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# Number of Spanning Trees of Sequence of Some Families of Graphs That Have the Same Average Degree and Their Entropies

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**Abstract**: The number of spanning trees is an important quantity characterizing the reliability of a network (graph). Generally, the number of spanning trees in a network can be obtained directly by calculating a related determinant corresponding to the network. However, for a large network, evaluating the relevant determinant is intractable. In this paper, we investigated the number of spanning trees in three sequences of families of graphs of the same average degree $\frac{16}{3}$. We used the electrically equivalent transformations and rules of weighted generating function which avoids the laborious computation of the determinant for counting the number of spanning trees. Finally, we determined the entropy of our studied graphs.

**Key Words**: Number of spanning trees, electrically equivalent transformations, entropy.

**AMS(2010)**: 05C30, 05C50, 05C63.

## §1. Introduction

The counting spanning trees in networks (graphs) is a fascinating and central issue in statistical physics, because of its inherent relevance to diverse aspects in related fields. For instance, the number of spanning trees is an important measure of reliability of a network [1], [2]. Again, for example, it is exactly the number of recurrent configurations of the Abelian sand-pile models [3],[4], which have been studied extensively in the past two decades as a paradigm of the self-organized criticality [5]. On the other hand, the problem of spanning trees has numerous connections with other interesting problems associated with networks, such as dimer coverings [8], Potts model [7] random walks [8], origin of fractality for fractal scale-free networks [8, 9] and many others.

The number of spanning trees $\tau(G)$ of a finite connected undirected graph $G$ is an acyclic $(n-1)$ - edge spanning sub-graph.

There exist various methods for finding this number. Kirchhoff's matrix tree theorem named after Gustav Kirchhoff[10] is a theorem about the number of spanning trees in a graph,

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showing that this number can be computed in polynomial time from the determinant of a submatrix of the Laplacian matrix of the graph; specifically, the number is equal to any co-factor of the Laplacian matrix.

Another method to count the complexity of a graph is using Laplacian eigenvalues. Let $G$ be a connected graph with $k$ vertices. Kelmans and Chelnoknoy [11] derived the following formula:

$$\tau(G) = \frac{1}{k}\prod_{i=1}^{k-1}\mu_i, \tag{1.1}$$

where $k = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k = 0$ are the eigenvalues of the Kirchhoff matrix $L$.

Degenerating the graph through successive elimination of contraction of its edges represent the core of another way to compute the complexity of a graph [12, 13, 14]. If $G = (V, E)$ is a multigraph with $e \in E$, then $G.e$ is the graph obtained from $G$ by contracting the degree until its endpoints are a single vertex. The formula for computing the number of spanning trees of a multigraph $G$ is given by:

$$\tau(G) = \tau(G - e) + \tau(G.e). \tag{1.2}$$

This formula is beautiful but not practically useful (grows exponentially with the size of the graph- may be as many as $2^{|E(G)|}$ terms. For a summary of further results for calculating umber of spanning trees of graphs, see [15, 16, 17, 18].

## §2. Electrically Equivalent Transformations

To begin with, we briefly review the electrically equivalent transformation technique introduced in [19, 20, 21, 22]. An edge-weighted graph G (with the weight function $\omega$ : E(G) $\rightarrow [0, \infty)$) can be considered as an electrical network with the weights being the conductances of the corresponding edges. The weighted number of spanning trees in G is defined as follows:

Let $G$ be an edge weighted graph, $G'$ be the corresponding electrically equivalent graph, $\tau(G)$ denotes the weighted number of spanning trees $G$.

(1) Parallel edges: If two parallel edges with conductances $u$ and $v$ in $G$ are merged into a single edge with conductances $u + v$ in $G'$, then $\tau(G') = \tau(G)$.

(2) Serial edges: If two serial edges with conductances $u$ and $v$ in $G$ are merged into a single edge with conductance $\frac{uv}{u+v}$ in $G'$, then $\tau(G') = \frac{1}{u+v}\tau(G)$.

(3) $\sqcup - Y$ transformation: If a triangle with conductances $u, v$ and $w$ in $G$ is changed into an electrically equivalent star graph with conductances

$$x = \frac{uv + vw + wu}{u}, y = \frac{uv + vw + wu}{v} \text{ and } z = \frac{uv + vw + wu}{w}$$

in $H'$, then $\tau(G') = \frac{(uv+vw+wu)^2}{uvw}\tau(G)$.

(4) $Y - \sqcup$ transformation: If a star graph with conductances $u, v$ and $w$ in $G$ is changed into an electrically equivalent triangle with conductances

$$x = \frac{vw}{u + v + w}, y = \frac{uw}{u + v + w} \text{ and } z = \frac{uv}{u + v + w}$$

in $G'$, then $\tau(G') = \frac{1}{u+v+w}\tau(G)$.

In this work, we compute the number of spanning trees of three sequences of graphs of the same average degree we named it $\mathrm{E}_n$, $\mathrm{F}_n$ and $\mathrm{H}_n$ respectively.

## §3. Number of Spanning Trees in the Sequences of $E_n$ Graph

Consider the sequence of graphs $\mathrm{E}_1, \mathrm{E}_2, \cdots, \mathrm{E}_n$ constructed as shown in Figure 1. According to this construction, the number of total vertices $|V(\mathrm{E}_n)|$ and edges $|E(\mathrm{E}_n)|$ are $|V(\mathrm{E}_n)| = 9n-6$ and $|E(\mathrm{E}_n)| = 24n-21, n = 1, 2, \cdots$. The average degree of $\mathrm{E}_n$ is $16/3$ in the large $n$ limit.

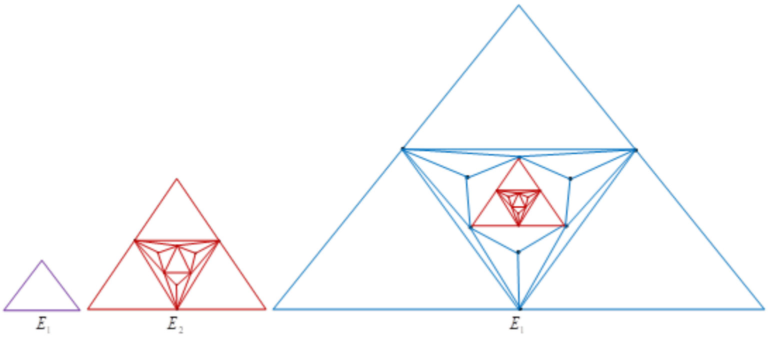


**Figure 1.** Some sequences of graph $\mathrm{E}_n$

**Theorem** 3.1 *For $n \geq 1$, the number of spanning trees in sequence of the graph $\mathrm{E}_n$ is given by*

$$\frac{1}{27} \times 16^{n-4}\left(256 - 13 \times 64^n\right)^2.$$

*Proof* We use the electrically equivalent transformation to transform $\mathrm{E}_i$ to $\mathrm{E}_{i-1}$. Figures 2-6 illustrate the transformation process from $\mathrm{E}_2$ to $\mathrm{E}_1$.

