International J.Math. Combin. Vol.1(2023), 54-61

Monophonic Graphoidal Covering Number of Corona Product Graph of Some Standard Graphs with the Wheel

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Abstract: A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called a monophonic path if it is a chordless path. A monophonic graphoidal cover of a graph G is a collection ψ_m of monophonic paths in G such that every vertex of G is an internal vertex of at most one monophonic path in ψ_m and every edge of G is in exactly one monophonic path in ψ_m. The minimum cardinality of a monophonic graphoidal cover of G is called the monophonic graphoidal covering number of G and is denoted by $\eta_m(G)$. In this paper, we find the monophonic graphoidal covering number of corona product of wheel with some standard graphs.

Key Words: Graphoidal cover, Smarandachely graphoidal cover, monophonic path, monophonic graphoidal cover, monophonic graphoidal covering number.

AMS(2010): 05C70.

§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [6]. The concept of graphoidal cover was introduced by Acharya and Sampathkumar [2] and further studied in [1, 3, 7, 8].

A *graphoidal cover* of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:

(i) Every path in ψ has at least two vertices;

(ii) Every vertex of G is an internal vertex of at most one path in ψ;

(iii) Every edge of G is in exactly one path in ψ.

The minimum cardinality of a graphoidal cover of G is called the *graphoidal covering number* of G and is denoted by $\eta(G)$.

The collection ψ is called an *acyclic graphoidal cover* of G if no member of ψ is cycle; it

[1]Received February 16, 2022, Accepted March 17, 2023.

is called a *geodesic graphoidal cover* if every member of ψ is a shortest path in G. The minimum cardinality of an acyclic (geodesic) graphoidal cover of G is called the *acyclic (geodesic) graphoidal covering number* of G and is denoted by $\eta_a(\eta_g)$. The acyclic graphoidal covering number and geodesic graphoidal covering number are studied in [4,5]. Generally, a *Smarandachely graphoidal cover* $\mathscr{C}(G,k,\mathscr{P})$ of graph G is the union of subgraphs with property $\mathscr{P}$, hold with every vertex $v \in V(G)$ is in at most k subgraphs and every edge is in exactly one subgraph with property $\mathscr{P}$. Certainly, let $\mathscr{P} = \{\text{path}, \text{cycle}\}$ or $\mathscr{P} = \{\text{path}\}$ and $k = 1$. Then, a Smarandachely graphoidal cover $\mathscr{C}(G,1,\mathscr{P})$ is respectively the graphoidal cover of G or acyclic graphoidal cover of G.

A *chord* of a path P is an edge joining any two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u,v)$ from u to v is defined as the length of a longest $u-v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ *of a vertex* v *in* G is $e_m(v) = max\{d_m(v,u) : u \in V(G)\}$. The *monophonic radius* is $rad_m(G) = min\{e_m(v) : v \in V(G)\}$ and the *monophonic diameter* is $diam_m(G) = max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [10, 11].

A *monophonic graphoidal cover* of a graph G is a collection ψ_m of monophonic paths in G such that every vertex of G is an internal vertex of at most one monophonic path in ψ_m and every edge of G is in exactly one monophonic path in ψ_m. The minimum cardinality of a monophonic graphoidal cover of G is called the *monophonic graphoidal covering number* of G and is denoted by $\eta_m(G)$. The monophonic graphoidal covering number was introduced [12] and studied in [13,14].

Product graphs have been used to generate mathematical models of complex networks which inherits properties of real networks. By using basic graphs, corona graphs are defined by taking corona product of the basic graphs.

Definition 1.1 *The corona of two graphs G and H is the graph $G \circ H$ formed from one copy of G and $|V(G)|$ copies of H, where the i^{th} vertex of G is adjacent to every vertex in the i^{th} copy of H.*

§2. Monophonic Graphoidal Covering Number on Corona Product of Wheel with Some Standard Graphs

Theorem 2.1 *For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 5)$, $\eta_m(W_n) = n$.*

Proof Let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v\}$ and $V(C_{n-1}) = \{u_1, u_2, \cdots, u_{n-1}\}$ and let $P_1 : u_1, u_2, \cdots, u_{n-2}$, $P_2 : u_1, u_{n-1}, u_{n-2}$, $P_3 : u_1, v, u_{n-2}$ and $P_{i+2} : v, u_i$ $(2 \leq i \leq n-3$ and $i = n-1)$. It is clear that $\psi_m = \{P_1, P_2, \cdots, P_{n-1}, P_{n+1}\}$ is a minimum monophonic graphoidal cover of W_n. Hence $\eta_m(W_n) = n$. □

Theorem 2.2 *(i) If $G = P_r \circ W_n$, then $\eta_m(G) = 2nr - 1$;*

(ii) If $G = W_n \circ P_r$, then $\eta_m(G) = n(r+2) - 2$.

