International J.Math. Combin. Vol.1(2020), 01-09

On k-Type Slant Helices Due to Bishop Frame in Euclidean 4-Space $\mathbb{E}^4$

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Abstract: In this work, we study $k-$type ($k \in \{0, 1, 2, 3\}$) slant helices with non-zero Bishop curvature functions due to Bishop frame in $\mathbb{E}^4$. General helix is a $0-$type slant helix within the notation of this study. We characterize all of slant helices in terms of Bishop curvatures in $\mathbb{E}^4$.

Key Words: Bishop frame, regular curves, general helix, slant helix, Euclidean 4-space.

AMS(2010): 53A04.

§1. Introduction

In the local differential geometry of space curves, it is well-known that a general helix is a curve whose tangent makes a constant angle with a non-zero constant vector field (the axis of the helix). Moreover, the necessary and sufficient condition for a curve is a general helix if and only if the ratio of the curvature and the torsion of that curve is constant. A slant helix is defined as a curve whose principal normal vector makes a constant angle with a fixed direction by Izumiya and Takeuchi in $\mathbb{E}^3$ [7]. Some characterizations of a slant helix are investigated in [9]. Ali and Turgut have generalized the slant helix to Euclidean n-space $\mathbb{E}^n$ and have given some properties for a non-degenerate slant helix [2]. Öztürk et.al. have considered the focal representation and some properties of focal curves with their curvatures of k-slant helices in $\mathbb{E}^{m+1}$ [11]. Further, some characterizations of slant helices in different spaces such as Minkowski and Galilean are studied [12, 13, 14, 16].

Most of the study of curves are done by using Frenet-Serret frame in classical differential geometry in Euclidean space. In [4], Bishop defined an alternative over Frenet frame for a curve and called it Bishop frame. The advantage of Bishop frame is well-defined when the curve has a vanishing second derivative in 3-dimensional Euclidean space $\mathbb{E}^3$ unlike Frenet frame. Also, Bishop frame is used in many applications such as engineering, computer aided design, DNA analysis etc. After defining this useful alternative frame, many studies have been done by

[1]Received August 23, 2019, Accepted March 2, 2020.

mathematicians using it [3, 8, 15]. Özçelik et. al. have been introduced the parallel transport frame of the curve in 4-dimensional Euclidean space $\mathbb{E}^4$ [10].

The present study aims to determine the characterize all of slant helices in terms of Bishop curvatures in $\mathbb{E}^4$ with the help of the literature.

§2. Preliminaries

Here, the basic definitions and theorems for the theory of curves in Euclidean 4-space $\mathbb{E}^4$ are given for the next section (A more complete elementary treatment can be found in [5], [6]).

The standard flat metric in Euclidean 4-space $\mathbb{E}^4$ is given by

$$\langle \quad , \quad \rangle = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of Euclidean 4-space $\mathbb{E}^4$. The norm of an arbitrary vector $a \in \mathbb{E}^4$ is given by $\|a\| = \sqrt{\langle a, a \rangle}$. The curve α is called an unit speed curve if a velocity vector v of α satisfies $\|v\| = 1$. For vectors $v, w \in \mathbb{E}^4$, it is said to be orthogonal if and only if $\langle v, w \rangle = 0$. Let $\alpha = \alpha(s)$ be a regular curve in Euclidean 4-space $\mathbb{E}^4$. If the tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called a general helix or an inclined curve.

Denote by $\{T, N, B, E\}$ the moving Frenet-Serret frame along the curve α in the space $\mathbb{E}^4$. For an arbitrary curve α in Euclidean 4-space $\mathbb{E}^4$, the following Frenet-Serret formulae is given with respect to the first curvature κ, the second curvature τ and the third curvature σ in [6]

$$\begin{bmatrix} T' \\ N' \\ B' \\ E' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \\ E \end{bmatrix},$$

where T, N, B and E are called the tangent, the principal normal, the first and the second binormal vectors of the curve α, respectively.

Theorem 2.1([14]) *Let $\alpha = \alpha(t)$ be an arbitrary curve in Euclidean 4-space $\mathbb{E}^4$ with above Frenet-Serret equations. Frenet-Serret apparatus of α can be written as follows:*

$$T = \frac{\alpha'}{\|\alpha'\|}, \tag{2.1}$$

$$N = \frac{\|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha'}{\left\| \|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha' \right\|}, \tag{2.2}$$

$$B = \mu N \wedge T \wedge B_2, \tag{2.3}$$

$$E = \mu \frac{T \wedge N \wedge \alpha'''}{\|T \wedge N \wedge \alpha'''\|}, \tag{2.4}$$

