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On Infinitesimal Transformation in a Finsler Space

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Abstract: In the present communication studies have been carried out with special reference to infinitesimal projective and special projective transformations in a Finsler space and accordingly results have been derived in the form of theorems in a projective symmetric and non-symmetric Finsler space.

Key Words: Finsler spaces, Projective transformation, affine and non-affine infinitesimal projective transformation, Lie-derivative.

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§1. Introduction

Berwald introduced a connection coefficient $C^i_{jk}(x,\dot{x})$ defined by

$$C^i_{jk}(x,\dot{x}) \overset{\text{def}}{=} \frac{\partial^2 G^i}{\partial x^j \partial x^k} \tag{1.1}$$

and accordingly the covariant derivative of an arbitrary covariant vector i X in the sense of Berwald is given by Rund [4]

$$X^i_{(j)} = \frac{\partial X^i}{\partial x^j} - \frac{\partial X^i}{\partial \dot{x}^h}\frac{\partial G^h}{\partial \dot{x}^j} + G^i_{jh}X^h. \tag{1.2}$$

The functions G^i appearing in (1.2) are positively homogeneous of degree two in its directional arguments $\dot{x}^j$ and satisfies the following identities

$$G^i_{hkr}\dot{x}^r = G^i_{hkr}\dot{x}^k = G^i_{kkr}\dot{x}^h, \quad G^i_{hk}\dot{x}^h = 0 \ \text{ and } \ G^i_k\dot{x}^k = 2G^i. \tag{1.3}$$

The geodesic deviation has been defined in the following form

$$\frac{\partial^2 Z^j}{\partial u^2} + H^j_k(x,\dot{x})x^k = 0, \tag{1.4}$$

where the vector Z^i is called the variation vector and the tensor $H^i_k(x,\dot{x})$ is being defined by

$$H^j i_k = 2\partial_k G^i - \partial_h \dot{\partial}_k G^i \dot{x}^h + 2G^i_{kl}G^l - \dot{\partial}_l G^i \dot{\partial}_k G^l. \tag{1.5}$$

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The tensors defined by

$$H^i_{jk}(x,\dot{x}) = \frac{1}{3}\left(\frac{\partial H^i_k}{\partial \dot{x}^j} - \frac{\partial H^i_j}{\partial \dot{x}^k}\right) \quad \text{and} \quad H^i_{jkl} = \frac{\partial H^i_{jl}}{\partial \dot{x}^k} \tag{1.6}$$

are respectively termed as Berwalds deviation tensor and Berwalds curvature tensor and they satisfy the following

$$H^k_{khj} = H_{jh} - H_{hj}, \quad H_i\dot{x}^i = (n-1)H, \quad H^j_{ki}\dot{x}^k = H^j_i = H^j_{ik}\dot{x}^k. \tag{1.7}$$

The projective covariant derivative of an arbitrary tensor $T^i_j(x,\dot{x})$ is given by Misra [2] as

$$T^i_{j((k))} = \partial_k T^i_j - \dot{\partial}_s T^s_j \Pi^i_{rk}\dot{x}^r + T^h_j \Pi^i_{hk} - T^i_h \Pi^h_{jk}, \tag{1.8}$$

where

$$\Pi^i_{jk}(x,\dot{x}) \overset{\text{def}}{=} G^i_{jk} - \frac{1}{n+1}\left(2\delta^i_{(j} G^r_{<r>k)} + \dot{x}^i G^i_{rkg}\right) \tag{1.9}$$

are called projective connection coefficient and these coefficients are symmetric in its lower indices. Involving the projective covariant derivative ,we have the following commutation formulae

$$\begin{aligned} &\partial_h\left(T^i_{j((k))}\right) - \left(\partial_h T^i_j\right)_{((k))} = T^s_j \Pi^i_{shk} - T^i_s \Pi^s_{jhk}, \\ &2T^i_{[((h))((k))]} = -\dot{\partial}_r T^i_j Q^r_{shk}\dot{x}^s + T^s_j Q^i_{shk} - T^i_s Q^s_{jhk}. \end{aligned} \tag{1.10}$$

where,

$$Q^i_{hjk} \overset{\text{def}}{=} 2\left\{\partial_{[k}\Pi^i_{j]h} - \Pi^i_{rh[j}\Pi^r_{k]} + \Pi^r_{h[j}\Pi^i_{k]r}\right\}. \tag{1.11}$$

