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Note on Full Signed Graphs and Full Line Signed Graphs

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Abstract: In this paper, we introduced the new notions full signed graph and full line signed graph of a signed graph and its properties are obtained. Also, we obtained the structural characterizations of these notions. Further, we presented some interesting switching equivalent characterizations.

Key Words: Signed graphs, neutrosophic signed graph, balance, switching, full signed graph, full line signed graph.

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§1. Introduction

For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [1]. The non-standard will be given in this paper as and when required.

To model individuals' preferences towards each other in a group, Harary [2] introduced the concept of signed graphs in 1953. A signed graph $S = (G, \sigma)$ is a graph $G = (V, E)$ whose edges are labeled with positive and negative signs (i.e., $\sigma : E(G) \to \{+, -\}$). The vertices of a graph represent people and an edge connecting two nodes signifies a relationship between individuals. The signed graph captures the attitudes between people, where a positive (negative edge) represents liking (disliking). A neutrosophic signed graph $S^N = (G, \sigma, H)$ for a subgraph $H \subset G$ with property $\mathscr{P}$ is such a graph that $G \setminus H$ is a signed graph but H is indefinite for those of uncertainties in reality. Certainly, if there are no indefinite subgraph in G, it must be a signed graph. An unsigned graph is a signed graph with the signs removed. Similar to an unsigned graph, there are many active areas of research for signed graphs.

The sign of a cycle (this is the edge set of a simple cycle) is defined to be the product of the signs of its edges; in other words, a cycle is positive if it contains an even number of negative edges and negative if it contains an odd number of negative edges. A signed graph S is said to be balanced if every cycle in it is positive. A signed graph S is called totally unbalanced if every cycle in S is negative. A chord is an edge joining two non adjacent vertices in a cycle.

A *marking* of S is a function $\zeta : V(G) \to \{+, -\}$. Given a signed graph S one can easily

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define a marking ζ of S as follows: For any vertex $v \in V(S)$,

$$\zeta(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking ζ of S is called *canonical marking* of S. For more new notions on signed graphs refer the papers (see [6, 8, 9, 13-17, 17-26]).

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1 *A signed graph S is balanced if and only if either of the following equivalent conditions is satisfied:*

(i) Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary [2]).

(ii) There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$ (Sampathkumar [6]).

Switching S with respect to a marking ζ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs.

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be *weakly isomorphic* (see [28]) or *cycle isomorphic* (see [29]) if there exists an isomorphism $\phi : G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known.

Theorem 1.2(T. Zaslavsky [29]) *Given a graph G, any two signed graphs in $\psi(G)$, where $\psi(G)$ denotes the set of all the signed graphs possible for a graph G, are switching equivalent if and only if they are cycle isomorphic.*

§2. **Full Signed Graph of a Signed Graph**

Let $G = (V, E)$ be a graph and the full graph $\mathcal{FG}(G)$ of G is a graph whose vertex is the union of vertices, edges and blocks of G in which two vertices are adjacent if the corresponding members of G are adjacent or incident (see [4]). Let $G = (V, E)$ be a graph. Then G is a connected graph if and only if $\mathcal{FG}(G)$ is connected.

Motivated by the existing definition of complement of a signed graph, we now extend the notion of full graphs to signed graphs as follows: The *full signed graph* $\mathcal{FS}(S) = (\mathcal{FG}(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{FG}(G)$ and sign of any edge uv is $\mathcal{FS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called a full signed graph, if $S \cong \mathcal{FS}(S')$ for some signed graph S'. The following result restricts the class of full signed graphs.

Theorem 2.1 *For any signed graph $S = (G, \sigma)$, its full signed graph $\mathcal{FS}(S)$ is balanced.*

Proof Since sign of any edge $e = uv$ in $\mathcal{FS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S, by Theorem 1.1, $\mathcal{FS}(S)$ is balanced. □

For any positive integer k, the k^{th} iterated full signed graph, $\mathcal{FS}^k(S)$ of S is defined as follows:

$$\mathcal{FS}^0(S) = S,\ \mathcal{FS}^k(S) = \mathcal{FS}(\mathcal{FS}^{k-1}(S)).$$

Corollary 2.2 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $\mathcal{FS}^k(S)$ is balanced.*

Corollary 2.3 *For any two signed graphs S_1 and S_2 with the same underlying graph, $\mathcal{FS}(S_1) \sim \mathcal{FS}(S_2)$.*

The following result characterize signed graphs which are full signed graphs.

Theorem 2.4 *A signed graph $S = (G, \sigma)$ is a full signed graph if, and only if, S is balanced signed graph and its underlying graph G is a full graph.*

Proof Suppose that S is balanced and G is a full graph. Then there exists a graph G' such that $\mathcal{FG}(G') \cong G$. Since S is balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge uv in S satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $\mathcal{FS}(S') \cong S$. Hence S is a full signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a full signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that

$$\mathcal{FS}(S') \cong S.$$

Hence, G is the full graph of G' and by Theorem 2.1, S is balanced. □

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [3] as follows:

$\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the $\mathcal{FS}(S)$ is balanced (Theorem 2.1). We now examine, the conditions under which negation $\eta(S)$ of $\mathcal{FS}(S)$ is balanced.

