International J.Math. Combin. Vol.3(2023), 89-94

On Equitable Associate Symmetric n-Sigraphs

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Abstract: In this paper we introduced a new notion equitable associate symmetric n-sigraph of a symmetric n-sigraph and its properties are obtained. Further, we discuss structural characterization of equitable associate symmetric n-sigraphs.

Key Words: Symmetric n-sigraphs, Smarandachely symmetric n-marked graph, symmetric n-marked graphs, Smarandachely symmetric n-marked graph, balance, switching, equitable associate n-sigraphs, Smarandachely equitable dominating set, complementation.

AMS(2010): 05C22.

§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [2]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \rightarrow H_n$ ($\mu : V \rightarrow H_n$) is a function. Generally, a *Smarandachely symmetric n-sigraph* (*Smarandachely symmetric n-marked graph)* for a subgraph H is such a graph that $G - E(H)$ is *symmetric n-sigraph (symmetric n-marked graph)*. For example, let H be a path $P_2 \succ G$ or a cycle $C_3 \prec G$. Certainly, if $H = \emptyset$, a Smarandachely symmetric n-sigraph (or Smarandachely symmetric n-sigraph) is nothing else but a symmetric n-sigraph (or symmetric n-marked graph).

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

[1]Received June 18, 2023, Accepted August 24, 2023.

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An n-tuple $(a_1, a_2, \cdots, a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*. Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [2], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P.S.K.Reddy [4]):

Definition 1.1 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then,*

(i) S_n is identity balanced *(or* i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and*

(ii) S_n is balanced*, if every cycle in S_n contains an even number of non-identity edges.*

Observation 1.2 An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [8].

Theorem 1.3 (E. Sampathkumar et al. [8]) *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S_n defined as follows: each vertex $v \in V$, $\mu(v)$ is the n-tuple which is the product of the n-tuples on the edges incident with v. *Complement* of S_n is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma^c)$, where for any edge $e = uv \in \overline{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Proposition 1 in [10].

In [8], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows (See also [3, 5 - 7, 10 - 20]):

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, \cdots, a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, \cdots, a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*. Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n.

We make use of the following known result (see [8]).

Theorem 1.4 (E. Sampathkumar et al. [8]) *Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S defined as follows: each vertex $v \in V$, $\mu(v)$ is the product of the n-tuples on the edges incident at v. *Complement* of S is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma')$, where for any edge $e = uv \in \overline{G}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Theorem 1.3.

§2. Equitable Associate n-Sigraph of an n-Sigraph

A subset D of $V(\Gamma)$ is called an *equitable dominating set* of a graph Γ, if for every $v \in V - D$ there exists a vertex $v \in D$ such that $uv \in E(\Gamma)$ and $|d(u) - d(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by γ_e and is called equitable domination number of Γ. A vertex $u \in V$ is said to be *degree equitable* with a vertex $v \in V$ if $|deg(u) - deg(v)| \leq 1$ (see [21]) and to be *Smarandachely degree equitable* if $|deg(u) - deg(v)| \geq 2$.

Generally, a subset D of V is called an *equitable dominating set* if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$ and a *Smarandachely equitable dominating set* if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \geq 2$. Further, a vertex $u \in V$ is said to be *degree equitable* with a vertex $v \in V$ if $|deg(u) - deg(v)| \leq 1$ and *Smarandachely degree equitable* if $|deg(u) - deg(v)| \geq 1$.

In [1], Dharmalingam introduced a new class of intersection graphs in the field of domination theory. The equitable associate graphs is denoted by $\mathcal{E}(G)$ is the graph which has the same vertex set as G with two vertices u and v are adjacent if and only if u and v are adjacent and degree equitable in G.

Motivated by the existing definition of complement of an n-sigraph, we extend the notion of equitable associate graphs to n-sigraphs as follows:

The *equitable associate n-sigraph* $\mathcal{E}(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $\mathcal{E}(G)$ and the n-tuple of any edge uv is $\mathcal{E}(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n. Further, an n-sigraph $S_n = (G, \sigma)$ is called equitable associate n-sigraph, if $S_n \cong \mathcal{E}_t(S'_n)$ for some n-sigraph S'_n. The following result indicates the limitations of the notion $\mathcal{E}(S_n)$ as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be equitable associate n-sigraphs.

Theorem 2.1 *For any n-sigraph $S_n = (G, \sigma)$, its equitable associate n-sigraph $\mathcal{E}(S_n)$ is i-balanced.*

Proof Since the n-tuple of any edge uv in $\mathcal{E}(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n, by Theorem 1.1, $\mathcal{E}(S_n)$ is i-balanced. □

For any positive integer k, the k^{th} iterated equitable associate n-sigraph $\mathcal{E}(S_n)$ of S_n is defined as follows:

$$(\mathcal{E})^0(S_n) = S_n, \quad (\mathcal{E})^k(S_n) = \mathcal{E}((\mathcal{E})^{k-1}(S_n)).$$

Corollary 2.2 *For any n-sigraph $S_n = (G, \sigma)$ and any positive integer k, $(\mathcal{E})^k(S_n)$ is i-balanced.*

The following result characterize n-sigraphs which are equitable associate n-sigraphs.

