International J.Math. Combin. Vol.1(2013), 38-45

m^{th}-Root Randers Change of a Finsler Metric

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Abstract: In this paper, we introduce a m^{th}-root Randers changed Finsler metric as

$$\bar{L}(x,y) = L(x,y) + \beta(x,y),$$

where $L = \{a_{i_1 i_2 \cdots i_m}(x)y^{i_1}y^{i_2}\cdots y^{i_m}\}^{\frac{1}{m}}$ is a m^{th}-root metric and β-is one form. Further we obtained the relation between the v- and hv- curvature tensor of m^{th}-root Finsler space and its m^{th}-root Randers changed Finsler space and obtained some theorems for its S3 and S4-likeness of Finsler spaces and when this changed Finsler space will be Berwald space (resp. Landsberg space). Also we obtain T-tensor for the m^{th}-root Randers changed Finsler space $\bar{F}^n$.

Key Words: Randers change, m^{th}-root metric, Berwald space, Landsberg space, S3 and S4-like Finsler space.

AMS(2010): 53B40, 53C60

§1. Introduction

Let $F^n = (M^n, L)$ be a n-dimensional Finsler space, whose M^n is the n-dimensional differentiable manifold and $L(x,y)$ is the Finsler fundamental function. In general, $L(x,y)$ is a function of point $x = (x^i)$ and element of support $y = (y^i)$, and positively homogeneous of degree one in y. In the year 1971 Matsumoto [6] introduced the transformations of Finsler metric given by

$$L^{'}(x,y) = L(x,y) + \beta(x,y)$$
$$L^{''2}(x,y) = L^2(x,y) + \beta^2(x,y),$$

where, $\beta = b_i(x)y^i$ is a one-form [1] and $b_i(x)$ are components of covariant vector which is a function of position alone. If $L(x,y)$ is a Riemannian metric, then the Finsler space with a metric $L(x,y) = \alpha(x,y) + \beta(x,y)$ is known as Randers space which is introduced by G.Randers [5]. In papers [3, 7, 8, 9], Randers spaces have been studied from a geometrical viewpoint and various theorem were obtained. In 1978, Numata [10] introduced another β-change of Finsler metric given by $L(x,y) = \mu(x,y) + \beta(x,y)$ where $\mu = \{a_{ij}(y)y^iy^j\}^{\frac{1}{2}}$ is a Minkowski metric and β as above. This metric is of similar form of Randers one, but there are different tensor

[1]Received October 25, 2012. Accepted March 4, 2013.

properties, because the Riemannian space with the metric α is characterized by $C^i_{jk} = 0$ and on the other hand the locally Minkowski space with the metric μ by $R_{hijk} = 0,\ C_{hij|k} = 0$.

In the year 1979, Shimada [4] introduced the concept of m^{th} root metric and developed it as an interesting example of Finsler metrics, immediately following M.Matsumoto and S.Numatas theory of cubic metrics [2]. By introducing the regularity of the metric various fundamental quantities as a Finsler metric could be found. In particular, the Cartan connection of a Finsler space with m-th root metric could be discussed from the theoretical standpoint. In 1992-1993, the m-th root metrics have begun to be applied to theoretical physics [11, 12], but the results of the investigations are not yet ready for acceding to the demands of various applications.

In the present paper we introduce a m^{th}-root Randers changed Finsler metric as

$$\bar{L}(x, y) = L(x, y) + \beta(x, y)$$

where $L = \{a_{i_1 i_2 \cdots i_m}(x) y^{i_1} y^{i_2 \cdots y^{i_m}}\}^{\frac{1}{m}}$ is a m^{th}-root metric. This metric is of the similar form to the Randers one in the sense that the Riemannian metric is replaced with the m^{th}-root metric, due to this we call this change as m^{th}-root Randers change of the Finsler metric. Further we obtained the relation between the v-and hv-curvature tensor of m^{th}-root Finsler space and its m^{th}-root Randers changed Finsler space and obtained some theorems for its S3 and S4-likeness of Finsler spaces and when this changed Finsler space will be Berwald space (resp. Landsberg space). Also we obtain T-tensor for the m^{th}-root Randers changed Finsler space $\bar{F}^n$.