$$\kappa = \frac{\left\| \|\alpha'\|^2 \alpha'' - \langle \alpha', \alpha'' \rangle \alpha' \right\|}{\|\alpha'\|^4} \tag{2.5}$$

$$\tau = \frac{\|T \wedge N \wedge \alpha'''\| \, \|\alpha'\|}{\left\| \|\alpha'\|^2 \alpha'' - g(\alpha', \alpha'')\alpha' \right\|} \tag{2.6}$$

and

$$\sigma = \frac{\left\langle \alpha^{(IV)}, E \right\rangle}{\|T \wedge N \wedge \alpha'''\| \, \|\alpha'\|}, \tag{2.7}$$

where μ is taken -1 or $+1$ to make $+1$ the determinant of the matrix $[T, N, B, E]$.

Bishop frame is also referred to as parallel transport that is an orthonormal frame formed by transporting in parallel each component of the frame. The parallel transport is formed with tangent vector and any convenient arbitrary basis for the remainder of the frame (for details, see [4], [10]). Then, the relations between Frenet-Serret frame and parallel transport frame for the curve $\alpha : I \subset R \to \mathbb{E}^4$ are given as follows:

$$\begin{aligned}
T(s) &= T(s), \\
N(s) &= \cos\theta(s)\cos\psi(s)M_1 + (-\cos\phi(s)\sin\psi(s) + \sin\phi(s)\sin\theta(s)\cos\psi(s))M_2 \\
&\quad + (\sin\phi(s)\sin\psi(s) + \cos\phi(s)\sin\theta(s)\cos\psi(s))M_3, \\
B(s) &= \cos\theta(s)\sin\psi(s)M_1 + (\cos\phi(s)\cos\psi(s) + \sin\phi(s)\sin\theta(s)\sin\psi(s))M_2 \\
&\quad + (-\sin\phi(s)\cos\psi(s) + \cos\phi(s)\sin\theta(s)\sin\psi(s))M_3, \\
E(s) &= -\sin\theta(s)M_1 + \sin\phi(s)\cos\theta(s)M_2 + \cos\phi(s)\cos\theta(s)M_3.
\end{aligned}$$

The parallel transport frame equations are expressed as [10]

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}, \tag{2.8}$$

where k_1, k_2, k_3 are curvature functions according to parallel transport frame of the curve α, and their expression as follows:

$$\begin{aligned}
k_1 &= \kappa\cos\theta(s)\cos\psi(s), \\
k_2 &= \kappa(-\cos\phi(s)\sin\psi(s) + \sin\phi(s)\sin\theta(s)\cos\psi(s)), \\
k_3 &= \kappa(\sin\phi(s)\sin\psi(s) + \cos\phi(s)\sin\theta(s)\cos\psi(s)),
\end{aligned}$$

where

$$\theta' = \frac{\sigma}{\sqrt{\kappa^2+\tau^2}}, \quad \psi' = -\tau - \sigma\frac{\sqrt{\sigma^2-\theta'^2}}{\sqrt{\kappa^2+\tau^2}}, \quad \phi' = -\frac{\sqrt{\sigma^2-\theta'^2}}{\cos\theta},$$

and Frenet curvature functions are given as follows:

$$\kappa(s) = \sqrt{k_1^2+k_2^2+k_3^2}, \quad \tau(s) = -\psi' + \phi'\sin\theta, \quad \sigma(s) = \frac{\theta'}{\sin\psi},$$

and

$$\phi'\cos\theta + \theta'\cot\psi = 0,$$

in terms of the invariants of parallel transport frame.

§3. On k-Type Slant Helices Due to Bishop Frame in Euclidean 4-Space $\mathbb{E}^4$

Definition 3.1 *Let $\alpha = \alpha(s)$ be a curve parametrized by arc-length with $\{T, M_1, M_2, M_3\}$ a Bishop frame in $\mathbb{E}^4$. If there exists a non-zero constant vector field U in $\mathbb{E}^4$ such that $\langle M_k, U\rangle \neq 0$ is a constant for all $s \in I$, where $M_0 = T$, then α is said to be $k-$type ($k \in \{0,1,2,3\}$) slant helix, and U is called axis of α.*

Theorem 3.2 *Let $\alpha = \alpha(s)$ be a unit speed curve with non zero Bishop curvatures k_1, k_2, and k_3 due to Bishop frame in $\mathbb{E}^4$. There is no $0-$type slant helix (general helix) due to Bishop frame in $\mathbb{E}^4$.*