is called the projective entity and satisfies the following relations

$$\begin{aligned} &Q^i_{hjk} + Q^i_{jkh} + Q^i_{khj} = 0, \\ &Q^i_{hjk((s))} + Q^i_{hks((j))} + Q^i_{hsj((k))} = 0, \\ &Q^i_{ijk} = Q_{jk}, \quad Q^i_{jk} = \frac{2}{3}\dot{\partial}_{[j} Q_{k]^i}, \\ &Q^i_{hjk} = \dot{\partial}_h Q^i_{jk}, \quad Q^i_{ijk} = Q^i_{jk}, \quad Q^i_k \dot{x}_k = 0, \\ &Q^i_{jk} = -Q^i_{kj} \ \text{ and } \ Q^i_{hk}\dot{x}^h = Q^i_k. \end{aligned} \tag{1.12}$$

The projective connection coefficient $\Pi^i_{jk}(x,\dot{x})$ satisfies the following relations

$$\begin{aligned} &\Pi^i_{hkr} - \dot{\partial}_h \Pi^i_{kr}, \quad \Pi^i_{hk} = \dot{\partial}_h \Pi^i_k, \\ &\Pi^i_{hkr}\dot{x}^h = 0 \ \text{ and } \ \Pi^i_{hk}\dot{x}^h = \Pi^i_k. \end{aligned} \tag{1.13}$$

§2. NonCAffine Infinitesimal Projective Transformation

In view of the Berwalds covariant derivative [4], the Lie-derivative of a tensor field $T^i_j(x,\dot{x})$ and the connection parameter $G^i_{jk}(x,\dot{x})$ are given as under [7] following

$$\mathcal{L}_\nu T^i_j(x,\dot{x}) = T^i_{j(h)}\nu^h + \left(\dot{\partial}_h T^i_j\right)\nu^h_{(s)}\dot{x}^s + T^i_h \nu^h_{(j)}, \tag{2.1}$$

$$\mathcal{L}_\nu G^i_{jk}(x,\dot{x}) = \nu^i_{(j)(k)} H^i_{jkh}\nu^h + G^i_{sjk} v^s_{(r)}\dot{x}^r. \tag{2.2}$$

where $H^i_{jkh}(x,\dot{x})$ has been defined by (1.6).

We also have the following communication formula from [7]

$$\dot{\partial}_l\left(\mathcal{L}_\nu T^i_j\right) - \mathcal{L}_\nu\left(\dot{\partial}_l T^i_j\right) = 0, \tag{2.3}$$

$$\mathcal{L}_\nu T^i_{j(k)} - \left(\mathcal{L}_\nu T^i_j\right)_{(k)} = T^i_j \mathcal{L}_\nu G^i_{kh} - \left(\dot{\partial}_h T^i_j\right)\mathcal{L}_\nu G^h_{ks}\dot{x}^s, \tag{2.4}$$

$$\left(\mathcal{L}_\nu G^i_{jh}\right)_{(k)} - \left(\mathcal{L}_\nu G^i_{kj}\right)_{(j)} = \mathcal{L}_\nu H^i_{hjk} + \left(\mathcal{L}_\nu G^r_{kl}\right) G^i_{rjh}\dot{x}^l - \left(\mathcal{L}_\nu G^r_{jl}\right) G^i_{rkh}\dot{x}^l. \tag{2.5}$$

Now, we give the following definitions which will be used in the later discussions.

Definition 2.1 *A Finsler space F_n is said to admit an affine motion [3] provided there exists a vector $v^i(x)$ such that*

$$(L)_\nu G^i_{jk}(x,\dot{x}) \;=\; 0. \tag{2.6}$$

Definition 2.2 *A Finsler space is said to be symmetric [1] if the Berwalds curvature tensor field $H^i_{hjk}(x,\dot{x})$ satisfies the relation*

