On $n$-Polynomial $P$-Function and Related Inequalities

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Abstract: In this paper, we introduce and study the concept of $n$-polynomial $P$-function and establish Hermite-Hadamard’s inequalities for this type of functions. In addition, we obtain some new Hermite-Hadamard type inequalities for functions whose first derivative in absolute value is $n$-polynomial $P$-function by using Hölder and power-mean integral inequalities. Some applications to special means of real numbers are also given.

Key Words: $n$-polynomial convexity, $n$-polynomial $P$-function, Hermite-Hadamard inequality.


§1. Preliminaries

Let $f : I \rightarrow \mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$  \hspace{1cm} (1.1)

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function $f$ is concave. This double inequality is well known as the Hermite-Hadamard inequality \[^5\]. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping $f$.

In \[^4\], Dragomir et al. gave the following definition and related Hermite-Hadamard integral inequalities as follow:

**Definition 1.1** A nonnegative function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be $P$-function if the inequality

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

**Theorem 1.1** Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a, b]$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x) dx \leq 2\left[f(a) + f(b)\right].$$  \hspace{1cm} (1.2)

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In [10], Tekin et al. gave the following definition and related Hermite-Hadamard integral inequalities as follow:

**Definition 1.2** Let \( n \in \mathbb{N} \). A non-negative function \( f : I \subset \mathbb{R} \to \mathbb{R} \) is called \( n \)-polynomial convex function if for every \( x, y \in I \) and \( t \in [0, 1] \),

\[
 f (tx + (1 - t)y) \leq \frac{1}{n} \sum_{s=1}^{n} [1 - (1 - t)^s] f(x) + \frac{1}{n} \sum_{s=1}^{n} [1 - t^s] f(y) .
\]

**Theorem 1.2 ([10])** Let \( f : [a, b] \to \mathbb{R} \) be a \( n \)-polynomial convex function. If \( a < b \) and \( f \in L[a, b] \), then the following Hermite-Hadamard type inequalities hold:

\[
 \frac{1}{2} \left( \frac{n}{n + 2^{-n} - 1} \right) f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) dx \leq \left( \frac{f(a) + f(b)}{n} \right) \sum_{s=1}^{n} \frac{s}{s + 1} .
\]

The main purpose of this paper is to introduce the concept of \( n \)-polynomial \( P \)-function which is connected with the concepts of \( P \)-function and \( n \)-polynomial convex function and establish some new Hermite-Hadamard type inequality for these classes of functions. In recent years many authors have studied error estimations Hermite-Hadamard type inequalities; for refinements, counterparts, generalizations, for some related papers see [1, 2, 3, 4, 6, 7, 8, 9, 10].

§2. Definition of \( n \)-Polynomial \( P \)-Function

In this section, we introduce a new concept, which is called \( n \)-polynomial \( P \)-function and we give by setting some algebraic properties for the \( n \)-polynomial \( P \)-function, as follows:

**Definition 2.1** Let \( n \in \mathbb{N} \). A non-negative function \( f : I \subset \mathbb{R} \to \mathbb{R} \) is called \( n \)-polynomial \( P \)-function if for every \( x, y \in I \) and \( t \in [0, 1] \),

\[
 f (tx + (1 - t)y) \leq \frac{1}{n} \sum_{s=1}^{n} [2 - t^s - (1 - t)^s] [f(x) + f(y)] .
\]

We will denote by \( POLP (I) \) the class of all \( n \)-polynomial \( P \)-functions on interval \( I \). Notice that every \( n \)-polynomial \( P \)-function is a \( h \)-convex function with the function

\[
 h(t) = \frac{1}{n} \sum_{s=1}^{n} [2 - t^s - (1 - t)^s] .
\]

Therefore, if \( f, g \in POLC (I) \), then

(i) \( f + g \in POLC (I) \) and for \( c \in \mathbb{R} \ (c \geq 0) \) \( cf \in POLC (I) \) (see [11], Proposition 9).

