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On b-Chromatic Number of Some Line, Middle and Total Graph Families

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Abstract: A proper coloring of the graph assigns colors to the vertices, edges, or both so that proximal elements are assigned distinct colors. Concepts and questions of graph coloring arise naturally from practical problems and have found applications in many areas, including information theory and most notably theoretical computer science. A b-coloring of a graph G is a proper coloring of the vertices of G such that there exists a vertex in each color class joined to at least one vertex in each other color class. The b-chromatic number of a graph G, denoted by $\varphi(G)$, is the largest integer k such that G may have a b-coloring with k colors. In this paper, the authors obtain the b-chromatic number for line, middle and total graph of some families such as cycle, helm and gear graphs.

Key Words: b-coloring, Helm graph, gear Graph, middle graph, total graph, line graph.

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§1. Introduction

All graphs considered in this paper are non-trivial, simple and undirected. Let G be a graph with vertex set V and edge set E. A k-coloring of a graph G is a partition $P = \{V_1, V_2, \cdots, V_k\}$ of V into independent sets of G. The minimum cardinality k for which G has a k-coloring is the chromatic number $\chi(G)$ of G. The b-chromatic number $\varphi(G)$ ([16,19,20]) of a graph G is the largest positive integer k such that G admits a proper k-coloring in which every color class has a representative adjacent to at least one vertex in each of the other color classes. Such a coloring is called a b-coloring. The b-chromatic number was introduced by Irving and Manlove in [11] by considering proper colorings that are minimal with respect to a partial order defined

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on the set of all partitions of $V(G)$. They have shown that determination of $\varphi(G)$ is NP-hard for general graphs, but polynomial for trees. There has been an increasing interest in the study of b-coloring since the publication of [11]. They also proved the following upper bound of $\varphi(G)$

$$\varphi(G) \leq \Delta(G) + 1. \tag{1.1}$$

Kouider and Mahéo [12] gave some lower and upper bounds for the b-chromatic number of the cartesian product of two graphs. Kratochvíl et al. [13] characterized bipartite graphs for which the lower bound on the b-chromatic number is attained and proved the NP-completeness of the problem to decide whether there is a dominating proper b-coloring even for connected bipartite graphs with $k = \Delta(G) + 1$. Effantin and Kheddouci studied in [3, 5, 6] the b-chromatic number for the complete caterpillars, the powers of paths, cycles, and complete k-ary trees. Faik [7] was interested in the continuity of the b-coloring and proved that chordal graphs are b-continuous. Corteel et al. [2] proved that the b-chromatic number problem is not approximable within $120/133 - \epsilon$ for any $\epsilon > 0$, unless $P = NP$. Hoáng and Kouider characterized in [10], the bipartite graphs and the P_4-sparse graphs for which each induced subgraph H of G has $\varphi(H) = \chi(H)$. Kouider and Zaker [14] proposed some upper bounds for the b-chromatic number of several classes of graphs in function of other graph parameters (clique number, chromatic number, biclique number). Kouider and El Sahili proved in [15] by showing that if G is a d-regular graph with girth 5 and without cycles of length 6, then $\varphi(G) = d + 1$. Effantin and Kheddouci [4] proposed a discussion on relationships between this parameter and two other coloring parameters (the Grundy and the partial Grundy numbers). The property of the dominating nodes in a b-coloring is very interesting since they can communicate directly with each partition of the graph.

There have been lots of works on various properties of line graphs, middle graphs and total graphs of graphs [1, 9, 17, 18].

For any integer $n \geq 4$, the wheel graph W_n is the $n-$vertex graph obtained by joining a vertex v_1 to each of the $n-1$ vertices $\{w_1, w_2, \cdots, w_{n-1}\}$ of the cycle graph C_{n-1}. Where $V(W_n) = \{v, v_1, v_2, \cdots v_{n-1}\}$ and $E(W_n) = \{e_1, e_2, \cdots, e_n\} \cup \{s_1, s_2, \cdots, s_n\}$.

The Helm graph H_n is the graph obtained from an n-wheel graph by adjoining a pendent edge at each node of the cycle.

The Gear graph G_n, also known as a bipartite wheel graph, is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle.

