-s $-3, -4, -5$ respectively to the vertices of the path P_3 from bottom to top. Now assign the labels $-6, -7, -8, -9$ to the vertices of the path P_4 from top to bottom and assign the labels $-10, -11, -12, -13, -14$ to the vertices of the path P_5 from bottom to top. Proceeding like this until we reach the path $P_{\frac{n}{2}}$.

For illustration, DSt_8 is shown in Figure 2.

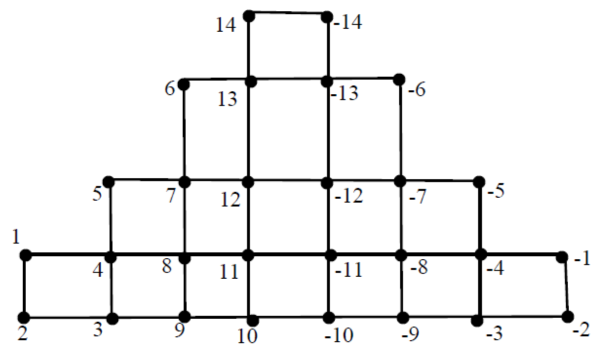

Figure 2

This completes the proof. □

Theorem 3.2 *A double cone graph DC_n is not pair difference cordial for all values of $n \geq 3$.*

Proof There are two cases arises.

Case 1. n is even.

The maximum possible number of edges get the label 1 is $\Delta f_1 = \frac{n}{2} + 2 + 2$ where $\frac{n}{2}$ edges from cycle, 2 edges from edges end with K_1 and next 2 edges from edges end with another K_1.

Therefore $\Delta f_1 = \frac{n}{2} + 4$ and $|E(DC_n)| = 3n$. This implies that $\Delta {f_1}^c = 3n - \frac{n}{2} + 4 = \frac{5n+8}{2}$. Hence $|\Delta f_1 - \Delta {f_1}^c| = 2n > 1$, which is a contradiction.

Case 2. n is odd.

The maximum possible number of edges get the label 1 is $\Delta f_1 = \frac{n+1}{2} + 2 + 2$ where $\frac{n+1}{2}$ edges from cycle, 2 edges from edges end with K_1 and next 2 edges from edges end with another K_1. Therefore $\Delta f_1 = \frac{n+1}{2} + 4$ and $|E(DC_n)| = 3n$. This gives that $\Delta {f_1}^c = 3n - \frac{n+1}{2} + 4 = \frac{5n+9}{2}$. Hence $|\Delta f_1 - \Delta {f_1}^c| = 2n > 1$, which is a contradiction.

Hence, a double cone graph DC_n is not pair difference cordial for all values of $n \geq 3$. □

Theorem 3.3 *A double arrow graph DA_n is pair difference cordial for all values of $n \geq 2$.*

Proof Take the vertex set and edge set from Definition 2.3. There are two cases arises.

Case 1. n is even.

Consider the $\frac{n}{2}^{th}$ row. That is consider the vertices $a_{\frac{n}{2},1}$, $a_{\frac{n}{2},2}$, $a_{\frac{n}{2},3}, \cdots, a_{\frac{n}{2},n}$. Assign the labels $1, 2, 3, \cdots, n$ to the vertices $a_{\frac{n}{2},1}$, $a_{\frac{n}{2},2}$, $a_{\frac{n}{2},3}, \cdots, a_{\frac{n}{2},n}$ respectively and next consider the $\frac{n-2}{2}^{th}$ row. Assign the labels $n+1, n+2, n+3, \cdots, 2n$ to the vertices $a_{\frac{n-2}{2},1}$, $a_{\frac{n-2}{2},2}$, $a_{\frac{n-2}{2},3}, \cdots, a_{\frac{n-2}{2},n}$. Now assign the labels $2n+1, 2n+2, 2n+3, \cdots, 3n$ to the vertices $a_{\frac{n-4}{2},1}$, $a_{\frac{n-4}{2},2}$, $a_{\frac{n-4}{2},3}, \cdots, a_{\frac{n-4}{2},n}$ respectively. Proceeding like this until we reach the first row. Note that the vertices $a_{1,1}$, $a_{1,2}$, $a_{1,3}, \cdots, a_{1,n}$ gets the labels $\frac{n}{2}+1, \frac{n}{2}+2, \frac{n}{2}+3, \cdots, \frac{n}{2}+n$.

Consider the $\frac{n+2}{2}^{th}$ row. That is consider the vertices $a_{\frac{n+2}{2},1}$, $a_{\frac{n+2}{2},2}$, $a_{\frac{n+2}{2},3}, \cdots, a_{\frac{n+2}{2},n}$. Assign the labels $-1, -2, -3, \cdots, -n$ to the vertices $a_{\frac{n+2}{2},1}$, $a_{\frac{n+2}{2},2}$, $a_{\frac{n+2}{2},3}, \cdots, a_{\frac{n+2}{2},n}$ respectively and next consider the $\frac{n+4}{2}^{th}$ row. Assign the labels $-(n+1), -(n+2), -(n+3), \cdots, -2n$ to the vertices $a_{\frac{n+4}{2},1}$, $a_{\frac{n+4}{2},2}$, $a_{\frac{n+4}{2},3}, \cdots, a_{\frac{n+4}{2},n}$. Now assign the labels $-(2n+1), -(2n+2), -(2n+3), \cdots, -3n$ to the vertices $a_{\frac{n+6}{2},1}$, $a_{\frac{n+6}{2},2}$, $a_{\frac{n+6}{2},3}, \cdots, a_{\frac{n+6}{2},n}$ respectively. Proceeding like this until we reach the n^{th} row. Note that the vertices $a_{n,1}$, $a_{n,2}$, $a_{n,3}, \cdots, a_{n,n}$ gets the labels

$$-(\frac{n}{2}+1), -(\frac{n}{2}+2), -(\frac{n}{2}+3), \cdots, -(\frac{n}{2}+n),$$

and finally assign the labels $\frac{n}{2}+n$,$-(\frac{n}{2}+n)$ to the vertices u, v respectively.