§2. The Fundamental Tensors of $\bar{F}^n$

We consider an n-dimensional Finsler space $\bar{F}^n$ with a metric $\bar{L}(x, y)$ given by

$$\bar{L}(x, y) = L(x, y) + b_i(x) y^i \tag{1}$$

where

$$L = \{a_{i_1 i_2 \cdots i_m}(x) y^{i_1} y^{i_2} \cdots y^{i_m}\}^{\frac{1}{m}} \tag{2}$$

By putting

$$\begin{aligned} &(\mathrm{I}).\quad L^{m-1} a_i(x, y) &&= a_{i i_2 \cdots i_m}(x) y^{i_2} y^{i_3} \cdots y^{i_m} \\ &(\mathrm{II}).\ \ L^{m-2} a_{ij}(x, y) &&= a_{ij i_3 i_4 \cdots i_m}(x) y^{i_3} y^{i_4} \cdots y^{i_m} \\ &(\mathrm{III}).\ L^{m-3} a_{ijk}(x, y) &&= a_{ijk i_4 i_5 \cdots i_m}(x) y^{i_4} y^{i_5} \cdots y^{i_m} \end{aligned} \tag{3}$$

Now differentiating equation (1) with respect to y^i, we get the normalized supporting element $\bar{l}_i = \dot{\partial}_i \bar{L}$ as

$$\bar{l}_i = a_i + b_i \tag{4}$$

where $a_i = l_i$ is the normalized supporting element for the m^{th}-root metric. Again differentiating above equation with respect to y^j, the angular metric tensor $\bar{h}_{ij} = \bar{L}\dot{\partial}_i \dot{\partial}_j \bar{L}$ is given as

$$\frac{\bar{h}_{ij}}{\bar{L}} = \frac{h_{ij}}{L} \tag{5}$$

where h_{ij} is the angular metric tensor of m^{th}-root Finsler space with metric L given by [4]

$$h_{ij} = (m-1)(a_{ij} - a_i a_j) \tag{6}$$

The fundamental metric tensor $\bar{g}_{ij} = \dot{\partial}_i \dot{\partial}_j \dfrac{\bar{L}^2}{2} = \bar{h}_{ij} + \bar{l}_i \bar{l}_j$ of Finsler space F^n are obtained from equations (4), (5) and (6), which is given by

$$\bar{g}_{ij} = (m-1)\tau a_{ij} + \{1-(m-1)\tau\} a_i a_j + (a_i b_j + a_j b_i) + b_i b_j \tag{7}$$

where $\tau = \dfrac{\bar{L}}{L}$. It is easy to show that

$$\dot{\partial}_i \tau = \frac{\{(1-\tau)a_i + b_i\}}{L}, \quad \dot{\partial}_j a_i = \frac{(m-1)(a_{ij} - a_i a_j)}{L}, \quad \dot{\partial}_k a_{ij} = \frac{(m-2)(a_{ijk} - a_{ij} a_k)}{L}$$

Therefore from (7), it follows (h)hv-torsion tensor $\bar{C}_{ijk} = \dot{\partial}_k \dfrac{\bar{g}_{ij}}{2}$ of the Cartan's connection $C\Gamma$ are given by

$$\begin{aligned} 2L\bar{C}_{ijk} \;=\; & (m-1)(m-2)\tau a_{ijk} + [\{1-(m-1)\tau\}(m-1)](a_{ij}a_k \\ & + a_{jk}a_i + a_{ki}a_j) + (m-1)(a_{ij}b_k + a_{jk}b_i + a_{ki}b_j) - \\ & (m-1)(a_i a_j b_k + a_j a_k b_i + a_i a_k b_j) + (m-1)\{(2m-1)\tau - 3\} a_i a_j a_k \end{aligned} \tag{8}$$