Proposition 2.5 *Let $S = (G, \sigma)$ be a signed graph. If $\mathcal{FG}(G)$ is bipartite then $\eta(\mathcal{FS}(S))$ is balanced.*

Proof Since, by Theorem 2.1, $\mathcal{FS}(S)$ is balanced, it follows that each cycle C in $\mathcal{FS}(S)$ contains even number of negative edges. Also, since $\mathcal{FG}(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{FS}(S)$ is also even. Hence $\eta(\mathcal{FS}(S))$ is balanced. □

§3. Full Line Signed Graph of a Signed Graph

Let $G = (V, E)$ be a graph and the full line graph $\mathcal{FLG}(G)$ of a graph G is a graph and $V(\mathcal{FLG}(G))$ is the union of the set of vertices, edges and blocks of G in which two vertices are joined by an edge in $\mathcal{SFL}(G)$ if the corresponding vertices and edges of G are adjacent or the corresponding members of G are incident (See [5]).

Motivated by the existing definition of complement of a signed graph, we now extend the notion of full line graphs to signed graphs as follows: The *full line signed graph* $\mathcal{FLS}(S) = (\mathcal{FLG}(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{FLG}(G)$ and sign of any edge uv is $\mathcal{FLS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called a full line signed graph, if $S \cong \mathcal{FLS}(S')$ for some signed graph S'. The following result restricts the class of full line signed graphs.

Theorem 3.1 *For any signed graph $S = (G, \sigma)$, its full line signed graph $\mathcal{FLS}(S)$ is balanced.*

Proof Since sign of any edge $e = uv$ in $\mathcal{FLS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S, by Theorem 1.1, $\mathcal{FLS}(S)$ is balanced. □

For any positive integer k, the k^{th} iterated full line signed graph, $\mathcal{FLS}^k(S)$ of S is defined as follows:

$$\mathcal{FLS}^0(S) = S, \, \mathcal{FLS}^k(S) = \mathcal{FLS}(\mathcal{FLS}^{k-1}(S)).$$

Corollary 3.2 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $\mathcal{FLS}^k(S)$ is balanced.*

Corollary 3.3 *For any two signed graphs S_1 and S_2 with the same underlying graph, $\mathcal{FLS}(S_1) \sim \mathcal{FLS}(S_2)$.*

The following result characterize signed graphs which are full line signed graphs.

Theorem 3.4 *A signed graph $S = (G, \sigma)$ is a full line signed graph if, and only if, S is balanced signed graph and its underlying graph G is a full line graph.*

Proof Suppose that S is balanced and G is a full line graph. Then there exists a graph G' such that $\mathcal{FLG}(G') \cong G$. Since S is balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge uv in S satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $\mathcal{FLS}(S') \cong S$. Hence S is a full line signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a full line signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $\mathcal{FLS}(S') \cong S$. Hence, G is the full line graph of G' and by Theorem 2.1, S is balanced. □

For a signed graph $S = (G, \sigma)$, the $\mathcal{FLS}(S)$ is balanced (Theorem 3.1). We now examine, the conditions under which negation $\eta(S)$ of $\mathcal{FLS}(S)$ is balanced.

Proposition 3.5 *Let $S = (G, \sigma)$ be a signed graph. If $\mathcal{FLG}(G)$ is bipartite then $\eta(\mathcal{FLS}(S))$ is balanced.*

Proof Since, by Theorem 3.1, $\mathcal{FLS}(S)$ is balanced, it follows that each cycle C in $\mathcal{FLS}(S)$ contains even number of negative edges. Also, since $\mathcal{FLG}(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{FLS}(S)$ is also even. Hence $\eta(\mathcal{FLS}(S))$ is balanced. □

§4. Switching Equivalence of Full Signed Graphs and Full Line Signed Graphs

In [5], the authors remarked that $\mathcal{FLG}(G)$ and $\mathcal{FG}(G)$ are isomorphic if and only if G is a block. We now give a characterization of signed graphs whose full signed graphs are switching equivalent to their full line signed graphs.

Theorem 4.1 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FS}(S) \sim \mathcal{FLS}(S)$ if and only if G is a block.*

Proof Suppose $\mathcal{FS}(S) \sim \mathcal{FLS}(S)$. This implies that $\mathcal{FG}(G) \cong \mathcal{FLG}(G)$ and hence, G is a block. Conversely, suppose that G is a block. Then

$$\mathcal{FG}(G) \cong \mathcal{FLG}(G).$$

Now, if S any signed graph with G is a block, by Theorems 2.1 and 3.1, $\mathcal{FS}(S)$ and $\mathcal{FLS}(S)$ are balanced and hence, the result follows from Theorem 1.2. This completes the proof. □

In view of the negation operator introduced by Harary [3], we have the following cycle isomorphic characterizations.

Corollary 4.2 *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $\eta(\mathcal{FS}(S_1)) \sim \eta(\mathcal{FS}(S_2))$ if G_1 and G_2 are isomorphic.*

Corollary 4.3 *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $\eta(\mathcal{FLS}(S_1)) \sim \eta(\mathcal{FLS}(S_2))$ if G_1 and G_2 are isomorphic.*

Corollary 4.4 *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $\mathcal{FS}(\eta(S_1))$ and $\mathcal{FS}(\eta(S_2))$ are cycle isomorphic if G_1 and G_2 are isomorphic.*

Corollary 4.5 *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $\mathcal{FLS}(\eta(S_1))$ and $\mathcal{FLS}(\eta(S_2))$ are cycle isomorphic if G_1 and G_2 are isomorphic.*

Corollary 4.6 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FS}(\eta(S)) \sim \mathcal{FLS}(S)$ if and only if G is a block.*

Corollary 4.7 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FS}(S) \sim \mathcal{FLS}(\eta(S))$ if and only if G is a block.*

Corollary 4.8 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FS}(\eta(S)) \sim \mathcal{FLS}(\eta(S))$ if and only if G is a block.*

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