Proof Let $P : u_1, u_2, \cdots, u_r$ be a path of order r and let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \cdots, v_n\}$.

(i) Let G be the corona product of P_r and W_n. The graph G is shown in Figure 1. Let $M_1 : v_{1,1}, u_1, u_2, \cdots, u_r, v_{r,1}$; $M_{i+1} : v_{i,2}, v_{i,3}, \cdots, v_{i,n-1}$ $(1 \le i \le r)$; $M'_i : v_{i,2}, v_{i,n}, v_{i,n-1}$ $(1 \le i \le r)$; $M''_i : v_{i,2}, v_{i,1}, v_{i,n-1}$ $(1 \le i \le r)$ and $S_1 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{n} \{(u_i, v_{i,j})\} - \{(u_1, v_{1,1}), (u_r, v_{r,1})\}$, $S_2 = \bigcup_{i=1}^{r} (\bigcup_{j=3}^{n} \{(v_{i,1}, v_{i,j})\} - \{(v_{i,1}, v_{i,n-1})\})$.

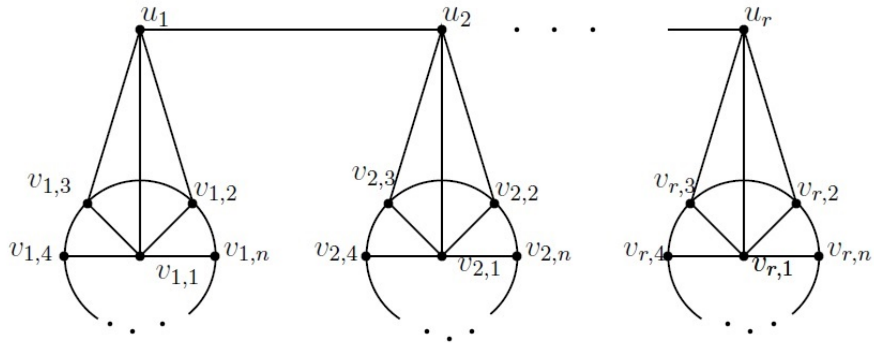

Figure 1

It is clear that $\psi_m = S_1 \cup S_2 \cup \{M_1, M_2, \ldots, M_{r+1}, M'_1, M'_2, \ldots, M'_r, M''_1, M''_2, \cdots, M''_r\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = (nr - 2) + r(n - 3) + (3r + 1) = 2nr - 1$.

(ii) Let G be the corona product of W_n and P_r. The graph G is shown in Figure 2.

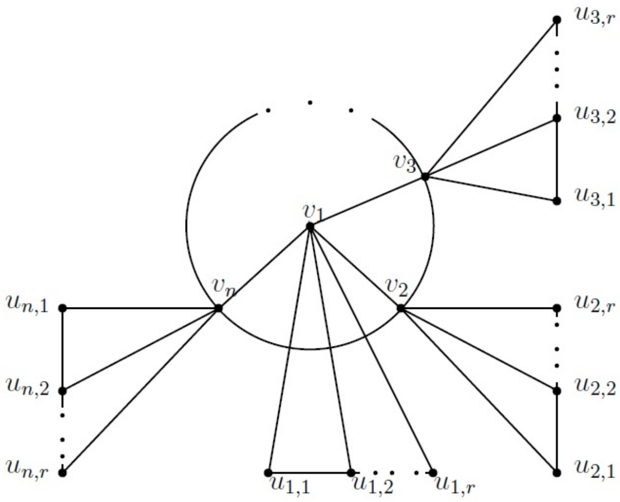

Figure 2

Let $M_1 : u_{2,1}, v_2, v_3, \cdots, v_{n-1}, u_{n-1,1}$; $M_2 : v_2, v_n, v_{n-1}$; $M_3 : v_2, v_1, v_{n-1}$; $M_{i+1} : v_1, v_i$ $(3 \le i \le n - 2)$; $M_n : v_1, v_n$; $M'_i : u_{i,1}, u_{i,2}, \cdots, u_{i,r}$ $(1 \le i \le n)$ and $S = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_{n-1}, u_{n-1,1})\}$.