In view of equation (6) the equation (8) may be written as

$$\bar{C}_{ijk} = \tau C_{ijk} + \frac{(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j)}{2L} \tag{9}$$

where $m_i = b_i - \dfrac{\beta}{L} a_i$ and C_{ijk} is the (h)hv-torsion tensor of the Cartan's connection $C\Gamma$ of the m^{th}-root Finsler metric L given by

$$2LC_{ijk} = (m-1)(m-2)\{a_{ijk} - (a_{ij}a_k + a_{jk}a_i + a_{ki}a_j) + 2a_i a_j a_k\} \tag{10}$$

Let us suppose that the intrinsic metric tensor $a_{ij}(x, y)$ of the m^{th}-root metric L has non-vanishing determinant. Then the inverse matrix (a^{ij}) of (a_{ij}) exists. Therefore the reciprocal metric tensor $\bar{g}^{ij}$ of $\bar{F}^n$ is obtain from equation (7) which is given by

$$\bar{g}^{ij} = \frac{1}{(m-1)\tau} a^{ij} + \frac{b^2 + (m-1)\tau - 1}{(m-1)\tau(1+q)^2} a^i a^j - \frac{(a^i b^j + a^j b^i)}{(m-1)\tau(1+q)} \tag{11}$$

where $a^i = a^{ij} a_j$, $b^i = a^{ij} b_j$, $b^2 = b^i b_i$, $q = a^i b_i = a_i b^i = \beta/L$.

Proposition 2.1 *The normalized supporting element l_i, angular metric tensor h_{ij}, metric tensor g_{ij} and (h)hv-torsion tensor C_{ijk} of Finsler space with m^{th}-root Randers changed metric are given by (4), (5), (7) and (9) respectively.*

§3. The v-Curvature Tensor of $\bar{F}^n$

From (6), (10) and definition of m_i and a^i, we get the following identities

$$\begin{aligned} & a^i a_i = 1, \quad a_{ijk} a^i = a_{jk}, \quad C_{ijk} a^i = 0, \quad h_{ij} a^i = 0, \\ & m_i a^i = 0, \quad h_{ij} b^j = 3m_i, \quad m_i b^i = (b^2 - q^2) \end{aligned} \tag{12}$$

To find the v-curvature tensor of F^n, we first find (h)hv-torsion tensor $\bar{C}^i_{jk} = \bar{g}^{ir}\bar{C}_{jrk}$

$$\begin{aligned}\bar{C}^i_{jk} &= \frac{1}{m-1}C^i_{jk} + \frac{1}{2(m-1)\bar{L}}(h^i_j m_k + h^i_k m_j + h_{jk}m^i) - \\ &\quad \frac{a^i}{\bar{L}(1+q)}\{m_j m_k + \frac{1}{(m-1)(m-2)}h_{jk}\} - \frac{1}{(m-1)(1+q)}a^i C_{jrk}b^r\end{aligned} \tag{13}$$

where $LC^i_{jk} = LC_{jrk}a^{ir} = (m-1)\{a^i_{jk} - (\delta^i_j a_k + \delta^i_k a_j + a^i a_{jk}) + 2a^i a_j a_k\}$,

$$\begin{aligned}h^i_j &= h_{jr}a^{ir} = (m-1)(\delta^i_j - a^i a_j) \\ m^i &= m_r a^{ir} = b^i - qa^i, \quad \textit{and} \quad a^i_{jk} = a^{ir}a_{jrk}\end{aligned} \tag{14}$$

From (12) and (14), we have the following identities

$$\begin{aligned}&C_{ijr}h^r_p = C^r_{ij}h_{pr} = (m-1)C_{ijp}, \quad C_{ijr}m^r = C_{ijr}b^r, \\ &m_r h^r_i = (m-1)m_i, \quad m_i m^i = (b^2 - q^2), \\ &h_{ir}h^r_j = (m-1)h_{ij}, \quad h_{ir}m^r = (m-1)m_i\end{aligned} \tag{15}$$