The line graph [8] of G, denoted by $L(G)$ is the graph with vertices are the edges of G with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph [8] of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G; (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [8] of G, denoted by $T(G)$ is defined in the following way. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one of the following holds:

(i) x, y are in $V(G)$ and x is adjacent to y in G; (ii) x, y are in $E(G)$ and x, y are adjacent in G; (iii) x is in $V(G), y$ is in $E(G)$, and x, y are incident in G.

§2. b-Coloring of Some Line, Middle and Total Graph Families

Lemma 2.1 *If $n \geq 8$ then b-chromatic number on middle graph of cycle $M(C_n)$ is $\varphi(M(C_n)) = 5$.*

Proof Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and let $V(M(C_n)) = \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ where u_i is the vertex of $T(C_n)$ corresponding to the edge $v_i v_{i+1}$ of C_n $(1 \leq i \leq n-1)$.

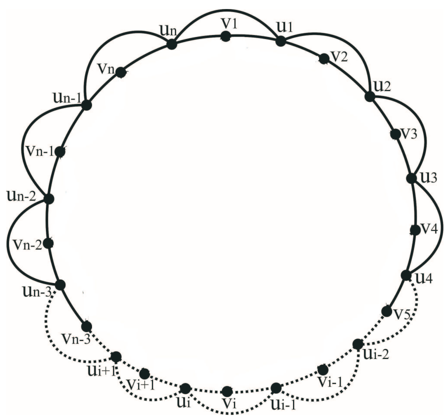

Fig.1 Middle Graph of Cycle $M(C_n)$

Consider the following 5-coloring $(c_1, c_2, c_3, c_4, c_5)$ of $M(C_n)$ as b-chromatic:

Assign the color c_1 to v_1, c_3 to u_1, c_4 to v_2, c_1 to u_2, c_5 to v_3, c_2 to u_3, c_4 to v_4, $c_3$to u_4, c_1 to v_5, c_5 to u_5, c_2 to v_6, c_4 to u_6, c_1 to v_7, c_3 to u_7. For $8 \leq i \leq n$, assign to vertex v_i one of the allowed colors-such color exists, because $\deg(v_i) = 2$. For $8 \leq i \leq n-1$, if any, assign to vertex u_i one of the allowed colors-such color exists, because $\deg(u_i) = 4$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(M(C_n)) \geq 5$. Since $\Delta(M(C_n)) = 4$, using (1.1) we get that $\varphi(M(C_n)) \leq 5$. Hence, $\varphi(M(C_n)) = 5$, $\forall\ n \geq 5$. □

Theorem 2.2 *If $n \geq 5$ then b-chromatic number on total graph of cycle $T(C_n)$ is $\varphi(T(C_n)) = 5$.*

Proof Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and let $V(T(C_n)) = \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ where u_i is the vertex of $T(C_n)$ corresponding to the edge $v_i v_{i+1}$ of C_n $(1 \leq i \leq n-1)$.

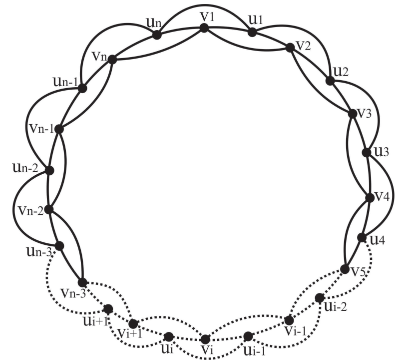

Fig.2 Total Graph of Cycle $T(C_n)$

Consider the following 5-coloring $(c_1, c_2, c_3, c_4, c_5)$ of $T(C_n)$ as b-chromatic: assign the color c_4 to v_1, c_5 to u_1, c_1 to v_2, c_2 to u_2, c_3 to v_3,c_4 to u_3, c_5 to v_4, c_1 to u_4, c_2 to v_5. For $6 \le i \le n$, assign to vertex v_i one of the allowed colors-such color exists, because $\deg(v_i) = 4$. For $5 \le i \le n-1$, if any, assign to vertex u_i one of the allowed colors-such color exists, because $\deg(u_i) = 4$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(T(C_n)) \ge 5$. Since $\Delta(T(C_n)) = 4$, using (1.1), we get that $\varphi(T(C_n)) \le 5$. Hence, $\varphi(T(C_n)) = 5$, $\forall\ n \ge 5$. □