Proof Let $\alpha = \alpha(s)$ be $0-$type slant helix in $\mathbb{E}^4$ and the axis of the curve be U. Then, we have that

$$\langle T, U\rangle = c_1(s) = constant \tag{3.1}$$

along the curve α. Differentiating (3.1) with respect to s and using Bishop frame, we know that

$$k_1\langle M_1, U\rangle + k_2\langle M_2, U\rangle + k_3\langle M_3, U\rangle = 0,$$

which implies that the unit vector U lies on the subspace spanned by $\{T\}$ and therefore, it can be written as

$$U = c_1(s)T. \tag{3.2}$$

Differentiation of (3.2) gives

$$c_1k_1M_1 + c_1(s)k_2M_2 + c_1(s)k_3M_3 = 0.$$

Since the vectors $\{M_1, M_2, M_3\}$ are linearly independent, we have $c_1 = 0$ which yields

$$U = 0. \tag{3.3}$$

Since the result (3.3) contradicts with the definition of U, we claim that there is no $0-$type slant helix (general helix) due to Bishop frame in $\mathbb{E}^4$. □

Theorem 3.3 *Let $\alpha = \alpha(s)$ be a unit speed curve with non zero Bishop curvatures k_1, k_2, and k_3 due to Bishop frame in $\mathbb{E}^4$. Then α is a $1-$type slant helix if and only if the function*

$$-c_0\frac{k_1}{k_3} - c_1\frac{k_2}{k_3} \tag{3.4}$$

is a constant, and $c_0 = const.$ and $c_1 = const.$

Proof Let $\alpha = \alpha(s)$ be $1-$type slant helix in $\mathbb{E}^4$ and U be a fixed non-zero direction. Then we have

$$\langle M_1, U\rangle = c_0(s), c_0(s) \in \mathbb{R} \tag{3.5}$$

along the curve α. Using (3.5) and Bishop frame formulae, we have

$$-k_1 \langle T, U\rangle = 0,$$

which implies that the unit vector U lies on the subspace spanned by $\{M_1, M_2, M_3\}$ and it can be written as

$$U = c_0(s)M_1 + a(s)M_2 + b(s)M_3. \tag{3.6}$$

Differentiation of (3.6) gives

$$(-c_0k_1 - ak_2 - bk_3)T + a'M_2 + b'M_3 = 0.$$

Since the vectors $\{T, M_2, M_3\}$ are linearly independent, we have

$$\begin{aligned} -c_0k_1 - ak_2 - bk_3 &= 0, \\ a' &= 0, \\ b' &= 0. \end{aligned} \tag{3.7}$$

From (3.7), we obtain

$$a = c_1, \quad b = -c_0\frac{k_1}{k_3} - c_1\frac{k_2}{k_3},$$

where c_1 is constant.

Conversely, if (3.4) holds, we can find a fixed non zero vector U satisfying $\langle M_1, U\rangle$ =constant. We consider the axis as

$$U = M_1 + M_2 - \left(\frac{k_1}{k_3} + \frac{k_2}{k_3}\right) M_3. \tag{3.8}$$

Differentiating U with the help of (3.4) gives $U' = 0$. This means that U is a constant vector. As a result, α is a $1-$type slant helix in $\mathbb{E}^4$. □

Using Theorem 3.3, we have the following result.

Corollary 3.4 *Let $\alpha = \alpha(s)$ be a $1-$type slant helix with non zero Bishop curvatures k_1, k_2,* *and* k_3 *due to Bishop frame in* $\mathbb{E}^4$. *Then the axes of* α *are obtained by*

$$U = c_0 M_1 + c_1 M_2 + (-c_0 \frac{k_1}{k_3} - c_1 \frac{k_2}{k_3}) M_3,$$

where c_0, c_1 *are constants.*

Theorem 3.5 *Let* $\alpha = \alpha(s)$ *be a unit speed curve with non zero Bishop curvatures* k_1, k_2, *and* k_3 *due to Bishop frame in* $\mathbb{E}^4$.*Then* α *is a* $2-$*type slant helix if and only if the function*

$$-c_0 \frac{k_2}{k_3} - c_1 \frac{k_1}{k_3} \tag{3.9}$$

is a constant, and $c_0 = const.$ *and* $c_1 = const.$

Proof Let $\alpha = \alpha(s)$ be $2-$type slant helix in $\mathbb{E}^4$ and U be a fixed non-zero constant direction. Then we have

$$\langle M_2, U \rangle = c_0(s), c_0(s) \in \mathbb{R} \tag{3.10}$$

along the curve α. Differentiating (3.10) with respect to s and using Bishop frame, we have

$$-k_2 \langle T, U \rangle = 0,$$

which implies that the unit vector U lies on the subspace spanned by $\{M_1, M_2, M_3\}$ and can be decomposed as

$$U = a(s) M_1 + c_0(s) M_2 + b(s) M_3. \tag{3.11}$$

Differentiation of (3.11) gives

$$(-ak_1 - c_0 k_2 - bk_3)T + a' M_2 + b' M_3 = 0.$$

Since the vectors $\{T, M_2, M_3\}$ are linearly independent, we have

$$\begin{aligned} -ak_1 - c_0 k_2 - bk_3 &= 0, \\ a' &= 0, \\ b' &= 0. \end{aligned} \tag{3.12}$$

From (3.12), we obtain

$$a = c_1, \quad b = -c_0 \frac{k_2}{k_3} - c_1 \frac{k_1}{k_3},$$

where c_1 is constant.