From (9) and (13), we get after applying the identities (15)

$$\begin{aligned}\bar{C}_{ijr}\bar{C}^r_{hk} &= \frac{\tau}{(m-1)}C_{ijr}C^r_{hk} + \frac{1}{2L}(C_{ijh}m_k + C_{ijk}m_h + C_{hjk}m_i + C_{hik}m_j) \\ &\quad + \frac{1}{2(m-1)}(C_{ijr}h_{hk} + C_{hrk}h_{ij})b^r + \frac{1}{4(m-1)L\bar{L}}(b^2 - q^2)h_{ij}h_{hk} \\ &\quad + \frac{1}{4L\bar{L}}(2h_{ij}m_h m_k + 2h_{kh}m_i m_j + h_{jh}m_i m_k \\ &\quad + h_{jk}m_i m_h + h_{ih}m_j m_k + h_{ik}m_j m_h)\end{aligned} \tag{16}$$

Now we shall find the v-curvature tensor $\bar{S}_{hijk} = \bar{C}_{ijr}\bar{C}^r_{hk} - \bar{C}_{ikr}\bar{C}^r_{hj}$. The tensor is obtained from (16) and given by

$$\begin{aligned}\bar{S}_{hijk} &= \Theta_{(jk)}\{\frac{\tau}{m-1}C_{ijr}C^r_{hk} + h_{ij}m_{hk} + h_{hk}m_{ij}\} \\ &= \frac{\tau}{(m-1)}S_{hijk} + \Theta_{(jk)}\{h_{ij}m_{hk} + h_{hk}m_{ij}\}\end{aligned} \tag{17}$$

where

$$m_{ij} = \frac{1}{2(m-1)L}\{C_{ijr}b^r + \frac{(b^2 - q^2)}{4\bar{L}}h_{ij} + \frac{(m-1)}{2}\bar{L}^{-1}m_i m_j\} \tag{18}$$

and the symbol $\Theta_{(jk)}\{\cdots\}$ denotes the exchange of j, k and subtraction.

Proposition 3.1 *The v-curvature tensor $\bar{S}_{hijk}$ of m^{th}-root Randers changed Finsler space $\bar{F}^n$ with respect to Cartan's connection $C\Gamma$ is of the form (17).*

It is well known [13] that the v-curvature tensor of any three-dimensional Finsler space is of the form

$$L^2 S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij}) \tag{19}$$

Owing to this fact M. Matsumoto [13] defined the S3-like Finsler space F^n $(n \geq 3)$ as such a Finsler space in which v-curvature tensor is of the form (19). The scalar S in (19) is a function of x alone.

The v-curvature tensor of any four-dimensional Finsler space may be written as [13]

$$L^2 S_{hijk} = \Theta_{(jk)}\{h_{hj}K_{ki} + h_{ik}K_{hj}\} \tag{20}$$

where K_{ij} is a (0, 2) type symmetric Finsler tensor field which is such that $K_{ij}y^j = 0$. A Finsler space $F^n (n \geq 4)$ is called S4-like Finsler space [13] if its v-curvature tensor is of the form (20).

From (17), (19), (20) and (5) we have the following theorems.