Lemma 2.3 *If $n \ge 6$ then b-chromatic number on helm graph H_n is $\varphi(H_n) = 5$.*

Proof Let H_n be the Helm graph obtained by attaching a pendant edge at each vertex of the cycle. Let $V(H_n) = \{v\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ where v_i's are the vertices of cycles taken in cyclic order and u_i's are pendant vertices such that each v_iu_i is a pendant edge and v is a hub of the cycle.

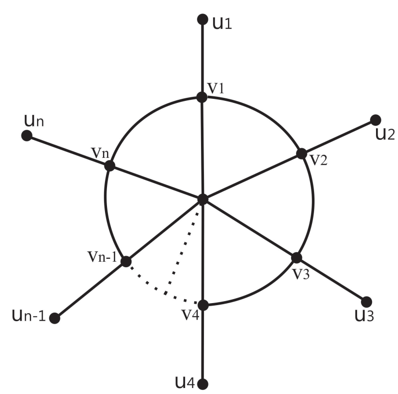

Fig.3 Helm Graph H_n

Consider the following 5-coloring $(c_1, c_2, c_3, c_4, c_5)$ of H_n as b-chromatic:

For $1 \le i \le 4$, assign the color c_i to v_i and assign the colors c_5 to v, c_1 to v_5, c_3 to v_n, c_4 to u_1, c_4 to u_2, c_1 to u_3, c_2 to u_4. For $6 \le i \le n$, assign to vertex v_i one of the allowed colors-such color exists, because $\deg(v_i) = 4$. For $5 \le i \le n$, if any, assign the color c_4 to the vertex u_i. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(H_n) \ge 5$.

Let us assume that $\varphi(H_n)$ is greater than 5, i.e. $\varphi(H_n) = 6$, $\forall\ n \geq 6$, there must be at least 6 vertices of degree 5 in H_n, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v, \{v_i : 1 \leq i \leq n\}$, since these are only ones with degree at least 4. This is the contradiction, b-coloring with 6 colors is impossible. Thus, we have $\varphi(H_n) \leq 5$. Hence, $\varphi(H_n) = 5$, $\forall\ n \geq 6$. □

Lemma 2.4 *If $n \geq 7$ then b-chromatic number on line graph of Helm graph $L(H_n)$ is $\varphi(L(H_n)) = n$.*

Proof Let $V(H_n) = \{v\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ and $E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n-1\} \cup \{e'_n\} \cup \{s_i : 1 \leq i \leq n\}$ where e_i is the edge vv_i $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq n-1)$, e'_n is the edge v_nv_1 and s_i is the edge v_iu_i $(1 \leq i \leq n)$. By the definition of line graph $V(L(H_n)) = E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

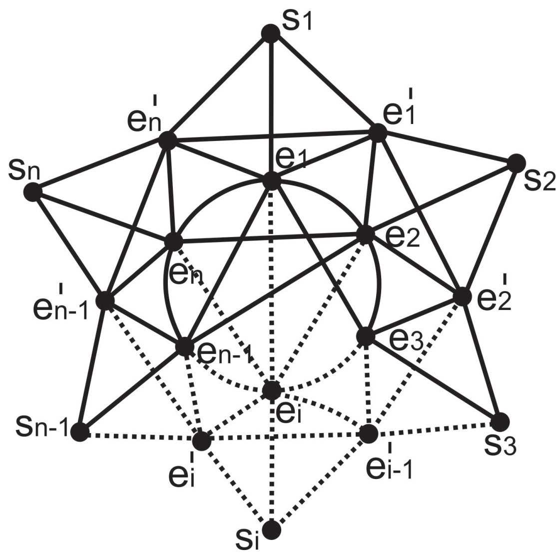

Fig.4 Line Graph of Helm Graph $L(H_n)$

Consider the following n-coloring of $L(H_n)$ as b-chromatic: For $1 \leq i \leq n$, assign the color c_i to e_i. For $1 \leq i \leq n$, assign to vertices s_i and e'_i, one of the allowed colors-such color exists, because $\deg(s_i) = 3$ and $\deg(e'_i) = 6$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(L(H_n)) \geq n$.

