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On Full Block Signed Graphs

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Abstract: In this paper we introduced the new notion called full block signed graph of a signed graph and its properties are studied. Also, we obtained the structural characterization of this new notion and presented some switching equivalent characterizations.

Key Words: Smarandachely signed graph, signed graphs, balance, switching, full signed graph, full line Signed graph, full block signed graph.

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§1. Introduction

For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [1]. The non-standard will be given in this paper as and when required.

To model individuals' preferences towards each other in a group, Harary [2] introduced the concept of signed graphs in 1953. A signed graph $S = (G, \sigma)$ is a graph $G = (V, E)$ whose edges are labeled with positive and negative signs (i.e., $\sigma : E(G) \to \{+, -\}$). The vertices of a graph represent people and an edge connecting two nodes signifies a relationship between individuals. The signed graph captures the attitudes between people, where a positive (negative edge) represents liking (disliking). An unsigned graph is a signed graph with the signs removed. Similar to an unsigned graph, there are many active areas of research for signed graphs.

The sign of a cycle (this is the edge set of a simple cycle) is defined to be the product of the signs of its edges; in other words, a cycle is positive if it contains an even number of negative edges and negative if it contains an odd number of negative edges. A signed graph S is said to be balanced if every cycle in it is positive. A signed graph S is called totally unbalanced if every cycle in S is negative. Otherwise, such a signed graph G is Smarandachely, i.e., both of the positive and negative cycles appeared in it. A chord is an edge joining two non adjacent vertices in a cycle.

A *marking* of S is a function $\zeta : V(G) \to \{+, -\}$. Given a signed graph S one can easily

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define a marking ζ of S as follows: For any vertex $v \in V(S)$,

$$\zeta(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking ζ of S is called *canonical marking* of S. For more new notions on signed graphs refer the papers (see [6, 8, 9, 13C17, 17C26]).

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1 *A signed graph S is balanced if and only if either of the following equivalent conditions is satisfied:*

(i) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2* (Harary [2]).

(ii) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$.* (Sampathkumar [10]).

A switching S with respect to a marking ζ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs.

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be *weakly isomorphic* (see [28]) or *cycle isomorphic* (see [29]) if there exists an isomorphism $\phi : G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (see [29]):

Theorem 1.2(T. Zaslavsky, [29]) *Given a graph G, any two signed graphs in $\psi(G)$, where $\psi(G)$ denotes the set of all the signed graphs possible for a graph G, are switching equivalent if and only if they are cycle isomorphic.*

§2. Full Block Signed Graph of a Signed Graph

The full block graph $\mathcal{FB}(G)$ of a graph G is the graph whose vertex set is the union of the set of vertices, edges and blocks of G in which two vertices are adjacent if the corresponding vertices and blocks of G are adjacent or the corresponding members of G are incident (See [5]).

Motivated by the existing definition of complement of a signed graph, we now extend the notion of full block graphs to signed graphs as follows: The *Full block signed graph* $\mathcal{FB}(S) = (\mathcal{FB}(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{FB}(G)$ and sign of any edge uv is $\mathcal{FB}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called a full block signed graph, if $S \cong \mathcal{FB}(S')$ for some signed graph S'. The following result restricts the class of full line signed graphs.

Theorem 2.1 *For any signed graph $S = (G, \sigma)$, its full block signed graph $\mathcal{FB}(S)$ is balanced.*

Proof Since sign of any edge $e = uv$ in $\mathcal{FB}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S, by Theorem 1.1, $\mathcal{FB}(S)$ is balanced. $\square$

For any positive integer k, the k^{th} iterated full block signed graph, $\mathcal{FB}^k(S)$ of S is defined as follows:

$$\mathcal{FB}^0(S) = S, \quad \mathcal{FB}^k(S) = \mathcal{FB}(\mathcal{FB}^{k-1}(S)).$$

Corollary 2.2 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $\mathcal{FB}^k(S)$ is balanced.*

Corollary 2.3 *For any two signed graphs S_1 and S_2 with the same underlying graph, $\mathcal{FB}(S_1) \sim \mathcal{FB}(S_2)$.*

The following result characterize signed graphs which are full line signed graphs.

Theorem 2.4 *A signed graph $S = (G, \sigma)$ is a full block signed graph if, and only if, S is balanced signed graph and its underlying graph G is a full block graph.*

Proof Suppose that S is balanced and G is a full block graph. Then there exists a graph G' such that $\mathcal{FB}(G') \cong G$. Since S is balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge uv in S satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $\mathcal{FB}(S') \cong S$. Hence S is a full block signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a full block signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $\mathcal{FB}(S') \cong S$. Hence, G is the full block graph of G' and by Theorem 2.1, S is balanced. $\square$

The notion of *negation* $\eta(S)$ of a given signed graph S defined to be $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S in [3]. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the $\mathcal{FB}(S)$ is balanced (Theorem 1.1). We now examine, the conditions under which negation $\eta(S)$ of $\mathcal{FB}(S)$ is balanced.