Theorem 3.1 *The m^{th}-root Randers changed S3-like or S4-like Finsler space is S4-like Finsler space.*

Theorem 3.2 *If v-curvature tensor of m^{th}-root Randers changed Finsler space $\bar{F}^n$ vanishes, then the Finsler space with m^{th}-root metric F^n is S4-like Finsler space.*

If v-curvature tensor of Finsler space with m^{th}-root metric F^n vanishes then equation (17) reduces to

$$\bar{S}_{hijk} = h_{ij}m_{hk} + h_{hk}m_{ij} - h_{ik}m_{hj} - h_{hj}m_{ik} \tag{21}$$

By virtue of (21) and (11) and the Ricci tensor $\bar{S}_{ik} = \bar{g}^{hk}\bar{S}_{hijk}$ is of the form

$$\bar{S}_{ik} = (-\frac{1}{(m-1)\tau})\{mh_{ik} + (m-1)(n-3)m_{ik}\},$$

where $m = m_{ij}a^{ij}$, which in view of (18) may be written as

$$\bar{S}_{ik} + H_1 h_{ik} + H_2 C_{ikr}b^r = H_3 m_i m_k, \tag{22}$$

where

$$H_1 = \frac{m}{(m-1)\tau} + \frac{(n-3)(b^2-q^2)}{8(m-1)\bar{L}^2},$$
$$H_2 = \frac{(n-3)}{2(m-1)\bar{L}},$$
$$H_3 = -\frac{(n-3)}{2\bar{L}^2}.$$

From (22), we have the following

Theorem 3.3 *If v-curvature tensor of m^{th}-root Randers changed Finsler space $\bar{F}^n$ vanishes then there exist scalar H_1 and H_2 in Finsler space with m^{th}-root metric $F^n (n \geq 4)$ such that matrix $\| \bar{S}_{ik} + H_1 h_{ik} + H_2 C_{ikr}b^r \|$ is of rank two.*

§4. The $(v)hv$-Torsion Tensor and hv-Curvature Tensor of $\bar{F}^n$

Now we concerned with $(v)hv$-torsion tensor P_{ijk} and hv-curvature tensor P_{hijk}. With respect to the Cartan connection $C\Gamma$, $L_{|i} = 0,\ l_{i|j} = 0,\ h_{ij|k} = 0$ hold good [13].

Taking h-covariant derivative of equation (9) and using (4) and $l_i = a_i = 0$ we have

$$\bar{C}_{ijk|h} = \tau C_{ijk|h} + \frac{b_{i|h}}{L}C_{ijk} + \frac{(h_{ij}b_{k|h} + h_{jk}b_{i|h} + h_{ki}b_{j|h})}{2L} \tag{23}$$

From equation (6) and using relation $h_{ij|h} = 0$ We have

$$a_{ij|h} = 0, \quad \text{and} \quad a_{ijk|h} = \frac{2LC_{ijk|h}}{(m-1)(m-2)} \tag{24}$$

The $(v)hv$-torsion tensor P_{ijk} and the hv-curvature tensor P_{hijk} of the Cartan connection $C\Gamma$ are written in the form, respectively

$$P_{ijk} = C_{ijk|0}, \tag{25}$$

$$P_{hijk} = C_{ijk|h} - C_{hjk|i} + P_{ikr}C^r_{jk} - P_{hkr}C^r_{ji}$$

where the subscript '0' means the contraction for the supporting element y^i. Therefore the $(v)hv$-torsion tensor $\bar{P}_{ijk}$ and the hv-curvature tensor $\bar{P}_{hijk}$ of the Cartan connection $C\Gamma$ for the Finsler space with m^{th}-root Randers metric by using (10), (23), (24) and (25) we have

$$\bar{P}_{ijk} = \frac{(m-1)(m-2)}{2L}\tau a_{ijk|0} + \frac{b_{i|0}}{L}C_{ijk} + \frac{(h_{ij}b_{k|0} + h_{jk}b_{i|0} + h_{ki}b_{j|0})}{2L} \tag{26}$$

and

$$\bar{P}_{hijk} = (m-1)(m-2)(2L)^{-1}\Theta_{(jk)}(a_{ijk|h} + \bar{P}_{ikr}\bar{C}^r_{jh}) \tag{27}$$

Definition 4.1([13]) *A Finsler space is called a Berwald space (resp. Landsberg space) if $C_{ijk|h} = 0$ (resp. $P_{ijk} = 0$) holds good.*