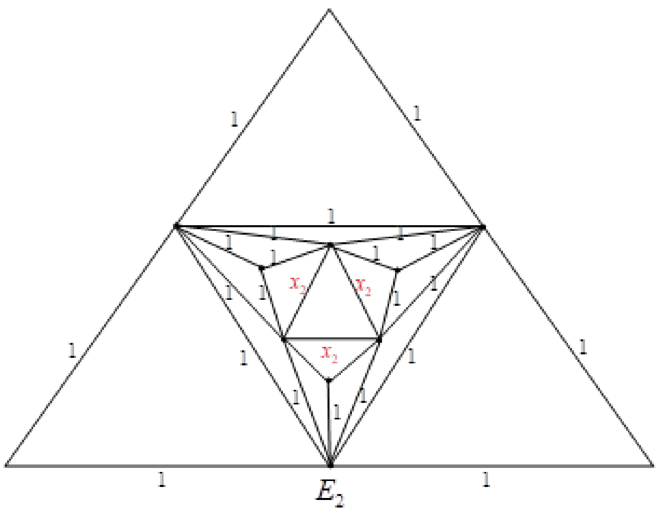


**Figure 2**

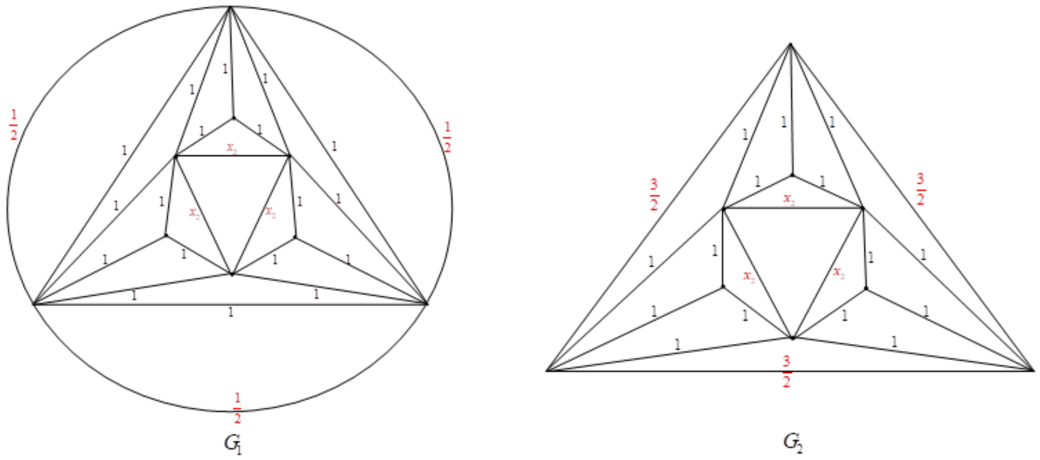



**Figure 3**

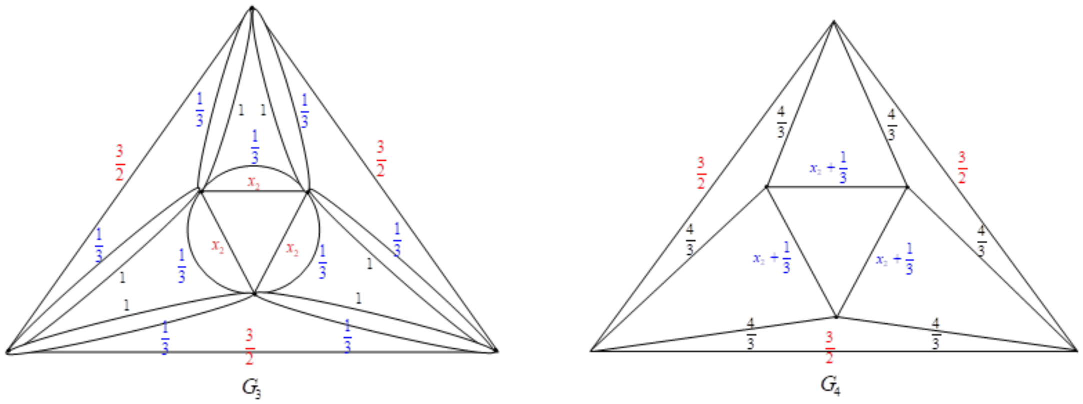



**Figure 4**

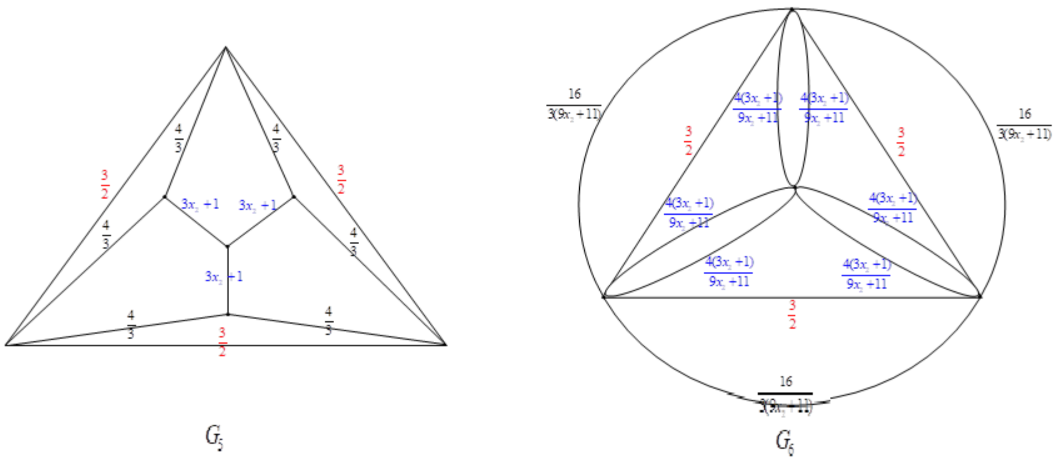



**Figure 5**

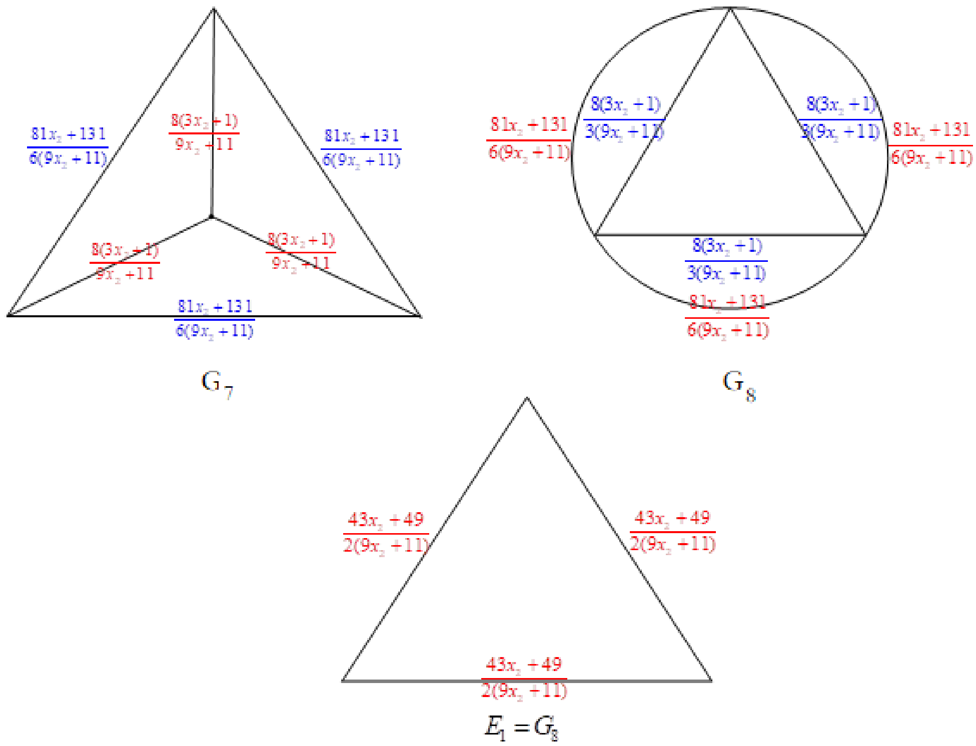



**Figure 6**

Using the properties given in Section 2, we have the following transformations:

$$\begin{aligned}
\tau(G_1) &= \left[\frac{1}{2}\right]^3 \tau(\mathrm{E}_2), \quad \tau(G_2) = \tau(G_1), \quad \tau(G_3) = \left[\frac{1}{3}\right]^3 \tau(G_2), \\
\tau(G_4) &= \tau(G_3), \quad \tau(G_5) = (9x_2+3)\,\tau(G_4), \quad \tau(G_6) = \left[\frac{3}{9x_2+11}\right]^3 \tau(G_5), \\
\tau(G_7) &= \tau(G_6), \quad \tau(G_8) = \frac{(9x_2+11)}{24(3x_2+1)}\tau(G_7) \text{ and } \tau(\mathrm{E}_1) = \tau(G_8).
\end{aligned}$$