Case 2. n is odd.

Consider the $\frac{n-1}{2}^{th}$ row. That is consider the vertices $a_{\frac{n-1}{2},1}$, $a_{\frac{n-1}{2},2}$, $a_{\frac{n-1}{2},3}, \cdots, a_{\frac{n-1}{2},n}$. Assign the labels $1, 2, 3, \cdots, n$ to the vertices $a_{\frac{n-1}{2},1}$, $a_{\frac{n-1}{2},2}$, $a_{\frac{n-1}{2},3}, \cdots, a_{\frac{n-1}{2},n}$ respectively and next consider the $\frac{n-3}{2}^{th}$ row. Assign the labels $n+1, n+2, n+3, \cdots, 2n$ to the vertices $a_{\frac{n-3}{2},1}$, $a_{\frac{n-3}{2},2}$, $a_{\frac{n-3}{2},3}, \cdots, a_{\frac{n-3}{2},n}$. Now assign the labels $2n+1, 2n+2, 2n+3, \cdots, 3n$ to the vertices $a_{\frac{n-5}{2},1}$, $a_{\frac{n-5}{2},2}$, $a_{\frac{n-5}{2},3}, \cdots, a_{\frac{n-5}{2},n}$ respectively. Proceeding like this until we reach the first row. Note that the vertices $a_{1,1}$, $a_{1,2}$, $a_{1,3}, \cdots, a_{1,n}$ gets the labels

$$\frac{n}{2}+1, \frac{n}{2}+2, \frac{n}{2}+3, \cdots, \frac{n}{2}+n.$$

Consider the $\frac{n+1}{2}^{th}$ row. That is consider the vertices $a_{\frac{n+1}{2},1}$, $a_{\frac{n+1}{2},2}$, $a_{\frac{n+1}{2},3}, \cdots, a_{\frac{n+1}{2},n}$. Assign the labels $-1, -2, -3, \cdots, -n$ to the vertices $a_{\frac{n+1}{2},1}$, $a_{\frac{n+1}{2},2}$, $a_{\frac{n+1}{2},3}, \cdots, a_{\frac{n+1}{2},n}$ respectively and next consider the $\frac{n+3}{2}^{th}$ row. Assign the labels $-(n+1), -(n+2), -(n+3), \cdots, -2n$ to the vertices $a_{\frac{n+3}{2},1}$, $a_{\frac{n+3}{2},2}$, $a_{\frac{n+3}{2},3}, \cdots, a_{\frac{n+3}{2},n}$. Now assign the labels $-(2n+1), -(2n+2), -(2n+3), \cdots, -3n$ to the vertices $a_{\frac{n+5}{2},1}$, $a_{\frac{n+5}{2},2}$, $a_{\frac{n+5}{2},3}, \cdots, a_{\frac{n+5}{2},n}$ respectively. Proceeding like this until we reach the $(n-1)^{th}$ row. Note that the vertices $a_{n-1,1}$, $a_{n-1,2}$, $a_{n-1,3}, \cdots, a_{n-1,n}$ gets the labels$-(\frac{n-1}{2}+1), -(\frac{n-1}{2}+2), -(\frac{n-1}{2}+3), \cdots, -(\frac{n-1}{2}+n)$. Finally assign the labels

$$(\frac{n-1}{2}+n+1), (\frac{n-1}{2}+n+2), (\frac{n-1}{2}+n+3), \cdots, (\frac{n-1}{2}+\frac{n-1}{2}),$$
$$-(\frac{n-1}{2}+n+1), -(\frac{n-1}{2}+n+2), -(\frac{n-1}{2}+n+3), \cdots, -(\frac{n-1}{2}+\frac{n-3}{2})$$

respectively and assign the labels $\frac{n}{2}+n-1, -(\frac{n-1}{2}+\frac{n-1}{2})$ to the vertices u, v.

Table 1 given below establishes that this vertex labeling is a pair difference cordial labeling of DA_n for all values of $n \geq 2$.

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
n is even	n^2	n^2
n is odd	n^2	n^2

Table 1

For illustration, DA_5 is shown in Figure 3.

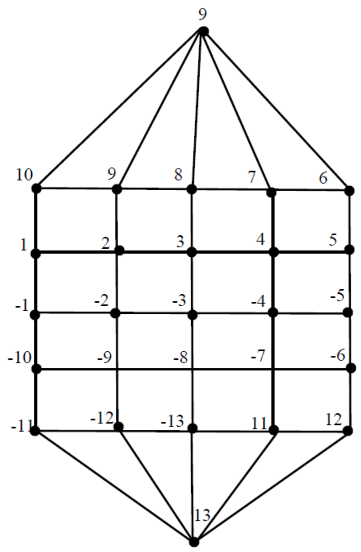

Figure 3

This completes the proof. □

Theorem 3.4 *A sunflower graph SF_n is pair difference cordial for all values of $n \geq 3$.*

Proof Take the vertex set and edge set from Definition 2.5. There are two cases arises.

Case 1. n is even.

Assign the label 2 to the vertex a. Next assign the labels $1, 3, 5, \cdots, n-1$ to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$ respectively and assign the labels $2, 4, 6, \cdots, n$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n}{2}}$.

Now we assign the labels $-1, -3, -5, \cdots, -(n-1)$ to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_n$ respectively and assign the labels $-2, -4, -6, \cdots, -n$ respectively to the vertices $b_{\frac{n+2}{2}}$, $b_{\frac{n+4}{2}}$, $b_{\frac{n+6}{2}}, \cdots, b_n$.

Case 2. n is odd.