Let us assume that $\varphi(L(H_n))$ is greater than n, $\varphi(L(H_n)) = n+1$, $\forall n \geq 7$, there must be at least $n+1$ vertices of degree n in $L(H_n)$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $e_i (1 \leq i \leq n)$, since these are only ones with degree at least n. This is the contradiction, b-coloring with $n+1$ colors is impossible. Thus, we have $\varphi(L(H_n)) \leq n$. Hence, $\varphi(L(H_n)) = n$, $\forall\ n \geq 7$. □

Theorem 2.5 *If $n \geq 8$ then b-chromatic number on middle graph of Helm graph $M(H_n)$ is $\varphi(M(H_n)) = n+1$.*

Proof Let $V(H_n) = \{v\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ and $E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n-1\} \cup \{e'_n\} \cup \{s_i : 1 \leq i \leq n\}$ where e_i is the edge vv_i $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq n-1)$, e'_n is the edge v_nv_1 and s_i is the edge v_iu_i $(1 \leq i \leq n)$. By the definition of middle graph $V(M(H_n)) = \{v\} \cup V(H_n) \cup E(H_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

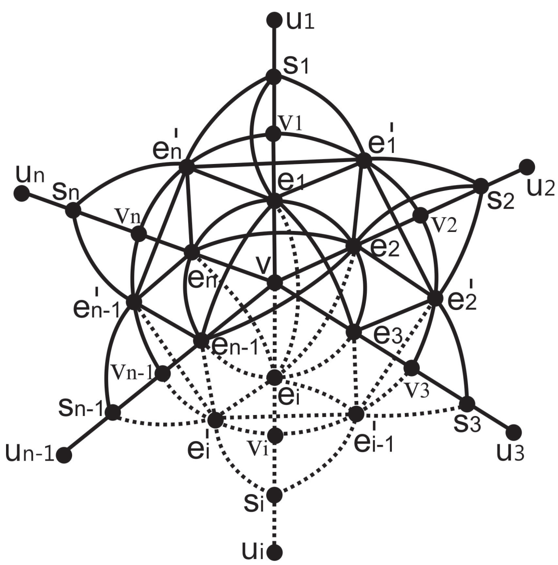

Fig.5 Middle Graph of Helm Graph $M(H_n)$

Consider the following $n+1$-coloring of $M(H_n)$ as b-chromatic: For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{n+1} to v. For $1 \leq i \leq n$, assign to vertices u_i, v_i, s_i, e'_i, one of the allowed colors-such color exists, because $deg(u_i) = 1$, $\deg(v_i) = 4$, $\deg(s_i) = 3$ and $\deg(e'_i) = 8$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(M(H_n)) \geq n+1$.

Let us assume that $\varphi(M(H_n))$ is greater than $n+1$, i.e. $\varphi(M(H_n)) = n+2$, $\forall\, n \geq 8$, there must be at least $n+2$ vertices of degree $n+1$ in $M(H_n)$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v, \{e_i : 1 \leq i \leq n\}$, since these are only ones with degree at least $(n-1)+3$. This is the contradiction, b-coloring with $n+2$ colors is impossible. Thus, we have $\varphi(M(H_n)) \leq n+1$. Hence, $\varphi(M(H_n)) = n+1$, $\forall\, n \geq 8$. □

Proposition 2.6 *If $n \geq 8$ then b-chromatic number on total graph of Helm graph $T(H_n)$ is $\varphi(T(H_n)) = n+1$.*

Proof Consider the coloring of $M(H_n)$ introduced on the proof of Theorem 5. An easy check shows that this coloring is a b-coloring of $T(H_n)$. Hence, $\varphi(T(H_n)) = n+1$, $\forall\, n \geq 8$. □