Combining these nine transformations, we get

$$\tau(\mathrm{E}_2) = 16\,(18x_2+22)^2\,\tau(\mathrm{E}_1), \tag{3.1}$$

where

$$\tau(\mathrm{E}_1) = 3 \times (16)^{n-1} x_1^2 \left[\prod_{i=2}^{n} (18x_i+22)\right]^2.$$

Further,

$$\tau(\mathrm{E}_n) = \prod_{i=2}^{n} 16\,(18x_2+22)^2, \tag{3.2}$$

where $x_{i-1} = \frac{43x_i+49}{18x_i+22}, i = 2, 3, \cdots, n$. Its characteristic equation is $18\mu^2 - 21\mu - 49 = 0$, which have two roots $\mu_1 = \frac{-7}{6}$ and $\mu_2 = \frac{7}{3}$. Subtracting these two roots into both sides of

$x_{i-1} = \frac{43x_i+49}{18x_i+22}$, we get

$$x_{i-1} + \frac{7}{6} = \frac{43x_i + 49}{18x_i + 22} + \frac{7}{6} = \frac{64\left(x_i + \frac{7}{6}\right)}{(18x_i + 22)}. \tag{3.3}$$

$$x_{i-1} - \frac{7}{3} = \frac{43x_i + 49}{18x_i + 22} - \frac{7}{3} = \frac{\left(x_i - \frac{7}{3}\right)}{(18x_i + 22)}. \tag{3.4}$$

Let $y_i = \frac{x_i+\frac{7}{6}}{x_i-\frac{7}{3}}$. Then by Eqs.(3.3) and (3.4), we get $y_{i-1} = (64)y_i$ and $y_i = (64)^{n-i}y_n$. Therefore,

$$x_i = \frac{(64)^{n-i}\left(\frac{7}{3}\right)y_n + \frac{7}{6}}{(64)^{n-i}y_{n-1}}.$$

Thus

$$x_1 = \frac{(64)^{n-1}\left(\frac{7}{3}\right)y_n + \frac{7}{6}}{(64)^{n-1}y_{n-1}}. \tag{3.5}$$

Using the expression $x_{n-1} = \frac{43x_n+49}{18x_n+22}$ and denoting the coefficients of $43x_n + 49$ and $18x_n + 22$ as $\sigma_n$ and $\delta_n$ we have

$$18x_n + 22 = \sigma_0\left(43x_n + 49\right) + \delta_0\left(18x_n + 22\right),$$

$$18x_{n-1} + 22 = \frac{\sigma_1\left(43x_n + 49\right) + \delta_1\left(18x_n + 22\right)}{\sigma_0\left(43x_n + 49\right) + \delta_0\left(18x_n + 22\right)},$$

$$\vdots$$

$$18x_{n-i} + 22 = \frac{\sigma_i\left(43x_n + 49\right) + \delta_i\left(18x_n + 22\right)}{\sigma_{i-1}\left(43x_n + 49\right) + \delta_{i-1}\left(18x_n + 22\right)'}, \tag{3.6}$$

$$18x_{n-(i+1)} + 22 = \frac{\sigma_{i+1}\left(43x_n + 49\right) + \delta_{i+1}\left(18x_n + 22\right)}{\sigma_i\left(43x_n + 49\right) + \delta_i\left(18x_n + 22\right)}, \tag{3.7}$$

$$\vdots$$

$$18x_2 + 22 = \frac{\sigma_{n-2}\left(43x_n + 49\right) + \delta_{n-2}\left(18x_n + 22\right)}{\sigma_{n-3}\left(43x_n + 49\right) + \delta_{n-3}\left(18x_n + 22\right)'}.$$

Substituting Eq.(3.6) into Eq.(3.2), we obtain

$$\tau\left(\mathrm{E}_n\right) = 3 \times (16)^{n-1}x_1^2\left[\sigma_{n-2}\left(43x_n + 49\right) + \sigma_{n-2}\left(18x_n + 22\right)\right]^2, \tag{3.8}$$

where $\sigma_0 = 0, \delta_0 = 1$ and $\sigma_1 = 18, \delta_1 = 22$.

By the expression $x_{n-1} = \frac{43x_n+49}{18x_n+22}$ and Eqs.(3.6) and (3.7), we have

$$\sigma_{i+1} = 65\sigma_i - 64\sigma_{i-1}; \delta_{i+1} = 65\delta_i - 64\delta_{i-1}. \tag{3.9}$$

The characteristic equation of Eq.(3.9) is $\gamma^2 - 65\gamma + 64 = 0$ which have two roots $\gamma_1 = 64$ and $\gamma_2 = 1$. The general solutions of Eq. (3.9) are $\sigma_i = a_1\gamma_1^i + a_2\gamma_2^i; \delta_i = b_1\gamma_1^i + b_2\gamma_2^i$. Using the initial conditions $\sigma_0 = 0, \delta_0 = 1$ and $\sigma_1 = 18, \delta_1 = 22$, yields

$$\sigma_i = \frac{2}{7}(64)^i - \frac{2}{7}; b_i = \frac{1}{3}(64)^i + \frac{2}{3}. \tag{3.10}$$

If $x_n = 1$, it means that $\mathrm{E}_n$ without any electrically equivalent transformation. Plugging Eq.(3.10) into Eq.(3.8), we have

$$\tau\left(\mathrm{E}_n\right) = 3 \times (16)^{n-1} x_1^2 \left[\frac{832}{21}(64)^{n-2} + \frac{8}{21}\right]^2, n \geq 2. \tag{3.11}$$

When $n = 1, \tau(E) = 3$ which satisfies Eq.(3.11). Therefore, the number of spanning trees in the sequence of the graph $\mathrm{E}_n$ is given by

$$\tau\left(\mathrm{E}_n\right) = 3 \times (16)^{n-1} x_1^2 \left[\frac{832}{21}(64)^{n-2} + \frac{8}{21}\right]^2, n \geq 1, \tag{3.12}$$

where

$$x_1 = \frac{91(64)^{n-1} - 28}{39(64)^{n-1} + 24}, \qquad n \geq 1. \tag{3.13}$$

Inserting Eq.(3.13) into Eq.(3.12) we obtain the result. □

## §4. Number of Spanning Trees in the Sequences of $F_n$ Graph

Consider the sequence of graphs $\mathrm{F}_1,\ \mathrm{F}_2, \cdots, \mathrm{F}_n$ constructed as shown in Figure 7. According to this construction, the number of total vertices $|V(\mathrm{F}_n)|$ and edges $|E(\mathrm{F}_n)|$ are $|\mathrm{V}(\mathrm{F}_n)| = 9n-6$ and $|E(\mathrm{F}_n)| = 24n-21, n = 1, 2, \cdots$. The average degree of $\mathrm{F}_n$ is $\frac{16}{3}$ in the large $n$ limit.

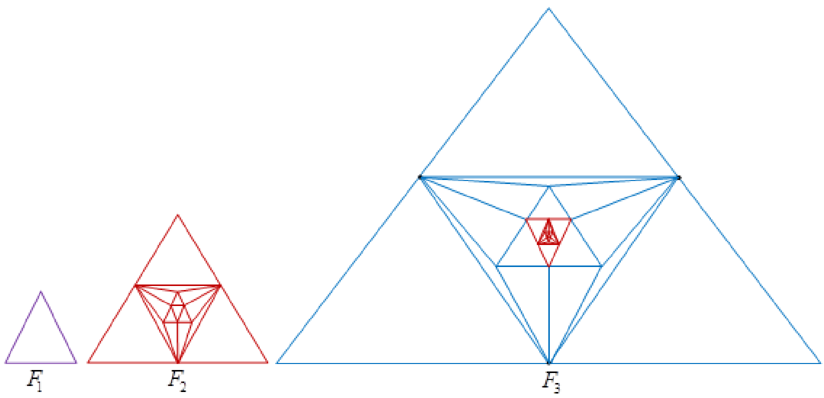



**Figure 7.** Some sequences of graph $\mathrm{F}_n$

**Theorem** 4.1 *For $n \geq 1$, the number of spanning trees in the sequence of $\mathrm{F}_n$ graph is given* *by* $\dfrac{A_n}{B_n}$, *where*

$$
\begin{aligned}
A_n &= \left(400^{n-3}\left((85-21\sqrt{15})(2(4+\sqrt{15}))^n+(8-2\sqrt{15})^n(85+21\sqrt{15})\right)^2\left(-61(321+83\sqrt{15})\right.\right.\\
&\qquad \left.\left.+(31+8\sqrt{15})^n(951+365\sqrt{15})\right)^2\right),\\
B_n &= 3\left(61(31+8\sqrt{15})+(64+5\sqrt{15})(31+8\sqrt{15})^n\right)^2.
\end{aligned}
$$

*Proof* We use the electrically equivalent transformation to transform $F_i$ to $F_{i-1}$. Figures 8-13 illustrate the transformation process from $F_2$ to $F_1$.