It is clear that $\psi_m = S \cup \{M_1, M_2, \ldots, M_n, M'_1, M'_2, \ldots, M'_n\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = (nr - 2) + 2n = n(r + 2) - 2$. □

Theorem 2.3 *(i) If $G = C_r \circ W_n$, then $\eta_m(G) = 2rn$;*

(ii) If $G = W_n \circ C_r$, then $\eta_m(G) = n(r + 3) - 2$.

Proof Let $C_r : u_1, u_2, \cdots, u_r, u_1$ be a cycle of order r and let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \cdots, v_n\}$.

(i) Let G be the corona product of C_r and P_n.

Case 1. $r = 3$.

The graph G in this case is shown in Figure 3.

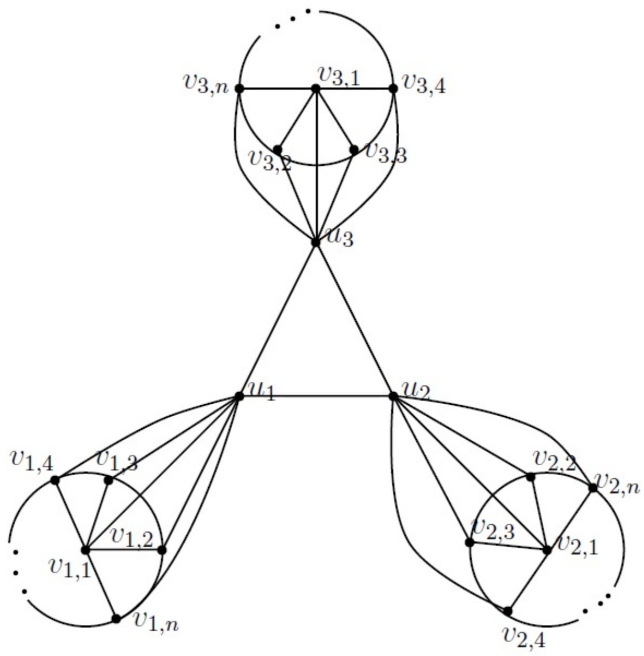

Figure 3

Let $M_1 : v_{1,1}, u_1, u_2$; $M_2 : v_{2,1}, u_2, u_3$; $M_3 : v_{3,1}, u_3, u_1$; $M_{i+3} : v_{i,2}, v_{i,3}, \cdots, v_{i,n-1}$ $(1 \le i \le 3)$; $M'_i : v_{i,2}, v_{i,n}, v_{i,n-1}$ $(1 \le i \le 3)$; $M'_{i+3} : v_{i,2}, v_{i,1}, v_{i,n-1}$ $(1 \le i \le 3)$ and $S_1 = \bigcup_{i=1}^{3}(\bigcup_{j=1}^{n}\{(u_i, v_{i,j})\} - \{(u_i, v_{i,1})\})$, $S_2 = \bigcup_{i=1}^{3}(\bigcup_{j=3}^{n}\{(v_{i,1}, v_{i,j})\} - \{(v_{i,1}, v_{i,n-1})\})$.

It is clear that every M_i $(1 \le i \le 6)$ and M'_i $(1 \le i \le 6)$ are monophonic paths and every element in $S_1 \cup S_2$ is a monophonic path. Hence $\psi_m = S_1 \cup S_2 \cup \{M_1, M_2, \cdots, M_6, M'_1, M'_2, \ldots, M'_6\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = 3n - 3 + 3(n - 3) + 12 = 6n$.

Case 2. $r > 3$.

