On Some Properties of Mixed Super Quasi Einstein Manifolds

Dipankar Debnath
(Department of Mathematics, Bamanpukur High School(H.S), Nabawip, India)

Nirabhra Basu
(Department of Mathematics, The Bhawanipur Education Society College, Kolkata-700020, West Bengal, India)
E-mail: dipankardebnath123@gmail.com, nirabhra.basu@thebges.edu.in

Abstract: In this paper we have studied Ricci-pseudo symmetric mixed super quasi Einstein manifolds and Ricci-semi symmetric mixed super quasi Einstein manifolds. Finally, we get relation mixed super quasi Einstein manifold.

Key Words: Mixed super quasi Einstein manifold, Ricci-pseudo symmetric, Ricci-semi symmetric manifold.


§1. Introduction

The notion of quasi Einstein manifold was introduced in a paper [8] by M.C.Chaki and R.K.Maity. According to them a non-flat Riemannian manifold \((M^n, g),(n \geq 3)\) is defined to be a quasi Einstein manifold if its Ricci tensor \(S\) of type \((0,2)\) satisfies the condition

\[
S(X, Y) = ag(X, Y) + bA(X)A(Y) \tag{1.1}
\]

and is not identically zero, where \(a, b\) are scalars, \(b \neq 0\) and \(A\) is a non-zero 1-form such that

\[
g(X, U) = A(X), \quad \forall X \in TM, \tag{1.2}
\]

\(U\) being a unit vector field. In such a case \(a, b\) are called the associated scalars. \(A\) is called the associated 1-form and \(U\) is called the generator of the manifold. Such an \(n\)-dimensional manifold is denoted by the symbol \((QE)_{n}\).

Again, in [15], U.C.De and G.C.Ghosh defined generalized quasi Einstein manifold. A non-flat Riemannian manifold is called a generalized quasi Einstein manifold if its Ricci-tensor \(S\) of type \((0,2)\) is non-zero and satisfies the condition

\[
S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y) \tag{1.3}
\]

\(^1\)Dr. N.Basu is supported by the project of The Bhawanipur Edu. Soc. College, Kolkata.
\(^2\)Received October 11, 2019, Accepted March 12, 2020.
where \( a, b, c \) are non-zero scalars and \( A, B \) are two 1-forms such that

\[
g(X, U) = A(X) \quad \text{and} \quad g(X, V) = B(X),
\]

(1.4)

where, \( U, V \) being unit vectors which are orthogonal, i.e,

\[
g(U, V) = 0.
\]

(1.5)

This type of manifold are denoted by \( G(QE)_n \).

Chaki introduced super quasi Einstein manifold [10], denoted by \( S(QE)_n \), where the Ricci tensor \( S \) of type (0,2) which is not identically zero satisfies the condition

\[
S(X, Y) = a g(X, Y) + b A(X) A(Y) + c [ A(X) B(Y) + A(Y) B(X) ] + d D(X, Y)
\]

(1.6)

where \( a, b, c, d \) are non-zero scalars of which, \( A, B \) are two non zero 1-forms defined as (1.4) and \( U, V \) being mutually orthogonal unit vector fields, \( D \) is a symmetric (0,2) tensor with zero trace which satisfies the condition

\[
D(X, U) = 0 \quad \forall \ X \in TM.
\]

(1.7)

In such case \( a, b, c, d \) are called the associated scalars, \( A, B \) are called the associated main and auxiliary 1-forms, \( D \) is called the associated tensor of the manifold. Such an \( n \)-dimensional manifold shall be denoted by the symbol \( S(QE)_n \).

In the recent papers [2],[4], A.Bhattacharyya and T.De introduced the notion of mixed generalized quasi Einstein manifold. A non-flat Riemannian manifold is called a mixed generalized quasi-Einstein manifold if its Ricci tensor \( S \) of type (0,2) is non-zero and satisfies the condition

\[
S(X, Y) = a g(X, Y) + b A(X) A(Y) + c B(X) B(Y) + d [ A(X) B(Y) + B(X) A(Y) ]
\]

(1.8)

where \( a, b, c, d \) are non-zero scalars,

\[
g(X, U) = A(X) \quad \text{and} \quad g(X, V) = B(X)
\]

(1.9)

\[
g(U, V) = 0
\]

(1.10)

where, \( A, B \) are two non-zero 1-forms, \( U \) and \( V \) are unit vector fields corresponding to the 1-forms \( A \) and \( B \) respectively. If \( d = 0 \), then the manifold reduces to a \( G(QE)_n \). This type of manifold is denoted by \( MG(QE)_n \).