Assign the label 2 to the vertex a. Next assign the labels $1, 3, 5, \cdots, n$ to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n+1}{2}}$ respectively and assign the labels $2, 4, 6, \cdots, n-1$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n-1}{2}}$.

Now we assign the labels $-1, -3, -5, \cdots, -(n)$ to the vertices $a_{\frac{n+3}{2}}$, $a_{\frac{n+5}{2}}$, $a_{\frac{n+7}{2}}, \cdots, a_n$ respectively and assign the labels $-2, -4, -6, \cdots, -(n-1)$ respectively to the vertices $b_{\frac{n+1}{2}}$, $b_{\frac{n+3}{2}}$, $b_{\frac{n+5}{2}}, \cdots, b_n$.

Notices that Table 2 given below establishes that this vertex labeling is a pair difference cordial labeling of SF_n for all values of $n \geq 3$.

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
n is even	$2n$	$2n$
n is odd	$2n$	$2n$

Table 2

This completes the proof. □

Theorem 3.5 *A shell graph $C_{(n,n-3)}$ is pair difference cordial for all values of $n \geq 3$.*

Proof Let us take vertex set and edge set from Definition 2.6 Assign the labels $1, 2, 3, \cdots, \frac{n}{2}$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$ and assign the labels $-1, -2, -3, \cdots, -\frac{n}{2}$ to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_n$ respectively. □

Theorem 3.6 *A butterfly graph with shell order m, m is pair difference cordial for all values of $m \geq 3$.*

Proof Assign the labels $1, 2, 3, \cdots, n$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_n$ and assign the labels $-1, -2, -3, \cdots, -n$ to the vertices $a_1, a_2, a_3, \cdots, a_n$ respectively. □

Theorem 3.7 *A graph obtained by joining two copies of shell graph by a path of arbitrary length is pair difference cordial.*

Proof Let G be the graph obtained by joining two copies of shell graph by a path of length. Let $a_1, a_2, a_3, \cdots, a_n$ be the successive vertices of 1^{st} copy of shell graph and let $b_1, b_2, b_3, \cdots, b_n$ be the successive vertices of 2^{nd} copy of shell graph. Let $c_1, c_2, c_3, \cdots, c_k$ be the successive vertices of path P_k with $c_1 = a_1$ and $c_k = b_1$.

There are two cases arises.

Case 1. k is odd.

Assign the labels $1, 2, -1, -2$ respectively to the vertices c_1, c_2, c_3, c_4 and assign the labels $3, 4, -3, -4$ to the vertices c_5, c_6, c_7, c_8 respectively. Next assign the labels $1, 2, -1, -2$ respectively to the vertices $c_9, c_{10}, c_{11}, c_{12}$ and assign the labels $5, 6, -5, -6$ to the vertices $c_{13}, c_{14}, c_{15}, c_{16}$ respectively. Proceeding like this until we reach c_{n-1}.

Now, we assign the labels

$$\frac{k+1}{2}, \frac{k+3}{2}, \frac{k+5}{2}, \cdots, \frac{2n+k-1}{2}$$

respectively to these vertices $a_2, a_3, a_4, \cdots, a_{n-1}$ and

$$-\frac{k+1}{2}, -\frac{k+3}{2}, -\frac{k+5}{2}, \cdots, -\frac{2n+k-1}{2}$$

respectively to the vertices $b_2, b_3, b_4, \cdots, b_{n-1}$. Finally assign the label $-\frac{k-1}{2}$ to the vertex c_n.

Case 2. k is even.

Assign the labels $1, 2, -1, -2$ respectively to the vertices c_1, c_2, c_3, c_4 and assign the labels $3, 4, -3, -4$ to the vertices c_5, c_6, c_7, c_8 respectively. Next assign the labels $1, 2, -1, -2$ respectively to the vertices $c_9, c_{10}, c_{11}, c_{12}$ and assign the labels $5, 6, -5, -6$ to the vertices $c_{13}, c_{14}, c_{15}, c_{16}$ respectively. Proceeding like this until we reach c_n.

Now, we assign the labels

$$\frac{k+2}{2}, \frac{k+4}{2}, \frac{k+6}{2}, \cdots, \frac{2n+k-2}{2}$$

respectively to these vertices $a_2, a_3, a_4, \cdots, a_{n-1}$ and

$$-\frac{k+2}{2}, -\frac{k+4}{2}, -\frac{k+6}{2}, \cdots, -\frac{2n+k-2}{2}$$

respectively to the vertices $b_2, b_3, b_4, \cdots, b_{n-1}$ and get the result. $\square$

Theorem 3.8 *An ice cream graph IC_n is pair difference cordial for $n \geq 3$.*

Proof Take the vertex set and edge set from Definition 2.5. There are two cases arises.

Case 1. n is even.

Assign the labels

$$1, 2, 3, \cdots, \frac{n}{2}$$

to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$ and assign the labels

$$-1, -2, -3, \cdots, -\frac{n}{2}$$

to the vertices $a_{\frac{n+2}{2}}$, $a_{\frac{n+4}{2}}$,$a_{\frac{n+6}{2}}, \cdots, a_n$. Finally assign the labels $\frac{n+2}{2}$,$-\frac{n+2}{2}$ to the vertices

v_0, v.

Case 2. n is odd.

Assign the labels

$$1, 2, 3, \cdots, \frac{n+1}{2}$$

to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n+1}{2}}$ and assign the labels

$$-1, -2, -3, \cdots, -\frac{n-1}{2}$$

to the vertices $a_{\frac{n+3}{2}}, a_{\frac{n+5}{2}}, a_{\frac{n+7}{2}}, \cdots, a_n$. Finally assign the labels $\frac{n+1}{2}$,1 to the vertices v_0, v.

For illustration, IC_5 is shown in Figure 4.