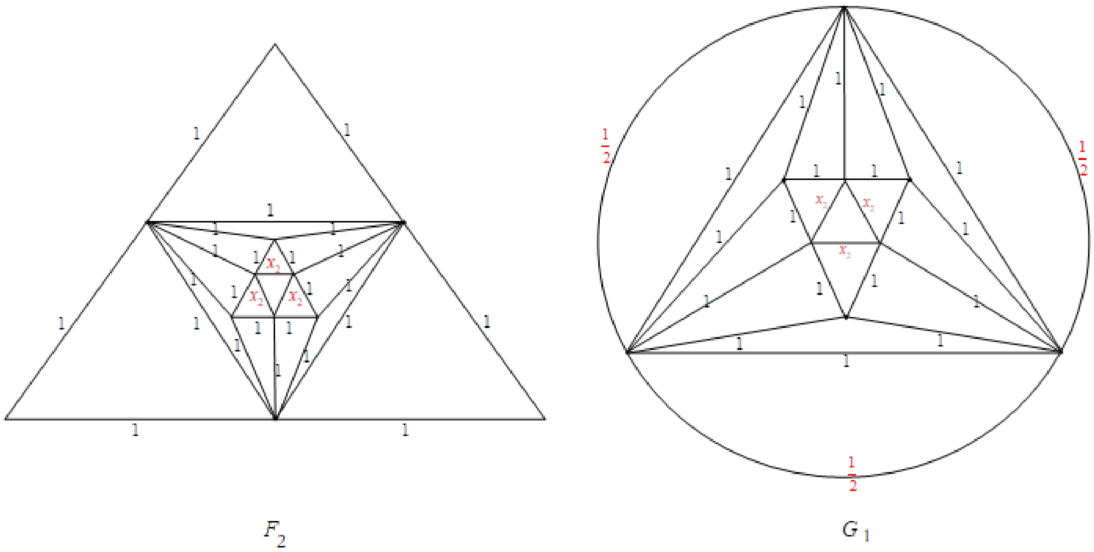



**Figure 8**

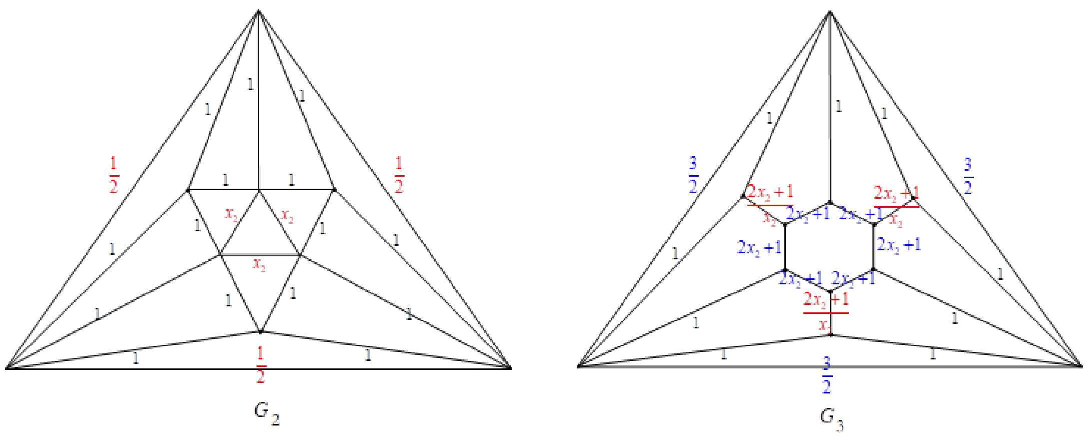
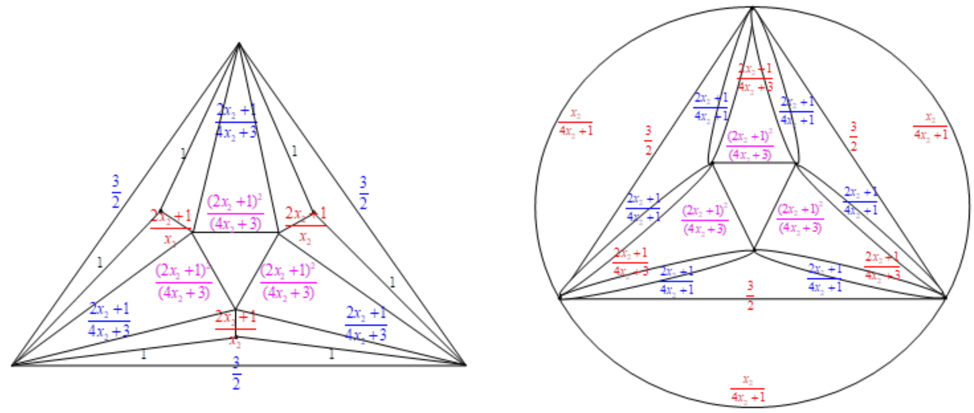
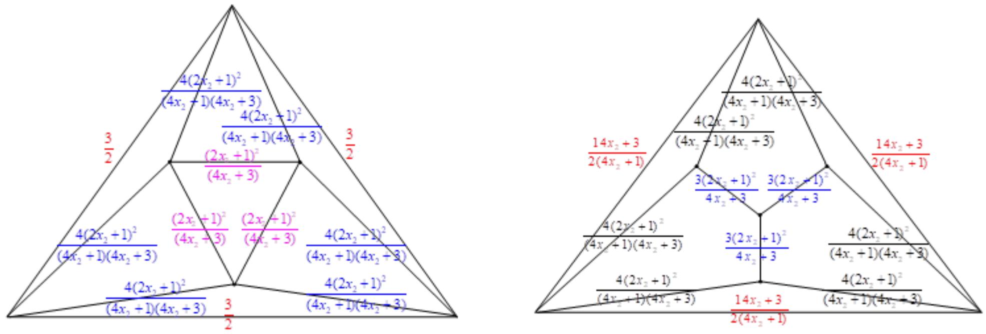