(ii) If \( f \) and \( g \) be a similarly ordered functions on \( I \), then \( fg \in POLCP (I) \). (see [11], Proposition 10).
Also, if \( f : I \to J \) is a convex and \( g \in \text{POLCP}(J) \) and nondecreasing, then \( g \circ f \in \text{POLCP}(I) \) (see [11], Theorem 15).

**Remark 2.1** We note that if \( f \) satisfies (2.1), then \( f \) is a nonnegative function. Indeed, if we rewrite the inequality (2.1) for \( t = 0 \), then

\[
f(y) \leq f(x) + f(y)
\]

for every \( x, y \in I \). Thus we have \( f(x) \geq 0 \) for all \( x \in I \).

**Proposition 2.1** Every nonnegative \( P \)-function is also a \( n \)-polynomial \( P \)-function.

**Proof** The proof is clear from the following inequalities

\[
t \leq \frac{1}{n} \sum_{s=1}^{n} [1 - (1 - t)^s] \quad \text{and} \quad 1 - t \leq \frac{1}{n} \sum_{s=1}^{n} [1 - t^s]
\]

for all \( t \in [0, 1] \) and \( n \in \mathbb{N} \). In this case, we can write

\[
1 \leq \frac{1}{n} \sum_{s=1}^{n} [2 - t^s - (1 - t)^s].
\]

Therefore, the desired result is obtained. \( \square \)

We can give the following corollary for every nonnegative convex function is also a \( P \)-function.

**Corollary 2.1** Every nonnegative convex function is also a \( n \)-polynomial \( P \)-function.

**Theorem 2.1** Let \( b > a \) and \( f_\alpha : [a, b] \to \mathbb{R} \) be an arbitrary family of \( n \)-polynomial \( P \)-function and let \( f(x) = \sup_\alpha f_\alpha(x) \). If \( J = \{ u \in [a, b] : f(u) < \infty \} \) is nonempty, then \( J \) is an interval and \( f \) is a \( n \)-polynomial \( P \)-function on \( J \).

**Proof** Let \( t \in [0, 1] \) and \( x, y \in J \) be arbitrary. Then

\[
f(tx + (1 - t)y) = \sup_\alpha f_\alpha(tx + (1 - t)y)
\]

\[
\leq \sup_\alpha \left[ \frac{1}{n} \sum_{s=1}^{n} (2 - t^s - (1 - t)^s) [f_\alpha(x) + f_\alpha(y)] \right]
\]

\[
\leq \frac{1}{n} \sum_{s=1}^{n} (2 - t^s - (1 - t)^s) \left[ \sup_\alpha f_\alpha(x) + \sup_\alpha f_\alpha(y) \right]
\]

\[
= \frac{1}{n} \sum_{s=1}^{n} (2 - t^s - (1 - t)^s) [f(x) + f(y)] < \infty.
\]

This shows simultaneously that \( J \) is an interval, since it contains every point between any two of its points, and that \( f \) is a \( n \)-polynomial \( P \)-function on \( J \). \( \square \)
§3. Hermite-Hadamard Inequality for \( n \)-Polynomial \( P \)-Functions

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for \( n \)-polynomial \( P \)-functions. In this section, we will denote by \( L[a, b] \) the space of (Lebesgue) integrable functions on \([a, b]\).

**Theorem 3.1** Let \( f : [a, b] \to \mathbb{R} \) be a \( n \)-polynomial \( P \)-function. If \( a < b \) and \( f \in L[a, b] \), then the following Hermite-Hadamard type inequalities hold

\[
\frac{1}{4} \left( \frac{n}{n + 2^{-n} - 1} \right) f\left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) dx \leq \left( \frac{f(a) + f(b)}{n} \right) \sum_{s=1}^n \frac{2s}{s+1}. \tag{3.1}
\]

**Proof** From the property of the \( n \)-polynomial \( P \)-function of \( f \), we get

\[
f\left( \frac{a + b}{2} \right) = f\left( \frac{[ta + (1-t)b] + [(1-t)a + tb]}{2} \right) = f\left( \frac{1}{2} [ta + (1-t)b] + \frac{1}{2} [(1-t)a + tb] \right) \leq \frac{1}{n} \sum_{s=1}^n \left[ 2 - 2 \left( \frac{1}{2} \right)^s \right] \left[ f\left( ta + (1-t)b \right) + f\left( (1-t)a + tb \right) \right].
\]