Consequently, from (24) and (26) we have

Theorem 4.1 *A Finsler space with the m^{th}-root Randers changed metric is a Berwald space (resp. Landsberg space), if and only if $a_{ijk|h} = 0$ (resp. $a_{ijk|0} = 0$ and $b_{i|h}$ is covariently constant.*

Proposition 4.1 *The $v(hv)$-torsion tensor and hv-curvature tensor $\bar{P}_{hijk}$ of m^{th}-root Randers changed Finsler space $\bar{F}^n$ with respect to Cartan's connection $C\Gamma$ is of the form (26) and (27).*

§5. **T-Tensor of $\bar{F}^n$**

Now, the T-tensor is given by [11,13]

$$T_{hijk} = LC_{hij}|_k + l_iC_{hjk} + l_jC_{hik} + l_kC_{hij} + l_hC_{ijk}$$

The above equation for m^{th}-root Randers changed Finsler space $\bar{F}^n$ is given as

$$\bar{T}_{hijk} = \bar{L}\bar{C}_{hij}|_k + \bar{l}_i\bar{C}_{hjk} + \bar{l}_j\bar{C}_{hik} + \bar{l}_k\bar{C}_{hij} + \bar{l}_h\bar{C}_{ijk} \tag{28}$$

The v-derivative of h_{ij} and L is given by [13]

$$h_{ij}|_k = -\frac{1}{L}(h_{ik}l_j + h_{jk}l_i), \quad \textit{and} \quad L|_i = l_i \tag{29}$$

Now using (29), the v-derivative of C_{ijk} is given as

$$\begin{aligned}\bar{L}\bar{C}_{ijk}|_h &= \tau\frac{(Lb_h-\beta l_h)}{L}C_{ijk}+\bar{L}\tau C_{ijk}|_h-\tau\frac{1}{2L}(h_{ih}l_jm_k+h_{jh}l_im_k \\ &+h_{jh}l_km_i+h_{kh}l_jm_i+h_{ih}l_km_j+h_{kh}l_im_j+h_{ij}l_hm_k \\ &+h_{jk}l_hm_i+h_{ki}l_hm_j)+\frac{\tau}{2}(h_{ij}m_k|_h+h_{jk}m_i|_h+h_{ki}m_j|_h)\end{aligned} \tag{30}$$

Using (4), (9) and (30), the T-tensor for m^{th}-root Randers changed Finsler space $\bar{F}^n$ is given by

$$\begin{aligned}\bar{T}_{hijk} &= \tau(T_{hijk}+B_{hijk})+\frac{\tau}{2L}(h_{jk}m_hl_i+h_{ik}m_hl_j+h_{ij}m_hl_k \\ &+h_{ki}m_jl_h)+\frac{1}{2L}(h_{hj}m_kb_i+h_{jk}m_hb_i+h_{kh}m_jb_i+h_{ik}m_hb_j \\ &+h_{ih}m_kb_j+h_{kh}m_ib_j+h_{ij}m_hb_k+h_{ih}m_jb_k+h_{jh}m_ib_k+h_{ij}m_kb_h \\ &+h_{ki}m_jb_h+h_{jk}m_ib_h)+\frac{\tau}{2}(h_{ij}m_k|_h+h_{jk}m_i|_h+h_{ki}m_j|_h) \\ &+\tau\frac{(Lb_h-\beta l_h)}{L}C_{ijk}\end{aligned} \tag{31}$$

where $B_{hijk}=\beta C_{hij}|_k+b_iC_{hjk}+b_jC_{hik}+b_kC_{hij}+b_hC_{ijk}$. Thus, we know

Proposition 5.1 *The T-tensor $\bar{T}_{hijk}$ for m^{th}-root Randers changed Finsler space $\bar{F}^n$ is given by (31).*

Acknowledgment

The first author is very much thankful to NBHM-DAE of government of India for their financial assistance as a Postdoctoral Fellowship.

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