**Figure 9**



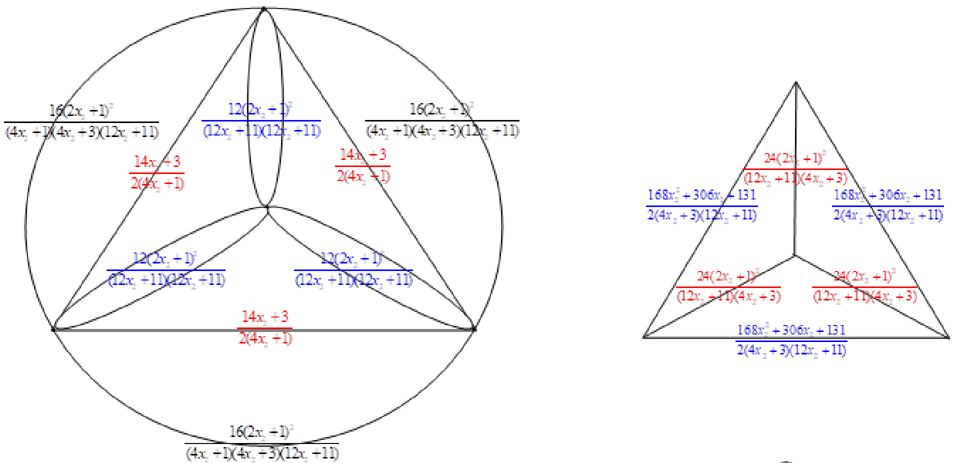

$G_4$ $G_5$

**Figure 10**




$G_6$ $G_7$

**Figure 11**




$G_8$ $G_9$

**Figure 12**

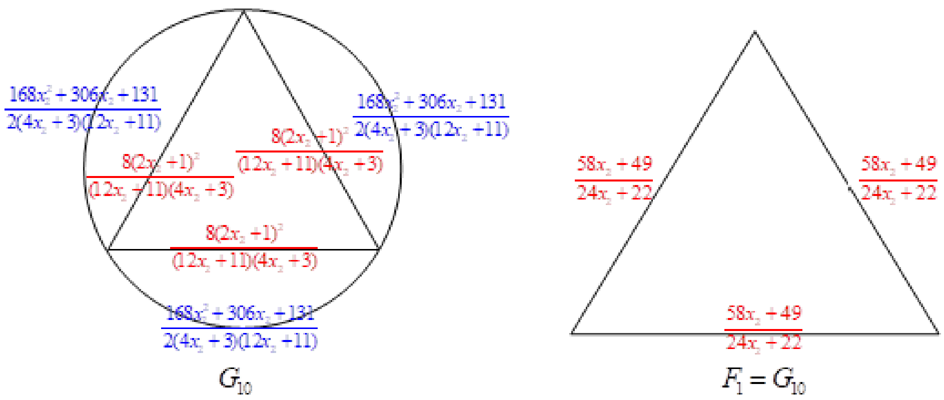



**Figure 13**

Using the properties given in Section 2, we have the following the transformations:

$$
\begin{aligned}
\tau(G_1) &= \left[\frac{1}{2}\right]^3 \tau(\mathrm{F}_2), \quad \tau(G_2) = \tau(G_1), \quad \tau(G_3) = \left[\frac{(2x_2+1)^2}{x_2}\right]^3 \tau(G_2), \\
\tau(G_4) &= \left[\frac{1}{4x_2+3}\right]^3 \tau(G_3), \quad \tau(G_5) = \left[\frac{x_2}{4x_2+3}\right]^3 \tau(G_4), \quad \tau(G_6) = \tau(G_5), \\
\tau(G_7) &= 9\left[\frac{(2x_2+1)^2}{4x_2+3}\right] \tau(G_6), \quad \tau(G_8) = \left[\frac{(4x_2+1)(4x_2+3)}{(2x_2+1)^2(12x_2+11)}\right]^3 \tau(G_7), \\
\tau(G_9) &= \tau(G_8), \quad \tau(G_{10}) = \frac{(12x_2+11)(4x_2+3)}{72(2x_2+1)^2}\tau(G_9) \text{ and } \tau(\mathrm{F}_1) = \tau(G_{10}).
\end{aligned}
$$

Combining these eleven transformations, we have

$$
\tau(\mathrm{F}_2) = 16(24x_2+22)^2 \tau(\mathrm{F}_1). \tag{4.1}
$$

Further

$$
\tau(\mathrm{F}_n) = \prod_{i=2}^{n} 16(24x_2+22)^2 \tau(\mathrm{F}_1) = 3 \times (16)^{n-1} x_1^2 \left[\prod_{i=2}^{n}(24x_i+22)\right]^2, \tag{4.2}
$$

where $x_{i-1} = \frac{58x_i+49}{24x_i+22}, i = 2, 3, \ldots, n$. Its characteristic equation is $24\mu^2 - 36\mu - 49 = 0$, which have two roots $\mu_1 = \frac{9-5\sqrt{15}}{12}$ and $\mu_2 = \frac{9+5\sqrt{15}}{12}$. Subtracting these two roots into both sides of $x_{i-1} = \frac{58x_i+49}{24x_i+22}$, we get

$$
x_{i-1} - \frac{9-5\sqrt{15}}{12} = \frac{58x_i+49}{24x_i+22} - \frac{9-5\sqrt{15}}{12} = 10(4+\sqrt{15})\frac{\left(x_i - \frac{9-5\sqrt{15}}{12}\right)}{(24x_i+22)}, \tag{4.3}
$$

$$
x_{i-1} - \frac{9+5\sqrt{15}}{12} = \frac{58x_i+49}{24x_i+22} - \frac{9+5\sqrt{15}}{12} = 10(4-\sqrt{15})\frac{\left(x_i - \frac{9+5\sqrt{15}}{12}\right)}{(24x_i+22)}. \tag{4.4}
$$

Let $y_i = \frac{x_i - \frac{9-5\sqrt{15}}{12}}{x_i - \frac{9+5\sqrt{15}}{12}}$. Then by Eqs.(4.3) and (4.4), we get $y_{i-1} = (31+8\sqrt{15})y_i$ and $y_i = (31+8\sqrt{15})^{n-i}y_n$. Therefore

$$x_i = \frac{(31+8\sqrt{15})^{n-i}\left(\frac{9+5\sqrt{15}}{12}\right)y_n - \frac{9-5\sqrt{15}}{12}}{(31+8\sqrt{15})^{n-i}y_n - 1}.$$

Thus

$$x_1 = \frac{(31+8\sqrt{15})^{n-1}\left(\frac{9+5\sqrt{15}}{12}\right)y_n - \frac{9-5\sqrt{15}}{12}}{(31+8\sqrt{15})^{n-1}y_n - 1}. \tag{4.5}$$

Using the expression $x_{n-1} = \frac{58x_n+49}{24x_n+22}$ and denoting the coefficients of $58x_n + 49$ and $24x_n + 22$ as $\sigma_n$ and $\delta_n$ we have

$$24x_n + 22 = \sigma_0\left(58x_n+49\right) + \delta_0\left(24x_n+22\right),$$

$$24x_{n-1} + 22 = \frac{\sigma_1\left(58x_n+49\right) + \delta_1\left(24x_n+22\right)}{\sigma_0\left(58x_n+49\right) + \delta_0\left(24x_n+22\right)},$$

$$\vdots$$

$$24x_{n-i} + 22 = \frac{\sigma_i\left(58x_n+49\right) + \delta_i\left(24x_n+22\right)}{\sigma_{i-1}\left(58x_n+49\right) + \delta_{i-1}\left(24x_n+22\right)}, \tag{4.6}$$

$$24x_{n-(i+1)} + 22 = \frac{\sigma_{i+1}\left(58x_n+49\right) + \delta_{i+1}\left(24x_n+22\right)}{\sigma_i\left(58x_n+49\right) + \delta_i\left(24x_n+22\right)}, \tag{4.7}$$

$$\vdots$$

$$24x_2 + 22 = \frac{\sigma_{n-2}\left(58x_n+49\right) + \delta_{n-2}\left(24x_n+22\right)}{\sigma_{n-3}\left(58x_n+49\right) + \delta_{n-3}\left(24x_n+22\right)}$$

Substituting Eq.(4.6) into Eq.(4.2), we obtain

$$\tau\left(\mathrm{F}_n\right) = 3 \times (16)^{n-1}x_1^2\left[\sigma_{n-2}\left(58x_n+49\right) + \sigma_{n-2}\left(24x_n+22\right)\right]^2, \tag{4.8}$$

where $\sigma_0 = 0, \delta_0 = 1$ and $\sigma_1 = 24, \delta_1 = 22$.

By the expression $x_{n-1} = \frac{58x_n+49}{24x_n+22}$ and Eqs.(4.6) and (4.7), we have

$$\sigma_{i+1} = 80\sigma_i - 100\sigma_{i-1}; \delta_{i+1} = 80\delta_i - 100\delta_{i-1}. \tag{4.9}$$

The characteristic equation of Eq.(4.9) is $\gamma^2 - 80\gamma + 100 = 0$ which have two roots $\gamma_1 = 10(4+\sqrt{15})$ and $\gamma_2 = 10(4-\sqrt{15})$.