We introduced mixed super quasi Einstein manifolds [5],[11],[18]. A non-flat Riemannian manifold \((M^n, g),(n \geq 3)\) is called mixed super quasi Einstein manifold if its Ricci tensor \( S \) of type (0,2) is not identically zero and satisfies the condition

\[
S(X, Y) = a g(X, Y) + b A(X) A(Y) + c B(X) B(Y) + d [ A(X) B(Y) + B(X) A(Y) ] + e D(X, Y)
\]

(1.11)
where \(a, b, c, d, e\) are non-zero scalars, \(A, B\) are two non zero 1-forms such that
\[
g(X, U) = A(X) \quad \text{and} \quad g(X, V) = B(X) \quad \forall X \in TM,
\]
and \(U, V\) being mutually orthogonal unit vector fields, \(D\) is a symmetric (0,2) tensor with zero trace which satisfies the condition
\[
D(X, U) = 0 \quad \forall X.
\]

In such case \(a, b, c, d, e\) are called the associated scalars, \(A, B\) are called the associated main and auxiliary 1-forms, \(D\) is called the associated tensor of the manifold. Such an \(n\)-dimensional manifold shall be denoted by the symbol \(MS(QE)_n\).

§2. Preliminaries

We know in a \(n\)-dimensional \((n > 2)\) Riemannian manifold the covariant quasi conformal curvature tensor is defined as \((3,7,17)\)
\[
\tilde{C}(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + \tilde{S}(Y, Z)g(X, W) - S(X, Z)g(Y, W) + g(Y, Z)g(X, W) - g(X, Z)g(Y, W) - \frac{r}{n} \left[ \frac{\alpha}{n-1} \right]
\]
(2.1)
where
\[
g(C(X, Y)Z, W) = \tilde{C}(X, Y, Z, W).
\]
(2.2)

The projective curvature tensor is denoted by \(\tilde{P}(X, Y, Z, W)\) and in a \(V_n(n > 2)\) it is defined as
\[
\tilde{P}(X, Y, Z, W) = R(X, Y, Z, W) - \frac{1}{n-1} [S(Y, Z)g(X, W) - S(Y, W)g(X, W)].
\]
(2.3)

§3. Ricci-Pseudo Symmetric Mixed Super Quasi Einstein Manifold

An \(n\)-dimensional semi-Riemannian manifold \((M^n, g)\) is called Ricci -pseudo symmetric [12] if the tensor \(R.S\) and \(Q(g, S)\) are linearly dependent, where
\[
(R(X, Y), S)(Z, W) = -S(R(X, Y)Z, W) - S(Z, R(X, Y)W),
\]
(3.1)
\[
Q(g, S)(Z, W; X, Y) = -S((X \wedge Y)Z, W) - S(Z, (X \wedge Y)W)
\]
(3.2)
and
\[
(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y
\]
(3.3)
for vector fields \(X, Y, Z, W\) on \(M^n\), \(R\) denotes the curvature tensor of \(M^n\) [19].
The condition of Ricci-pseudo symmetricity is equivalent to the relation

\[(R(X,Y).S)(Z,W) = L_s Q(g,S)(Z,W;X,Y),\]  

which holds on the set

\[U_s = \{ x \in M : S \neq \frac{r}{n} g \text{ at } x \},\]

where \(L_s\) is some function on \(U_s\) \cite{19}. If \(R.S = 0\) then the manifold is called Ricci-semi symmetric. Every Ricci-semi symmetric manifold is Ricci-pseudo symmetric but the converse is not true \cite{1,6,9,12,13,19}.

**Theorem 3.1** In a Ricci-pseudo symmetric mixed super quasi Einstein manifold the following relation holds

\[R(X,Y,U,V) = L_s[A(Y)B(X) - A(X)B(Y)].\]

**Proof** Let the manifold be Ricci-pseudo symmetric. Then from (3.1) and (3.4) we get

\[L_s Q(g,S)(Z,W;X,Y) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W).\]

Using (3.2) in (3.5), we get

\[S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = L_s[g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] + g(Y,W)S(X,Z) - g(X,W)S(Y,Z).\]

Since the manifold is a mixed super quasi Einstein manifold, Using the well-known properties of curvature tensor \(R\) we obtain

\[+ e[D(R(X,Y)Z)W + D(R(X,Y)W)Z]]\]
\[= L_s[b[g(Y,Z)A(X)A(W) - g(X,Z)a(Y)A(W)] + g(Y,W)A(X)A(Z) - g(X,W)A(Y)A(Z) + c[g(Y,Z)B(X)B(W) - g(X,Z)b(Y)B(W)] + g(Y,W)B(X)B(Z) - g(X,W)B(Y)B(Z)\]
\[+ d[g(Y,Z)A(X)B(W) + g(Y,Z)B(X)A(W) - g(X,Z)A(Y)B(W) - g(X,Z)B(Y)A(W)\]
\[+ g(Y,W)A(X)B(Z) + g(Y,W)A(Z)B(X) - g(X,W)A(Y)B(Z) - g(X,W)A(Z)B(Y)]\]
\[+ e[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) + g(Y,W)D(X,Z) - g(X,W)D(Y,Z)].\]

Putting \(Z = W = U\) in (3.7) we obtain

\[2d\{R(X,Y,U,V)\} = L_s[2d[A(Y)B(X) - A(X)B(Y)]] = 0.\]

Since \(d \neq 0\), we get

\[R(X,Y,U,V) = L_s[A(Y)B(X) - A(X)B(Y)].\]

Hence the theorem follows.
§4. Ricci-Semi Symmetric Mixed Super Quasi-Einstein Manifold

An n-dimensional manifold \((M^n, g)\) is called semi-symmetric \([13]\) if \(R(X; Y) \cdot S = 0, \forall X, Y\), where \(R(X; Y)\) denotes the curvature operator.