Let $M_1 : v_{i,1}, u_1, u_2, \ldots, u_{r-1}, v_{r-1,1}$; $M_2 : u_1, u_r, u_{r-1}$; $M_{i+2} : v_{i,2}, v_{i,3}, \cdots, v_{i,n-1}$ $(1 \le i \le r)$; $M'_i : v_{i,2}, v_{i,n}, v_{i,n-1}$ $(1 \le i \le r)$; $M''_i : v_{i,2}, v_{i,1}, v_{i,n-1}$ $(1 \le i \le r)$ and $S_1 = (\bigcup_{i=1}^{r}\bigcup_{j=1}^{n}(u_i, v_{i,j})) - \{(u_1, v_{1,1}), (u_{r-1}, v_{r-1,1})\}$, $S_2 = \bigcup_{i=1}^{r}(\bigcup_{j=3}^{n}(v_{i,1}, v_{i,j}) - \{(v_{i,1}, v_{i,n-1})\})$.

It is clear that every M_i $(1 \leq i \leq r+2)$, M'_i $(1 \leq i \leq r)$ and M''_i $(1 \leq i \leq r)$ is a monophonic path and every element in $S_1 \cup S_2$ is a monophonic path. Hence $\psi_m = S_1 \cup S_2 \cup \{M_1, M_2, \cdots, M_{r+2}, M'_1, M'_2, \ldots, M'_r, M''_1, M''_2, \cdots, M''_r\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = (rn-2) + r(n-3) + (3r+2) = 2rn$.

(ii) Let G be the corona product of W_n and C_r. The graph G in this case is shown in Figure 4. Let $M_1 : u_{2,1}, v_2, v_3, \cdots, v_{n-1}, u_{n-1,1}$; $M_2 : v_2, v_n, v_{n-1}$; $M_3 : v_2, v_1, v_{n-1}$; $M'_i : u_{i,1}, u_{i,2}, \cdots, u_{i,r-1}$ $(1 \leq i \leq n)$; $M''_i : u_{i,1}, u_{i,r}, u_{i,r-1}$ $(1 \leq i \leq n)$ and $S_1 = \bigcup_{i=3}^{n}(v_1, v_i) - \{(v_1, v_{n-1})\}$, $S_2 = \bigcup_{i=1}^{n}\bigcup_{j=1}^{r}(v_i, u_{i,j}) - \{(v_2, u_{2,1})(v_{n-1}, u_{n-1,1})\}$.

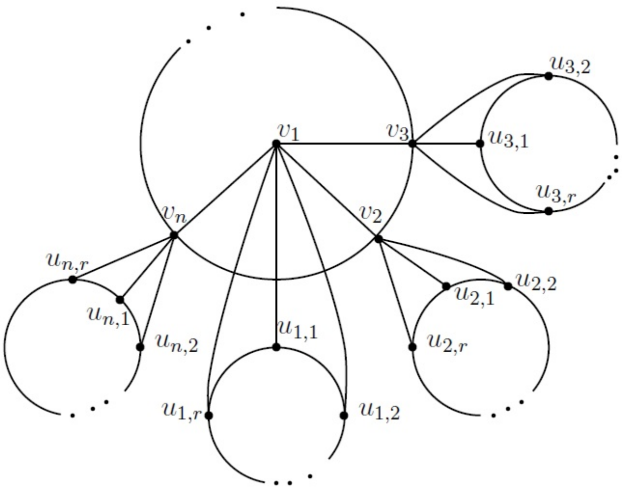

Figure 4

It is clear that every M_i $(1 \leq i \leq 3)$, M'_i $(1 \leq i \leq n)$ and M''_i $(1 \leq i \leq n)$ are monophonic paths and every element in $S_1 \cup S_2$ is a monophonic path in G. Hence $\psi_m = S_1 \cup S_2 \cup \{M_1, M_2, M_3, M'_1, M'_2, M'_3, \cdots, M'_n, M''_1, M''_2, \cdots, M''_n\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = (n-3) + (nr-2) + (2n+3) = n(r+3) - 2$. □

Theorem 2.4 (i) *If* $G = K_r \circ W_n$, *then* $\eta_m(G) = \frac{r}{2}(r + 4n - 11)$;

(ii) *If* $G = W_n \circ K_r$, *then* $\eta_m(G) = n(r^2 + r + 2) - 10$.

Proof Let K_r be the complete graph of order r with the vertex set $\{u_1, u_2, \cdots, u_r\}$ and let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \ldots, v_n\}$.