The general solutions of Eq.(4.9) are $\sigma_i = a_1\gamma_1^i + a_2\gamma_2^i; \delta_i = b_1\gamma_1^i + b_2\gamma_2^i$. Using the initial conditions $\sigma_0 = 0, \delta_0 = 1$ and $\sigma_1 = 24, \delta_1 = 22$, yields

$$\sigma_i = \frac{2\sqrt{15}}{25}(10(4+\sqrt{15}))^i - \frac{2\sqrt{15}}{25}(10(4-\sqrt{15}))^i,$$

$$b_i = \left(\frac{25-3\sqrt{15}}{50}\right)(10(4+\sqrt{15}))^i + \left(\frac{25+3\sqrt{15}}{50}\right)(10(4-\sqrt{15}))^i. \tag{4.10}$$

If $x_n = 1$, it means that $\mathrm{F}_n$ without any electrically equivalent transformation. Plugging

Eq.(4.10) into Eq.(4.8), we have

$$\begin{aligned}\tau(\mathrm{F}_n) &= 3\times(16)^{n-1}x_1^2\left[\left(\frac{115+29\sqrt{15}}{5}\right)(40+10\sqrt{15})^{n-2}\right.\\ &\quad\left.+\left(\frac{115-29\sqrt{15}}{5}\right)(40-10\sqrt{15})^{n-2}\right]^2 \qquad (4.11)\end{aligned}$$

for integer $n\geq 2$. When $n=1, \tau(F_1)=3$ which satisfies Eq.(4.11). Therefore, for $,n\geq 1$, the number of spanning trees in the sequence of the graph $\mathrm{F}_n$ is given by

$$\begin{aligned}\tau(\mathrm{F}_n) &= 3\times(16)^{n-1}x_1^2\left[\left(\frac{115+29\sqrt{15}}{5}\right)(40+10\sqrt{15})^{n-2}\right.\\ &\quad\left.+\left(\frac{115-29\sqrt{15}}{5}\right)(40-10\sqrt{15})^{n-2}\right]^2, \qquad (4.12)\end{aligned}$$

where

$$x_1=\frac{(31+8\sqrt{15})^{n-1}(951+365\sqrt{15})+61(9-5\sqrt{15})}{(31+8\sqrt{15})^{n-1}(64+5\sqrt{15})+732}, n\geq 1. \qquad (4.13)$$

Inserting Eq.(4.13) into Eq.(4.12), we obtain the result. □

## §5. Number of Spanning Trees in the Sequences of $\mathrm{H}_n$ Graph

Consider the sequence of graphs $\mathrm{H}_1, \mathrm{H}_2, \cdots, \mathrm{H}_n$ constructed as shown in Figure 14. According to this construction, the number of total vertices $|V(\mathrm{H}_n)|$ and edges $|E(\mathrm{H}_n)|$ are $|\mathrm{V}(\mathrm{H}_n)|=9n-6$ and $|E(\mathrm{H}_n)|=24n-21$ for integers $n=1,2,\cdots$. The average degree of $\mathrm{H}_n$ is in the large $n$ limit which is $\dfrac{16}{3}$.

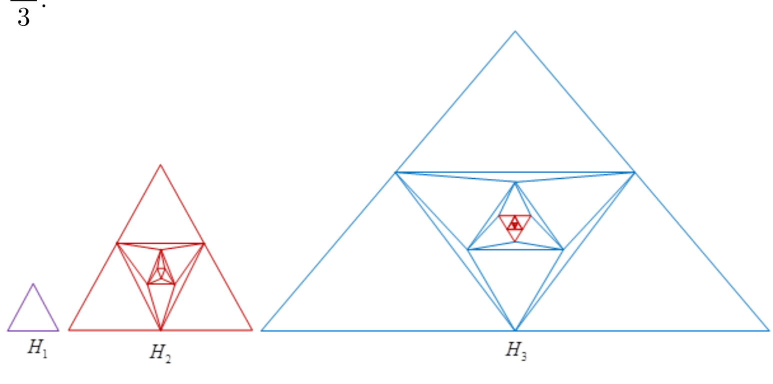



**Figure 14.** Some sequences of $\mathrm{H}_n$

**Theorem** 5.1 *For $n\geq 1$, the number of spanning trees in the sequence of $\mathrm{H}_n$ is given by*

$$\begin{aligned}2^{n-15}(115+\sqrt{13209})^{2n} &\times \left(-76(-63+\sqrt{13209})+\left(\frac{1}{8}(13217-115\sqrt{13209})\right)^{1-n}(8421+97\sqrt{13209})\right)^2\\ &\times \frac{\left(29563-257\sqrt{13209}+\left(\frac{1}{8}(13217-115\sqrt{13209})\right)^{n}(29563+257\sqrt{13209})\right)^2}{\left(58159227\left(38+8^{-n}(325+\sqrt{13209})(13217+115\sqrt{13209})^{n-1}\right)^2\right)}.\end{aligned}$$

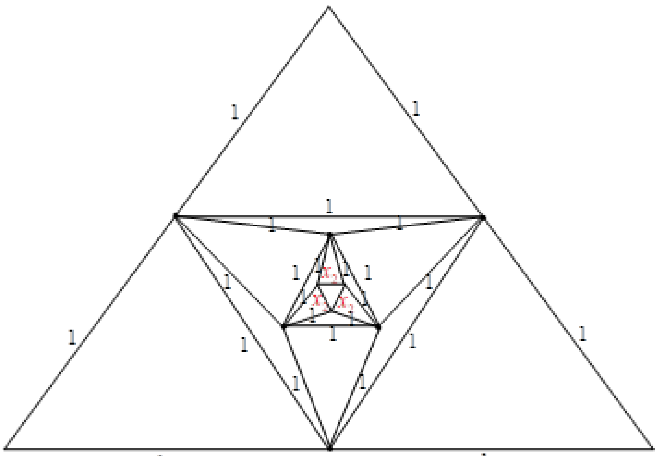
*Proof* We use the electrically equivalent transformation to transform $\mathrm{H}_i$ to $\mathrm{H}_{i-1}$. Figures

15-19 illustrate the transformation process from $H_2$ to $H_1$. Using the properties given in Section 2, we have the following the transformations:

$$\tau(G_1)=\left[\frac{1}{2}\right]^3\tau(\mathrm{H}_2),\ \tau(G_2)=\tau(G_1),\ \tau(G_3)=9x_2\tau(G_2),\ \tau(G_4)=\left[\frac{1}{3x_2+2}\right]^3\tau(G_3),$$

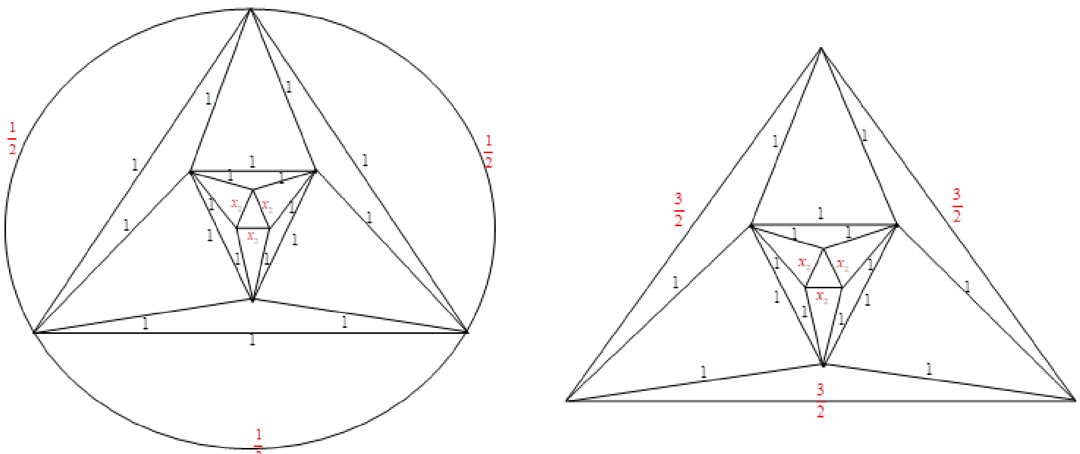
$$\tau(G_5)=\tau(G_4),\ \tau(G_6)=\left(\frac{3x_2+2}{18x_2}\right)\tau(G_5),\ \tau(G_7)=\tau(G_6),\ \tau(G_8)=9\left(\frac{5x_2+3}{3x_2+2}\right)\tau(G_7),$$

$$\tau(G_9)=\left[\frac{3x_2+2}{21x_2+13}\right]^3\tau(G_8),\ \tau(G_{10})=\tau(G_9),\tau(G_{11})=\left[\frac{21x_2+13}{18(5x_2+3)}\right]\tau(G_{10}),\ \tau(\mathrm{H}_1)=\tau(G_{11}).$$