$$H^i_{hjk(m)} \;=\; 0 \tag{2.7}$$

The following relations also hold good in such a symmetric Finsler space

$$H^i_{jk(m)} = 0, \quad H^i_{j(m)} = 0 \text{ and } H_{(m)} = 0. \tag{2.8}$$

We now consider an infinitesimal point transformation

$$\overline{x}^i = x^i + v^i(x)dt \tag{2.9}$$

where, $v^i(x)$ stands for a non-zero contravariant vector field defined over the domain of the space and dt is an infinitesimal point constant. If such a transformation transforms the system of geodesics into the same system then such a transformation in F_n is termed as infinitesimal projective transformation. It has been mentioned in [3] that the necessary and sufficient condition in order that the infinitesimal point transformation given by (2.9) be an infinitesimal projective transformation is given by the following equation

$$\mathcal{L}_\rho G^i_{jk} = \overline{G}^i_{jk} - G^i_{jk} = \delta^i_k p_k + \delta^i_k p_j - g_{jk}g^{il}d_l, \tag{2.10}$$

where, $p_k(x,\dot{x})$ and $d_l(x,\dot{x})$ are covariant vectors and satisfy the following identities

$$\dot{\partial}_j p = p_j, \quad p_{hk} = \dot{\partial}_h\dot{\partial}_k p, \quad p_{hk}\dot{x}^h = p_k,$$
$$p_{hk}\dot{x}^h\dot{x}^k = p, \quad \dot{\partial}_j d = d_j, \quad d_{hk} = \dot{\partial}_h\dot{\partial}_k d,$$
$$d_{hk}\dot{x}^h = d_k \quad \text{and} \quad d_{hk}\dot{x}^h\dot{x}^k = d, \tag{2.11}$$

Keeping in mind the formula (2.5), the Lie-derivative of H^i_{hjk} can be expressed in the following form

$$\mathcal{L}_\rho H^i_{hjk} = \left(\mathcal{L}_\rho G^i_{jh}\right)_{(k)} - \left(\mathcal{L}_\rho G^i_{kh}\right)_{(j)} + \left(\mathcal{L}_\rho G^r_{il}\right)\dot{x}^l G^i_{rhk}. \tag{2.12}$$

Using (2.10) and (1.3) in (2.12), we get

$$\begin{aligned}\mathcal{L}_\rho H^i_{hjk} &= \delta^i_j p_{h(k)} - \delta^i_k p_{h(j)} + \delta^i_h p_{j(k)} - \delta^i_h p_{k(j)} - g_{jh}g^{il}d_{l(k)} + g_{kh}g^{il}d_{l(j)} \\ &\quad + g_{kl}g^{rm}G^i_{rjh}d_m\dot{x}^l - g_{ij}g^{rm}G^i_{rkh}d_m\dot{x}^l. \end{aligned} \tag{2.13}$$

We multiply (2.13) by $\dot{x}^h\dot{x}^j$ and thereafter note (2.11) and the homogeneity property of $H^i_{hjk}(x,\dot{x})$ and get

$$\mathcal{L}_\rho H^i_k = 2\dot{x}^i p_{(k)} - \delta^i_k p_{(j)}\dot{x}^j - \dot{x}^i p_{k(j)}\dot{x}^j - g_{jh}g^{il}d_{l(k)}\dot{x}^h\dot{x}^j + g_{kh}g^{il}d_{l(j)}\dot{x}^h\dot{x}^j. \tag{2.14}$$

Now, allow a contraction in (2.14) with respect to the indices i, k and thereafter use equations (1.7), (2.11) and get

$$\mathcal{L}_\rho H = -p_{(j)}\dot{x}^j + \frac{1}{n-1}\left(d_{(j)}\dot{x}^j - g_{jh}g^{il}d_{l(i)}\dot{x}^h\dot{x}^j\right). \tag{2.15}$$

With the help of (2.15) and (2.14), we get

$$\begin{aligned}\left(\mathcal{L}_\rho H^i_k - \mathcal{L}_\rho H\delta^i_k\right) &= 3\dot{x}^i p_{(k)} - \delta^i_k p_{k(j)}\dot{x}^j + g_{kh}g^{il}d_{l(j)}\dot{x}^h\dot{x}^j \\ &\quad - \frac{1}{n-1}\left\{d_k\dot{x}^i + (2-n)g_{jh}g^{il}d_{l(k)}\dot{x}h\dot{x}^j\right\}. \end{aligned} \tag{2.16}$$