Proposition 2.5 *Let $S = (G, \sigma)$ be a signed graph. If $\mathcal{FB}(G)$ is bipartite then $\eta(\mathcal{FB}(S))$ is balanced.*

Proof Since, by Theorem 1.1, $\mathcal{FB}(S)$ is balanced, it follows that each cycle C in $\mathcal{FLS}(S)$ contains even number of negative edges. Also, since $\mathcal{FB}(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{FB}(S)$ is also even. Hence $\eta(\mathcal{FB}(S))$ is balanced. $\square$

§3. Switching Equivalence of Full Block Signed Graphs and Full Signed Graphs

In [27], we defined the full signed graph of a signed graph as follows: The *full signed graph* $\mathcal{FS}(S) = (\mathcal{FG}(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{FG}(G)$ and sign of any edge uv is $\mathcal{FS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called a full signed graph, if $S \cong \mathcal{FS}(S')$ for some signed graph S'. The following result restricts the class of full signed graphs.

Theorem 3.1(Swamy et al., [27]) *For any signed graph $S = (G, \sigma)$, its full signed graph $\mathcal{FS}(S)$ is balanced.*

In [5], the authors remarked that $\mathcal{FB}(G)$ and $\mathcal{FG}(G)$ are isomorphic if and only if G is a P_2. We now give a characterization of signed graphs whose full block signed graphs are switching equivalent to their full signed graphs.

Theorem 3.2 *For any nontrivial connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(S) \sim \mathcal{FS}(S)$ if and only if G is a P_2.*

Proof Suppose $\mathcal{FB}(S) \sim \mathcal{FS}(S)$. This implies, $\mathcal{FB}(G) \cong \mathcal{FG}(G)$ and hence G is a P_2.

Conversely, suppose that G is a P_2. Then $\mathcal{FB}(G) \cong \mathcal{FG}(G)$. Now, if S any signed graph with G is a P_2, By Theorem 2.1 and 3.1, $\mathcal{FB}(S)$ and $\mathcal{FS}(S)$ are balanced and hence, the result follows from Theorem 1.2. This completes the proof. □

§4. Switching Equivalence of Full Block Signed Graphs and Full Line Signed Graphs

In [27], we defined the full line signed graph of a signed graph as follows: The *full line signed graph $\mathcal{FLS}(S) = (\mathcal{FLG}(G), \sigma')$* of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{FLG}(G)$ and sign of any edge uv is $\mathcal{FLS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called a full line signed graph, if $S \cong \mathcal{FLS}(S')$ for some signed graph S'. The following result restricts the class of full line signed graphs.

Theorem 4.1(Swamy et al., [27]) *For any signed graph $S = (G, \sigma)$, its full line signed graph $\mathcal{FLS}(S)$ is balanced.*

In [5], the authors remarked that $\mathcal{FB}(G)$ and $\mathcal{FLG}(G)$ are isomorphic if and only if G is a tree. We now give a characterization of signed graphs whose full block signed graphs are switching equivalent to their full line signed graphs.

Theorem 4.2 *For any nontrivial connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(S) \sim \mathcal{FLS}(S)$ if and only if G is a P_2.*

Proof Suppose $\mathcal{FB}(S) \sim \mathcal{FLS}(S)$. This implies, $\mathcal{FB}(G) \cong \mathcal{FLG}(G)$ and hence G is a tree.

Conversely, suppose that G is a tree. Then $\mathcal{FB}(G) \cong \mathcal{FLG}(G)$. Now, if S any signed graph with G is a tree, By Theorem 2.1 and 4.1, $\mathcal{FB}(S)$ and $\mathcal{FLS}(S)$ are balanced and hence, the result follows from Theorem 1.2. This completes the proof. □

In view of the negation operator introduced by Harary [3], we have the following cycle isomorphic characterizations.

Corollary 4.3 *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $\eta(\mathcal{FB}(S_1)) \sim \eta(\mathcal{FB}(S_2))$, if G_1 and G_2 are isomorphic.*

Corollary 4.4 *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $\mathcal{FB}(\eta(S_1))$ and $\mathcal{FB}(\eta(S_2))$ are cycle isomorphic, if G_1 and G_2 are isomorphic.*

Corollary 4.5 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(\eta(S)) \sim \mathcal{FS}(S)$ if and only if G is a P_2.*

Corollary 4.6 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(S) \sim \mathcal{FS}(\eta(S))$ if and only if G is a P_2.*

Corollary 4.7 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(\eta(S)) \sim \mathcal{FS}(\eta(S))$ if and only if G is a P_2.*

Corollary 4.8 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(\eta(S)) \sim \mathcal{FLS}(S)$ if and only if G is a tree.*

Corollary 4.9 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(S) \sim \mathcal{FLS}(\eta(S))$ if and only if G is a tree.*

Corollary 4.10 *For any connected signed graph $S = (G, \sigma)$, $\mathcal{FB}(\eta(S)) \sim \mathcal{FLS}(\eta(S))$ if and only if G is a tree.*

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