Theorem 2.3 *An n-sigraph $S_n = (G, \sigma)$ is an equitable associate n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is an equitable associate graph.*

Proof Suppose that S_n is i-balanced and G is a $\mathcal{E}(G)$. Then there exists a graph H such that $\mathcal{E}(H) \cong G$. Since S_n is i-balanced, by Theorem 1.3, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $\mathcal{E}(S'_n) \cong S_n$. Hence S_n is an equitable associate n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is an equitable associate n-sigraph. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $\mathcal{E}(S'_n) \cong S_n$. Hence G is the $\mathcal{E}(G)$ of H and by Theorem 2.1, S_n is i-balanced. □

In [1], the author characterized graphs for which $\overline{\mathcal{E}(G)} \cong \mathcal{E}(\overline{G})$.

Theorem 2.4 (K. M. Dharmalingam [1]) *For any graph $G = (V, E)$, $\overline{\mathcal{E}(G)} \cong \mathcal{E}(\overline{G})$ if and only if every edge of G is equitable.*

We now characterize n-sigraphs whose complementary equitable associate n-sigraphs and equitable associate n-sigraphs are switching equivalent.

Theorem 2.5 *For any n-sigraph $S_n = (G, \sigma)$, $\overline{\mathcal{E}(S_n)} \sim \mathcal{E}(\overline{S_n})$ if and only if every edge of G is equitable.*

Proof Suppose $\overline{\mathcal{E}(S_n)} \sim \mathcal{E}(\overline{S_n})$. This implies, $\overline{\mathcal{E}(G)} \cong \mathcal{E}(\overline{G})$ and hence by Theorem 2.4, every edge of G is equitable.

Conversely, suppose that every edge of G is equitable. Then $\overline{\mathcal{E}(G)} \cong \mathcal{E}(\overline{G})$ by Theorem 2.4. Now, if S_n is an n-sigraph with each edge of G is equitable, by the definition of complementary n-sigraph and Theorem 2.1, $\overline{\mathcal{E}(S_n)}$ and $\mathcal{E}(\overline{S_n})$ are i-balanced and hence, the result follows from Theorem 1.4. □

Theorem 2.6 *For any two n-sigraphs S_n and S'_n with the same underlying graph, their equitable associate n-sigraphs are switching equivalent.*

Proof Suppose $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ be two n-sigraphs with $G \cong G'$. By Theorem 2.1, $\mathcal{E}(S_n)$ and $\mathcal{E}(S'_n)$ are i-balanced and hence, the result follows from Theorem 1.4. □

For any $m \in H_n$, the *m-complement* of $a = (a_1, a_2, \cdots, a_n)$ is: $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the *m-complement* of M is $M^m = \{a^m : a \in M\}$.

For any $m \in H_n$, the *m-complement* of an n-sigraph $S_n = (G, \sigma)$, written (S_n^m), is the same graph but with each edge label $a = (a_1, a_2, \cdots, a_n)$ replaced by a^m.

For an n-sigraph $S_n = (G, \sigma)$, the $\mathcal{E}(S_n)$ is i-balanced. We now examine, the condition under which m-complement of $\mathcal{E}(S_n)$ is i-balanced, where for any $m \in H_n$.

Theorem 2.7 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $m \in H_n$, if $\mathcal{E}(G)$ is bipartite then $(\mathcal{E}(S_n))^m$ is i-balanced.*

Proof Since, by Theorem 2.1, $\mathcal{E}(S_n)$ is i-balanced, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $\mathcal{E}(S_n)$ whose k^{th} co-ordinate are $-$ is even. Also, since $\mathcal{E}(G)$ is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $\mathcal{E}(S_n)$ whose k^{th} co-ordinate are $+$ is also even. This implies that the same thing is true in any m-complement, where for any $m, \in H_n$. Hence $(\mathcal{E}(S_n))^t$ is i-balanced. □

Notice that Theorem 2.6 provides an easy solutions to other n-sigraph switching equivalence relations, which are given in the following results.

Corollary 2.8 *For any two n-sigraphs S_n and S'_n with the same underlying graph, $\mathcal{E}(S_n)$ and $\mathcal{E}((S'_n)^m)$ are switching equivalent.*

Corollary 2.9 *For any two n-sigraphs S_n and S'_n with the same underlying graph, $\mathcal{E}((S_n)^m)$ and $\mathcal{E}(S'_n)$ are switching equivalent.*

Corollary 2.10 *For any two n-sigraphs S_n and S'_n with the same underlying graph, $\mathcal{E}((S_n)^m)$ and $\mathcal{E}((S'_n)^m)$ are switching equivalent.*

Corollary 2.11 *For any two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ with the $G \cong G'$ and G, G' are bipartite, $(\mathcal{E}(S_n))^m$ and $\mathcal{E}(S'_n)$ are switching equivalent.*

Corollary 2.12 *For any two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ with the $G \cong G'$ and G, G' are bipartite, $\mathcal{E}(S_n)$ and $(\mathcal{E}(S'_n))^m$ are switching equivalent.*

Corollary 2.13 *For any two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ with the $G \cong G'$ and G, G' are bipartite, $(\mathcal{E}(S_1))^m$ and $(\mathcal{E}(S_2))^m$ are switching equivalent.*

Corollary 2.14 *For any n-sigraph $S_n = (G, \sigma)$, $S_n \sim \mathcal{E}((S_n)^m)$ if and only if G is K_n and S_n is i-balanced.*

Acknowledgements

The authors would like to thank the referees for their invaluable comments and suggestions which led to the improvement of the manuscript.

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