**Theorem 4.1** In a Ricci-semi symmetric mixed super quasi Einstein manifold satisfy the condition

\[(a + b)\hat{R}(X, Y, U, U) + d\hat{R}(X, Y, U, V) = 0.\]

**Proof** We know

\[(R(X, Y) \cdot S)(Z, W) = -S(R(X, Y)Z, W) - S(Z, R(X, Y)W) \quad \text{and} \quad R(X, Y) \cdot S = 0\]

which implies

\[S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0.\]

i.e.,

\[ag(R(X, Y)Z, W) + bA(R(X, Y)Z)A(W) + cB(R(X, Y)Z)B(W)\]
\[+ d[A(R(X, Y)Z)B(W) + B(R(X, Y)Z)A(W)] + eD(R(X, Y)Z, W)\]
\[+ ag(R(X, Y)W, Z) + bA(R(X, Y)W)A(Z) + cB(R(X, Y)W)B(Z)\]
\[+ d[A(R(X, Y)W)B(Z) + B(R(X, Y)W)A(Z)] + eD(R(X, Y)W, Z) = 0. \quad (4.1)\]

Putting \(Z = W = U\) in (4.1) we get

\[(a + b)A(R(X, Y)U) + dB(R(X, Y)U) = 0.\]
\[(a + b)\hat{R}(X, Y, U, U) + d\hat{R}(X, Y, U, V) = 0. \quad (4.2)\]

Hence the theorem follows. \(\Box\)

§5. Mixed Super Quasi Manifolds Satisfying the Condition \(\hat{C} \cdot S = 0\)

**Theorem 5.1** In a mixed super quasi Einstein manifold with the condition \(\hat{C} \cdot S = 0\) satisfy

\[(a + b)\hat{R}(X, Y, U, U) + d\hat{R}(X, Y, U, V) + d(a_1 + b_1 + c_1)[A(X)B(Y) - A(Y)B(X)]\]
\[+ de_1(a_1 + b_1 + c_1)[A(X)B(Y) - A(Y)B(X)] = 0.\]

**Proof** From condition \(\hat{C} \cdot S = 0\) we get

\[S(\hat{C}(X, Y)Z, W) + S(Z, \hat{C}(X, Y)W) = 0\]
for all vector fields $X, Y, Z, W$ on $(M^n, g)$, i.e.,

$$
ag(\tilde{C}(X,Y)Z,W) + bA(\tilde{C}(X,Y)Z)A(W) + cB(\tilde{C}(X,Y)Z)B(W) \\
+ d\{A(\tilde{C}(X,Y)W,Z) + bA(\tilde{C}(X,Y)W)A(Z) + cB(\tilde{C}(X,Y)W)B(Z) \\
$$

(5.1)

Putting $Z = W = U$ in (2.7) we obtain

$$
(a + b)\dot{\tilde{C}}(X,Y,U,U) + d\dot{\tilde{C}}(X,Y,U,V) = 0.
$$

(5.2)

Using (1.11) in (2.1) and putting $Z = W = U$ we get

$$
\dot{\tilde{C}}(X,Y,U,U) = \dot{a}\hat{R}(X,Y,U,U).
$$

(5.3)

Using (1.11) in (2.1) and putting $Z = U$ and $W = V$ we obtain

$$
\dot{\tilde{C}}(X,Y,U,V) = \dot{a}\hat{R}(X,Y,U,V) + (a_1 + b_1 + c_1)[A(X)B(Y) - A(Y)B(X)] \\
+ e_1[A(X)D(Y,V) - A(Y)D(X,V)],
$$

(5.4)

where

$$
a_1 = -\left\{\frac{r}{n}n - 1 + 2\hat{b} - 2a\hat{b}\right\}, \quad b_1 = \hat{b}, c_1 = \hat{c}, d_1 = \hat{d} \text{ and } e_1 = \hat{e}.
$$

Using (5.3) and (5.4) in (5.2) we get

$$
(a + b)\dot{a}\hat{R}(X,Y,U,U) + d\dot{a}\hat{R}(X,Y,U,V) + d(a_1 + b_1 + c_1)[A(X)B(Y) - A(Y)B(X)] \\
+ d\dot{e}_1(a_1 + b_1 + c_1)[A(X)B(Y) - A(Y)B(X)] = 0.
$$

(5.5)

**Corollary 5.1** In a mixed super quasi Einstein manifold with the condition $\tilde{P}.S = 0$ satisfying the condition $(a + b)\hat{R}(X,Y,U,U) + d\hat{R}(X,Y,U,V) = 0$.

**References**