By taking integral in the last inequality with respect to \( t \in [0,1] \), we deduce that

\[
f\left( \frac{a + b}{2} \right) \leq \frac{1}{n} \sum_{s=1}^n \left[ 2 - 2 \left( \frac{1}{2} \right)^s \right] \left[ \int_0^1 f\left( ta + (1-t)b \right) dt + \int_0^1 f\left( (1-t)a + tb \right) dt \right] = \frac{4}{b - a} \left( \frac{n + 2^{-n} - 1}{n} \right) \int_a^b f(x) dx.
\]

By using the property of the \( n \)-polynomial \( P \)-function of \( f \), if the variable is changed as \( x = ta + (1-t)b \), then

\[
\frac{1}{b - a} \int_a^b f(x) dx = \int_0^1 f\left( ta + (1-t)b \right) dt \leq \int_0^1 \left[ \frac{1}{n} \sum_{s=1}^n [2 - t^s - (1-t)^s] \left[ f(a) + f(b) \right] \right] dt = \frac{f(a) + f(b)}{n} \sum_{s=1}^n \int_0^1 [2 - t^s - (1-t)^s] dt = \left[ \frac{f(a) + f(b)}{n} \right] \sum_{s=1}^n \frac{2s}{s+1},
\]

where

\[
\int_0^1 [2 - t^s - (1-t)^s] dt = \frac{2s}{s+1}.
\]

This completes the proof of theorem. \( \square \)
Remark 3.1 In case of $n = 1$, the inequality (3.1) coincides with the the inequality (1.2).

§4. New Inequalities for $n$-Polynomial $P$-Functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value is $n$-polynomial $P$-function. Dragomir and Agarwal [3] used the following lemma.

Lemma 4.1[3] Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^o$, $a, b \in I^o$ with $a < b$. If $f' \in L[a, b]$, then

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx = \frac{b-a}{2} \int_0^1 (1 - 2t) f'(ta + (1-t)b) dt.$$ 

Theorem 4.1 it Let $f : I \to \mathbb{R}$ be a differentiable mapping on $I^o$, $a, b \in I^o$ with $a < b$ and assume that $f' \in L[a, b]$. If $|f'|$ is $n$-polynomial $P$-function on interval $[a, b]$, then the following inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{n} \sum_{s=1}^n \left\{ \frac{(s^2 + s + 2)^2 - 2}{(s + 1)(s + 2)^2} \right\} A (|f'(a)|, |f'(b)|)$$

holds for $t \in [0, 1]$.

Proof Using Lemma 4.1 and the inequality

$$|f'(ta + (1-t)b)| \leq \frac{1}{n} \sum_{s=1}^n (2 - t^s - (1 - t)^s) ||f'(a)|| + ||f'(b)||,$$

we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \int_0^1 (1 - 2t) f'(ta + (1-t)b) dt$$

$$\leq \frac{b-a}{2} \int_0^1 |1 - 2t| \left( \frac{1}{n} \sum_{s=1}^n (2 - t^s - (1 - t)^s) ||f'(a)|| + ||f'(b)|| \right) dt$$

$$\leq \frac{b-a}{2n} [||f'(a)|| + ||f'(b)||] \sum_{s=1}^n \int_0^1 |1 - 2t| (2 - t^s - (1 - t)^s) dt$$

$$= \frac{b-a}{n} \sum_{s=1}^n \left\{ \frac{(s^2 + s + 2)^2 - 2}{(s + 1)(s + 2)^2} \right\} A (|f'(a)|, |f'(b)|),$$

where

$$\int_0^1 |1 - 2t| (2 - t^s - (1 - t)^s) dt = \frac{(s^2 + s + 2)^2 - 2}{(s + 1)(s + 2)^2}.$$
and $A$ is the arithmetic mean. This completes the proof of theorem. \hfill \Box