(i) Let G be the corona product of K_r and W_n. The graph G is shown in Figure 5. Let $M_i : v_{i,1}, u_i, u_{i+1}$ $(1 \leq i \leq r-1)$; $M_r : v_{r,1}, u_r, u_1$; $N_i : v_{i,2}, v_{i,3}, \cdots, v_{i,n-1}$ $(1 \leq i \leq r)$; $N'_i : v_{i,2}, v_{i,n}, v_{i,n-1}$ $(1 \leq i \leq r)$; $N''_i : v_{i,2}, v_{i,1}, u_{i,n-1}$ $(1 \leq i \leq r)$ and

$$S_1 = \bigcup_{i=1}^{r}\bigcup_{j=2}^{n}(u_i, v_{i,j}),$$

$$S_2 = \bigcup_{i=1}^{r}(\bigcup_{j=3}^{n}(v_{i,1}, v_{i,j}) - \{(v_{i,1}, v_{i,n-1})\},$$

$$S_3 = E(K_r) - \{(u_1, u_2), (u_2, u_3), \cdots, (u_{r-1}, u_r), (u_r, u_1)\}.$$

It is clear that every M_i, N_i, N_i', N_i'', for $1 \leq i \leq r$, are monophonic paths and every element in $S_1 \cup S_2 \cup S_3$ is a monophonic path. Hence,

$$\psi_m = S_1 \bigcup S_2 \bigcup S_3 \bigcup \{M_1, M_2, \cdots, M_r, N_1, N_2, \cdots, N_r, N_1', N_2', \cdots, N_r', N_1'', N_2'', \cdots, N_r''\}$$

is a minimum monophonic graphoidal cover of G and hence

$$\eta_m(G) = r(n-1) + r(n-3) + \frac{r(r-1)}{2} - r + 4r = \frac{r}{2}(r + 4n - 11).$$

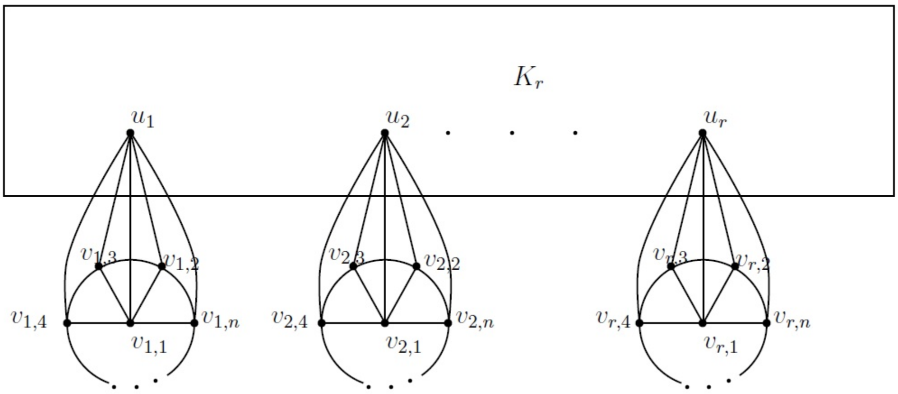

Figure 5

(ii) Let G be the corona product of W_n and K_r, which is shown in Figure 6.

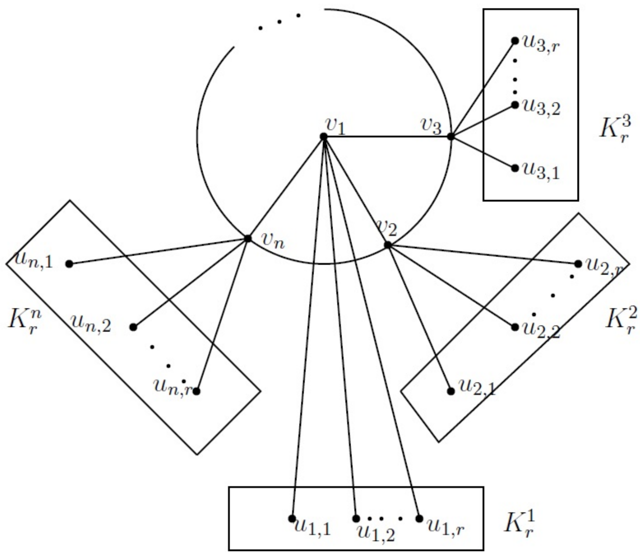

Figure 6

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{n-1}, u_{n-1,1}$; $M_2 : v_2, v_n, v_{n-1}$; $M_3 : v_2, v_1, v_{n-1}$ and

$$S_1 = \bigcup_{i=3}^{n}(v_1, v_i) - \{(v_1, v_{n-1})\},$$
$$S_2 = \bigcup_{i=1}^{n}\bigcup_{j=1}^{r}(v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_{n-1}, u_{n-1,1})\},$$
$$S_3 = \bigcup_{i=1}^{n} E(K_r^i).$$