Differentiate (2.16) partially with respect to $\dot{x}^r$ and thereafter allow a contraction in the resulting equation with respect to the indices i and r, we get the following

$$\begin{aligned}\mathcal{L}_\rho\dot{\partial}_r H^r_k - \mathcal{L}_\rho\dot{\partial}_k H &= (3n+2)p_k - (n+3)p_{k(j)} + d_{k(j)}\dot{x}^j + g_{kh}g^{rl}\dot{x}^h\{d_{rl(j)} + d_{lr}\} \\ &\quad + \frac{5-n}{n-1}\times d_k + 2\dot{x}^h\dot{x}^j \times \frac{C^l_{sl}}{g_{rs}} \times \left\{\frac{2-n}{n-1}\times g_{rh}d_{l(k)} - g_{kh}d_{l(j)}\right\} \end{aligned} \tag{2.17}$$

after making use of (1.7) and (2.11).

The underlined equation

$$\overline{G}^i(x,\dot{x}) = G^i(a,\dot{x}) - P(s,\dot{x})\dot{x}^i \tag{2.18}$$

represents the most general modification of the function i G which will leave (2.18) unchanged. Thus, we say that the equation (2.18) defines the projective change [4] of the function $G^i(x,\dot{x})$. The tensor defined by

$$W_k^j(x,\dot{x}) = H_k^j - H\delta_k^j - \frac{1}{n+1}\left(\dot{\partial}_l H_k^j - \delta_k^j H\right)\dot{x}^l \tag{2.19}$$

is invariant under the projective change (2.18) and therefore it is regarded as projective deviation tensor. This deviation tensor also satisfies the following identities

$$W_j^j = 0, \quad \dot{\partial}_k W_h^j \dot{x}^h = -W_k^j \quad \text{and} \quad \dot{\partial}_i W_k^i = 0. \tag{2.20}$$

The Lie-derivative of the projective deviation tensor $W_j^i(x,\dot{x})$ in view of (2.16) and (2.17) can be written in the following form

$$\begin{aligned} \mathcal{L}_\rho W_k^i \quad = \quad & \frac{1}{n+1}\Bigg\{p_{(k)}\dot{x}^i + 2p_{k(j)}\dot{x}^i\dot{x}^j + \frac{4-n}{n-1}d_{(k)}\dot{x}^i \\ & -\dot{x}^i\left[d_{k(j)}\dot{x}^j + g_{kh}g^{rl}(d_{rl(j)} + d_{l(r)}) + 2\dot{x}^h\dot{x}^j\frac{C_{sr}^l}{g_{rs}}\left(\frac{2-n}{n-1}g_{rh}d_{l(k)} - g_{kh}d_{l(j)}\right)\right]\Bigg\} \\ & -\delta_k^i p_{(j)}\dot{x}^j + g_{kh}g^{il}d_{l(j)}\dot{x}^h\dot{x}^j + \frac{2-n}{n-1}g_{jh}g^{il}d_{l(k)}\dot{x}^h\dot{x}^j. \end{aligned} \tag{2.21}$$

We now apply the commutation formula given by (2.4) to the projective deviation tensor $W_j^i(x,\dot{x})$ and get

$$\mathcal{L}_\rho W_{j(r)}^i - \left(\mathcal{L}_\rho W_j^i\right)_{(r)} = W_j^h\mathcal{L}_\rho G_{rh}^i - W_h^i\mathcal{L}_\rho G_{jr}^h - \left(\dot{\partial}_h W_j^i\right)\left(\mathcal{L}_\rho G_{rs}^h\right)\dot{x}^s. \tag{2.22}$$

Using (2.2) and (2.3) in (2.22), we get

$$\begin{aligned} \mathcal{L}_\rho W_{j(r)}^i - \left(\mathcal{L}_\rho W_j^i\right)_{(r)} \quad = \quad & W_j^i\left(\delta_r^i p_r - g_{rh}g^{il}d_l\right) - W_r^i p_j - 2W_j^i p_r \\ & +g^{hl}d_l\left[W_h^i g_{jr} + \left(\dot{\partial}_h W_j^i\right)g_{rs}\dot{x}^s - \left(\dot{\partial}_r W_j^i\right)p\right]. \end{aligned} \tag{2.23}$$