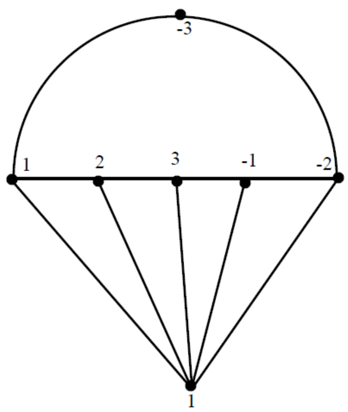

Figure 4

This completes the proof. □

References

[1] Andar M., Boxwala S., Limaye N., New families of cordial graphs, *J.combin. Math.Combin. Comput.*, 53 (2005), 117–154.

[2] Andar M., Boxwala S., Limaye N., On cordiality of corona graphs, *Ars combin.*, 78 (2006), 179–199.

[3] Basker Babujee J., Shobana L., Prime cordial llabelings, *Int. Review on Pure and Appl. Math.*, 5 (2009),277–282.

[4] Basker Babujee J., Shobana L., Prime and prime cordial llabeling for some special graphs, *Int. J. Contemp. Math. Sci.*, 5 (2010), 2347–2356.

[5] Cahit I., Cordial Graphs : A weaker version of graceful and harmonious graphs, *Ars Combin.*, 23 (1987), 201–207.

[6] Cahit I., On cordial and 3-equitable labelings of graphs, *Util. Math.*, 37 (1990), 189–198.

[7] Cahit I., H-Cordial graphs : A weaker version of graceful and harmonious graphs, *Bull. Inst. combin. Appl.*, 18 (1996), 87–101.

[8] Cichacz S., Gorich A., Tuza Z., Cordial labeling of hypertrees, *Disc. Math.*, 313(22) (2013), 2514–2524.

[9] Diab A.T., On cordial labelings of wheels with other graphs, *Ars Combin.*, 100 (2011), 265–279.

[10] Diab A.T., Generalization of some results on cordial graphs, *Ars Combin.*, 99 (2011), 161–173.

[11] Diab A.T., Elsakhawi E., Some results on cordial graphs, *Proc. Math. Phys. Soc. Egypt*, 77 (2002), 67–87.

[12] Gallian J.A., A Dynamic survey of graph labeling, *The Electronic Journal of Combinatorics.*, 19(2016).

[13] Harary F., *Graph Theory*, Addision Wesley, New Delhi, 1969.

[14] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordial labeling of graphs, *J.Math. Comp.Sci.*, Vol.11(3) (2021), 2551–2567.

[15] Ponraj R., Gayathri A., and Somasundaram S., Pair difference cordiality of some snake and butterfly graphs, *Journal of Algorithms and Computation*, Vol.53(1) (2021), 149–163.

[16] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordial graphs obtained from the wheels and the paths, *J. Appl. and Pure Math.*, Vol.3, No. 3-4(2021), 97–114.

[17] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordiality of some graphs derived from ladder graph, *J.Math. Comp.Sci.*, Vol.11, No 5 (2021), 6105–6124.

[18] Ponraj R., Gayathri A. and Somasundaram S., Some pair difference cordial graphs, *Ikonion Journal of Mathematics*, Vol.3(2) (2021), 17–26.

[19] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordial labeling of planar grid and mangolian tent, *Journal of Algorithms and Computation*, Vol.53(2) (2021), 47–56.

[20] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordiality of some special graphs, *J. Appl. and Pure Math.*, Vol.3 No. 5-6(2021), 263–274.

[21] Ponraj R., Gayathri A. and Soma Sundaram S., Pair difference cordiality of mirror graph, shadow graph and splitting graph of certain graphs, Maltepe Journal of Mathematics, Vol.4, Issue 1(2022), 24–32.

[22] Ponraj R., Gayathri A. and Soma Sundaram S., Pair difference cordial Labeling of $m-$ copies of path, cycle , star and laddar graphs, *Journal of Algorithms and Computation*, Vol.54, Issue 2 (2022), 37–47.

[23] Ponraj R., Gayathri A. and Soma Sundaram S., Pair difference cordial labeling of some star related graphs, *Journal of Mahani Mathematical Research*, Vol.12, Issue 2(2023), 255–266.

[24] Ponraj R., Gayathri A and Soma Sundaram S., Pair difference cordial labeling of certain broken wheel graphs, *Journal of Universal Mathematics*, Vol.6, No 2 (2023), 39–48.

[25] Swedha V.P., Vanithashree R., A study on combination of sheel graph and path P_2 graph, *Journal of Emerging Tech. and Inno. Resea.*, Vol.6, No 5(2019), 578–580.

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A Mathematical Modal for Bingham Flow Properties of Blood in Narrow Tapered Tube

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Abstract: The stenosis and non-Newtonian property of the fluid in the blood flow represent the behavior of Herschel- Buckley fluid. In a tapered tube model all the vessels which carry blood towards the tissues are considered as long, slowly tapering cones rather than cylinders. Since the blood flow consist of two regions in which one is central region, consist of concentrated blood cells and its behavior is non-Newtonian and other region is peripheral layer of plasma which represent the Newtonian behavior of fluid motion. In present paper, we have considered the flow of blood through a uniform tapered tube which obeys the Bingham fluid model and obtained the condition for the wall shear stress and pressure gradient. Further in various graphs we represent the variation of shear stress at the wall and pressure gradient with respect to suspension concentration and tapered angle over the flow rate range 0.01 to 0.1 cc/sec.

Key Words: Newtonian fluid, blood flow, wall shear stress, tapered vessel, stenosis, Bingham model.

AMS(2010): 76D03, 76A05.