$H_2$

**Figure 15**



$G_1$ $G_2$

**Figure 16**

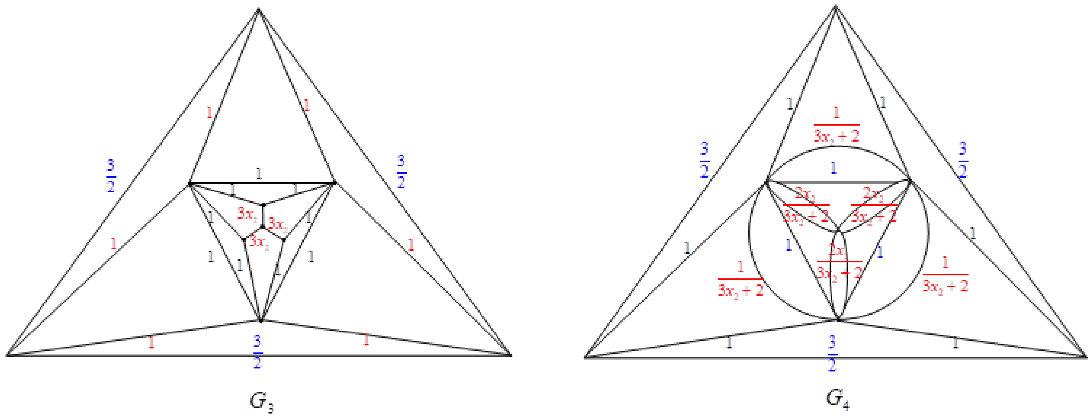



**Figure 17**

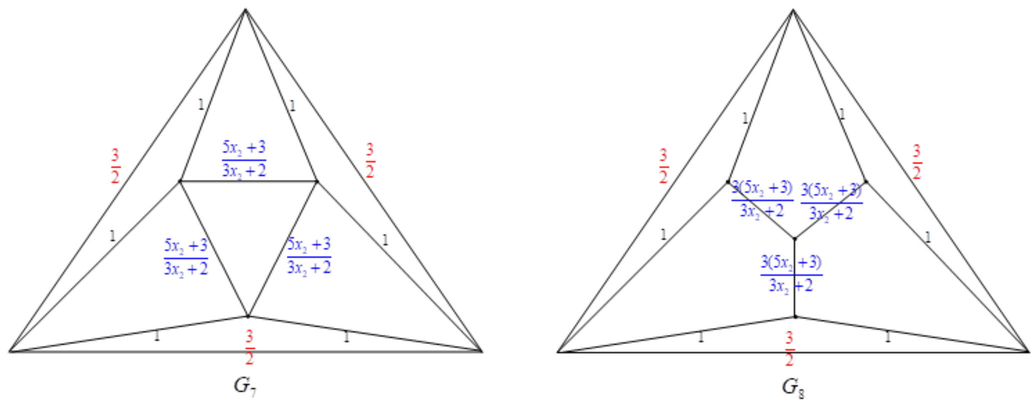



**Figure 18**

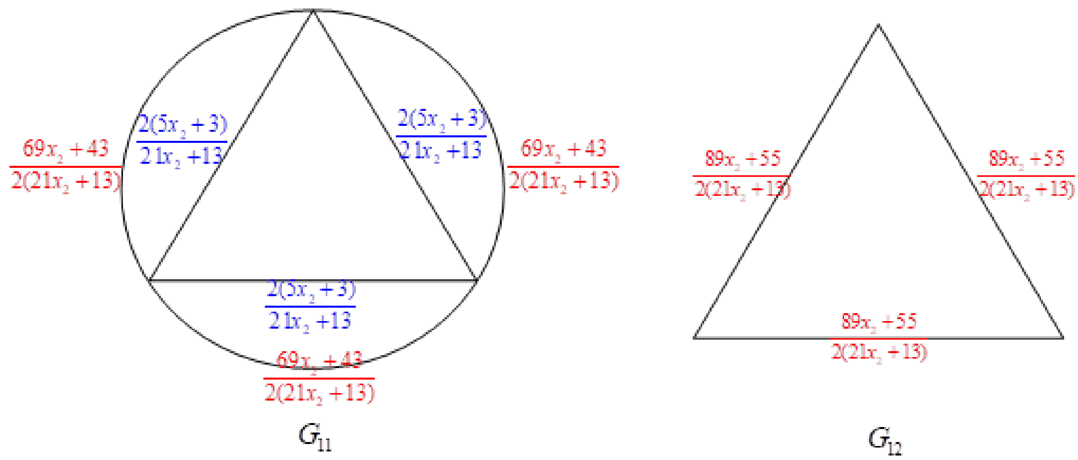



**Figure 19**

Combining these twelve transformations, we get

$$\tau\left(\mathrm{H}_2\right)=8\left(42 x_2+26\right)^2 \tau\left(\mathrm{H}_1\right). \tag{5.1}$$

Further

$$\tau\left(\mathrm{H}_n\right)=\prod_{i=2}^n 8\left(42 x_2+26\right)^2 \tau\left(\mathrm{H}_1\right)=3 \times(8)^{n-1} x_1^2\left[\prod_{i=2}^n\left(42 x_i+26\right)\right]^2, \tag{5.2}$$

where $x_{i-1}=\frac{89 x_i+55}{42 x_i+26}, i=2,3, \ldots, n$. Its characteristic equation is $42 \mu^2-63 \mu-55=0$, which have two roots $\mu_1=\frac{63-\sqrt{13209}}{84}$ and $\mu_2=\frac{63+\sqrt{13209}}{84}$. Subtracting these two roots into both sides of $x_{i-1}=\frac{89 x_i+55}{42 x_i+26}$, we get

$$x_{i-1}-\frac{63-\sqrt{13209}}{84}=(115+\sqrt{13209}) \frac{\left(x_i-\frac{68-\sqrt{13209}}{84}\right)}{2\left(42 x_i+26\right)}, \tag{5.3}$$

$$x_{i-1}-\frac{63+\sqrt{13209}}{84}=(115-\sqrt{13209}) \frac{\left(x_i-\frac{68+\sqrt{13209}}{84}\right)}{2\left(42 x_i+26\right)}. \tag{5.4}$$

Let $y_i=\frac{x_i-\frac{63-\sqrt{18209}}{84}}{x_i-\frac{63+\sqrt{18209}}{84}}$. Then by Eqs.(5.3) and (5.4), we get $y_{i-1}=\left(\frac{13217+115 \sqrt{13209}}{8}\right) y_i$ and $y_i=\left(\frac{13217+115 \sqrt{13209}}{8}\right)^{n-i} y_n$. Therefore

$$x_i=\frac{\left(\frac{13217+115 \sqrt{13209}}{B}\right)^{n-i}\left(\frac{63+\sqrt{13209}}{84}\right) y_n-\frac{63-\sqrt{13209}}{84}}{\left(\frac{13217+115 \sqrt{13209}}{8}\right)^{n-i} y_n-1}.$$

Thus

$$x_1=\frac{\left(\frac{13217+115 \sqrt{13209}}{8}\right)^{n-1}\left(\frac{63+\sqrt{13209}}{84}\right) y_n-\frac{63-\sqrt{13209}}{84}}{\left(\frac{13217+115 \sqrt{13209}}{8}\right)^{n-1} y_n-1}. \tag{5.5}$$