We now allow a contraction in (2.23) with respect to the indices i and r and thereafter use (2.20) and get

$$\mathcal{L}_\rho W_{j(i)}^i - \left(\mathcal{L}_\rho W_j^i\right)_{(i)} = (n-2)W_j^h p_h - W_j^h d_h + g^{hl}d_l\left\{W_h^i g_{ji} + \left(\dot{\partial}_h W_j^i\right)g_{is}\dot{x}^s\right\}. \tag{2.24}$$

Now, transvect $\dot{x}^r$ in (2.23) and thereafter use (2.3) and (2.20), we get

$$\begin{aligned} \left[\mathcal{L}_\rho W_{j(r)}^i - \left(\mathcal{L}_\rho W_j^i\right)_{(r)}\right]\dot{x}^r \quad = \quad & W_j^h\dot{x}^i p_h - 4W_j^i p - W_j^h g_{rh}g^{il}d_l\dot{x}^r + g^{hl}d_l\dot{x}^r \\ & +g^{hl}d_l\dot{x}^r\left[W_h^i g_{jr} + (\dot{\partial}_h W_j^i)g_{rs}\dot{x}^s\right]). \end{aligned} \tag{2.25}$$

We now make an assumption that the space under consideration is symmetric one, i.e., $W_{j(r)}^i = 0$ and as such under this assumption the equations (2.24) and (2.25) can alternatively be written in the following forms

$$\left(\mathcal{L}_\rho W^i_j\right)_{(r)} = (2-n)W^i_j p_r + W^i_j d_r - g^{hl} d_l \left[W^i_h d_{ri} + \left(\dot{\partial}_h W^i_j\right) g_{rs}\dot{x}^s\right] \tag{2.26}$$

and

$$\left(\mathcal{L}_\rho W^i_j\right)_{(r)} \dot{x}^r = W^i_j p - W^h_j \dot{x}^i p_h + W^h_j g_{rh} g^{il} d_l \dot{x}^r - g_{hl} d^l \dot{x}^r \left[W^i_h g_{jr} + \left(\dot{\partial}_h W^i_j\right) g_{rs}\dot{x}^s\right]. \tag{2.27}$$

We propose to eliminate the term $W^h_j p_h$ with the help of (2.26) and (2.27) and the result of elimination will give the following

$$M^i_j = \left\{W^h_j d_h - g^{hl} d_l \left[W^r_h g_{jr} + \left(\dot{\partial} W^i_j\right) g_{rs}\dot{x}^s\right]\right\} \dot{x}^i, \tag{2.28}$$

where,

$$M^i_j = \left(\mathcal{L}_\rho W^i_j\right)_{(r)} \dot{x}^r + (2-n)\left(\mathcal{L}_\rho W^i_j\right)_{(r)} \dot{x}^r. \tag{2.29}$$

At this stage, if we assume that the Finsler space F_n admits a projective motion which will be characterized by

$$\mathcal{L}_\rho G^i_{jk} \;=\; 0. \tag{2.30}$$

Therefore, in such a case, with the help of (2.10) and (2.30) we shall easily arrive at the conclusion that the vectors $p(x,\dot{x})$ and $d(x,\dot{x})$ should separately vanish.

With the help of all these observations, we can therefore state the following conclusions.

Theorem 2.1 *In a Finsler space F_n, the equation (2.28) always holds provided the space under consideration admits a nonCaffine infinitesimal transformation such that the Berwalds covariant derivative of W^i_j remains an invariant.*

Theorem 2.2 *In a Finsler space F_n, $M^i_j = 0$ (where M^i_j has been given by (2.29)) provided the space under consideration admits an affine infinitesimal transformation such that the Berwalds covariant derivative of W^i_j remains an invariant.*

Theorem 2.3 *In a Finsler space F_n, the equation (2.28) necessarily holds provided the space under consideration is symmetric one and it admits a non-affine infinitesimal transformation.*

Theorem 2.4 *In a Finsler space F_n, the equation (2.26) necessarily holds provided the space under consideration is symmetric.*