§1. Introduction

Womersley [14] introduced the concept of a tapered tube model for blood vessel and considered that all the vessels which carry blood towards the tissues should be long, slowly tapering cones rather than cylinders. Further Charm and Kurland [4] have examined the nature of blood flow in non-uniform capillary tubes which are relatively large diameter where the influence of a marginal gap is negligible, experimental values agree well with the anticipated value. But in cylindrical tubes where the influence of a marginal gap becomes important, the calculated and anticipated values diverge unless the probable gap width based on formulas validated in straight tubes. This conditions strongly suggests that marginal layers develop in tapered tubes similar to those in straight tubes.

Oka [9] has calculated the pressure development in a non-Newtonian flow through a tapered tube and obtained the distribution of pressure through tapered tube for Power law, Bingham

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body and Casson fluids. Chaturani and Pralhad [5] and Kumar and Kumar [5] have studied a steady laminar flow of blood in a uniform tapered tube by assuming blood as a polar fluid and obtained the analytical expressions for wall shear stress, pressure drop, total angular and axial velocities. Bagchi [1], Bugliarello and Sevilla [2], Chan [3], Chaturani and Palanisamy [6], Jianncing [7], Pappu and Bagchi [10], Pries and Secomb [11], Sakamoto et al. [12] and Singh and Kumar [13] have discussed the role of plasma peripheral layer on blood flow in capillaries.

In the present paper we have considered an anomalous behavior of blood flow through uniform tapered tubes and to understanding the complex rheological characteristics of blood flow we also considered Bingham blood model and obtained the expressions for the wall shear stress and pressure gradient. Further in various graphs we represent the variation of shear stress at the wall and pressure gradient with respect to suspension concentration and tapered angle over the flow rate range 0.01 to 0.1 cc/sec.

§2. The Mathematical Model

We considered a steady laminar flow of incompressible viscous non-Newtonian fluid model in a uniformly tapered tube of circular cross-section and the problem investigated under following assumptions:

(i) Taper angle is very small;
(ii) The motion is steady axisymmetric and in the z-direction;
(iii) No body forces act in the fluid;
(iv) The motion is so slow that inertia term can be neglected;
(v) Pressure gradient is a function of axial co-ordinates only.

Further a section of tapered vessel is shown in Figure 1.

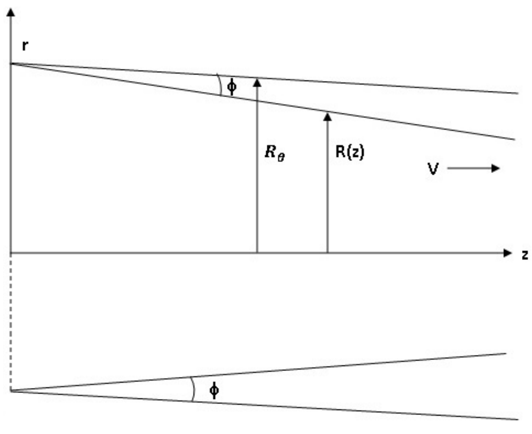

Figure 1. Geometry of the vessel

The radius of the tapered tube $R(z)$ is given by

$$R(z) = R_\theta - z \tan\phi$$

where R_θ is the tube radius at $z = 0$, ϕ is the tapered angle and z the axis of the tapered tube.

2.1. Governing Equations. The governing equations in cylindrical co-ordinate system (r, z, θ), which mathematically describe the laminar flow problem of an incomressible fluid are given by the continuity equation

$$\frac{\partial V_z}{\partial z} + \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0 \tag{2.1}$$

and the momentum equations

$$\rho\frac{DV_z}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial z}(2\mu\frac{\partial V_z}{\partial z}) + \frac{1}{r}\frac{\partial}{\partial \theta}[\mu(\frac{1}{r}\frac{\partial V_z}{\partial \theta} + \frac{\partial V_\theta}{\partial z})] + \frac{\partial}{\partial r}[\mu(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r})] + \frac{\mu}{r}(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r}), \tag{2.2}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + V_r\frac{\partial}{\partial r} + V_z\frac{\partial}{\partial z} + \frac{V_\theta}{r}\frac{\partial}{\partial \theta}$.

Making use of the assumptions (ii), we have

$$\frac{\partial}{\partial t} = 0, \quad \frac{\partial}{\partial \theta} = 0, \quad V_r = V_\theta = 0, \quad V_z = V(r). \tag{2.3}$$

Using equation (2.3) in equation (2.2), we find the equations of motion and continuity for fully developed steady viscous incompressible laminar flow under no-body forces as

$$0 = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}), \tag{2.4}$$

$$0 = \frac{\partial p}{\partial r}, \tag{2.5}$$

$$\frac{\partial V}{\partial z} = 0, \tag{2.6}$$

where p is the pressure, V is the axial velocity and $\tau_{rz} = (\mu\frac{\partial V}{\partial r})$ the shear stress normal to r in z-direction.

2.2. Constitutive Equation. The constitutive equation for the shear stress τ and strain rate $\dot{\gamma}$ is given by

$$\tau = \tau_0 + \mu\dot{\gamma}; \quad \tau \geq \tau_0, \qquad and \quad \dot{\gamma} = 0; \quad \tau \leq \tau_0, \tag{2.7}$$

where τ_0 is the yield stress, μ the coefficient of viscosity and $\dot{\gamma}$ the shear strain rate.

2.3. Boundary Conditions. The appropriate boundary conditions are given by

$$V = 0 \ \ at \ \ r = R(z), \tag{2.8}$$

$$\tau_{rz} = \tau_w \ \ at \ \ r = R(z), \tag{2.9}$$

$$V = V_p \ \ at \ \ r = R_p, \tag{2.10}$$

$$\tau_{rz} \ \ is \ \ finite \ \ at \ \ r = 0, \tag{2.11}$$

where R_p is the plug radius and V_p the plug velocity.

