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On Some Properties of Mixed Super Quasi Einstein Manifolds

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Abstract: In this paper we have studied Ricci-pseudo symmetric mixed super quasi Einstein manifolds and Ricci-semi symmetric mixed super quasi Einstein manifolds. Finally, we get relation mixed super quasi Einstein manifold.

Key Words: Mixed super quasi Einstein manifold, Ricci-pseudo symmetric, Ricci-semi symmetric manifold.

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§1. Introduction

The notion of quasi Einstein manifold was introduced in a paper [8] by M.C.Chaki and R.K.Maity. According to them a non-flat Riemannian manifold (M^n, g),$(n \geq 3)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type (0,2)satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) \tag{1.1}$$

and is not identically zero,where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

$$g(X,U) = A(X), \qquad \forall X \in TM, \tag{1.2}$$

U being a unit vector field. In such a case a, b are called the associated scalars. A is called the associated 1-form and U is called the generator of the manifold. Such an n-dimensional manifold is denoted by the symbol$(QE)_n$.

Again, in [15], U.C.De and G.C.Ghosh defined generalized quasi Einstein manifold. A non-flat Riemannian manifold is called a generalized quasi Einstein manifold if its Ricci-tensor S of type (0,2) is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \tag{1.3}$$

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where a, b, c are non-zero scalars and A, B are two 1-forms such that

$$g(X,U) = A(X) \qquad and \qquad g(X,V) = B(X), \tag{1.4}$$

where, U, V being unit vectors which are orthogonal, i.e,

$$g(U,V) = 0. \tag{1.5}$$

This type of manifold are denoted by $G(QE)_n$.

Chaki introduced super quasi Einstein manifold [10], denoted by S$(QE)_n$, where the Ricci tensor S of type (0,2) which is not identically zero satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] + dD(X,Y) \tag{1.6}$$

where a, b, c, d are non-zero scalars of which, A, B are two non zero 1-forms defined as (1.4) and U, V being mutually orthogonal unit vector fields, D is a symmetric (0,2) tensor with zero trace which satisfies the condition

$$D(X,U) = 0 \qquad \forall\ X \in TM. \tag{1.7}$$

In such case a, b, c, d are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, D is called the associated tensor of the manifold. Such an n-dimensional manifold shall be denoted by the symbol $S(QE)_n$.

In the recent papers [2],[4], A.Bhattacharyya and T.De introduced the notion of mixed generalized quasi Einstein manifold. A non-flat Riemannian manifold is called a mixed generalized quasi-Einstein manifold if its Ricci tensor S of type (0,2) is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + d[A(X)B(Y) + B(X)A(Y)] \tag{1.8}$$

where a, b, c, d are non-zero scalars,

$$g(X,U) = A(X) \;\text{ and }\; g(X,V) = B(X) \tag{1.9}$$

$$g(U,V) = 0 \tag{1.10}$$

where, A, B are two non-zero 1-forms, U and V are unit vector fields corresponding to the 1-forms A and B respectively. If $d = 0$, then the manifold reduces to a $G(QE)_n$. This type of manifold is denoted by $MG(QE)_n$.

We introduced mixed super quasi Einstein manifolds [5],[11],[18]. A non-flat Riemannian manifold (M^n, g),(n≥3) is called mixed super quasi Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) &= ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ &\quad + d[A(X)B(Y) + B(X)A(Y)] + eD(X,Y) \end{aligned} \tag{1.11}$$

where a, b, c, d, e are non-zero scalars, A, B are two non zero 1-forms such that

$$g(X,U) = A(X) \quad \textit{and} \quad g(X,V) = B(X) \qquad \forall X \in TM, \tag{1.12}$$

and U, V being mutually orthogonal unit vector fields, D is a symmetric (0,2) tensor with zero trace which satisfies the condition

$$D(X,U) = 0. \qquad \forall \;\; X. \tag{1.13}$$

In such case a, b, c, d, e are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, D is called the associated tensor of the manifold.Such an n-dimensional manifold shall be denoted by the symbol $MS(QE)_n$.