§3. Infinitesimal Special Projective Transformation

In view of the projective covariant derivative as has been given by (1.8) and the projective connection coefficient $\Pi^i_{jk}(x,\dot{x})$ as has been given by (1.9), the Lie-derivatives of an arbitrary tensor $T^i_j(x,\dot{x})$ and the projective connection coefficient are respectively given by

$$\mathcal{L}_\rho T^i_j(x,\dot{x}) = T^i_{j((r))} v^r + \left(\dot{\partial}_s T^i_j\right) v^s_{((r))}\dot{x}^r - T^r_j v^i_{((r))} + T^i_r v^r_{((j))} \tag{3.1}$$

and

$$\mathcal{L}_v \Pi^i_{mk}(x,\dot{x}) = v^i_{((m))((k))} + Q^i_{mkr} v^r + \left(\dot{\partial}_r \Pi^i_{mk}\right) v^r_{((s))} \dot{x}^s. \tag{3.2}$$

In the operators $\mathcal{L}_v$, $\dot{\partial}$ and (( )), we have the following commutation formulae

$$\dot{\partial}_\rho \left(\mathcal{L}_\nu T^i_j\right) - \mathcal{L}_\nu \left(\dot{\partial}_\rho T^i_j\right) = 0,$$

$$\left(\mathcal{L}_\nu T^i_j\right)_{((r))} - \mathcal{L}_\nu T^i_{j((r))} = T^i_j \mathcal{L}_\nu \Pi^l_{lr} - T^i_l \mathcal{L}_\nu \Pi^l_{rj} - \left(\dot{\partial}_l T^i_j\right) \mathcal{L}_\nu \Pi^l_{rm} \dot{x}^m \quad \text{and}$$

$$\left(\mathcal{L}_\nu \Pi^i_{hj}\right)_{((k))} - \left(\mathcal{L}_\nu \Pi^i_{hk}\right)_{((j))} = \mathcal{L}_\nu Q^i_{hkj} + \left(\mathcal{L}_\nu \Pi^l_{jb}\right) \Pi^i_{hkl} \dot{x}^b + \left(\mathcal{L}_\nu \Pi^l_{kb}\right) \Pi^i_{jhl} \dot{x}^b. \tag{3.3}$$

In order that the infinitesimal point transformation given by (2.9) may define an infinitesimal special projective transformation, it is necessary and sufficient that [3]

$$\mathcal{L}_\nu \Pi^i_{jk} = \overline{\Pi}^i_{jk} - \Pi^i_{jk} = \delta^i_j b_k + \delta^i_k b_j - g_{jk} g^{il} c_l, \tag{3.4}$$

where, $b_k(x,\dot{x})$ and $c_l(x,\dot{x})$ are covariant vectors and they satisfy the following relations

$$\begin{aligned} &\dot{\partial}_j b = b_j, \quad b_{hk} = \dot{\partial}_h \dot{\partial}_k b, \quad b_{hk}\dot{x}^h = b_k, \\ &b_{hk}\dot{x}^h \dot{x}^k = b, \quad \dot{\partial}_j = c_j, \quad c_{jk} = \dot{\partial}_j \dot{\partial}_k c, \\ &c_{hk}\dot{x}^h = c_k, \quad \text{and} \quad c_{hk}\dot{x}^h \dot{x}^k = c. \end{aligned} \tag{3.5}$$

Using (3.4), (3.5) and the commutation formula given by (3.3), the Lie-derivative of the projective entity $Q^i_{hjk}(x, \dot{x}0$ can be written in the following form

$$\begin{aligned} \mathcal{L}_\nu Q^i_{hjk} = {} & \delta^i_j b_{h((k))} + \delta^i_h b_{j((k))} - g_{jh} g^{il} c_{l((k))} - g_{jh((k))} g^{il} c_l \\ & - g_{jh} g^{il}_{((k))} c_l - \delta^i_k b_{h((j))} - \delta^i_h b_{k((j))} + g_{kh} g^{il} c_{l((j))} \\ & + g_{kh((j))} g^{il} c_l + g_{kh} g^{il}_{((j))} c_l - \delta^r_k b \Pi^i_{rjh} + g_{kl} g^{rm} c_m \Pi^i_{rjh} \dot{x}^l \\ & + b \delta^r_j \Pi^i_{rkh} - g_{jl} g^{rm} c_m \dot{x}^i \Pi^i_{rhk}. \end{aligned} \tag{3.6}$$