Conversely, if (3.9) holds, we can find a fixed non zero vector U satisfying $\langle M_1, U \rangle$ =constant. We consider the axis as

$$U = M_1 + M_2 + \left(\frac{k_2}{k_3} + \frac{k_1}{k_3} \right) M_3. \tag{3.13}$$

Differentiating U with the help of (3.9) gives $U' = 0$. This means that U is a constaant vector. As a result, α is a $2-$type slant helix in $\mathbb{E}^4$. □

Using Theorem 3.5, we have the following result.

Corollary 3.6 *Let $\alpha = \alpha(s)$ be a $2-$type slant helix with non zero Bishop curvatures k_1, k_2, and k_3 due to Bishop frame in $\mathbb{E}^4$. Then the axes of α are obtained by*

$$U = c_1 M_1 + c_0 M_2 + (-c_0 \frac{k_2}{k_3} - c_1 \frac{k_3}{k_3}) M_3,$$

where c_0, c_1 are constants.

Theorem 3.7 *Let $\alpha = \alpha(s)$ be a unit speed curve with non zero Bishop curvatures k_1, k_2, and k_3 due to Bishop frame in $\mathbb{E}^4$. Then α is a $3-$type slant helix if and only if the function*

$$-c_1 \frac{k_1}{k_2} - c_0 \frac{k_3}{k_2} \tag{3.14}$$

is a constant, and $c_0 = const.$ and $c_1 = const.$

Proof Let $\alpha = \alpha(s)$ be $3-$type slant helix in $\mathbb{E}^4$ and U be a fixed non-zero direction. Then we have

$$\langle M_3, U \rangle = c_0(s), c_0(s) \in \mathbb{R} \tag{3.15}$$

along the curve α. Using (3.15) and Bishop frame formulae, we have

$$-k_3 \langle T, U \rangle = 0,$$

which implies that the vector U lies on the subspace spanned by $\{M_1, M_2, M_3\}$ and can be written as

$$U = a(s) M_1 + b(s) M_2 + c_0(s) M_3. \tag{3.16}$$

Differentiation of (3.16) gives

$$(-c_0 k_1 - a k_2 - b k_3) T + a' M_2 + b' M_3 = 0.$$

Since the vectors $\{T, M_2, M_3\}$ are linearly independent, we have

$$\begin{aligned} -a k_1 - b k_2 - c_0 k_3 &= 0, \\ a' &= 0, \\ b' &= 0. \end{aligned} \tag{3.17}$$

From (3.17), we obtain

$$a = c_1, \quad b = -c_1 \frac{k_1}{k_2} - c_0 \frac{k_3}{k_2},$$

where c_1 is constant.

Conversely, if (3.4) holds, we can find a fixed non zero vector U satisfying $\langle M_1, U \rangle$ =constant.

We consider the axis as

$$U = M_1 + \left(-\frac{k_1}{k_2} - \frac{k_3}{k_2} \right) M_2 + M_3. \tag{3.18}$$

Differentiating U with the help of (3.14) gives $U' = 0$. This means that U is a constant vector. As a result, α is a $3-$type slant helix in $\mathbb{E}^4$. □

From the above theorem, we have the following result.

Corollary 3.8 *Let $\alpha = \alpha(s)$ be a $3-$type slant helix with non zero Bishop curvatures k_1, k_2, and k_3 due to Bishop frame in $\mathbb{E}^4$.Then the axes of α are obtained by*

$$U = c_1 M_1 + \left(-c_1 \frac{k_1}{k_2} - c_0 \frac{k_3}{k_2} \right) M_2 + c_0 M_3,$$

where c_0, c_1 are constants.

§4. Conclusion

The properties of $k-$type ($k \in \{0, 1, 2, 3\}$) slant helices with non-zero Bishop curvature functions with Bishop frame in $\mathbb{E}^4$ are obtained. General helix ($0-$type slant helix) that does not exist according to Bishop frame in $\mathbb{E}^4$ is given. All of slant helices are characterized in terms of Bishop curvatures in $\mathbb{E}^4$ in this paper.

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