§2. Preliminaries

We know in a n-dimensional ($n > 2$) Riemannian manifold the covariant quasi conformal curvature tensor is defined as ([3],[7],[17])

$$\begin{aligned}\tilde{C}(X,Y,Z,W) \;\; = \;\; & \acute{a}\acute{R}(X,Y,Z,W) + \acute{b}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ & + g(Y,Z)g(QX,W) - g(X,W)g(QY,W)] - \frac{r}{n}[\frac{\acute{a}}{n-1} \\ & + 2\acute{b}][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\end{aligned} \tag{2.1}$$

where

$$g(C(X,Y)Z,W) = \tilde{C}(X,Y,Z,W). \tag{2.2}$$

The projective curvature tensor is denoted by $\tilde{P}(X,Y,Z,W)$ and in a $V_n(n > 2)$ it is defined as

$$\tilde{P}(X,Y,Z,W) = \acute{R}(X,Y,Z,W) - \frac{1}{n-1}[S(Y,Z)g(X,W) - S(Y,W)g(X,W)]. \tag{2.3}$$

§3. Ricci-Pseudo Symmetric Mixed Super Quasi Einstein Manifold

An n-dimensional semi-Riemannian manifold (M^n, g) is called Ricci -pseudo symmetric [12] if the tensor $R.S$ and $Q(g,S)$ are linearly dependent, where

$$(R(X,Y).S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W), \tag{3.1}$$

$$Q(g,S)(Z,W;X,Y) = -S((X \wedge Y)Z,W) - S(Z,(X \wedge Y)W) \tag{3.2}$$

and

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y \tag{3.3}$$

for vector fields X, Y, Z, W on M^n, R denotes the curvature tensor of M^n [19].

The condition of Ricci-pseudo symmetricity is equivalent to the relation

$$(R(X,Y).S)(Z,W) = L_s Q(g,S)(Z,W;X,Y), \tag{3.4}$$

which holds on the set

$$U_s = \{x \in M : S \neq \frac{r}{n} g \ at \ x\},$$

where L_s is some function on U_s [19]. If $R.S = 0$ then the manifold is called Ricci-semi symmetric. Every Ricci-semi symmetric manifold is Ricci-pseudo symmetric but the converse is not true [1],[6],[9],[12],[13],[19].

Theorem 3.1 *In a Ricci-pseudo symmetric mixed super quasi Einstein manifold the following relation holds*

$$R(X,Y,U,V) = L_s[A(Y)B(X) - A(X)B(Y)].$$

Proof Let the manifold be Ricci-pseudo symmetric. Then from (3.1) and (3.4) we get

$$L_s Q(g,S)(Z,W;X,Y) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W). \tag{3.5}$$

Using (3.2) in (3.5), we get

$$\begin{aligned} S(R(X,Y)Z,W) + S(Z,R(X,Y)W) &= L_s[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &\quad + g(Y,W)S(X,Z) - g(X,W)S(Y,Z)]. \end{aligned}$$

Since the manifold is a mixed super quasi Einstein manifold, Using the well - known properties of curvature tensor R we obtain

$$\begin{aligned} & b[A(R(X,Y)Z)A(W) + A(R(X,Y)W)A(Z)] + c[B(R(X,Y)Z)B(W) + B(R(X,Y)W)B(Z)] \\ & + d[A(R(X,Y)Z)B(W) + B(R(X,Y)Z)A(W) + A(R(X,Y)W)B(Z) + B(R(X,Y)W)A(Z)] \\ & + e[D(R(X,Y)Z,W) + D(R(X,Y)W,Z)] \\ & = L_s\{b[g(Y,Z)A(X)A(W) - g(X,Z)a(Y)A(W) + g(Y,W)A(X)A(Z) - g(X,W)A(Y)A(Z)] \\ & + c[g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W) + g(Y,W)B(X)B(Z) - g(X,W)B(Y)B(Z)] \\ & + d[g(Y,Z)A(X)B(W) + g(Y,Z)B(X)A(W) - g(X,Z)A(Y)B(W) - g(X,Z)B(Y)A(W) \\ & + g(Y,W)A(X)B(Z) + g(Y,W)A(Z)B(X) - g(X,W)A(Y)B(Z) - g(X,W)A(Z)B(Y)] \\ & + e[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) + g(Y,W)D(X,Z) - g(X,W)D(Y,Z)]\}. \end{aligned} \tag{3.7}$$

Putting $Z = W = U$ in (3.7) we obtain

$$2d\{R(X,Y,U,V)\} = L_s\{2d[A(Y)B(X) - A(X)B(Y)]\} = 0.$$

Since $d \neq 0$, we get

$$R(X,Y,U,V) = L_s[A(Y)B(X) - A(X)B(Y)]. \tag{3.8}$$

Hence the theorem follows. □

§4. Ricci-Semi Symmetric Mixed Super Quasi-Einstein Manifold

An n-dimensional manifold (M^n, g) is called semi-symmetric [13] if $R(X;Y) \cdot S = 0, \forall\ X, Y$, where $R(X;Y)$ denotes the curvature operator.