2.4. Solution for Velocities, Volume Flow Rate and Wall Shear Stress.

(1) **Velocities.** Integrating equation (2.4) with the boundary condition (2.11), we get

$$\tau_{rz} = \frac{r}{2}\frac{\partial p}{\partial z}. \tag{2.12}$$

Making use of the equation (7) in equation (12), we have velocity equations as

$$\frac{dV}{dr} = \frac{1}{\mu}(\frac{\partial p}{\partial z}\frac{r}{2} - \tau_0); \quad R_p \leq r \leq R(z), \tag{2.13}$$

$$\frac{dV_p}{dr} = 0; \quad 0 \leq r \leq R_p. \tag{2.14}$$

The plug flow exists whenever the shear stress does not exceed yield stress. Solving equations (2.13) and (2.14) with boundary conditions (2.8) to (2.10), we get

$$V = \frac{\tau_w(z)}{2\mu}R(z)[1 - \frac{r^2}{R^2(z)} - 2\beta(1 - \frac{r}{R(z)})], \tag{2.15}$$

$$V_p = \frac{\tau_w(z)}{2\mu}R(z)(1-\beta)^2, \tag{2.16}$$

where $\beta = \frac{\tau_0}{\tau_w(z)}$.

(2) **Volume Flow Rate and Wall Shear Stress.** The volume flow rate Q is given by

$$Q = Q_1 + Q_2, \tag{2.17}$$

where Q_1 and Q_2 are given by

$$Q_1 = \int_0^{R_p} 2\pi V_p r dr = \pi V_p R_p^2, \tag{2.18}$$

$$Q_2 = \int_{R_p}^{R(z)} 2\pi V r dr. \tag{2.19}$$

Now substituting the values of V_p and V from equations (2.15) and (2.16) in equations (2.18) and (2.19), we obtain

$$Q_1 = \frac{\pi}{2}\frac{\tau_w(z)}{\mu}R^3(z)(1-\beta)^2, \tag{2.20}$$

$$Q_2 = \frac{\pi}{2\mu}\tau_w(z)R^3(z)[\frac{1}{2} - \frac{2}{3}\beta - \beta^2 + 2\beta^3 - \frac{5}{6}\beta^4]. \tag{2.21}$$

Using equations (2.20) and (2.21) in equation (2.17), we have

$$Q = \frac{\pi}{4\mu}\tau_w(z)R^3(z)(1 - \frac{4}{3}\beta) \tag{2.22}$$

in where higher order of β are neglected.

Now from equation (2.12) and (2.22) with boundary condition (2.9), the pressure gradient is obtained as

$$\frac{\partial p}{\partial z} = \frac{8\mu Q}{\pi R^4(z)}(1 + \frac{4}{3}\beta). \tag{2.23}$$

From equation (2.22) we have the shear stress at the wall as

$$\tau_w(z) = \frac{4\mu Q}{\pi R^3(z)}(1 + \frac{4}{3}\beta). \tag{2.24}$$

Now using equations (2.23) and (2.24), we have

$$\tau_w(z) = \frac{R(z)}{2}\frac{\partial p}{\partial z}. \tag{2.25}$$

Therefore, by the equations (2.23) and (2.24) we obtain that the pressure and the shear stress at the wall increases when $R(z)$ decreases.

§3. Results and Discussion

The pressure gradient and shear stress at the wall are given by the equation (2.23) and (2.25) respectively in which we observe that $\tau_w z$ and $\partial p/\partial z$ changes with z (along the tube axis), i.e. pressure gradient and wall shear stress increases with decrease in the radius of the tapered tube. Therefore we should not take the pressure gradient to be constant. Some authors have proposed a micro polar fluid model for blood flow through a small tapered tube and have assumed pressure gradient to be constant throughout the investigation, which is not true.

We consider the radius of tapered vessel $R_\theta = 100\mu m$. The variation of pressure gradient and shear stress at the wall are calculated, with the help of equations (2.23) and (2.24) for the flow rate over the range 0.02 to 0.10 cc/sec, for different tapered angles ($1^0 \leqslant \theta \leqslant 2^0$) and the suspension concentrations 20%, 30% and 40% .

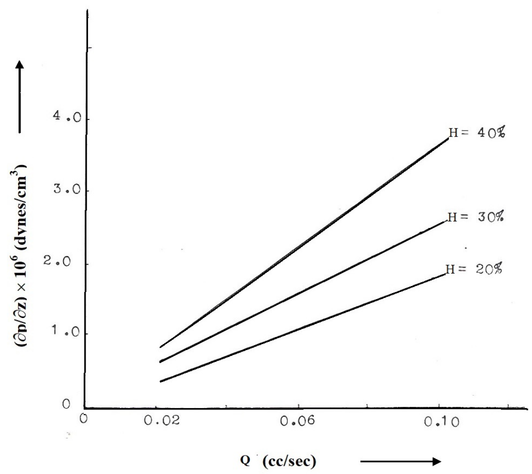

Figure 2. Variation of pressure gradient with flow rate for different suspension concentrations ($Z = 0.10cm, \phi = 1.4^0$, $R_\theta = 0.01cm$)

Figures 2, 3 and 4 show the variation of pressure gradient with flow rate for different suspension concentration, different tapered angle and different axial distance. From these Figures it is clear that pressure gradient increases with increase in axial distance, tapered angle and suspension concentration.