It is clear that every M_1, M_2 and M_3 are monophonic paths and every element in $S_1 \cup S_2 \cup S_3$ is a monophonic path. Hence $\psi_m = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3\}$ is a minimum monophonic graphoidal cover of G and hence

$$\eta_m(G) = (n-3) + (nr-2) + n(\frac{r(r-1)}{2}) = n(r^2 + r + 2) - 10. \qquad \square$$

Theorem 2.5 *If $G = W_r \circ W_s$, then $\eta_m(G) = r(2s+4) - 2$.*

Proof Let $W_r = K_1 + C_{r-1}$ be a wheel with $V(K_1) = \{u_1\}$ and $V(C_{r-1}) = \{u_1, u_2, \cdots, u_r\}$ and let $W_s = K_1 + C_{s-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{s-1}) = \{v_1, v_2, \cdots, v_s\}$. The graph G is shown in Figure 7. Let $M_1 : v_{2,1}, u_2, u_3, \cdots, u_{r-1}, v_{r-1,1}$; $M_2 : u_2, u_r, u_{r-1}$; $M_3 : u_2, u_1, u_{r-1}$; $M_i' : v_{i,2}, v_{i,3}, \cdots, v_{i,s-1}$ $(1 \le i \le r)$; $M_i'' : v_{i,2}, v_{i,s}, v_{i,s-1}$ $(1 \le i \le r)$; $M_i''' : v_{i,2}, v_{i,1}, v_{i,s-1}$ $(1 \le i \le r)$ and $S_1 = \bigcup_{i=3}^{r}(u_1, u_i) - \{(u_1, u_{r-1})\}$, $S_2 = \bigcup_{i=1}^{r}(\bigcup_{j=3}^{s}(v_{i,1}, v_{i,j}) - \{(v_{i,1}, v_{i,s-1})\})$, $S_3 = \bigcup_{i=1}^{r}\bigcup_{j=1}^{s}(u_i, v_{i,j}) - \{(u_2, v_{2,1}), (u_{r-1}, v_{r-1,1})\}$.

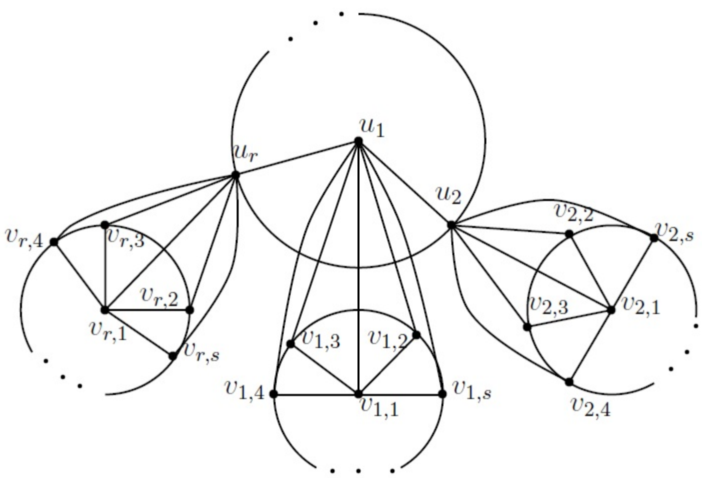

Figure 7

It is clear that $\psi_m = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M_1', M_2', \cdots, M_r', M_1'', M_2'', \cdots, M_r'', M_1''', M_2''', \cdots, M_r'''\}$ is a minimum monophonic graphoidal cover of G and so

$$\eta_m(G) = (3r+3) + (r-3) + r(s-3) + (rs-2) = r(2s+4) - 2. \qquad \square$$

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