Theorem 4.1 *In a Ricci-semi symmetric mixed super quasi Einstein manifold satisfy the condition*

$$(a+b)\acute{R}(X,Y,U,U) + d\acute{R}(X,Y,U,V) = 0.$$

it Proof We know

$$(R(X,Y).S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W) \ \ \textit{and}\ R(X,Y).S = 0$$

which implies

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W = 0.$$

i.e.,

$$\begin{aligned}
&ag(R(X,Y)Z,W) + bA(R(X,Y)Z)A(W) + cB(R(X,Y)Z)B(W)\\
&+d\{A(R(X,Y)Z)B(W) + B(R(X,Y)Z)A(W)\} + eD(R(X,Y)Z,W)\\
&+ag(R(X,Y)W,Z) + bA(R(X,Y)W)A(Z) + cB(R(X,Y)W)B(Z)\\
&+d\{A(R(X,Y)W)B(Z) + B(R(X,Y)W)A(Z)\} + eD(R(X,Y)W,Z) = 0. \qquad (4.1)
\end{aligned}$$

Putting $Z = W = U$ in (4.1) we get

$$(a+b)A(R(X,Y)U) + dB(R(X,Y)U) = 0.$$

$$(a+b)\acute{R}(X,Y,U,U) + d\acute{R}(X,Y,U,V) = 0. \qquad (4.2)$$

Hence the theorem follows. □

§5. Mixed Super Quasi Manifolds Satisfying the Condition $\tilde{C} \cdot S = 0$

Theorem 5.1 *In a mixed super quasi Einstein manifold with the condition* $\tilde{C} \cdot S = 0$ *satisfy*

$$\begin{aligned}
&(a+b)\acute{a}\acute{R}(X,Y,U,U) + d\acute{a}\acute{R}(X,Y,U,V) + d(a_1+b_1+c_1)[A(X)B(Y) - A(Y)B(X)]\\
&+de_1(a_1+b_1+c_1)[A(X)B(Y) - A(Y)B(X)] = 0.
\end{aligned}$$

Proof From condition $\tilde{C}.S = 0$ we get

$$S(\tilde{C}(X,Y)Z,W) + S(Z,\tilde{C}(X,Y)W = 0$$

for all vector fields X, Y, Z, W on (M^n, g), i.e.,

$$\begin{aligned}
&ag(\tilde{C}(X,Y)Z,W) + bA(\tilde{C}(X,Y)Z)A(W) + cB(\tilde{C}(X,Y)Z)B(W) \\
&+d\{A(\tilde{C}(X,Y)Z)B(W) + B(\tilde{C}(X,Y)Z)A(W)\} + eD(\tilde{C}(X,Y)Z,W) \\
&+ag(\tilde{C}(X,Y)W,Z) + bA(\tilde{C}(X,Y)W)A(Z) + cB(\tilde{C}(X,Y)W)B(Z) \\
&+d\{A(\tilde{C}(X,Y)W)B(Z) + B(\tilde{C}(X,Y)W)A(Z)\} + eD(\tilde{C}(X,Y)W,Z) = 0. \qquad (5.1)
\end{aligned}$$

Putting $Z = W = U$ in (2.7) we obtain

$$(a+b)\acute{C}(X,Y,U,U) + d\acute{C}(X,Y,U,V) = 0. \qquad (5.2)$$

Using (1.11) in (2.1) and putting $Z = W = U$ we get

$$\acute{C}(X,Y,U,U) = \acute{a}\acute{R}(X,Y,U,U). \qquad (5.3)$$

Using (1.11) in (2.1) and putting $Z = U$ and $W = V$ we obtain

$$\begin{aligned}
\acute{C}(X,Y,U,V) &= \acute{a}\acute{R}(X,Y,U,V) + (a_1+b_1+c_1)[A(X)B(Y) - A(Y)B(X)] \\
&\quad +e_1[A(X)D(Y,V) - A(Y)D(X,V)], \qquad (5.4)
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= -\{\frac{r}{n}[\frac{\acute{a}}{n-1} + 2\acute{b}] - 2a\acute{b}\}, \\
b_1 &= b\acute{b}, c_1 = c\acute{b}, d_1 = d\acute{b} \text{ and } e_1 = e\acute{b}.
\end{aligned}$$

Using (5.3) and (5.4) in (5.2) we get

$$\begin{aligned}
(a+b)\acute{a}\acute{R}(X,Y,U,U) &+ d\acute{a}\acute{R}(X,Y,U,V) + d(a_1+b_1+c_1)[A(X)B(Y) - A(Y)B(X)] \\
&+ de_1(a_1+b_1+c_1)[A(X)B(Y) - A(Y)B(X)] = 0. \qquad (5.5)
\end{aligned}$$

Corollary 5.1 *In a mixed super quasi Einstein manifold with the condition $\tilde{P}.S = 0$ satisfying the condition $(a+b)\acute{R}(X,Y,U,U) + d\acute{R}(X,Y,U,V) = 0$.*

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