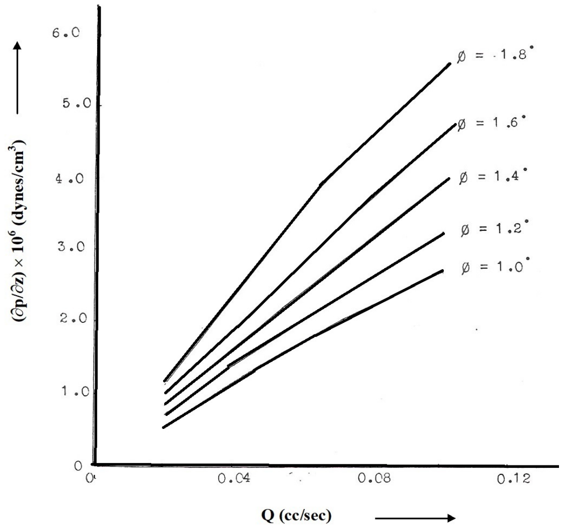

Figure 3. Variation of pressure gradient with flow rate for different tapered angles $(H = 4\%, R_\theta = 0.01cm, z = 0.10cm)$

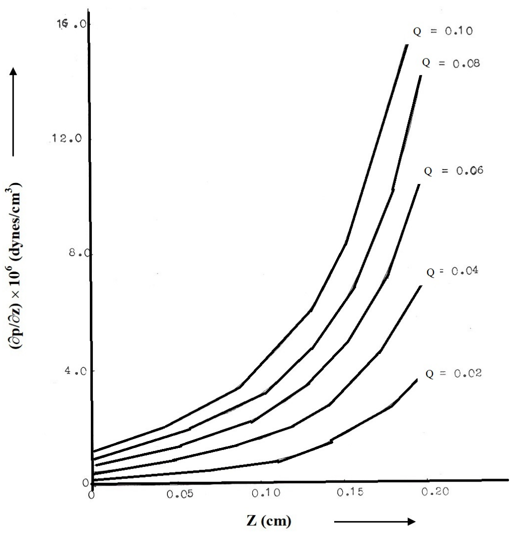

Figure 4. Variation of pressure gradient with axial distance for different flow rates $(R_\theta = 0.01cm, H = 40\%, \phi = 1.4^0)$

Now, Figures 5, 6 and 7 show the variation of shear stress at the wall with flow rate Q for different angles, axial distances and concentrations.

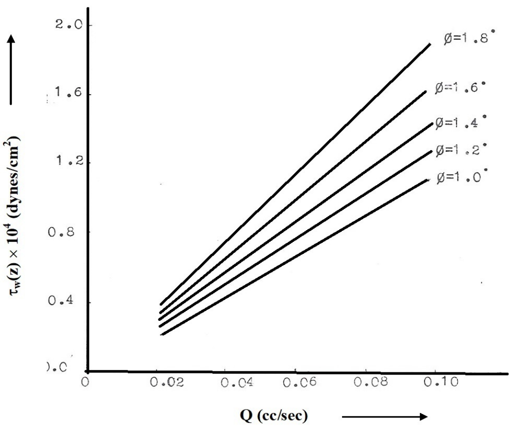

Figure 5. Variation of wall shear stress with flow rate for different tapered angles ($H = 40\%, R_\theta = 0.01cm, z = 0.10cm$)

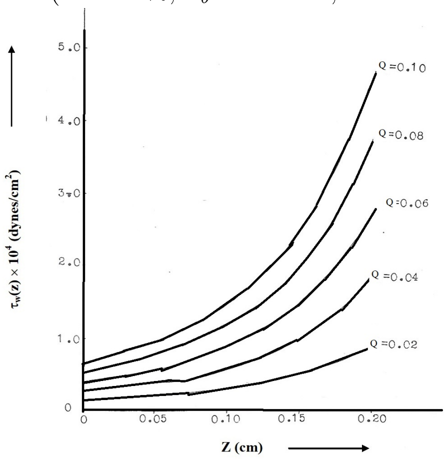

Figure 6. Variation of wall shear stress with axial distance for different flow rates ($H = 40\%, \phi = 1.4^0, R_\theta = 0.01cm$)

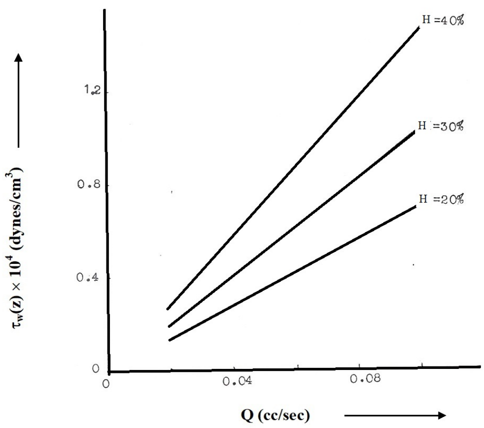

Figure 7. Variation of wall shear stress with flow rate for different hematocrit ($\phi = 1.4^0, z = 0.1cm, R_\theta = 0.01cm$)

From the above figures, it is clear that shear stress at wall increases with suspension concentration and tapered angles. $\tau_w z$ is an increasing function of axial distance. Thus, for known flow rate, the shear stress can be calculated at any point of the tapered tube. These fluid dynamics results could be very useful in understanding the vascular fluid mechanics. Now from Figures 8, 9 and 10 we have, for Newtonian fluid $\beta = 0$, the variation of pressure gradient with flow rate Q for different suspension concentrations, tapered angles and axial distances. We observe that the values of pressure gradient are less than those for Bingham fluid model. From these Figures, the same trends for pressure gradient are obtained as for Bingham fluids.