**Corollary 4.1** If we take $n = 1$ in the inequality (4.1), we get the following inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2} A(|f'(a)|, |f'(b)|).$$  \hspace{1cm} (4.2)

**Theorem 4.2** Let $f : I \to \mathbb{R}$ be a differentiable mapping on $I$, $a, b \in I$ with $a < b$ and assume that $f' \in L[a, b]$. If $|f'|^q$, $q > 1$, is an $n$-polynomial $P$-function on interval $[a, b]$, then the following inequality holds for $t \in [0, 1]$.

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{4}{n} \sum_{s=1}^{n} \frac{s}{s+1} \right)^{\frac{1}{q}} \left( A^{\frac{1}{q}}(|f'(a)|^q, |f'(b)|^q) \right),$$  \hspace{1cm} (4.3)

where $\frac{1}{p} + \frac{1}{q} = 1$ and $A$ is the arithmetic mean.

**Proof** Using Lemma 4.1, Hölder’s integral inequality and the following inequality

$$|f'(ta + (1-t)b)|^q \leq \frac{1}{n} \sum_{s=1}^{n} (2 - t^s - (1-t)^s) \left[ |f'(a)|^q + |f'(b)|^q \right]$$

which is the $n$-polynomial $P$-function of $|f'|^q$, we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2} \left( \int_0^1 |1 - 2t|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{4}{n} \sum_{s=1}^{n} \frac{s}{s+1} \right)^{\frac{1}{q}} \left( \frac{1}{n} \sum_{s=1}^{n} (2 - t^s - (1-t)^s) \right)^{\frac{1}{q}}$$

$$= \frac{b-a}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{4}{n} \sum_{s=1}^{n} \frac{s}{s+1} \right)^{\frac{1}{q}} A^{\frac{1}{q}}(|f'(a)|^q, |f'(b)|^q)$$

where

$$\int_0^1 |1 - 2t|^p dt = \frac{1}{p+1}, \quad \int_0^1 (2 - t^s - (1-t)^s) dt = \frac{2s}{s+1}.$$

This completes the proof of theorem. \hfill \Box
Corollary 4.2 If we take $n = 1$ in the inequality (4.3), we get the following inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq (b-a) \left[ \frac{1}{2(2^{q+1})} \right]^{\frac{1}{q}} A^{\frac{1}{q}} (|f'(a)|^q, |f'(b)|^q).$$

Theorem 4.3 Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I$, $a,b \in I$ with $a < b$ and assume that $f' \in L[a,b]$. If $|f'|^q, q \geq 1$, is an $n$-polynomial $P$-function on the interval $[a,b]$, then the following inequality holds for $t \in [0,1]$.

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2^{\frac{q}{2}} - \frac{1}{4}} \left( \sum_{s=1}^{n} \left( \frac{s^2 + s + 2}{s(s + 1)(2^s - 1)} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} \leq \frac{b-a}{2^{\frac{q}{2}} - \frac{1}{4}} \left( \sum_{s=1}^{n} \left( \frac{s^2 + s + 2}{s(s + 1)(2^s - 1)} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} A^{\frac{1}{q}} (|f'(a)|^q, |f'(b)|^q).$$

Proof From Lemma 4.1, well known power-mean integral inequality and the property of the $n$-polynomial $P$-function of $|f'|^q$, we obtain

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2^{\frac{q}{2}} - \frac{1}{4}} \left( \sum_{s=1}^{n} \left( \frac{s^2 + s + 2}{s(s + 1)(2^s - 1)} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2^{\frac{q}{2}} - \frac{1}{4}} \left( \sum_{s=1}^{n} \left( \frac{s^2 + s + 2}{s(s + 1)(2^s - 1)} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} A^{\frac{1}{q}} (|f'(a)|^q, |f'(b)|^q)$$

where

$$\int_0^1 |1 - 2t|dt = \frac{1}{2},$$

$$\int_0^1 |1 - (1-t)| dt = \frac{(s^2 + s + 2)2^s - 2}{(s+1)(s+2)2^s}.$$ 

This completes the proof of theorem.