Now, transvect $\dot{x}^h \dot{x}^j$ in (3.6) and therefore use (1.12) and (1.13) together, we get

$$\begin{aligned} \mathcal{L}_\nu Q^i_k = {} & 2\dot{x}^i b_{((k))} - \delta^i_k b_{((j))} \dot{x}^j + \dot{x}^h \dot{x}^j \Big[g_{kh} \left(g^{il}_{((j))} c_l + g^{il} c_{l((j))} \right) \\ & - g_{jh} \left(g^{il}_{((k))} c_l + g^{il} c_{l((k))} \right) - g_{jh((k))} g^{il} c_l \Big]. \end{aligned} \tag{3.7}$$

We allow a contraction in (3.6) with respect to the indices i and k and thereafter transvecting the equation thus obtained by $\dot{x}^h \dot{x}^j$, we get

$$\begin{aligned} \mathcal{L}_\nu Q_{hj} \dot{x}^h \dot{x}^j = {} & (1-n) b_{((j))} \dot{x}^j + c_{((j))} \dot{x}^j + g^{il} c_l \dot{x}^h \dot{x}^j \left(g_{ih((j))} - g_{jh((i))} \right) \\ & - g_{jh} \dot{x}^h \dot{x}^j \left(g^{il} c_{l((i))} + g^{il}_{((i))} c_l \right) + g_{ih} g^{il}_{((j))} c_l \dot{x}^h \dot{x}^j. \end{aligned} \tag{3.8}$$

where we have taken into account (1.12).

We now eliminate $b_{((j))}\dot{x}^j$ using (3.7), (3.8) and get

$$\begin{aligned} L^i_k(x,\dot{x}) &= 2(1-n)b_{((k))}\dot{x}^i - b_{k((j))}\dot{x}^i\dot{x}^j \\ &+g_{kh}\dot{x}^h\dot{x}^j\left(g^{il}_{((j))}c_l + g^{il}c_{l((j))}\right) \\ &-g^{il}_{((k))}c_l - g^{il}c_{l((k))} + c_{((j))}\dot{x}^j\delta^i_k + g^{il}c_l\dot{x}^h\dot{x}^j\delta^i_k\left(g_{ih((j))} - g_{jh((i))}\right) \\ &-g_{jh}\dot{x}^h\dot{x}^j\delta^i_k\left(g^{il}c_{l((i))} - g^{il}_{((i))}c_l\right) + g_{ih}g^{il}_{((j))}c_l\dot{x}^h\dot{x}^j, \end{aligned} \tag{3.9}$$

where,

$$L^i_k \stackrel{\text{def}}{=} \mathcal{L}_\nu Q^i_k + \delta^i_k \mathcal{L}_\nu Q_{hj}\dot{x}^h\dot{x}^j. \tag{3.10}$$

We apply the commutation formula (3.36) to the projective deviation tensor $W^i_j(x,\dot{x})$ and thereafter use (3.4) and (3.5) to get

$$\begin{aligned} \left(\mathcal{L}_\nu W^i_j\right)_{((r))} - \mathcal{L}_\nu W^i_{j((r))} &= W^l_j\delta^i_r b_l - W^l_j g_{rl}g^{ip}c_p - W^i_r b_j + W^i_l g_{rj}g^{lp}c_p \\ &- \left(\dot{\partial}_r W^i_j\right)b - 2W^i_j b_r - \left(\dot{\partial}_r W^i_j\right)g_{lm}g^{lp}c_p\dot{x}^m. \end{aligned} \tag{3.11}$$

Allow a contraction in (3.11) with respect to the indices i and r, we get

$$\left(\mathcal{L}_\nu W^i_j\right)_{((i))} - \mathcal{L}_\nu W^i_{j((i))} = (n-2)W^l_j b_l - W^l_j c_l + g^{lp}c_p\left(W^i_l g_{ij} - \left(\dot{\partial} W^i_j\right)g_{im}\dot{x}^m\right). \tag{3.12}$$

Now, transvect (3.11) by $\dot{x}^r$ and thereafter use (3.5), we get

$$\begin{aligned} \left(\left(\mathcal{L}_\nu W^i_j\right)_{((i))} - \mathcal{L}_\nu W^i_{j((r))}\right)\dot{x}^r &= W^l_j b_l\dot{x}^i - 4W^i_j b - W^l_j g_{rl}g^{ip}c_p\dot{x}^r \\ &+W^i_l g_{rj}g^{lp}c_p\dot{x}^r - \left(\dot{\partial}_l W^i_j\right)g_{rm}g^{lp}c_p\dot{x}^r\dot{x}^m. \end{aligned} \tag{3.13}$$