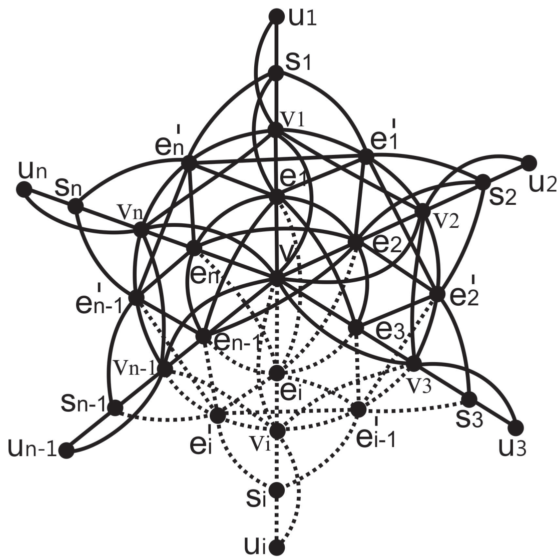

Fig.6 Total Graph of Helm Graph $T(H_n)$.

Lemma 2.7 *If $n \geq 4$ then b-chromatic number of gear graph G_n is $\varphi(G_n) = 4$.*

Proof Let $V(G_n) = \{v\} \cup \{v_1, v_2, \cdots, v_{2n}\}$ where v_i's are the vertices of cycles taken in cyclic order and v is adjacent with $v_{2i-1} (1 \leq i \leq n)$.

Consider the following 4-coloring (c_1, c_2, c_3, c_4) of G_n as b-chromatic:

Assign the colors c_1 to v_1, c_3 to v_2, c_2 to v_3, c_1 to v_4, c_3 to v_5, c_2 to v_6, c_4 to v and c_2 to v_{2n}. For $7 \leq i \leq 2n-1$, if any, assign to vertex v_i one of the allowed colors-such color exists, because $2 \leq \deg(v_i) \leq 3$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(G_n) \geq 4$.

Let us assume that $\varphi(G_n)$ is greater than 4, i.e. $\varphi(G_n) = 5$, $\forall\ n \geq 4$, there must be at least 5 vertices of degree 4 in G_n, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v, \{v_{2i-1} : 1 \leq i \leq n\}$, since these are only ones with degree at least 3. This is the contradiction, b-coloring with 5 colors is impossible. Thus, we have $\varphi(G_n) \leq 4$. Hence, $\varphi(G_n) = 4$, $\forall\ n \geq 4$. □

Lemma 2.8 *If $n \geq 4$ then b-chromatic number on line graph of Gear graph $L(G_n)$ is $\varphi(L(G_n)) = n$.*

Proof Let $V(G_n) = \{v\} \cup \{v_1, v_2, \ldots, v_{2n}\}$ and $E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n-1\} \cup \{e'_n\}$ where e_i is the edge vv_{2i-1} $(1 \leq i \leq n)$, e'_i is the edge v_iv_{i+1} $(1 \leq i \leq 2n-1)$, and e'_{2n} is the edge $v_{2n-1}v_1$. By the definition of line graph $V(L(G_n)) = E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\}$.

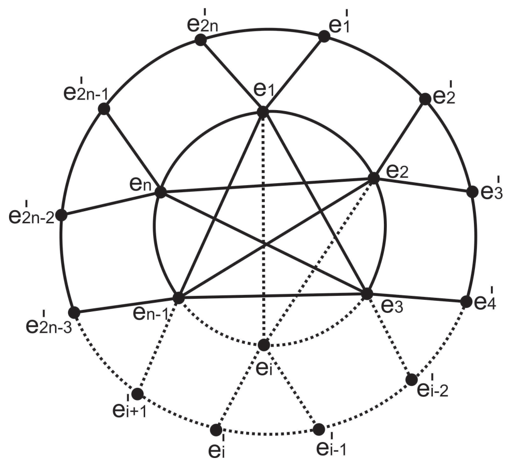

Fig.7 Line graph of Gear Graph $L(G_n)$.

Consider the following n-coloring of $L(G_n)$ as b-chromatic: For $1 \leq i \leq n$, assign the color c_i to e_i. For $1 \leq i \leq 2n$, assign to vertices e'_i, one of the allowed colors-such color exists, because $\deg(e'_i) = 3$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(L(G_n)) \geq n$.