Corollary 4.3 Under the assumption of Theorem 4.3, If we take $q = 1$ in the inequality (4.4), then we get the following inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{n} \left( \sum_{s=1}^{n} \left( \frac{s^2 + s + 2}{s(s + 1)(2^s)} \right) \right) A (|f'(a)|, |f'(b)|)$$

This inequality coincides with the inequality (4.1).
Corollary 4.4 Under the assumption of Theorem 4.3, If we take $n = 1$ in the inequality (4.4), then we get the following inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2^{2-q}} A^\frac{1}{q} \left( |f'(a)|^q, |f'(b)|^q \right).$$

which is identical to the inequality in [1, Theorem 2.3].

Corollary 4.5 Under the assumption of Theorem 4.3, If we take $n = 1$ and $q = 1$ in the inequality (4.4), then we get the following inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2} A(|f'(a)|, |f'(b)|).$$

This inequality coincides with the inequality (4.2).

§5. Applications for Special Means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers $a, b$ with $b > a$.

1. The arithmetic mean

$$A := A(a, b) = \frac{a + b}{2}, \quad a, b \geq 0,$$

2. The geometric mean

$$G := G(a, b) = \sqrt{ab}, \quad a, b \geq 0,$$

3. The harmonic mean

$$H := H(a, b) = \frac{2ab}{a + b}, \quad a, b > 0,$$

4. The logarithmic mean

$$L := L(a, b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b \\ a, & a = b \end{cases}; \quad a, b > 0$$

5. The $p$-logarithmic mean

$$L_p := L_p(a, b) = \begin{cases} \left( \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right) \right)^\frac{1}{p}, & a \neq b, p \in \mathbb{R} \setminus \{-1, 0\} \\ a, & a = b \end{cases}; \quad a, b > 0.$$
6. The identric mean

\[ I := I(a, b) = \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}}, \quad a, b > 0, \]

These means are often used in numerical approximation and in other areas. However, the following simple relationships

\[ H \leq G \leq L \leq I \leq A. \]

are known in the literature. It is also known that \( L_p \) is monotonically increasing over \( p \in \mathbb{R} \), denoting \( L_0 = I \) and \( L_{-1} = L \).

**Proposition 5.1** Let \( a, b \in [0, \infty) \) with \( a < b \) and \( n \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\} \). Then, the following inequalities are obtained:

\[ \frac{1}{4} \left( \frac{n}{n + 2^{-n} - 1} \right) A^n(a, b) \leq L^n_n(a, b) \leq A(a^n, b^n) \frac{2^n}{n} \sum_{s=1}^{n} \frac{2s}{s + 1}. \]

*Proof* The assertion follows from the inequalities (3.1) for the function \( f(x) = x^n, \; x \in [0, \infty). \)

**Proposition 5.2** Let \( a, b \in (0, \infty) \) with \( a < b \). Then, the following inequalities are obtained

\[ \frac{1}{4} \left( \frac{n}{n + 2^{-n} - 1} \right) A^{-1}(a, b) \leq L^{-1}(a, b) \leq \frac{2}{n} H^{-1}(a, b) \sum_{s=1}^{n} \frac{2s}{s + 1}. \]

*Proof* The assertion follows from the inequalities (3.1) for the function \( f(x) = x^{-1}, \; x \in (0, \infty). \)

**Proposition 5.3** Let \( a, b \in (0, 1] \) with \( a < b \). Then, the following inequalities are obtained

\[ \frac{2 \ln G(a, b)}{n} \sum_{s=1}^{n} \frac{2s}{s + 1} \leq \ln I(a, b) \leq \frac{1}{4} \left( \frac{n}{n + 2^{-n} - 1} \right) \ln A(a, b). \]

*Proof* The assertion follows from the inequalities (3.1) for the function \( f(x) = -\ln x, \; x \in (0, 1]. \)

**References**

(2012), 129-134.