We make the supposition that the infinitesimal special projective transformation given by (3.4) leaves invariant the projective covariant derivative of the projective deviation tensor, i.e.,

$$\mathcal{L}_\nu W^i_{j((r))} = 0. \tag{3.14}$$

As a result of this supposition, the equations (3.12) and (3.13) can respectively be expressed in the following alternative form

$$\left(\mathcal{L}_\nu W^i_j\right)_{((i))} = (n-2)W^l_j b_l - W^l_j c_l + g^{lp}c_p\left(g_{ij}W^i_l - \left(\dot{\partial}_l W^i_j\right)g_{im}\dot{x}^m\right) \tag{3.15}$$

and

$$\left(\mathcal{L}_\nu W^i_j\right)_{((r))} = W^l_j b_l\dot{x}^i - 4W^i_j b - W^l_j g_{rl}g^{ip}c_p\dot{x}^r + W^i_l g_{rj}g^{ip}c_p\dot{x}^r - \left(\dot{\partial} W^i_j\right)g_{rm}g^{lp}c_p\dot{x}^r\dot{x}^m. \tag{3.16}$$

We now propose to eliminate $W^l_j b_l$ with the help of (3.15) and (3.16), the result of elimination gives the following

$$\begin{aligned} B^i_j(x,\dot{x}) &= \dot{x}^j \left\{-W^l_j c_l + g^{lp} c_p \left[W^k_l g_{kj} - \left(\dot{\partial}_l W^r_j\right) g_{rm}\dot{x}^m\right]\right\} + (n-2) \\ &\times \left[4W^i_j b + W^l_j g_{rl} g^{ip} c_p \dot{x}^r - W^i_l g_{rj} g^{lp} c_p \dot{x}^r + \left(\dot{\partial} W^i_j\right) g_{rm} g^{lp} c_p \dot{x}^r \dot{x}^m\right], \end{aligned} \tag{3.17}$$

where,

$$B^i_j(x,\dot{x}) \overset{\text{def}}{=} \left(\mathcal{L}_\nu W^i_j\right)_{((r))} \dot{x}^r - (n-2)\left(\mathcal{L}_\nu W^i_j\right)_{((r))} \dot{x}^r. \tag{3.18}$$

In order that the space under consideration may admit a special projective affine motion, we always have

$$\mathcal{L}_\nu \Pi^i_{jh} = 0. \tag{3.19}$$

Using (3.4) and (3.19), we easily arrive at the conclusion that the vectors $b(x,\dot{x})$ and $c(x,\dot{x})$ must separately vanish.

In the light of all these observations, we can therefore state results following.

Theorem 3.1 *In a Finsler space F_n, the equation (3.17) always holds provided the space under consideration admits a non-affine infinitesimal special projective transformation such that the projective covariant derivative of projective deviation tensor W^i_j remains an invariant.*

Theorem 3.2 *In a Finsler space F_n, $B^i_j(x,\dot{x})$ given by (3.18) always vanishes provided the space under consideration admits an affine infinitesimal special projective transformation such that the projective covariant derivative of the projective deviation tensor W^i_j remains an invariant.*

If the Finsler space F_n under consideration be assumed to be symmetric one i.e., $W^i_{j((r))} = 0$, then under such an assumption the equation (3.14) will always hold. Therefore, we can state the result following.

Theorem 3.3 *In a symmetric Finsler space F_n, the equation (3.17) always holds provided the space under consideration admits a non-affine infinitesimal special projective transformation characterized by (3.4).*

Theorem 3.4 *In a symmetric Finsler space F_n, B^i_j characterized by (3.18) always vanishes provided the space under consideration admits an affine infinitesimal special projective transformation.*

§4. **Conclusion**

The present communication has been divided into three sections of which the first section is introductory, the second section deals with non-affine infinitesimal transformations, and in this section, we have derived conditions which will hold when the space under consideration admits non-affine as well as an affine infinitesimal transformation and in the sequel have established the conditions which will hold when the space is symmetric and it admits an affine as well as non-affine infinitesimal transformation. The third section deals with infinitesimal special projective transformation. Like the previous section, in this section we have established the conditions which will hold when the space under consideration is symmetric and it admits a non-affine as well as an affine infinitesimal special projective transformation too.

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