Let us assume that $\varphi(L(G_n))$ is greater than n, $\varphi(L(G_n)) = n + 1$, $\forall\ n \geq 4$, there must be at least $n + 1$ vertices of degree n in $L(G_n)$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $e_i (1 \leq i \leq n)$, since these are only ones with degree at least n. This is the contradiction, b-coloring with $n + 1$ colors is impossible. Thus, we have $\varphi(L(G_n)) \leq n$. Hence, $\varphi(L(G_n)) = n$, $\forall\ n \geq 4$. □

Theorem 2.9 *If $n \geq 5$ then b-chromatic number on middle graph of Gear graph $M(G_n)$ is $\varphi(M(G_n)) = n + 1$.*

Proof Let $V(G_n) = \{v\} \cup \{v_1, v_2, \cdots, v_{2n}\}$ and $E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n - 1\} \cup \{e'_n\}$ where e_i is the edge vv_{2i-1} $(1 \leq i \leq n)$, e'_i is the edge $v_i v_{i+1}$ $(1 \leq i \leq 2n - 1)$, and e'_{2n} is the edge $v_{2n-1}v_1$. By the definition of middle graph $V(M(G_n)) = V(G_n) \cup E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\}$.

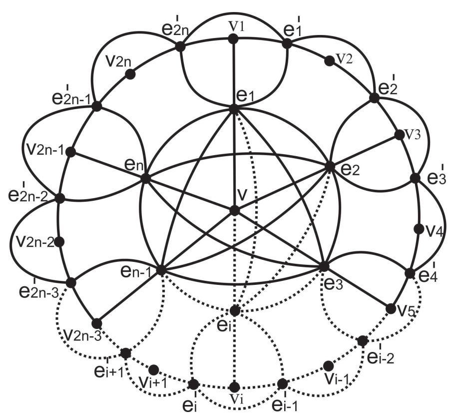

Fig.8 Middle graph of Gear Graph $M(G_n)$.

Consider the following $n+1$-coloring of $M(G_n)$ as b-chromatic:

For $1 \le i \le n$, assign the color c_i to e_i and assign the color c_{n+1} to v. For $1 \le i \le 2n$, assign to vertices v_i and e_i', one of the allowed colors-such color exists, because $2 \le \deg(v_i) \le 3$ and $\deg(e_i') = 5$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(M(G_n)) \ge n+1$.

Let us assume that $\varphi(M(G_n))$ is greater than $n+1$, i.e. $\varphi(M(G_n)) = n+2$, $\forall\ n \ge 5$, there must be at least $n+2$ vertices of degree $n+1$ in $M(G_n)$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v, \{e_i : 1 \le i \le n\}$, since these are only ones with degree at least $n+1$. This is the contradiction, b-coloring with $n+2$ colors is impossible. Thus, we have $\varphi(M(G_n)) \le n+1$. Hence, $\varphi(M(G_n)) = n+1$, $\forall\ n \ge 5$. □

Proposition 2.10 *If $n \ge 6$ then b-chromatic number on total graph of Gear graph $T(G_n)$ is $\varphi(T(G_n)) = n+1$.*

Proof Consider the coloring of $M(G_n)$ introduced on the proof of Theorem 9. An easy check shows that this coloring is a b-coloring of $T(G_n)$. Hence, $\varphi(T(G_n)) = n+1$, $\forall\ n \ge 6$. □

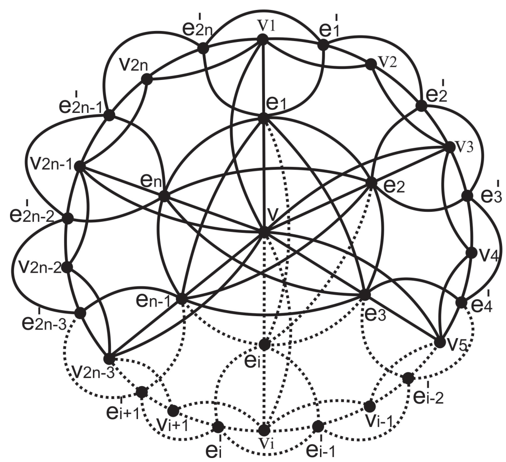

Fig.9 Total graph of Gear Graph $T(G_n)$.

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