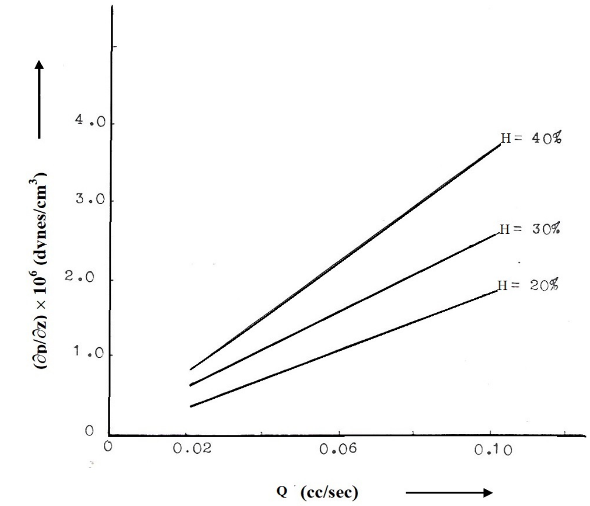

Figure 8. Variation of pressure gradient with flow rate different suspension concentration (Newtonian fluid with $\phi = 1.4^0, z = 0.10cm, R_\theta = 0.01cm, \beta = 0.0$)

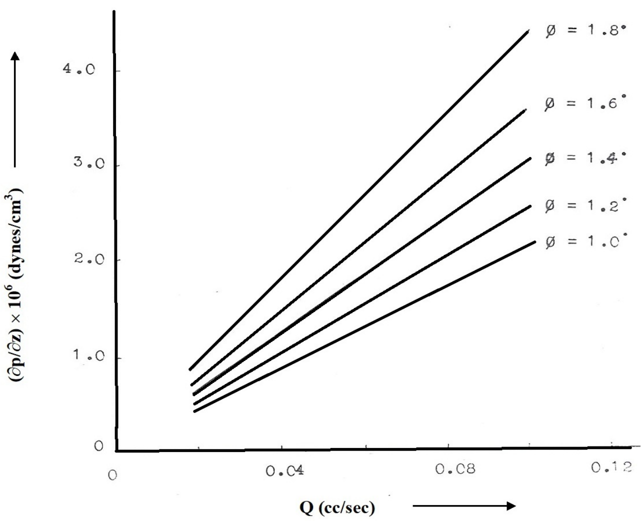

Figure 9. Variation of pressure gradient with flow rate for different tapered angles (Newtonian fluid with $H = 40\%, R_\theta = 0.01cm, z = 0.10cm, \beta = 0.0$)

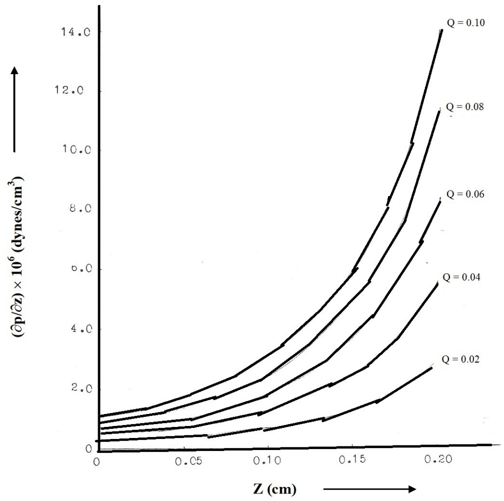

Figure 10. Variation of pressure gradient with axial distance for different flow rates (Newtonian fluid with $H = 40\%, R_\theta = 0.01cm, \phi = 1.4^0, \beta = 0.0$)

References

[1] Bagchi P., Mesoscale simulation of blood flow in small vessels, *J. Biophysics* 92(6) (2007), 1858 C 1877, doi: 10.1529/biophysj.106.095042.

[2] Bugliarello G. and Sevilla J., Velocity distribution and other characteristics of steady and pulsatile blood flow in fine glass tubes, *Biorheology*, 7(2)(1970), 85 C 107, doi: 10.3233/BIR-1970-7202.

[3] Chan W. Y., *Simulation of Arterial Stenosis Incorporating Fluid Structural Interaction and Non-Newtonian Blood Flow*, Masters Thesis, RMIT University, Australia, 2006.

[4] Charm S. E. and Kurland G. S., Blood flow in non-uniform tapered capillaries tubes, *Biorheology*, 4(4)(1967), 175-183, doi:10.3233/bir-1967-4404.

[5] Chaturani P. and Pralhad R. N., Blood flow in tapered tube with bio-rheological spplications, *Biorheology*, 22(4)(1985), 303-314.

[6] Chaturani P. and Palanisamy V., Casson fluid model for pulsatile flow of blood with periodic body acceleration, *Biorheology*, 27(5)(1990), 619 C 630, doi:3233/bir1990-27501.

[7] Jiannong Fang and Owens R. G., Numerical simulation of pulsatile blood flow using a new constitutive model, *Biorheology*, 43(5)(2006), 637 C 660.

[8] Sanjeev Kumar and Sanjeet Kumar, A mathematical model for Newtonian and non-Newtonian flow through tapered tubes,*Indian Journal of Biomechanics*, Special Issue (NCBM 7-8) (2009), 191-195.

[9] Oka S., Pressure development in a non-Newtonian flow through a tapered tube, *Biorheology*, 10(2)(1973), 207-212.

[10] Pappu V. and Bagchi P., Hydrodynamic interaction between erythrocytes and leukocytes affects rheology of blood in microvessels, *Biorheology*, 44(3)(2007), 191-215.

[11] Pries A. R. and Secomb T. W., Resistance to blood flow in vivo, from poiseuille to the in vivo viscosity law, *Biorheology*, 34(4)(1997), 369-373.

[12] Sakamoto K., Sunaga R., Nakamara K. and Sato Y., Study of the relation between fluid distribution change in tissue and impedance change in during hemodialysis by frequency characteristics of the flowing blood, *Annals N. Y. Acad. Sci.*, 873(1) (1999), 77-88, doi: 10.1111/j.1749-6632.1999.tb09452.x.

[13] Singh N. L. and Kumar J., Mathematical study of steady blood flow in narrow vessel with plasma layer near the wall, *Indian Journal Theoretical Physics*, 50(2002), 155-160.

[14] Womersley J. R., Method for the calculation of velocity rate of flow and viscous drug in the arteries when the pressure gradient is known, *J. Physiol*, 127(3)(1955), 553-563, doi: 10.1113/jphysiol.1955.sp005276.

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