Using the expression $x_{n-1}=\frac{89 x_n+55}{42 x_n+26}$ and denoting the coefficients of $89 x_n+55$ and $42 x_n+26$ as $\sigma_n$ and $\delta_n$ we have

$$\begin{aligned}
42 x_n+26 & =\sigma_0\left(89 x_n+55\right)+\delta_0\left(42 x_n+26\right), \\
42 x_{n-1}+26 & =\frac{\sigma_1\left(89 x_n+55\right)+\delta_1\left(42 x_n+26\right)}{\sigma_0\left(89 x_n+55\right)+\delta_0\left(42 x_n+26\right)}, \\
42 x_{n-2}+26 & =\frac{\sigma_2\left(89 x_n+55\right)+\delta_2\left(42 x_n+26\right)}{\sigma_1\left(89 x_n+55\right)+\delta_1\left(24 x_n+26\right)}, \\
& \vdots
\end{aligned}$$

$$42 x_{n-i}+26=\frac{\sigma_i\left(89 x_n+55\right)+\delta_i\left(42 x_n+26\right)}{\sigma_{i-1}\left(89 x_n+55\right)+\delta_{i-1}\left(42 x_n+26\right)}, \tag{5.6}$$

$$42 x_{n-(i+1)}+26=\frac{\sigma_{i+1}\left(89 x_n+55\right)+\delta_{i+1}\left(42 x_n+26\right)}{\sigma_i\left(89 x_n+55\right)+\delta_i\left(42 x_n+26\right)}, \tag{5.7}$$

$$\vdots$$

$$42x_2+26=\frac{\sigma_{n-2}\left(89x_n+55\right)+\delta_{n-2}\left(42x_n+26\right)}{\sigma_{n-3}\left(89x_n+55\right)+\delta_{n-3}\left(42x_n+26\right)}. \tag{5.8}$$

Substituting Eq.(5.6) into Eq.(5.2), we obtain

$$\tau\left(\mathrm{H}_n\right)=3\times(8)^{n-1}x_1^2\left[\sigma_{n-2}\left(89x_n+55\right)+\sigma_{n-2}\left(42x_n+26\right)\right]^2, \tag{5.9}$$

where $\sigma_0=0,\delta_0=1$ and $\sigma_1=42,\delta_1=26$.

By the expression $x_{n-1}=\frac{89x_n+55}{42x_n+26}$ and Eqs.(5.6), (5.7), we have

$$\sigma_{i+1}=115\sigma_i-4\sigma_{i-1};\delta_{i+1}=115\delta_i-4\delta_{i-1}$$

The characteristic equation of Eq.(5.9) is $\gamma^2-115\gamma+4=0$ which have two roots

$$\gamma_1=\left(\frac{115+\sqrt{13209}}{2}\right) \text{ and } \gamma_2=\left(\frac{115-\sqrt{13209}}{2}\right).$$

The general solutions of Eq.(5.9) are $\sigma_i=a_1\gamma_1^i+a_2\gamma_2^i;\delta_i=b_1\gamma_1^i+b_2\gamma_2^i$. Using the initial conditions $\sigma_0=0,\delta_0=1$ and $\sigma_1=42,\delta_1=26$, yields

$$\sigma_i=\frac{2\sqrt{13209}}{629}\left(\frac{115+\sqrt{13209}}{2}\right)^i-\frac{2\sqrt{13209}}{629}\left(\frac{115-\sqrt{13209}}{2}\right)^i,$$

$$\delta_i=\left(\frac{629-3\sqrt{13209}}{1258}\right)\left(\frac{115+\sqrt{13209}}{2}\right)^i+\left(\frac{629+3\sqrt{13209}}{1258}\right)\left(\frac{115-\sqrt{13209}}{2}\right)^i. \tag{5.10}$$

If $x_n=1$, it means that $\mathrm{H}_n$ without any electrically equivalent transformation. Plugging Eq.(5.10) into Eq.(5.8), we have

$$\begin{aligned}\tau\left(\mathrm{H}_n\right) &= 3\times(8)^{n-1}x_1^2\times\left[\left(\frac{21386+186\sqrt{13209}}{629}\right)\left(\frac{115+\sqrt{13209}}{2}\right)^{n-2}\right.\\ &\left.+\left(\frac{21386-186\sqrt{13209}}{629}\right)\left(\frac{115-\sqrt{13209}}{2}\right)^{n-2}\right]^2\end{aligned} \tag{5.11}$$

for integers $n\geq 2$. When $n=1,\tau\left(H_1\right)=3$ which satisfies Eq.(5.11). Therefore, the number of spanning trees in the sequence of the graph $\mathrm{H}_n$ is given by

$$\begin{aligned}\tau\left(\mathrm{H}_n\right) &= 3\times(8)^{n-1}x_1^2\left[\left(\frac{21386+186\sqrt{13209}}{629}\right)\left(\frac{115+\sqrt{13209}}{2}\right)^{n-2}\right.\\ &\left.+\left(\frac{21386-186\sqrt{13209}}{629}\right)\left(\frac{115-\sqrt{13209}}{2}\right)^{n-2}\right]^2\end{aligned} \tag{5.12}$$

for integers $n \geq 1$, where

$$x_1 = \frac{\left(\frac{13217+115\sqrt{13209}}{B}\right)^{n-1}(8421+97\sqrt{13209})+76(63-\sqrt{13209})}{21\left(\frac{13217+115\sqrt{13209}}{8}\right)^{n-1}(325+\sqrt{13209})+6384}, \quad n \geq 1. \tag{5.13}$$

Inserting Eq.(5.13) into Eq.(5.12) we obtain the result. □

## §6. Numerical Results

An illustration on the numbers of spanning trees in graphs $\mathrm{E}_n$, $\mathrm{F}_n$ and $\mathrm{H}_n$ are listed in Table 1 following.

| $n$ | $\tau(E_n)$ | $\tau(F_n)$ | $\tau(H_n)$ |
|---|---|---|---|
| 1 | 3 | 3 | 3 |
| 2 | 406272 | 549552 | 497664 |
| 3 | 26879275008 | 54966988800 | 52627418112 |
| 4 | 1761820718333952 | 5452053012480000 | 5564612377337856 |
| 5 | 115462949411396517888 | 540704118669312000000 | 588379800446293966848 |
| 6 | 7566980125843657045573632 | 53623893196800000000000000 | 62212920881826474870964224 |

**Table 1**

## §7. Spanning Tree Entropy

After having explicit Formulas for the number of spanning trees of the sequence of the three families of graphs $\mathrm{E}_n$, $\mathrm{F}_n$ and $\mathrm{H}_n$, we can calculate its spanning tree entropy $Z$ which is a finite number and a very interesting quantity characterizing the network structure, defined as in [23, 24] as

$$Z(G) = \lim_{n\to\infty} \frac{\ln \tau(G)}{|V(G)|}. \tag{7.1}$$

for a graph $G$. Particularly, we know that

$$Z(E_n) = \frac{16}{9}(\ln 2) = 1.232261654$$

$$Z(F_n) = \frac{1}{9}(\ln[1600] + 2\ln[4+\sqrt{15}]) = 1.278292561,$$

$$Z(H_n) = \ln[2] - \frac{2}{9}(\ln[115-\sqrt{13209}]) = 1.285411179.$$

Now we compare the value of entropy in our graphs with other graphs. The entropy of the graph $\mathrm{H}_n$ is larger than the entropy of the graph $\mathrm{E}_n$ and the graph $\mathrm{F}_n$. In addition the entropy of the families $\mathrm{E}_n, F_n$ and $\mathrm{H}_n$ which have average degree 16/3 is larger than the entropy of fractal scale free lattice [25] which has the entropy1.040 and 3-prism graph of average degree 4 which has entropy1.0445 [26] and two dimensional Sierpinski gasket [27] which has the entropy

1.166 of the same average degree 4 but the entropy of the families $\mathrm{E}_n, E_n$ and $\mathrm{H}_n$ is smaller than the entropy of Apollonian graph [28] which has the entropy 1.3540 of average degree 6.

## §8. Conclusions

In this work, we enumerated the number of spanning trees in the sequences of three sequences of graphs of average degree 16/3 using electrically equivalent transformations. An advantage of this method lies in the avoidance of laborious computation of Laplacian spectra that is needed for a generic method for determining spanning trees.

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